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ERRATUM TO "MORE RESULTS ON HERMITE-HDAMARD TYPE INEQUALITIES THROUGH (α,m)-PREINVEXITY"

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Abstract In this paper, we present some corrections to definitions of m-preinvex, (α, m)-preinvex functions and statements of the theorems of the results proved in [7].

Keywords Hermite-Hadamard's inequality, invex set, preinvex function, m-preinvex function, (α, m)-preinvex function, Hölder's integral inequality, power-mean inequality.

MSC(2010) 26D15, 26D20, 26D07.

1. Introduction

In a very recent article [7], Hussain and Qaisar have proved some Hermite-Hadamard type inequalities by using (α, m)-preinvexity, two already existing identities from literature and mathematical analysis. However, there are some vital errors in the statements of these results because of the deficiencies in the definition of (α, m)-preinvexity.

Here, we will give some corrections to the definition of (α, m)-preinvexity and then corrections to the statements of the results given in [7].

To this end, we first quote some necessary definitions from the literature.

It is well-know in literature that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex in classical sense if

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for every x, $y \in I$ and $\lambda \in [0, 1]$.

The classical convexity stated above was generalized as m-convexity by G. Toader in [16] as follows:

Definition 1.1 ([16]). A function $f : [0, b^*] \to \mathbb{R}$, $b^* > 0$, is said to be m-convex, if

$$f(\lambda x + m(1-\lambda) y) \leq \lambda f(x) + m(1-\lambda) f(y)$$

for all $x, y \in [0, b^*]$, $\lambda \in [0, 1]$ and $m \in [0, 1]$. The function f is said to be m-concave if $-f$ is m-convex.

Obviously, for $m = 1$ the Definition 1.1 recaptures the concept of standard convex functions on $[0, b^*]$.

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The notion of m-convexity has been further generalized in [12] and it is stated in the following definition.

Definition 1.2 ([12]). A function $f : [0, b^*] \to \mathbb{R}$, $b^* > 0$ is said to be (α, m)-convex, if

$$f(\lambda x + m(1-\lambda) y) \leq \lambda^\alpha f(x) + m(1-\lambda^\alpha) f(y)$$

for all $x, y \in [0, b^*]$, $\lambda \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$.

It can easily be seen that for $\alpha = 1$, the class of m-convex functions are derived from the above definition and for $\alpha = m = 1$ a class of convex functions are derived.

Remark 1.1. It can be observed from 1.1 and 1.2 that the domain of m-convex and (α, m)-convex functions must be a subset of $[0, \infty)$ of the form $[0, b^*]$, $b^* > 0$.

A number of mathematicians have attempted to generalize the concept of classical convexity. For example in [8], Hason gave the notion of invexity as significant generalization of classical convexity. Ben-Israel and Mond [2] introduced the concept of preinvex functions, which is a special case of invex functions.

Let us first restate the definition of preinvexity as follows.

Definition 1.3 ([17]). Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions. The set K is said to be invex at $x \in K$ with respect to $\eta(\cdot, \cdot)$, if

$$x + \lambda\eta(y, x) \in K, \forall x, y \in K, \lambda \in [0, 1].$$

The set K is said to be an invex set with respect to η if f is invex at each $x \in K$. The invex set K is also called an η-connected set.

Definition 1.4 ([17]). A function f on an invex set K is said to be preinvex with respect to η, if

$$f(x + \lambda\eta(y, x)) \leq (1-\lambda) f(x) + \lambda f(y), \forall x, y \in K, \lambda \in [0, 1].$$

The function f is said to be preincave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(y, x) = y - x$ but the converse is not true see for instance [17].

In [10], the author has given the generalizations of Definition 1.1 and Definition 1.2 as follows.

Definition 1.5 ([10]). Let $K \subseteq [0, b^*]$, $b^* > 0$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}$. A function $f : K \to \mathbb{R}$ is said to be m-preinvex with respect to η on K if

$$f(x + \lambda\eta(y, x)) \leq (1-\lambda) f(x) + m\lambda f\left(\frac{y}{m}\right)$$

holds for all $x, y \in K$, $\lambda \in [0, 1]$ and $m \in (0, 1]$. The function f is said to be m-preincave if and only if $-f$ is m-preinvex.

Definition 1.6 ([10]). Let $K \subseteq [0, b^*]$, $b^* > 0$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}$. A function $f : K \to \mathbb{R}$ is said to be (α, m)-preinvex with respect to η on K if

$$f(x + \lambda\eta(y, x)) \leq (1-\lambda^\alpha) f(x) + m\lambda^\alpha f\left(\frac{y}{m}\right)$$

holds for all $x, y \in K$, $\lambda \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$. The function f is said to be (α, m)-preincave if and only if $-f$ is (α, m)-preinvex.

Remark 1.2. The Definition 1.5 and Definition 1.6 have some weaknesses. Since $K \subseteq [0, b^*]$, $b^* > 0$, the set K may not contain 0 (for an m-preinvex and (α, m)-preinvex functions the domain must be an interval of the form $[0, b^*]$, $b^* > 0$) and if $0 < m < 1$, the point $\frac{y}{m}$ may not belong to the set K and hence the right hand sides of Definition 1.5 and Definition 1.6 are meaningless.

In [7], Hussain and Qaisar claimed that the following definition of (α, m)-preinvex was given in [2].

Definition 1.7. Let $K \subseteq \mathbb{R}$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be (α, m)-preinvex with respect to η, if for all $x, y \in K$, $\lambda \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$

$$f(x + \lambda\eta(y, x)) \le (1 - \lambda^\alpha) f(x) + m\lambda^\alpha f\left(\frac{y}{m}\right).$$

The function f is said to be (α, m)-preconcave if and only if $-f$ is (α, m)-preinvex.

Remark 1.3. Indeed, Definition 1.7 has never been given in [2]. Moreover, in this definition $\eta : K \times K \to \mathbb{R}^n$ has to be $\eta : K \times K \to \mathbb{R}$ and the domain of the function f cannot be a subset of the set of real numbers. Suppose if $K = [-1, 1] \subseteq \mathbb{R}$, $m = \frac{1}{2}$, $y = 1$, then $\frac{y}{m} = 2 \notin [-1, 1]$ and hence the right hand side in Definition 1.7 is meaningless.

Hussain and Qaisar [7] have also claimed that the following lemmas will be used to prove their results.

Lemma 1.1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}+$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \ge 1$ and $f^{(n)}$. If is $f^{(n)}$ integrable on $[a, a + \eta(b, a)]$, then for every a, $b \in K$ with $\eta(b, a) > 0$, the following inequality holds:*

$$\begin{aligned} &-\frac{f(a) + f(a + \eta(b, a))}{2} + \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\,dx \\ &+ \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \\ =& \frac{(-1)^{n-1} (\eta(b, a))^n}{2n!} \int_0^1 \lambda^{n-1} (n - 2\lambda) f^{(n)}(a + \lambda\eta(b, a))\,d\lambda. \end{aligned} \tag{1.1}$$

Lemma 1.2. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}+$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \ge 1$ and $f^{(n)}$. If is $f^{(n)}$ integrable on $[a, a + \eta(b, a)]$, then for every a, $b \in K$ with $\eta(b, a) > 0$, the following inequality holds:*

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b, a)\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\,dx \\ &\qquad = \frac{(-1)^{n+1} (\eta(b, a))^n}{n!} \int_0^1 P_n(\lambda) f^{(n)}(a + \lambda\eta(b, a))\,d\lambda, \end{aligned} \tag{1.2}$$

where

$$P_n(\lambda) = \begin{cases} \lambda^n, & \lambda \in \left[0, \frac{1}{2}\right], \\ (\lambda - 1)^n, & \lambda \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

These two lemmas have not been cited and the function η has not been defined correctly as well. The range of the function η must be the set of real numbers instead of the set of positive real numbers. In fact, these two lemmas were proved by the author in [10]. In Lemma 1.1 and Lemma 1.2, (1.1) and (1.2) are equalities but not the inequalities.

The main aim of this erratum is to provide corrections to the definitions of m-preinvex and (α, m)-preinvex and hence the corrections to the statements of the theorems given in [7].

2. Corrections

In this section we give corrections to the definitions of m-preinvex and (α, m)-preinvex functions and then corrections to the statements of theorems proved in [7].

Definition 2.1. Let $\mathbb{R}_0 = [0, +\infty)$ be an invex set with respect to $\eta : \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_0$. A function $f : \mathbb{R}_0 \to \mathbb{R}$ is said to be m-preinvex on $\left[0, \frac{y^*}{m}\right] \subseteq \mathbb{R}_0$ with respect to η if

$$f(x + \lambda\eta(y, x)) \leq (1 - \lambda) f(x) + m\lambda f\left(\frac{y}{m}\right)$$

holds for all x, $y \in [0, y^*]$, $\lambda \in [0, 1]$ and $m \in (0, 1]$. The function f is said to be m-preconcave if and only if $-f$ is m-preinvex.

Definition 2.2. Let $\mathbb{R}_0 = [0, +\infty)$ be an invex set with respect to $\eta : \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_0$. A function $f : \mathbb{R}_0 \to \mathbb{R}$ is said to be (α, m)-preinvex on $\left[0, \frac{y^*}{m}\right]$ with respect to η if

$$f(x + \lambda\eta(y, x)) \leq (1 - \lambda^\alpha) f(x) + m\lambda^\alpha f\left(\frac{y}{m}\right)$$

holds for all x, $y \in [0, y^*]$, $\lambda \in [0, 1]$ and $(\alpha, m) \in (0, 1]^2$. The function f is said to be (α, m)-preconcave if and only if $-f$ is (α, m)-preinvex.

Remark 2.1. If in Definition 2.1, $m = 1$, then one obtain the usual definition of preinvexity. If $\alpha = m = 1$, then Definition 2.2 recaptures the usual definition of the the preinvex functions. It is to be noted that every m-preinvex function and (α, m)-preinvex functions are m-convex and (α, m)-convex with respect to $\eta(y, x) = y - x$ respectively.

The following example illustrates that m-preinvex functions are different from m-convex functions.

Example 2.1. Let the mapping $f : \mathbb{R}_0 \to \mathbb{R}$ be defined as

$$f(x) = -x^2.$$

Let the function $\eta : \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_0$ be defined as

$$\eta(v, u) = \frac{v}{\sqrt{\lambda m}} + u, 0 < m, \lambda \leq 1.$$

Then

$$f\left(x+\lambda\eta(y,x)\right) = -\left(\left(1+\lambda\right)x + y\sqrt{\frac{\lambda}{m}}\right)^2$$
$$= -\left(1+\lambda\right)^2 x^2 - \frac{\lambda}{m}y^2 - 2\left(1+\lambda\right)\sqrt{\frac{\lambda}{m}}xy$$
$$= \phi_{\lambda,m}\left(x,y\right)$$

and

$$\left(1-\lambda\right)f\left(x\right) + m\lambda f\left(\frac{y}{m}\right) = -\left(1-\lambda\right)x^2 - \frac{\lambda}{m}y^2 = \varphi_{\lambda,m}\left(x,y\right).$$

It is obvious that

$$\phi_{\lambda,m}\left(x,y\right) \le \varphi_{\lambda,m}\left(x,y\right)$$

for x, $y \in \mathbb{R}_0$, $\lambda \in [0,1]$ and $m \in (0,1]$. Hence the function f is an m-preinvex with respect to η on $\mathbb{R}_0$ for every $m \in (0,1]$. However, the same function is not an m-convex for any $m \in (0,1]$. For instance, let $x = 1$, $y = 3$, $\lambda = \frac{1}{2}$ and $m = \frac{3}{4}$. Then

$$f\left(\lambda x + \left(1-\lambda\right)y\right) = -4$$

and

$$\lambda f\left(x\right) + m\left(1-\lambda\right)f\left(\frac{y}{m}\right) = -6.5.$$

That is

$$f\left(\lambda x + \left(1-\lambda\right)y\right) > \lambda f\left(x\right) + m\left(1-\lambda\right)f\left(\frac{y}{m}\right).$$

Remark 2.2. A similar example can be constructed to show that (α, m)-preinvex functions are different from (α, m)-convex functions.

Correction to the statement of Theorem 2.1 from [7]

Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $\mathbb{R}_0 \subseteq K$. Suppose that $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}$, $n \ge 2$, a, $b \in K$, $0 \le a < b < \infty$ with $\eta(b,a) > 0$. If $\left|f^{(n)}\right|$ is (α, m)-preinvex on $\left[0, \frac{b}{m}\right]$, the following inequality holds:

$$\left|\frac{f\left(a\right)+f\left(a+\eta\left(b,a\right)\right)}{2} - \frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)} f\left(x\right)dx - \sum_{k=2}^{n-1}\frac{\left(-1\right)^k\left(k-1\right)\left(\eta\left(b,a\right)\right)^k}{2\left(k+1\right)!}f^{(k)}(a+\eta\left(b,a\right))\right| \le \frac{\left(\eta\left(b,a\right)\right)^n}{2n!}\left[U_2\left|f^{(n)}\left(a\right)\right| + U_1 m\left|f^{(n)}\left(\frac{b}{m}\right)\right|\right], \tag{2.1}$$

where

$$U_1 = \frac{n\left(n-1\right)+\alpha\left(n-2\right)}{\left(n+\alpha\right)\left(n+\alpha+1\right)} \text{ and } U_2 = \frac{n\alpha\left(n+\alpha\right)-\alpha\left(\alpha+1\right)}{\left(n+1\right)\left(n+\alpha\right)\left(n+\alpha+1\right)}.$$

Correction to the statement of Corollary 2.1 from [7]
If $n = 2$, in Theorem 2.1, the following inequality holds:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^2}{4}\left[\frac{\alpha}{3(\alpha+2)}\left|f''(a)\right|+\frac{2m}{(\alpha+2)(\alpha+3)}\left|f''\left(\frac{b}{m}\right)\right|\right], \quad (2.2)$$

Correction to the statement of Theorem 2.2 from [7]
Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $\mathbb{R}_0 \subseteq K$. Suppose that $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 2$, a, $b \in K$, $0 \leq a < b < \infty$ with $\eta(b,a) > 0$. If $\left|f^{(n)}\right|^q$, for $q \geq 1$ is (α, m)-preinvex on $\left[0, \frac{b}{m}\right]$, the following inequality holds:

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right| \quad (2.3)\\ &\leq \frac{(\eta(b,a))^n}{2n!}(n-1)^{1-\frac{1}{q}}\left\{U_3\left|f^{(n)}(a)\right|^q+mU_4\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right\}^{\frac{1}{q}},\end{aligned}$$

where

$$U_3 = \frac{n}{nq-q+1}-\frac{2}{nq-q+2}-U_4 \text{ and } U_4 = \frac{2}{1+nq-q+\alpha}-\frac{2}{2+nq-q+\alpha}.$$

Correction to the statement of Corollary 2.2 from [7]
If $n = 2$ in Theorem 2.2, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^2}{4}\left\{U_3\left|f''(a)\right|^q+mU_4\left|f''\left(\frac{b}{m}\right)\right|^q\right\}^{\frac{1}{q}}, \quad (2.4)$$

where

$$U_3 = \frac{2}{q+1}-\frac{2}{q+2}-U_4 \text{ and } U_4 = \frac{2}{1+q+\alpha}-\frac{2}{2+q+\alpha}.$$

Correction to the statement of Corollary 2.3 from [7]
If we take $q = 1$, $\alpha = 1$ and $m = 1$ in Corollary 2.2 we get,

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^2}{24}\left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}. \quad (2.5)$$

Correction to the statement of Theorem 2.3 from [7]

Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $\mathbb{R}_0\subseteq K$. Suppose that $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\geq 1$, a, $b\in K$, $0\leq a<b<\infty$ with $\eta(b,a)>0$. If $\left|f^{(n)}\right|^q$, for $q>1$ is (α,m)-preinvex on $\left[0,\frac{b}{m}\right]$, the following inequality holds

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^n}{2^n n!\,(np+1)^{\frac{1}{p}}}\left[\frac{\alpha\left|f^{(n)}(a)\right|^q+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}},\quad(2.6)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Correction to the statement of Corollary 2.4 from [7]

If $n=2,\alpha=1$ and $m=1$, in Theorem 2.3, then we have the following inequality:

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{8(2p+1)^{\frac{1}{p}}}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}},\quad(2.7)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Correction to the statement of Theorem 2.4 from [7]

Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $\mathbb{R}_0\subseteq K$. Suppose that $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\geq 1$, a, $b\in K$, $0\leq a<b<\infty$ with $\eta(b,a)>0$. If $\left|f^{(n)}\right|^q$, for $q>1$ is (α,m)-preinvex on $\left[0,\frac{b}{m}\right]$, the following inequality holds

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^n}{2^{n+\frac{1}{p}}n!\,(np+1)^{\frac{1}{p}}}\left[\left(V_1\left|f^{(n)}(a)\right|^q+mV_2\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(V_3\left|f^{(n)}(a)\right|^q+mV_4\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right],\quad(2.8)$$

where

$$V_1=\frac{2^\alpha(\alpha+1)-1}{2^{\alpha+1}(\alpha+1)},\quad V_2=\frac{1}{2^{\alpha+1}(\alpha+1)},$$
$$V_3=\frac{\alpha\cdot 2^\alpha-2^\alpha(\alpha+1)+1}{2^{\alpha+1}(\alpha+1)},\quad V_4=\frac{2^{\alpha+1}-1}{2^{\alpha+1}(\alpha+1)}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Correction to the statement of Corollary 2.5 from [7]

If $\alpha = 1, m = 1$ and $n = 2$ in Theorem 2.4, then we have the following inequality:

$$\left| f\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right|$$
$$\leq \frac{(\eta(b,a))^2}{2^{3+\frac{1}{p}}(2p+1)^{\frac{1}{p}}} \left[\left(\frac{3}{8}\left|f''(a)\right|^q + \frac{1}{8}\left|f''(b)\right|^q \right)^{\frac{1}{q}} + \left(\frac{1}{8}\left|f''(a)\right|^q + \frac{3}{8}\left|f''(b)\right|^q \right)^{\frac{1}{q}} \right], \tag{2.9}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Correction to the statement of Theorem 2.5 from [7]

Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $\mathbb{R}_0 \subseteq K$. Suppose that $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 1$, a, $b \in K$, $0 \leq a < b < \infty$ with $\eta(b,a) > 0$. If $\left|f^{(n)}\right|^q$, for $q \geq 1$ is (α, m)-preinvex on $\left[0, \frac{b}{m}\right]$, the following inequality holds

$$\left| \sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right|$$
$$\leq \frac{(\eta(b,a))^n}{n!} \left(\frac{1}{2^{n+1}(n+1)} \right)^{1-\frac{1}{q}} \left[\left(D\left|f^{(n)}(a)\right|^q + mE\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} + \left(F\left|f^{(n)}(a)\right|^q + mG\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \right], \tag{2.10}$$

where

$$D = \frac{1}{2^{n+1}(n+1)} - E, E = \frac{1}{(n+\alpha+1)\,2^{n+\alpha+1}},$$
$$F = \frac{1}{2^{n+1}(n+1)} - G, G = B\left(\frac{1}{2}; n+1, \alpha+1\right)$$

and $B(z; x, y) = \int_0^z t^{x-1}(1-t)^{1-y}\,dt$, $0 \leq z \leq 1$ for $x, y > 0$ is the incomplete Beta function.

Correction to the statement of Corollary 2.6 from [7]

If $\alpha = 1, m = 1$ and $n = 2$ in Theorem 2.5, then we have the following inequality:

$$\left| f\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right|$$
$$\leq \frac{(\eta(b,a))^2}{2} \left(\frac{1}{24}\right)^{1-\frac{1}{q}} \left[\left(\frac{5}{192}\left|f''(a)\right|^q + \frac{1}{64}\left|f''\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} + \left(\frac{1}{64}\left|f''(a)\right|^q + \frac{5}{192}\left|f''\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \right]. \tag{2.11}$$

Remark 2.3. There are number of typos in the proofs of the theorems given in [7] as well.

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