NUMERICAL APPROXIMATION OF THE
PHASE-FIELD TRANSITION SYSTEM WITH
NON-HOMOGENEOUS CAUCHY-NEUMANN
BOUNDARY CONDITIONS IN BOTH UNKNOWN
FUNCTIONS VIA FRACTIONAL STEPS METHOD*

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Abstract The paper concerns with the proof of the convergence for an iterative scheme of fractional steps type associated to the phase-field transition system endowed with non-homogeneous Cauchy-Neumann boundary conditions, in both unknown functions. The advantage of such method consists in simplifying the numerical computation necessary to be done in order to approximate the solution of nonlinear parabolic system. On the basis of this approach, a numerical algorithm in 2D case is introduced and an industrial implementation is made.

Keywords Boundary value problems for nonlinear parabolic PDE, stability and convergence of numerical method, finite element method, thermodynamics and heat transfer.

MSC(2000) 35K55, 65N12, 65N30, 80AXX.

1. Introduction

Consider the following nonlinear parabolic system in $Q = (0, T] \times \Omega$, where $T > 0$ and $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \leq 3$) whose boundary is smooth enough:

$$\begin{cases}
C_p \frac{\partial}{\partial t} u + \ell \frac{\partial}{\partial t} \varphi = k \Delta u + f(t, x) & \text{on } Q = (0, T] \times \Omega \\
\alpha \xi \frac{\partial}{\partial \nu} \varphi = \xi \Delta \varphi + \frac{1}{2\xi} (\varphi - \varphi^3) + s \xi u + g(t, x) & \text{on } Q,
\end{cases} \tag{1.1}$$

subject to the non-homogeneous Cauchy-Neumann boundary conditions, in both unknown functions $u$ and $\varphi$:

$$\begin{cases}
k \frac{\partial}{\partial \nu} u + h u = w_1(t, x) & \text{on } \Sigma = (0, T] \times \partial \Omega \\
\xi \frac{\partial}{\partial \nu} \varphi + c_0 \varphi = w_2(t, x) & \text{on } \Sigma,
\end{cases} \tag{1.2}$$

and the initial conditions:

$$u(0, x) = u_0(x), \quad \varphi(0, x) = \varphi_0(x) \quad \text{on } \Omega, \tag{1.3}$$

where

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*The work is supported by the project “Bourses Eugen Ionescu”.
• $u(t, x)$ represents the reduced temperature distribution on $Q$, i.e. $u(t, x) = \theta(t, x) - \theta_M$, with $\theta(t, x)$ representing the temperature of the material at $(t, x) \in Q$ and $\theta_M$ the melting temperature (the temperature at which solid and liquid may coexist in equilibrium, separated by a planar interface);

• $\varphi(t, x)$ is the phase function (the order parameter) used to distinguish between the states (phases) of material which occupies the region $\Omega$ at every moment of time $t \in [0, T]$;

• $C_p = \rho c$; $\rho$ - the density, $c$ - the specific heat capacity;

• $\ell, \kappa, \alpha, \xi, h, c_0$ are physical parameters representing: the latent heat, the thermal conductivity, the relaxation time, the measure of the interface thickness, the heat transfer coefficient, a positive constant, respectively;

• $s_\xi = \frac{m[S]_E}{2\sigma} T_E$ is a bounded and positive quantity, expressed by positive and bounded physical values: $m = \int_{-1}^{1} (2F(s))^{\frac{1}{2}} ds$, $F(s) = \frac{1}{4} (s^2 - 1)^2$, $[S]_E$ - the entropy difference between phases per volume, $\sigma$ - the interfacial tension, $T_E$ - the equilibrium melting temperature (see [11]);

• $p, q$ are given numbers assumed to satisfy

$$q \geq p \geq 2;$$

(1.4)

• $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ are given functions (also, can be interpreted as distributed control);

• $w_1(t, x), w_2(t, x) \in W^{1, \frac{2}{p}}(\Sigma)$ - are given functions depending on two variables which also can be interpreted as boundary control;

• $u_0(x) \in W^{2, \frac{2}{p}}(\Omega), \varphi_0(x) \in W^{2, \frac{2}{p}}(\Omega)$, provided $k \frac{\partial}{\partial n} u_0(x) + hu_0(x) = w_1(0, x), \xi \frac{\partial}{\partial n} \varphi_0(x) + c_0 \varphi_0(x) = w_2(0, x)$ on $\partial \Omega$.

In the formulation of problem (1.1) we have started from the phase field equations describing the phenomenon of solidification (see [11]) to which we have added some new physics parameters, as well as appropriate boundary conditions, in order to cover a wide variety of industrial applications (see [18–20]).

The non-homogeneous Cauchy-Neumann boundary conditions in both unknown functions $u$ and $\varphi$ (see relation (1.2)), untreated until now in mathematical literature, include a broad class of complex phenomena at $\partial \Omega$ and will thus allow the formulation of new boundary optimal control problems.

At the moment $t$ the material is considered to be liquid if the phase function $\varphi$ is close to $+1 + \delta_1$ and $u(t, x) \geq 1 + \delta_2$, while it is considered to be solid if the phase function $\varphi$ is close to $-1 - \delta_1$ and $u(t, x) \leq -1 - \delta_2$, with $\delta_1, \delta_2$ prescribed positive numbers.

We define the separating region (the interface at the moment $t$) as being the set:

$$\Omega_t = \{x \in \Omega; |u(t, x)| \leq 1 + \delta_2, |\varphi(t, x)| \leq 1 + \delta_1\}.$$

Regarding the existence and regularity of solutions in (1.1)-(1.3), we have
Theorem 1.1. Problem (1.1)-(1.3) has a unique solution \((u, \varphi)\) with \(u \in W^{1,2}_p(Q)\) and \(\varphi \in W^{1,2}_p(Q)\), where \(\nu = \min\{p, \mu\}\). In addition \((u, \varphi)\) satisfies

\[
\|u\|_{W^{1,2}_p(Q)} + \|\varphi\|_{W^{1,2}_p(Q)} \leq C \left\{ 1 + \|u_0\|_{W^{2-\frac{2}{p}}_p(\Omega)} + \|\varphi_0\|_{W^{2-\frac{2}{q}}_\infty(\Omega)} + \|w_1\|_{W^{1-\frac{2}{p}, \frac{2}{p} - \frac{2}{q}}(\Sigma)} \right\} \quad (1.5)
\]

where the constant \(C\) depends on \(\|\Omega\|\) (the measure of \(\Omega\)), \(T\), \(n\), \(p\), \(q\) and physical parameters.

Moreover, given any number \(M > 0\), if \((u_1, \varphi_1)\) and \((u_2, \varphi_2)\) are solutions of (1.1)-(1.3) for the same initial conditions, corresponding to the dates

\[
(f_1, g_1, w_1^1, w_2^1), (f_2, g_2, w_1^2, w_2^2) \in L^p(Q) \times L^q(Q) \times \left(W^{1-\frac{2}{p}, \frac{2}{p} - \frac{2}{q}}(\Sigma)\right)^2,
\]

such that \(\|\varphi_1\|_{L^p(Q)}, \|\varphi_2\|_{L^q(Q)} \leq M\), then the estimate below holds

\[
\|u_1 - u_2\|_{W^{1,2}_p(Q)} + \|\varphi_1 - \varphi_2\|_{W^{1,2}_p(Q)} \leq C \left\{ \|f_1 - f_2\|_{L^p(Q)} + \|g_1 - g_2\|_{L^q(Q)} + \|w_1^1 - w_2^1\|_{W^{1-\frac{2}{p}, \frac{2}{p} - \frac{2}{q}}(\Sigma)} \right\} \quad (1.6)
\]

where the constant \(C\) depends on \(\|\Omega\|\), \(T\), \(M\), \(n\), \(p\), \(q\) and physical parameters.

Remark 1.1. The result established by Theorem 1.1 is also valid for the linear system (1.7)-(1.9).

The sketch proof of Theorem 1.1 can be found in [9].

The phase-field transition system (1.1)-(1.2) with constant physical parameters, subject to the non-homogeneous Cauchy-Neumann boundary conditions for unknown \(u\), namely: \(\frac{\partial}{\partial u} u + hu = w(t, x)\) on \(\Sigma\), has been analyzed in [8] and [9]. Endowed with dynamic boundary conditions and singular potentials, the system (1.1)-(1.2) was treated in [12].

Numerical investigations of the phase-field subject to different boundary conditions, can be found in [2, 13, 15, 21], where finite difference scheme is used, and [8, 16, 18-20] which authors use finite element method (fem).

For other detailed discussions on the phase-field transition system we refer to [2, 8-13, 15-24] and references there in.

In order to approximate the nonlinear problem let’s associate to (1.1)-(1.3) for every \(\varepsilon > 0\) the following approximating scheme (see also [8, 15, 16, 22]):

\[
\begin{cases}
C_p \frac{\partial u^\varepsilon}{\partial t} + \frac{\ell}{2} \frac{\partial u^\varepsilon}{\partial t} = k \Delta u^\varepsilon + f(t, x) & \text{in } Q^\varepsilon_i = (i\varepsilon, (i + 1)\varepsilon) \times \Omega \\
\alpha \Delta \varphi^\varepsilon + \Delta \varphi^\varepsilon = \Delta \varphi^\varepsilon + s \varphi^\varepsilon + w_2(t, x) & \text{in } Q^\varepsilon_i.
\end{cases}
\]

\[
\begin{cases}
k \frac{\partial}{\partial \nu} w^\varepsilon + hu^\varepsilon = w_1(t, x) & \text{on } \Sigma^\varepsilon_i = (i\varepsilon, (i + 1)\varepsilon) \times \partial \Omega \\
\xi \frac{\partial}{\partial \nu} \varphi^\varepsilon + c_0 \varphi^\varepsilon = w_2(t, x) & \text{on } \Sigma^\varepsilon_i.
\end{cases}
\]

\[
\begin{cases}
C_p \frac{\partial u^\varepsilon}{\partial t} + \frac{\ell}{2} \frac{\partial u^\varepsilon}{\partial t} = k \Delta u^\varepsilon + f(t, x) & \text{in } Q^\varepsilon_i = (i\varepsilon, (i + 1)\varepsilon) \times \Omega \\
\alpha \Delta \varphi^\varepsilon + \Delta \varphi^\varepsilon = \Delta \varphi^\varepsilon + s \varphi^\varepsilon + w_2(t, x) & \text{in } Q^\varepsilon_i.
\end{cases}
\]
\begin{equation}
\begin{cases}
  u_+^\varepsilon(i\varepsilon, x) = u_+^\varepsilon(i\varepsilon, x) & \text{on } \Omega \\
  \varphi_+^\varepsilon(i\varepsilon, x) = z((i+1)\varepsilon, \varphi_-^\varepsilon(i\varepsilon, x)) & \text{on } \Omega,
\end{cases}
\end{equation}

below where \( z(\cdot, \varphi_-^\varepsilon(i\varepsilon, x)) \) is the solution of Cauchy problem:

\begin{equation}
\begin{cases}
  z' + \frac{1}{2\varepsilon}z^3 = 0, & s \in (i\varepsilon, (i+1)\varepsilon), \\
  z(i\varepsilon) = \varphi_-^\varepsilon(i\varepsilon, x), & \varphi_-^\varepsilon(0, x) = \varphi_0(x),
\end{cases}
\end{equation}

for \( i = 0, 1, \cdots, M_\varepsilon - 1 \), with \( M_\varepsilon = \left[ \frac{T}{\varepsilon} \right], Q_{M_\varepsilon - 1}^\varepsilon = [(M_\varepsilon - 1)\varepsilon, T] \times \Omega, \varphi_+^\varepsilon(i\varepsilon, x) = \lim_{t \uparrow t_\varepsilon} \varphi^\varepsilon(t, x) \) and \( \varphi_-^\varepsilon(i\varepsilon, x) = \lim_{t \downarrow t_\varepsilon} \varphi^\varepsilon(t, x) \).

**Definition 1.1.** By weak solution of the nonlinear system (1.1)-(1.3) we mean a pair of functions \( u, \varphi \in L^2([0, T]; H^1(\Omega)) \cap L^\infty(Q) \) which satisfies (1.1)-(1.3) in the following sense:

\begin{equation}
\int_Q \left( \frac{C_p}{\varepsilon} \frac{\partial}{\partial t} u + \frac{\ell}{2} \frac{\partial}{\partial t} \varphi \right) \phi dx dt + k \int \nabla u \nabla \phi dx dt + h \int \psi d\gamma dt
\end{equation}

\begin{equation}
= \int_\Sigma w_1 \phi d\gamma dt + \int f \phi dx dt,
\end{equation}

\begin{equation}
\alpha \xi \int_Q \left( \frac{\partial}{\partial t} \varphi \right) \psi dx dt + \xi \int \nabla \varphi \nabla \psi dx dt + c_0 \int \varphi \psi d\gamma dt
\end{equation}

\begin{equation}
= \int_\Sigma w_2 \psi d\gamma dt + \frac{1}{2\varepsilon} \int (\varphi - \varphi^3) \psi dx dt + s_\xi \int u \psi dx dt + \int g \psi dx dt,
\end{equation}

\( \forall \phi, \psi \in L^2([0, T]; H^1(\Omega)) \) and \( u(0, x) = u_0(x), \varphi(0, x) = \varphi_0(x) \) in \( \Omega \).

**Definition 1.2.** By weak solution of the linear system (1.7)-(1.9) we mean a pair of functions \( u, \varphi \in L^2([0, T]; H^1(\Omega)) \cap L^\infty(Q) \) which satisfies (1.7)-(1.9) in the following sense:

\begin{equation}
\int_Q \left( \frac{C_p}{\varepsilon} \frac{\partial}{\partial t} u + \frac{\ell}{2} \frac{\partial}{\partial t} \varphi \right) \phi dx dt + k \int \nabla u \nabla \phi dx dt + h \int u^\varphi \phi d\gamma dt
\end{equation}

\begin{equation}
= \int_\Sigma w_1 \phi d\gamma dt + \int f \phi dx dt,
\end{equation}

\begin{equation}
\alpha \xi \int_Q \left( \frac{\partial}{\partial t} \varphi \right) \psi dx dt + \xi \int \nabla \varphi \nabla \psi dx dt + c_0 \int \varphi \psi d\gamma dt
\end{equation}

\begin{equation}
= \int_\Sigma w_2 \psi d\gamma dt + \frac{1}{2\varepsilon} \int \varphi \psi dx dt + s_\xi \int u^\varphi \psi dx dt + \int g \psi dx dt,
\end{equation}

\( \forall \phi, \psi \in L^2([0, T]; H^1(\Omega)) \) and \( u_\varepsilon(0, x) = u_0(x), \varphi_\varepsilon(0, x) = \varphi_0(x) \) in \( \Omega \).

The symbol \( \int_Q \) above denotes the duality between \( L^2([0, T]; H^1(\Omega)) \) and \( L^2([0, T]; H^1(\Omega)') \).
**Remark 1.2.** We choose \( u, \varphi \in L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q) \) in Definitions 1.1 and 1.2 making use of the continuous embedding \( W^{1,2}_p(Q) \subset L^\infty(Q) \) when \( n = 2, \ p > 2 \) (see Theorem 1.1) which is relevant for industrial applications. Such a choice can be made also in the case \( n = 3, \ p > \frac{5}{2} \) (see also [9]).

Throughout this paper when is not clearly precised, we will denote by \( C \) a constant which may change from line to line.

This paper is divided as follows: we start by giving the convergence of the linear approximating scheme (1.7)-(1.9) associated to the nonlinear transition system (1.1)-(1.3) and finish by a numerical algorithm in the 2D case and industrial implementation.

### 2. Convergence and weak stability of the approximating scheme

In this section, we will prove the convergence of the iterative scheme (1.7)-(1.10) of fractional steps type for the phase-field transition system (1.1)-(1.3). We have

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^n \) \((n \leq 3)\) be a bounded domain with a smooth boundary. Assume that \( u_0(x) \in W^{2-\frac{2}{p}}_p(\Omega), \ \varphi_0(x) \in W^{2-\frac{2}{q}}_q(\Omega), \) satisfying \( k \frac{\partial^2}{\partial x^2} u_0(x) + hu_0(x) = w_1(0,x), \ \xi \frac{\partial}{\partial x} \varphi_0(x) + c_0 \varphi_0(x) = w_2(0,x) \) on \( \partial \Omega \) and \( w_1(t,x), w_2(t,x) \in W^{1-\frac{2}{p},\frac{2}{q}-\frac{2}{p}}_p(\Omega) \). Let \((u^\varepsilon, \varphi^\varepsilon)\) be the solution of the approximating scheme (1.7)-(1.9). Then for \( \varepsilon \to 0, \) one has

\[
(u^\varepsilon, \varphi^\varepsilon) \to (u^*, \varphi^*) \text{ strongly in } L^2(\Omega) \text{ for any } t \in (0,T],
\]

where \( u^*, \varphi^* \in L^2([0,T]; H^1(\Omega)) \) is the weak solution to the nonlinear phase transition system (1.1)-(1.3).

The following lemmas, which targets the Cauchy problem (1.10) and which are very useful in the proof of the main result of this Section (Theorem 2.1) were established for the first time in the work [16]. For reader convenience we fully reproduce their proofs.

**Lemma 2.1.** If \( \varphi^-((i\varepsilon, x) \in L^\infty(\Omega), \ i = 0, 1, ..., M_\varepsilon - 1, \) then \( z((i+1)\varepsilon, x) \in L^\infty(\Omega). \)

**Proof.** From the Cauchy problem (1.10), using the method of separation of variables and integrating on \((i\varepsilon, (i+1)\varepsilon)\), we get

\[
z^2((i+1)\varepsilon, x) = \frac{\xi z^2(i\varepsilon, x)}{\varepsilon z^2(i\varepsilon, x) + \xi},
\]
i.e, by (1.10)_2

\[
z^2((i+1)\varepsilon, x) = \frac{\xi \varphi^-((i\varepsilon, x)^2}{\varepsilon \varphi^-((i\varepsilon, x)^2 + \xi}.
\]

This gives us

\[
z^2((i+1)\varepsilon, x) \leq \varphi^-((i\varepsilon, x)^2, \ a.e \ x \in \Omega.
\]

Since \( \varphi^-((i\varepsilon, x) \in L^\infty(\Omega), \) we conclude that \( z((i+1)\varepsilon, x) \in L^\infty(\Omega) \) for all \( i \in \{0, 1, ..., M_\varepsilon - 1\}. \) \( \Box \)
Lemma 2.2. For $i = 0, 1, \ldots, M_{\varepsilon} - 1$, the estimate below holds

$$
\| \varphi^+_{\varepsilon} (i\varepsilon, x) \|_{L^2(\Omega)}^2 \leq \| \varphi^-_{\varepsilon} (i\varepsilon, x) \|_{L^2(\Omega)}^2.
$$

(2.4)

Proof. The proof follows directly to Lemma 2.1. In fact, using (2.3) and relation (1.9), we deduce

$$
\| \varphi^+_{\varepsilon} (i\varepsilon, x) \|_{L^2(\Omega)}^2 \leq \| \varphi^-_{\varepsilon} (i\varepsilon, x) \|_{L^2(\Omega)}^2
$$

as claimed.

Lemma 2.3. For $i = 0, 1, \ldots, M_{\varepsilon} - 1$, the estimate below holds

$$
\| \nabla \varphi^+_{\varepsilon} (i\varepsilon, x) \|_{L^2(\Omega)} \leq \| \nabla \varphi^-_{\varepsilon} (i\varepsilon, x) \|_{L^2(\Omega)}.
$$

(2.5)

Proof. Let us set $\theta(t, x) = \nabla z(t, x)$. Thus (1.10) becomes

$$
\begin{align*}
\theta'(s, x) + \frac{3}{2\xi} \varphi_k^2(s, x) = 0, & s \in (i\varepsilon, (i+1)\varepsilon), \\
\theta(i\varepsilon, x) = \nabla \varphi^-_{\varepsilon} (i\varepsilon, x).
\end{align*}
$$

(2.6)

The solution of (2.6) is then

$$
\theta((i+1)\varepsilon, x) = e^{(i+1)\varepsilon} \int_{i\varepsilon}^{(i+1)\varepsilon} \frac{1}{2\xi} \varphi_k^2(t, x) dt \theta(i\varepsilon, x).
$$

(2.7)

After a small majoration, the proof is well done.

Lemma 2.4. The following estimate holds

$$
\| z(\varepsilon, x) - \varphi^-_{\varepsilon} (i\varepsilon, x) \|_{L^2(\Omega)} \leq \varepsilon L,
$$

(2.8)

where $L > 0$ is a constant depending on $\Omega$, $\| \varphi^-_{\varepsilon} \|_{L^\infty(\Omega)}$ and on the parameter $\xi$.

Proof. From (1.10), using the inequality $(a^3 - b^3)(a - b) \geq 0 \ \forall a, b \in \mathbb{R}$, we have

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} |z(t, x) - z(i\varepsilon, x)|^2 & \leq - \frac{1}{2\xi} \varphi_k^3(t, x) (z(t, x) - z(i\varepsilon, x)) \\
& \leq - \frac{1}{2\xi} \varphi_k^3(i\varepsilon, x) (z(t, x) - z(i\varepsilon, x)).
\end{align*}
$$

(2.9)

Integrating (2.9) on $(i\varepsilon, (i+1)\varepsilon)$, we get

$$
|z((i+1)\varepsilon, x) - z(i\varepsilon, x)| \leq \frac{\varepsilon}{2\xi} |\varphi_k^3(i\varepsilon, x)| = \frac{\varepsilon}{2\xi} |\varphi^-_{\varepsilon} (i\varepsilon, x)|^3.
$$

(2.10)

Hence

$$
\| z(\varepsilon, x) - \varphi^-_{\varepsilon} (i\varepsilon, x) \|_{L^2(\Omega)} \leq \varepsilon L,
$$

where $L > 0$ is a constant depending on $\Omega$, $\| \varphi^-_{\varepsilon} \|_{L^\infty(\Omega)}$ and on the parameter $\xi$.

Proof of Theorem 2.1. Consider $i = 0$. In this case, from lemma 2.1 we derive that the solution of the Cauchy problem (1.10) $z((i+1)\varepsilon, x)$ belongs to $L^\infty(\Omega)$. Since $W^{2, \frac{2}{3}}(\Omega) \subset W^1(\Omega)$ then $z((i+1)\varepsilon, x) \in W^1(\Omega)$. Using Remark 1.1 to the problem (1.7)-(1.9) we ensure the existence of a solution $u^\varepsilon$, $\varphi^\varepsilon \in W^{1,2}(Q_0) \times W^{1,2}(Q_0) \cap L^\infty(Q_0)$. Thus, by induction $\varphi^-_{\varepsilon} (i\varepsilon, x) \in L^\infty(\Omega)$,
Using Hölder’s inequality
\begin{equation}
\frac{2}{\ell} s_{\xi} \frac{d}{dt} \int |u^{\varepsilon}|^2 dx + s_{\xi} \int \phi^{\varepsilon} u^{\varepsilon} dx + \frac{2k}{\ell} s_{\xi} \int |\nabla u^{\varepsilon}|^2 dx + \frac{2h}{\ell} s_{\xi} \int |u^{\varepsilon}|^2 d\gamma \leq \frac{2h}{\ell} s_{\xi} \int |u^{\varepsilon}|^2 d\gamma + \frac{s_{\xi}}{2\ell h} \int |w_1|^2 d\gamma,
\end{equation}
\begin{equation}
\int g \phi^{\varepsilon} dx \leq \frac{\alpha_{\varepsilon} \xi}{2} \int |\phi^{\varepsilon}|^2 dx + \frac{1}{2\alpha_{\varepsilon} \xi} \int |g|^2 dx.
\end{equation}

Hence, adding (2.11)-(2.12) and making use (2.13)-(2.14), we obtain
\begin{equation}
\frac{2}{\ell} C_p \frac{d}{dt} \int |u^{\varepsilon}|^2 dx + \frac{s_{\xi}}{2\ell h} \int |w_1|^2 d\gamma + \int \frac{\alpha_{\varepsilon} \xi}{2} |\phi^{\varepsilon}|^2 dx + \frac{1}{2\alpha_{\varepsilon} \xi} \int |g|^2 dx.
\end{equation}

Multiplying (1.7) by \( \frac{1}{\alpha_{\varepsilon} \xi} \phi^{\varepsilon} \), integrating over \( \Omega \) and using Green’s formula we get
\begin{equation}
\frac{1}{2\xi} \frac{d}{dt} \int |\phi^{\varepsilon}|^2 dx + \frac{1}{\alpha_{\varepsilon} \xi} \int |\nabla \phi^{\varepsilon}|^2 dx + \frac{co}{\alpha_{\varepsilon} \xi^2} \int |\phi^{\varepsilon}|^2 d\gamma
\end{equation}
\begin{equation}
= \frac{1}{\alpha_{\varepsilon} \xi^2} \int w_2 \phi^{\varepsilon} d\gamma + \frac{1}{2\alpha_{\varepsilon} \xi^3} \int |\phi^{\varepsilon}|^2 dx + \frac{s_{\xi}}{\alpha_{\varepsilon} \xi^2} \int u^{\varepsilon} \phi^{\varepsilon} dx + \frac{1}{\alpha_{\varepsilon} \xi^2} \int g \phi^{\varepsilon} dx.
\end{equation}

Using again Hölder’s inequality
\begin{equation}
\int w_2 \phi^{\varepsilon} d\gamma \leq \frac{co}{\alpha_{\varepsilon} \xi} \int |\phi^{\varepsilon}|^2 d\gamma + \frac{1}{4co\alpha} \int |w_2|^2 d\gamma,
\end{equation}
from (2.16) and (2.17) we obtain,

$$\frac{1}{2\xi} \frac{d}{dt} \int_{\Omega} |\varphi|^2 dx + \frac{1}{\alpha \xi} \int_{\Omega} |\nabla \varphi|^2 dx$$

$$\leq \frac{1}{4\alpha \xi^2} \int_{\partial \Omega} |\varphi|^2 d\gamma + \frac{1}{2\alpha \xi^3} \int_{\Omega} |\varphi|^2 dx + \frac{s \xi}{\alpha \xi^2} \int_{\Omega} u^2 \varphi dx + \frac{1}{\alpha \xi^2} \int_{\Omega} g \varphi dx. \tag{2.18}$$

Adding (2.15) and (2.18) and performing with Cauchy’s inequality, we obtain

$$\frac{\partial}{\partial t} \left[ \frac{2C^p}{\ell} s \xi \int_{\Omega} |u|^2 dx + \frac{\xi}{2} \int_{\Omega} |\nabla \varphi|^2 dx + \frac{1}{4\xi} \int_{\Omega} |\varphi|^2 dx + \frac{c_0}{2} \int_{\partial \Omega} |\varphi|^2 d\gamma \right]$$

$$+ \frac{2k}{\ell} s \xi \int_{\Omega} |\nabla u|^2 dx + \frac{\alpha \xi}{2} \int_{\Omega} |\varphi|^2 dx + \frac{1}{\alpha \xi} \int_{\Omega} |\nabla \varphi|^2 dx$$

$$\leq \int_{\partial \Omega} w_2 \varphi_t \, d\gamma + C \left\{ \int_{\Omega} |\varphi|^2 dx + \int_{\Omega} |u|^2 dx + \int_{\partial \Omega} |w_1|^2 d\gamma \right. \left. + \int_{\partial \Omega} |f|^2 dx + \int_{\partial \Omega} |g|^2 dx \right\}. \tag{2.19}$$

Integrating over $(0, \varepsilon)$ gives

$$\frac{2C^p}{\ell} s \xi \|u(\varepsilon)\|^2_{L^2(\Omega)} + \frac{1}{4\xi} \|\varphi(\varepsilon)\|^2_{L^2(\Omega)} + \frac{\xi}{2} \|\nabla \varphi(\varepsilon)\|^2_{L^2(\Omega)} + \frac{c_0}{2} \|\varphi(\varepsilon)\|^2_{L^2(\partial \Omega)}$$

$$+ \frac{2k}{\ell} s \xi \int_0^\varepsilon \|\nabla u\|^2_{L^2(\Omega)} ds + \frac{\alpha \xi}{2} \int_0^\varepsilon \|\varphi\|^2_{L^2(\Omega)} ds + \frac{1}{\alpha \xi} \int_0^\varepsilon \|\varphi\|^2_{L^2(\partial \Omega)} ds$$

$$\leq \frac{2C^p}{\ell} s \xi \|u_0\|^2_{L^2(\Omega)} + \frac{1}{4\xi} \|\varphi_0\|^2_{L^2(\Omega)} + \frac{\xi}{2} \|\nabla \varphi_0\|^2_{L^2(\Omega)} + \frac{c_0}{2} \|\varphi_0\|^2_{L^2(\partial \Omega)}$$

$$+ \int_{\Sigma_0} w_2 \varphi_t^2 \, d\gamma_1 + C \left\{ \int_0^\varepsilon \|\varphi(s)\|^2_{L^2(\Omega)} ds + \int_0^\varepsilon \|u(s)\|^2_{L^2(\Omega)} ds \right. \left. + \|w_1\|^2_{L^2(\Sigma_0)} + \|w_2\|^2_{L^2(\Sigma_0)} + \|f\|^2_{L^2(Q_0^2)} + \|g\|^2_{L^2(Q_0^2)} \right\}. \tag{2.20}$$

We now focus on the right term $\int_{\Sigma_0} w_2 \varphi_t^2 \, d\gamma_1$ in the previous inequality. We have

$$\int_{\Sigma_0} w_2 \varphi_t^2 \, d\gamma_1 = \int_{\Sigma_0} \frac{\partial}{\partial t} (w_2 \varphi^2) \, d\gamma_1 ds - \int_{\Sigma_0} w_2' \varphi^2 \, d\gamma_1 ds \tag{2.21}$$

and

$$\int_{\partial \Omega} w_2 \varphi^2 \, d\gamma_1 \leq \frac{1}{c_0} \int_{\partial \Omega} |w_2|^2 \, d\gamma + \frac{c_0}{4} \int_{\partial \Omega} |\varphi|^2 \, d\gamma,$$

$$\int_{\Sigma_0} w_2^2 \varphi^2 \, d\gamma_1 \leq \frac{1}{2} \int_{\Sigma_0} |\varphi|^2 \, d\gamma + \frac{1}{2} \int_{\Sigma_0} |w_2|^2 \, d\gamma ds. \tag{2.22}$$
Combining (2.20), (2.21) and (2.22), we obtain
\[
\frac{2C_p}{\xi} s_\xi \|u^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{4\xi} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} + \frac{\xi}{2} \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega)} + \frac{\xi^2}{2} \int_\Omega \nabla^2 \varphi^\varepsilon \varphi^\varepsilon - \varphi^\varepsilon T \nabla \varphi^\varepsilon \|_{L^2(\Omega)}
\]
\[
+ \frac{2k}{\xi} s_\xi \int_0^\varepsilon \|\nabla u^\varepsilon\|^2_{L^2(\Omega)} ds + \frac{1}{4\xi} \int_0^\varepsilon \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \frac{\xi}{2} \int_0^\varepsilon \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds \leq C \left\{ \int_0^\varepsilon \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \int_0^\varepsilon \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \int_0^\varepsilon \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds \right\}.
\]
(2.23)

Similarly for $Q_\varepsilon^i$, $i = 1, 2, ..., M_\varepsilon - 2$,
\[
\frac{2C_p}{\xi} s_\xi \|u^\varepsilon(\varphi + (i + 1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{1}{4\xi} \|\varphi^\varepsilon(\varphi + (i + 1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{\xi}{2} \|\nabla \varphi^\varepsilon(\varphi + (i + 1)\varepsilon)\|^2_{L^2(\Omega)}
\]
\[
+ \frac{co}{\xi} \|\varphi^\varepsilon(\varphi + (i + 1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{2k}{\xi} s_\xi \int_\varepsilon^{(i+1)\varepsilon} \|\nabla u^\varepsilon\|^2_{L^2(\Omega)} ds
\]
\[
+ \frac{1}{4\xi} \int_\varepsilon^{(i+1)\varepsilon} \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \frac{\xi}{2} \int_\varepsilon^{(i+1)\varepsilon} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds \leq C \left\{ \int_\varepsilon^{(i+1)\varepsilon} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \int_\varepsilon^{(i+1)\varepsilon} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \int_\varepsilon^{(i+1)\varepsilon} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds \right\}.
\]
(2.24)

Again to $Q_{M_\varepsilon - 1}^T$,
\[
\frac{2C_p}{\xi} s_\xi \|u^\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{1}{4\xi} \|\varphi^\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{\xi}{2} \|\nabla \varphi^\varepsilon(T)\|^2_{L^2(\Omega)}
\]
\[
+ \frac{co}{\xi} \|\varphi^\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{2k}{\xi} s_\xi \int_0^{\varepsilon\varepsilon} \|\nabla u^\varepsilon\|^2_{L^2(\Omega)} ds + \frac{1}{4\xi} \int_0^{\varepsilon\varepsilon} \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega)} ds
\]
\[
+ \frac{\xi}{2} \int_0^{\varepsilon\varepsilon} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds \leq C \left\{ \int_0^{\varepsilon\varepsilon} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \int_0^{\varepsilon\varepsilon} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \int_0^{\varepsilon\varepsilon} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds \right\}.
\]
\[
\begin{align*}
\leq \frac{2C_p}{\ell} s_{\xi}^2 u^\varepsilon((M_\varepsilon - 1)\varepsilon)\|u^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{4\xi}\|\varphi_+^\varepsilon((M_\varepsilon - 1)\varepsilon)\|_{L^2(\Omega)}^2 \\
+ \frac{\xi}{2}\|\varphi_+^\varepsilon((M_\varepsilon - 1)\varepsilon)\|_{L^2(\Omega)}^2 + \frac{c_0}{4}\|\varphi_+^\varepsilon((M_\varepsilon - 1)\varepsilon)\|_{L^2(\Omega)}^2 \\
+ C\left\{ \int_{M_\varepsilon - 1}^T \|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)} ds + \int_{M_\varepsilon - 1}^T \|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)} ds + \int_{M_\varepsilon - 1}^T \|u^\varepsilon(s)\|^2_{L^2(\Omega)} ds \\
+ \|w_1\|^2_{L^2(\Sigma_{M_\varepsilon - 1})} + \|w_2\|^2_{L^2(\Sigma_{M_\varepsilon - 1})} + \|w_3\|^2_{L^2(\Sigma_{M_\varepsilon - 1})} + \|f\|^2_{L^2(Q_{M_\varepsilon - 1}^T)} \\
+ \|g\|^2_{L^2(Q_{M_\varepsilon - 1}^T)} \right\}.
\end{align*}
\]  

(2.25)

Considering inequalities (2.4) and (2.5) given respectively by Lemmas 2.2 and 2.3, we remark

\[
\begin{align*}
E_i^1 &+ \frac{1}{4\xi}\|\varphi_+^\varepsilon((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{c_0}{4}\|\varphi_+^\varepsilon((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{\xi}{2}\|\nabla \varphi_+^\varepsilon((i+1)\varepsilon)\|^2_{L^2(\Omega)} \\
&\leq E_i^1 + \frac{1}{4\xi}\|\varphi_+^\varepsilon((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{c_0}{4}\|\varphi_+^\varepsilon((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{\xi}{2}\|\nabla \varphi_+^\varepsilon((i+1)\varepsilon)\|^2_{L^2(\Omega)} \\
&\leq E_i^2 + \|\varphi_+^\varepsilon(i\varepsilon)\|^2_{L^2(\Omega)} + \|\varphi_+^\varepsilon(i\varepsilon)\|^2_{L^2(\Omega)} + \|\nabla \varphi_+^\varepsilon(i\varepsilon)\|^2_{L^2(\Omega)},
\end{align*}
\]  

(2.26)

where

\[
E_i^1 = \frac{2C_p}{\ell} s_{\xi}^2 u^\varepsilon((i+1)\varepsilon)\|u^\varepsilon\|_{L^2(\Omega)}^2 + \frac{2k}{\ell} s_{\xi} \int_{i\varepsilon}^{(i+1)\varepsilon} \|\nabla u^\varepsilon\|^2_{L^2(\Omega)} ds
\]  

(2.27)

and

\[
E_i^2 = \frac{2C_p}{\ell} s_{\xi}^2 \|u^\varepsilon(i\varepsilon)\|^2_{L^2(\Omega)} + C\left\{ \int_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi_+^\varepsilon\|^2_{L^2(\Omega)} ds \\
+ \int_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \int_{i\varepsilon}^{(i+1)\varepsilon} \|u^\varepsilon\|^2_{L^2(\Omega)} ds + \|w_1\|^2_{L^2(\Omega)} \\
+ \|w_2\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q_{i\varepsilon}^T)} + \|g\|^2_{L^2(Q_{i\varepsilon}^T)} \right\}.
\]  

(2.28)

Adding (2.23), (2.25), (2.26) and doing some calculations

\[
\begin{align*}
\frac{2C_p}{\ell} s_{\xi}^2 u^\varepsilon(T)\|u^\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{1}{4\xi}\|\varphi_+^\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{\xi}{2}\|\nabla \varphi_+^\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{c_0}{4}\|\varphi_+^\varepsilon(T)\|^2_{L^2(\Omega)} \\
+ \frac{2k}{\ell} s_{\xi} \int_0^T \|\nabla u^\varepsilon\|^2_{L^2(\Omega)} ds + \frac{1}{\alpha\xi} \int_0^T \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega)} ds + \frac{\alpha\xi}{2} \sum_{i=0}^{M_\varepsilon - 1} \int_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi^\varepsilon\|_{L^2(\Omega)} ds
\end{align*}
\]
Continuing by applying Gronwall inequality, we finally deduce

\[
\int_0^T \| \nabla u^\varepsilon \|_{L^2(\Omega)}^2 \, ds + \int_0^T \| \nabla \varphi^\varepsilon \|_{L^2(\Omega)}^2 \, ds + \int_0^T \| \varphi^\varepsilon \|_{L^2(\Omega)}^2 \, ds + \int_0^T \| \varphi^\varepsilon (s) \|_{L^2(\Omega)}^2 \, ds \leq C, \tag{2.30}
\]

where \( C \) does not depend on \( \varepsilon \) and \( M_\varepsilon \).

Furthermore multiplying (1.7) by \( u^\varepsilon_i \), using Green’s formula and integrating over \([i\varepsilon, (i+1)\varepsilon]_t \), \( i = 0 \cdots M_\varepsilon - 1 \), we get

\[
\begin{align*}
& C_p \int_{Q^*_t} |u^\varepsilon_i|^2 \, dx \, ds + \frac{k}{2} \int_{\Omega} |\nabla u^\varepsilon|^2 \, dx \, ds + \frac{h}{2} \int |u^\varepsilon|^2 \, d\gamma + \frac{\ell}{2} \int \varphi^\varepsilon_i u^\varepsilon_i \, dx \, ds \\
& = \int_{\Sigma_t} w_1 u^\varepsilon_i \, dx \, ds + \int_{Q^*_t} f u^\varepsilon_i \, dx \, ds. \tag{2.31}
\end{align*}
\]

Using Hölder’s inequality

\[
\frac{\ell}{2} \int \varphi^\varepsilon_i u^\varepsilon_i \, dx \, ds \leq \frac{C_p}{4} \int_{Q^*_t} |u^\varepsilon_i|^2 \, dx \, ds + \frac{\ell^2}{4C_p} \int |\varphi^\varepsilon_i|^2 \, dx \, ds,
\]

\[
\int_{Q^*_t} f u^\varepsilon_i \, dx \, ds \leq \frac{C_p}{4} \int_{Q^*_t} |u^\varepsilon_i|^2 \, dx \, ds + \frac{1}{C_p} \int_{Q^*_t} |f|^2 \, dx \, ds. \tag{2.32}
\]

From (2.30),(2.31),(2.32) and after summing

\[
\frac{C_p}{2} \int_Q |u^\varepsilon_i|^2 \, dx \, ds + \frac{k}{2} \int_{\Omega} |\nabla u^\varepsilon|^2 \, dx \, ds + \frac{h}{2} \int |u^\varepsilon|^2 \, d\gamma \leq C + \int_{\Sigma} w_1 u^\varepsilon_i \, dx \, ds. \tag{2.33}
\]

Since

\[
\int_{\Sigma} w_1 u^\varepsilon_i d\gamma \, ds = \int_{\Sigma} \frac{\partial}{\partial t} (w_1 u^\varepsilon) d\gamma \, ds - \int_{\Sigma} w_1' u^\varepsilon d\gamma \, ds \tag{2.34}
\]

and

\[
\begin{align*}
\int_{\partial \Omega} w_1 u^\varepsilon d\gamma \, ds & \leq \frac{1}{k} \int_{\partial \Omega} |w_1|^2 \, d\gamma + \frac{h}{4} \int_{\partial \Omega} |u^\varepsilon|^2 \, d\gamma \\
\int_{\Sigma} w_1' u^\varepsilon d\gamma \, ds & \leq \frac{1}{2} \int_{\Sigma} |u^\varepsilon|^2 \, d\gamma \, ds + \frac{1}{2} \int_{\Sigma} |w_1'|^2 \, d\gamma \, ds. \tag{2.35}
\end{align*}
\]
Performing (2.33) by using (2.32)-(2.34) and Gronwall's inequality, we obtain

$$\int_Q |u_1^\varepsilon| dxds + \int_\Omega |\nabla u_1^\varepsilon|^2 dxds + \int_{\partial\Omega} |u_1^\varepsilon|^2 d\gamma \leq C,$$  \hspace{1cm} (2.36)

where $C$ does not depend to $\varepsilon$ and $M_\varepsilon$. Due to estimate (2.8) we have

$$\sum_{i=0}^{M_\varepsilon-1} \|\varphi_+^{\varepsilon}(i\varepsilon, x) - \varphi_-^{\varepsilon}(i\varepsilon, x)\|_{L^2(\Omega)} \leq TL = C_1$$  \hspace{1cm} (2.37)

$$\sum_{i=0}^{M_\varepsilon-1} \|\varphi_+^{\varepsilon}(i\varepsilon, x) - \varphi_-^{\varepsilon}(i\varepsilon, x)\|_{L^2(\partial\Omega)} \leq C_2,$$

where $C_1$ and $C_2$ do not depend on $M_\varepsilon$ and $\varepsilon$.

Adding (2.30), (2.36) and (2.37), we deduce

$$\frac{1}{V_0} \int_0^T \varphi^\varepsilon + V_0^2 \varphi^\varepsilon + \int_0^T \|u_1^\varepsilon(t)\|^2_{L^2(\Omega)} dt + \int_0^T \|\varphi_1^\varepsilon(t)\|^2_{L^2(\Omega)} dt$$

$$+ \int_0^T \|\nabla u_1^\varepsilon\|^2_{L^2(\Omega)} dt + \int_0^T \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega)} dt \leq C, \quad \varepsilon > 0,$$  \hspace{1cm} (2.38)

where $V_0^{1T} \varphi^\varepsilon$ and $V_0^{2T} \varphi^\varepsilon$ stand respectively for the variation of $\varphi^\varepsilon : [0, T] \to L_2(\Omega)$ and $\varphi^\varepsilon : [0, T] \to L_2(\partial\Omega)$. Since the injection of $L_2(\Omega)$ into $H^{-1}(\Omega)$ is compact and the set $\{\varphi_1^\varepsilon(t)\}$ is bounded in $L_2(\Omega)$ for every $t \in [0, T]$, we deduce that there exists (Helly-Foiaş theorem) a bounded variation $\varphi^*(t) \in BV([0, T]; H^{-1}(\Omega))$ and a subsequence $\varphi^\varepsilon(t)$ such that

$$\varphi^\varepsilon(t) \to \varphi^*(t) \quad \text{strongly in} \quad H^{-1}(\Omega) \quad \text{for every} \quad t \in [0, T].$$  \hspace{1cm} (2.39)

In addition we deduce from (2.38)

$$\varphi^\varepsilon \to \varphi^* \quad \text{weakly in} \quad L^2(0, T; H^1(\Omega)).$$  \hspace{1cm} (2.40)

Remember that due to the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ then (see [5]) for each $\kappa > 0$, there exists some constant $c_\kappa$ depending on $\kappa$ (and on the spaces $H^1(\Omega), H^{-1}(\Omega), L^2(\Omega)$) such that

$$\|\varphi^\varepsilon(t) - \varphi^*(t)\|_{L^2(\Omega)} \leq \kappa \|\varphi^\varepsilon(t) - \varphi^*(t)\|_{H^1(\Omega)} + c_\kappa \|\varphi^\varepsilon(t) - \varphi^*(t)\|_{H^{-1}(\Omega)},$$  \hspace{1cm} (2.41)

$$\forall \varepsilon > 0 \quad \text{and} \quad \forall t \in [0, T],$$

where $c_\kappa \to 0$ as $\kappa \to 0$.

Using (2.39)-(2.41), we get

$$\varphi^\varepsilon \to \varphi^* \quad \text{strongly in} \quad L^2(\Omega) \quad \text{for any} \quad t \in [0, T].$$  \hspace{1cm} (2.42)

Therefore, by (1.7) and (2.38) we also have

$$\int_0^T \|\Delta \varphi^\varepsilon\|_{L^2(\Omega)} dt \leq C, \quad \forall t \in [0, T],$$

$$\int_0^T \|\Delta u_1^\varepsilon\|_{L^2(\Omega)} dt \leq C, \quad \forall t \in [0, T],$$  \hspace{1cm} (2.43)
Numerical approximation of the phase-field

and also

\[ \|\varphi^\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq C \]
\[ \|u^\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq C. \]  

(2.44)

Hence, due to the inclusion \( H^2(\Omega) \subset H^1(\Omega) \) is compact, we get that the sequence \( \{u^\varepsilon\} \) is compact in \( L^2(0,T;H^1(\Omega)) \). Thus, up to a subsequence denoted \( u^\varepsilon \), we get

\[ u^\varepsilon \to u^* \quad \text{strongly in} \quad L^2([0,T];H^1(\Omega)), \]
\[ u^\varepsilon \to u^* \quad \text{weakly in} \quad L^2([0,T];H^2(\Omega)), \]  

(2.45)

\[ u^\varepsilon_t \to u^*_t \quad \text{weakly in} \quad L^2([0,T];L^2(\Omega)) \]

(2.46)

and, by the Ascoli-Arzelà theorem

\[ u^\varepsilon \to u^* \quad \text{strongly in} \quad C([0,T];L^2(\Omega)). \]  

Our assertion (2.1) holds true from (2.42) and (2.46). This achieves the proof of Theorem 2.1.

3. Approximation of phase-field transition system in 2D by finite element method Algorithm Armelfracfem2D

In this Section we are concerned with the numerical approximation of the weak solution corresponding with (1.7)-(1.9) (see Definition 1.2) by finite element method (fem) i.e. with the numerical approximation of the weak solution of the following equations:

\[ \left( C_p u^\varepsilon_t + \frac{\ell}{2} \varphi^\varepsilon_t, \phi \right) + k(\nabla u^\varepsilon, \nabla \phi) + h \int_{\partial \Omega} u^\varepsilon \phi dxdy \]
\[ = \int_{\partial \Omega} w_1(\cdot, x, y) \phi dxdy + \int_{\Omega} f \phi dxdy \quad \forall \phi \in H^1(\Omega), \quad \text{a.e. in} \ [0,T], \]  

(3.1)

\[ \alpha \xi(\varphi^\varepsilon, \psi) + \xi(\nabla \varphi^\varepsilon, \nabla \psi) + c_0 \int_{\partial \Omega} u^\varepsilon \psi dxdy \]
\[ = \int_{\partial \Omega} w_2(\cdot, x, y) \psi dxdy + \frac{1}{2 \xi}(\varphi^\varepsilon, \psi) + s_\xi(u^\varepsilon, \psi) + \int_{\Omega} g \psi dxdy \quad \forall \psi \in H^1(\Omega), \quad \text{a.e. in} \ [0,T], \]  

(3.2)
together with the initial conditions

\[ u(0, x) = u_0(x), \quad \varphi(0, x) = \varphi_0(x), \quad x \in \Omega. \] (3.3)

In (3.1)-(3.2), \((\cdot, \cdot)\) denotes the scalar product in \(L^2(\Omega)\).

Considering \(M = M_\varepsilon = \frac{T}{\varepsilon}\) as the number of equidistant nodes in which is divided the time-interval \([0, T]\), we set

\[ \varepsilon = dt = \frac{T}{M}, \quad t_i = i dt, \quad i = 0, 1, 2, \ldots, M. \]

We assume that \(\Omega \subset \mathbb{R}^2\) is a polygonal domain. Let \(T_\rho\) be the triangulation (mesh) over \(\Omega\) and \(\bar{\Omega} = \bigcup\{K, \; K \in T_\rho\}\) and let \(N_j = (x_k, y_l)\), \(j = 1, nn\), be the nodes associated to \(T_\rho\). If we denote by \(V_{nn}\) the corresponding finite element space to \(T_\rho\) then, the basis functions \(\{b_j\}_{j=1}^{nn}\) of \(V_{nn}\) are defined by

\[ b_j(N_i) = \delta_{ji}, \quad i, j = 1, nn, \]

and

\[ V_{nn} = \text{SPAN} \{b_1, b_2, \ldots, b_{nn}\}. \]

We say that the function \(v(x, y)\) belongs to \(V_{nn}\) only if it can be expressed as

\[ v(x, y) = \sum_{l=1}^{nn} c_l b_l(x, y), \quad (x, y) \in \bar{\Omega}. \]

For \(i = 1, M\), we denote by \(u^i\) and \(\varphi^i\) the \(V_{nn}\) interpolant of \(u^\varepsilon\) and \(\varphi^\varepsilon\), respectively. Then \(u^i, \varphi^i \in V_{nn}\) and

\[ u^i(x, y) = \sum_{l=1}^{nn} u^i_l b_l(x, y), \quad i = 1, M, \]

\[ \varphi^i(x, y) = \sum_{l=1}^{nn} \varphi^i_l b_l(x, y), \quad i = 1, M, \]

where the unknowns \(u^i_l = u^\varepsilon(t_i, N_l)\), \(\varphi^i_l = \varphi^\varepsilon(t_i, N_l)\), \(i = 1, M, l = 1, nn\), represents the discrete solution of (3.10)-(3.11) below.

Let now \(U, \Phi \in V_{nn}\) be two arbitrary functions, i.e.

\[ U(x, y) = \sum_{l=1}^{nn} U_l b_l(x, y), \] (3.6)

\[ \Phi(x, y) = \sum_{l=1}^{nn} \Phi_l b_l(x, y). \] (3.7)

Using an implicit (backward) finite difference scheme in time and taking into account the above notations, we introduce the discrete equations corresponding to (3.1)-(3.2) as follows \((i = 1, M)\)

\[ C_p(u^i, U) + \frac{\ell}{2} (\varphi^i, U) + dt k (\nabla u^i, \nabla U) + dt h \int_{\partial \Omega} u^i U \, dx \, dy = dt \int_{\partial \Omega} u^i U \, dx \, dy + C_p(u^{i-1}, U) + \frac{\ell}{2} (\varphi^{i-1}, U) + dt \int_{\Omega} f U \, dx \, dy, \] (3.8)
\( \alpha \xi (\varphi^i, \Phi) + dt \xi (\nabla \varphi^i, \nabla \Phi) + dt c_0 \int \varphi^i \Phi \, dx \, dy \)

\[
= \int_{\partial \Omega} w_1^i \Phi \, dx \, dy + \frac{dt}{2 \xi} (\varphi^i, \Phi) + dt s_\xi (u^i, \Phi) + \alpha \xi (\varphi^{i-1}, \Phi) + dt \int_\Omega g \Phi \, dx \, dy, \\
\quad \text{∀ } U, \Phi \in V_{nn}, \text{ with } w_1^i = w_1(t_i, \cdot), w_2^i = w_2(t_i, \cdot). \text{ Replacing (3.4)-(3.7) in (3.8)-(3.9), we get}
\]

\[
C_p \sum_{k=1}^{nn} U_k \left( \sum_{l=1}^{nn} u_{tl}^i \int b_k b_l \, dx \, dy \right) + \frac{\ell}{2} \sum_{k=1}^{nn} U_k \left( \sum_{l=1}^{nn} \varphi_{tl}^i \int b_k b_l \, dx \, dy \right) \\
+ dt k \sum_{k=1}^{nn} U_k \left( \sum_{l=1}^{nn} u_{tl}^i \int \nabla b_k \nabla b_l \, dx \, dy \right) + dt h \sum_{k=1}^{nn} U_k \left( \sum_{l=1}^{nn} f_{tl}^i \int b_k b_l \, dx \, dy \right),
\]

\[
\alpha \xi \sum_{k=1}^{nn} \Phi_k \left( \sum_{l=1}^{nn} \varphi_{tl}^i \int b_k b_l \, dx \, dy \right) + dt \xi \sum_{k=1}^{nn} \Phi_k \left( \sum_{l=1}^{nn} \varphi_{tl}^{i-1} \int b_k b_l \, dx \, dy \right) \\
+ dt c_0 \sum_{k=1}^{nn} \Phi_k \left( \sum_{l=1}^{nn} \varphi_{tl}^i \int b_k b_l \, dx \, dy \right)
\]

\[
= dt \sum_{k=1}^{nn} \Phi_k \left( \sum_{l=1}^{nn} w_{tl}^i \int b_k b_l \, dx \, dy \right) + \frac{dt}{2 \xi} \sum_{k=1}^{nn} \Phi_k \left( \sum_{l=1}^{nn} \varphi_{tl}^i \int b_k b_l \, dx \, dy \right) \\
+ dt s_\xi \sum_{k=1}^{nn} \Phi_k \left( \sum_{l=1}^{nn} u_{tl}^i \int b_k b_l \, dx \, dy \right) + \alpha \xi \sum_{k=1}^{nn} \Phi_k \left( \sum_{l=1}^{nn} \varphi_{tl}^{i-1} \int b_k b_l \, dx \, dy \right) \\
+ dt \sum_{k=1}^{nn} \Phi_k \left( \sum_{l=1}^{nn} g_{tl}^i \int b_k b_l \, dx \, dy \right),
\]

for \( i = 1, \ldots, M \), where \( w_{tl}^i = w_1(t_i, N_l), w_{2l}^i = w_2(t_i, N_l), f_{tl}^i = f(t_i, N_l), g_{tl}^i = g(t_i, N_l), \) \( l = 1, \ldots, n \).

Setting

\[
b_{kl} = \int_{\Omega} b_k(x, y)b_l(x, y) \, dx \, dy, \quad g_{kl} = \int_{\Omega} \nabla b_k(x, y) \nabla b_l(x, y) \, dx \, dy
\]

\[
B = (b_{kl})_{k,l=1,nn}, \quad R = (r_{kl})_{k,l=1,nn}, \quad s_{kl} = \alpha \xi b_{kl} + dt \xi g_{kl} - \frac{dt}{2 \xi} b_{kl},
\]

\[
S = (s_{kl})_{k,l=1,nn}, \quad r_{kl} = C_p b_{kl} + dt k \cdot g_{kl}, \quad \frac{FR}{\partial \Omega} = (r_{kl})_{k,l=1,nn}, \quad f_{rkl} = \int_{\Omega} b_k(x, y)b_l(x, y) \, dx \, dy.
\]
the system (3.10)-(3.11) can be rewritten in the matrix form as

\[
\begin{cases}
R u_i^l + \frac{\ell}{2} B \varphi_i^l + h dt FR u_i^l = B (C_p u_i^{l-1} + \frac{\ell}{2} \varphi_i^{l-1} + dt f_i^l) + dt FR w_i^l \\
S \varphi_i^l - s \xi dt B u_i^l + c_0 dt FR \varphi_i^l = \alpha \xi B (\varphi_i^{l-1} + dt f_i^l) + dt FR w_{2l},
\end{cases}
\]

(3.12)

where \( u_i^l, \varphi_i^l, \) and \( l = 1, n \) are unknown vectors corresponding to time the level \( i \).

From the initial conditions (3.3), we have

\[
u_0^o(x, y) = u_0(x, y) = \sum_{l=1}^{n} u_0(N_l) b_l(x, y),
\]

\[
\varphi_0^o(x, y) = \varphi_0(x, y) = \sum_{l=1}^{n} \varphi_0(N_l) b_l(x, y)
\]

and then (see (3.4)-(3.5))

\[
u_l^o = u_0(N_l), \quad \varphi_l^o = \varphi_0(N_l), \quad l = 1, n.
\]

(3.13)

The numerical algorithm to calculate the approximate solution by fractional steps method can be obtained by the following sequence (\( i \) denotes the time level).

Begin Armel-frcfem2D

Choose \( T > 0 \) and \( \Omega \subset \mathbb{R}^2 \);

Choose \( M > 0, \ n > 0 \) and compute \( \varepsilon = dt, dx \);

Choose \( u_0, \varphi_0, w_1, w_2, f, g \);

\( i := 0 \rightarrow \) Compute \( u_i^0, \varphi_i^0, \ l = 1, n \) from (3.13)

For \( i := 1 \) to \( M \) do

Compute \( z_l := z(\varepsilon, N_l), \ l = 1, n \) from (2.2);

\( \varphi_i^{l-1} := z_l, \ l = 1, n \);

Compute \( u_i^l, \varphi_i^l, \ l = 1, n \), solving the linear system (3.12);

End-for;

End.

Before to give some details regarding the numerical implementation of the algorithm Armel_frcfem2D, we recall that the convergence result established by Theorem 2.1 in previous Section guarantees that the approximate solution of the linear system (1.7)-(1.9) can be viewed as approximate solution of the nonlinear phase-field transition solution of the nonlinear phase-field transition System (1.1)-(1.3).

As it is well known, the finite element method (fem) is a general method for approximating the solution of boundary value problems for partial differential equations. This method is derived from the Ritz (or Gelerkin) method, characteristic for the finite element method being the choice of the finite dimensional space, namely, the \( \text{SPAN} \) of a set of finite element basis functions. The steps in solving a boundary value problem using fem are:

P0. (D) The direct formulation of the problem;

P1. (V) A variational (weak) formulation for problem (D);

P2. The construction of a finite element mesh (triangulation);

P3. The construction of the finite dimensional space of test function, called finite element basis functions;
**P4.** $(V_{nn})$ A discrete analogous of $(V)$;

**P5.** Assemble the linear system of equations;

**P6.** Solve the system obtained in P5.

In our concrete case, on the position of (D) we have the problem given by relations (1.7)-(1.8) and, corresponding to (V), the relations (3.1)-(3.2). The next steps (P2-P4) ending with the discrete equations corresponding to (1.7)-(1.8), i.e. the linear system (3.12).

Corresponding to the physical parameters of the mathematical model (1.1)-(1.3), we have used industrial values indicated in the works [19,20]. To generate the triangulation $T_{p}$, we consider $\Omega$ a cross-section in a slab (thin) of $1300mm \times 220mm$. In Figure 1, the mesh can be seen in the directions of $x_1$ and $x_2$ - axis of a rectangular profile.

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**Figure 1.** The triangulation over $\Omega = [0,1300] \times [0,220]$  

**Figure 2.** The values $w_1(t,x)$ on the mobile part
Figure 3. The values $w_2(t, x)$ on the mobile part

Figure 4. The approximate temperature $u^2$

Figure 5. The approximate solution $\varphi^2$

The initial solution $u_1^0, \varphi_1^0$ in (3.13) was computed as solution of stationary equation $\varphi_t = \Delta \varphi = 0$ and as solution of Cauchy problem (1.10), respectively.

The values of $w_1(t, x)$ and $w_2(t, x) \in \Sigma$ are given as a spline interpolation (only the mobile part of the continuous casting machine is illustrated in Figures 2 and 3).
We shall present now the numerical experiments implementing the conceptual algorithm Armel_fracfem2D.

Figures 4 and 6 represent the approximate solutions $u^i$ ($i = 2$ and $i = M = 8$, respectively), while Figures 5 and 7 represent the approximate solutions $\varphi^i$ ($i = 2$ and $i = M = 8$, respectively).

The shape of the graphs shows the numerical stability and accuracy of the results obtained by implementing the fractional steps method (1.7) and (1.10). The most interesting aspect that we can observe while analyzing Figures 4-7 is the presence of supercooling and superheating phenomena (presence of solid fractions in the liquid, for example).

The numerical solution computed by this way can be considered as an admissible one for the corresponding boundary optimal control problem in order to improve the process optimization of continuous casting. Generally the fractional steps method considered here can used to approximate the solution of a nonlinear parabolic phase-field system containing a general nonlinear part.
Acknowledgements

This article was written while A. Andami Ovono was visiting the Universitatea Alexandru Ioan Cuza din Iasi. He wishes to thank Conf Dr Costica Morosanu for introduction to this subject and for its warm hospitality.

References

Numerical approximation of the phase-field


