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EXPLICIT EXACT PERIODIC WAVE SOLUTIONS AND THEIR LIMIT FORMS FOR A LONG WAVES-SHORT WAVES MODEL*

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Abstract A long waves-short waves model is studied by using the approach of dynamical systems. The sufficient conditions to guarantee the existence of solitary wave, kink and anti-kink waves, and periodic wave in different regions of the parametric space are given. All possible explicit exact parametric representations of above traveling waves are presented. When the energy of Hamiltonian system corresponding to this model varies, we also show the convergence of the periodic wave solutions, such as the periodic wave solutions converge to the solitary wave solutions, kink and anti-kink wave solutions, and periodic wave solutions, respectively.

Keywords A long waves-short waves model, solitary wave solution, kink and anti-kink wave solutions, periodic wave solution, limit form.

MSC(2010) 34C25, 34F10, 35C07, 35C08.

1. Introduction

Nonlinear partial differential equations (NPDEs) including the KdV equation, KP equation, K(m,n) equation, sine-Gordon equation and long waves-short waves model arising in plasma physics, chemistry, mechanics, biology and optics, etc. To investigate the exact solutions of NPDEs, lots of methods have been proposed, such as the inverse scattering method [12], Darboux and Bäcklund transformations [18], Hirota bilinear method [1, 13], Lie symmetry analysis method [4, 11, 16, 23, 31], ansatz method [8,30], G'/G-expansion method [2,3], Kudryashov method [21,28], approach of dynamical systems [14, 19, 22, 24, 29] and so on.

In this paper, we consider a long waves-short waves model

$$\begin{aligned} A_t &= 2\sigma\left(|B|^2\right)_x, \\ B_t &= iB_{xx} - A_xB + iA^2B - 2i\sigma B\,|B|^2, \end{aligned} \tag{1.1}$$

where $i^2 = -1, A = A(x,t)$ represents the amplitude of the long wave and $B = B(x,t)$ the envelope of the short wave. Equation (1.1) was presented by Newell [26, 27] and has been studied by some authors [6,7,9,10,15,17,20,25,32,33]. Chowdhury

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and Chanda [6,7] proved the complete integrability, took the Weiss-Tabor-Carnevale approach to the Painlevé analysis and considered its integrabilty and Bäcklund transformation. Equation (1.1) has important applications in plasma physics [25] and fluid mechanics [10]. For its physical relevance was shown in [9]. In [17], it was shown that (1.1) is associated with a model equation proposed by Yajima and Oikawa through a Muira transformation. The nonsmooth behaviors of solitary waves for this equation are investigated by qualitative techniques in dynamical systems [32]. A closed multi-soliton solution formula and some novel solutions are obtained by Darboux transformation method [15, 20]. Zhu and Kuang [33] also shown that the equation (1.1) for $\sigma = 1$ has the cusp solitons by using the $\bar{\partial}$-dressing method. Unfortunately, to our knowledge, the dynamic behaviors have not been studied and not much is known for the solutions of equation (1.1) in the previous literatures.

In this paper, we shall consider the dynamic behaviors, present some new exact explicit traveling wave solutions of equation (1.1), and investigate the relations of the traveling wave solutions using the approach of dynamical systems [14,19,22,24,29].

We consider the travelling wave solution of the form:

$$A(x,t) = A(\xi),\ B(x,t) = \phi(\xi)\mathrm{e}^{i(\psi(\xi)-\omega t)},\ \xi = x - ct, \tag{1.2}$$

where $A(\xi), \phi(\xi), \psi(\xi)$ are real-valued functions, $c\ (\neq 0)$ is the wave speed, ω is a parameter and $i^2 = -1$. Substituting (1.2) into the first equation of (1.1) and integrating the obtained equation once, we have

$$A(\xi) = g - \frac{2\sigma}{c}\phi^2(\xi), \tag{1.3}$$

where g is the integral constant. Substituting (1.2) and (1.3) into the second equation of (1.1) and decomposing the real and imaginary parts, we obtain

$$\begin{aligned} &c\phi' + 2\phi'\psi' + \phi\psi'' - \frac{4\sigma}{c}\phi^2\phi' = 0,\\ &\phi'' + (\omega + g^2)\phi - 2\sigma\left(1 + \frac{2g}{c}\right)\phi^3 + \frac{4\sigma^2}{c^2}\phi^5 + c\phi\psi' - \phi\left(\psi'\right)^2 = 0, \end{aligned} \tag{1.4}$$

where "$'$" is the derivative with respect to ξ.

Multiplying both sides of the first equation of (1.4) by ϕ and integrating with respect to ξ and setting the integral constant is zero, we have

$$\psi' = \frac{\sigma}{c}\phi^2 - \frac{1}{2}c. \tag{1.5}$$

Substituting (1.5) into the second equation of (1.4), yields equation

$$\phi'' = \rho\left(\phi^4 + \alpha\phi^2 + \beta\right)\phi, \tag{1.6}$$

where $\rho = -\frac{3\sigma^2}{c^2}, \alpha = -\frac{4cg}{3\sigma}, \beta = \frac{c^2}{3\sigma^2}\left(\omega + g^2 - \frac{3}{4}c^2\right)$.

From (1.6), we get the following planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \rho\left(\phi^4 + \alpha\phi^2 + \beta\right)\phi \tag{1.7}$$

with the first integral

$$H(\phi, y) = y^2 - \frac{1}{6}\rho\left(2\phi^4 + 3\alpha\phi^2 + 6\beta\right)\phi^2 = h. \tag{1.8}$$

For a fixed h, the level curve $H(\phi, y) = h$ defined by (1.8) determines a set of invariant curves of system (1.7) which contains different branches of curves. As h is varied, it defines different families of orbits of system (1.7) with different dynamical behaviors.

The reminder of this paper is organized as follows. In Section 2, we consider bifurcation sets and phase portraits of (1.7). In Section 3, we state and prove our main results for equation (1.1). Some numerical simulations for the main results and a short conclusion will be given in Section 4 and Section 5, respectively.

2. Bifurcation sets and phase portraits of (1.7)

Definition 2.1 (see [19]). Suppose that $\phi(\xi)$ is a continuous solution of system (1.7) for $\xi \in (-\infty, +\infty)$ and $\lim_{\xi\to+\infty} \phi(\xi) = p$, $\lim_{\xi\to-\infty} \phi(\xi) = q$. Recall that $\phi(\xi)$ is called a solitary wave solution if $p = q$, $\phi(\xi)$ is called a kink (or anti-kink) solution if $p \neq q$.

Usually, a solitary wave solution of (1.6) corresponds to a homoclinic orbit of system (1.7), a kink (or anti-kink) wave solution of (1.6) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of system (1.7). Similarly, a periodic orbit of system (1.7) corresponds to a periodic wave solution of (1.6). Thus, to investigate all possible solitary wave, periodic wave, kink and anti-kink wave solutions of (1.6), we need to find all periodic annuli, homoclinic and heteroclinic orbits of system (1.7), which depend on the system parameters ρ, α and β.

System (1.7) has three equilibrium points at $(0,0), (\pm\phi_1, 0)$ when $\beta < 0$, has three equilibrium points at $(0,0), \left(\pm\sqrt{-\alpha}, 0\right)$ when $\alpha < 0$ and $\beta = 0$, has three equilibrium points at $(0,0), \left(\pm\frac{1}{2}\sqrt{-2\alpha}, 0\right)$ when $\alpha < 0$ and $\beta = \frac{1}{4}\alpha^2$, has five equilibrium points at $(0,0), (\pm\phi_1, 0), (\pm\phi_2, 0)$ when $\alpha < 0$ and $0 < \beta < \frac{1}{4}\alpha^2$, otherwise, has only one equilibrium point at $(0,0)$, where $\phi_{1,2} = \frac{1}{2}\sqrt{2\left(-\alpha \pm \sqrt{\alpha^2 - 4\beta}\right)}$. From (1.8), we have

$$h_1 = H(\pm\phi_1, 0) = -\frac{\rho\left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right)\left(-\alpha^2 + \alpha\sqrt{\alpha^2 - 4\beta} + 8\beta\right)}{24},$$

$$h_2 = H(\pm\phi_2, 0) = -\frac{\rho\left(\alpha + \sqrt{\alpha^2 - 4\beta}\right)\left(\alpha^2 + \alpha\sqrt{\alpha^2 - 4\beta} - 8\beta\right)}{24}.$$

Let $M(\phi_e, 0)$ be the coefficient matrix of the linearized system of (1.7) at equilibrium point $(\phi_e, 0)$ and $J(\phi_e, 0) = \det\left(M(\phi_e, 0)\right)$, then we have

$$J(0,0) = -\rho\beta,$$

$$J\left(\pm\sqrt{-\alpha}, 0\right) = -\rho\left(2\alpha^2 + \beta\right),$$

$$J\left(\pm\frac{1}{2}\sqrt{-2\alpha}, 0\right) = \rho\left(\frac{1}{4}\alpha^2 - \beta\right),$$

$$J(\pm\phi_1, 0) = \rho\left(4\beta - \alpha^2 + \alpha\sqrt{\alpha^2 - 4\beta}\right),$$

$$J(\pm\phi_2, 0) = \rho\left(4\beta - \alpha^2 - \alpha\sqrt{\alpha^2 - 4\beta}\right).$$

For an equilibrium point $(\phi_e, 0)$ of system (1.7), we know that $(\phi_e, 0)$ is a saddle point if $J(\phi_e, 0) < 0$, a center point if $J(\phi_e, 0) > 0$, a cusp if $J(\phi_e, 0) = 0$ and the Poincaré index of $(\phi_e, 0)$ is zero.

By using the properties of equilibrium points and the approach of dynamical systems, we can show that bifurcation sets and phase portraits of (1.7) are as drawn in Fig. 1.

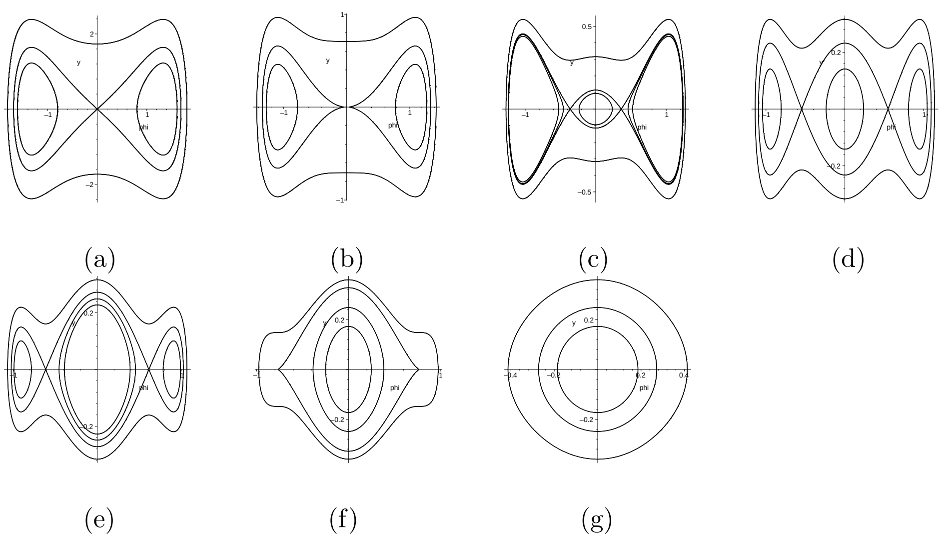


Figure 1. Bifurcation sets and phase portraits of (1.7) for $\rho < 0$. Parameters: (**a**) $\beta < 0$. (**b**) $\alpha < 0, \beta = 0$. (**c**) $\alpha < 0, 0 < \beta < \frac{3}{16}\alpha^2$. (**d**) $\alpha < 0, \beta = \frac{3}{16}\alpha^2$. (**e**) $\alpha < 0, \frac{3}{16}\alpha^2 < \beta < \frac{1}{4}\alpha^2$. (**f**) $\alpha < 0, \beta = \frac{1}{4}\alpha^2$. (**g**) $\alpha < 0, \beta > \frac{1}{4}\alpha^2$ or $\alpha \geq 0, \beta \geq 0$.

3. Main results and their proofs

Pay attention to that $\mathrm{sn}(\cdot, k), \Pi(\cdot, \cdot, k)$ are the Jacobian elliptic function and the normal elliptic integral of the third kind, respectively, with the modulus $k, \mathrm{am}(u_1, k)$ reads amplitude u_1 [5], we state some results of equation (1.1) as follows.

Proposition 3.1. *Let*

$$\Omega_1 = \frac{2}{3}\phi_3\phi_4\sqrt{-3\rho},\ \Omega_2 = \frac{1}{3}\gamma_1\sqrt{-3\rho\left(\gamma_2^2 + \gamma_3^2\right)},$$

$$k_1 = \sqrt{\frac{\gamma_3^2\left(\gamma_1^2 - \gamma_2^2\right)}{\gamma_1^2\left(\gamma_2^2 + \gamma_3^2\right)}},\ \alpha_1^2 = \frac{\gamma_2^2 - \gamma_1^2}{\gamma_2^2 + \gamma_3^2},\ \phi_{3,4} = \frac{1}{2}\sqrt{\mp 3\alpha + \sqrt{9\alpha^2 - 48\beta}},$$

$$\psi_1(x - ct) = \frac{\sigma\phi_3\phi_4}{c\Omega_1}\arcsin\left(\frac{\phi_3^2 + \phi_4^2 - \left(\phi_3^2 - \phi_4^2\right)\cosh\left(\Omega_1(x - ct)\right)}{\phi_3^2 - \phi_4^2 - \left(\phi_3^2 + \phi_4^2\right)\cosh\left(\Omega_1(x - ct)\right)}\right) - \frac{c}{2}(x - ct),$$

$$\psi_2(x-ct)=\frac{\sigma(\gamma_1^2+\gamma_3^2)}{c\Omega_2}\Pi\left(am\left(\Omega_2(x-ct),k_1\right),\alpha_1^2,k_1\right)-\left(\frac{\sigma\gamma_3^2}{c}+\frac{c}{2}\right)(x-ct),$$

and $\gamma_1,\gamma_2,\gamma_3$ $(\gamma_3<\gamma_2<\gamma_1)$ *satisfy equation*

$$(X^2-\gamma_1^2)(X^2-\gamma_2^2)(X^2+\gamma_3^2)=\left(X^4+\frac{3}{2}\alpha X^2+3\beta\right)X^2+\frac{3h}{\rho},\ h_1<h<0,$$

then when $\beta<0$, *equation (1.1) has two solitary wave solutions*

$$\begin{aligned}A(x,t)&=g-\frac{4\sigma\phi_3^2\phi_4^2}{c\left((\phi_3^2+\phi_4^2)\cosh(\Omega_1(x-ct))-(\phi_3^2-\phi_4^2)\right)},\\ B(x,t)&=\pm\frac{\sqrt{2}\phi_3\phi_4 e^{i(\psi_1(x-ct)-\omega t)}}{\sqrt{(\phi_3^2+\phi_4^2)\cosh(\Omega_1(x-ct))-(\phi_3^2-\phi_4^2)}},\end{aligned}\tag{3.1}$$

and has two periodic wave solutions

$$\begin{aligned}A(x,t)&=g-\frac{2\sigma\left(\gamma_1^2\left(\gamma_2^2+\gamma_3^2\right)-\gamma_3^2\left(\gamma_1^2-\gamma_2^2\right)sn^2\left(\Omega_2(x-ct),k_1\right)\right)}{c\left(\gamma_2^2+\gamma_3^2+(\gamma_1^2-\gamma_2^2)\,sn^2\left(\Omega_2(x-ct),k_1\right)\right)},\\ B(x,t)&=\pm\sqrt{\frac{\gamma_1^2\left(\gamma_2^2+\gamma_3^2\right)-\gamma_3^2\left(\gamma_1^2-\gamma_2^2\right)sn^2\left(\Omega_2(x-ct),k_1\right)}{\gamma_2^2+\gamma_3^2+(\gamma_1^2-\gamma_2^2)\,sn^2\left(\Omega_2(x-ct),k_1\right)}}\,e^{i(\psi_2(x-ct)-\omega t)}.\end{aligned}\tag{3.2}$$

Moreover, as $h\to0$, *the periodic wave solutions (3.2) converge to the solitary wave solutions (3.1).*

Proof. When $\beta<0$, from Fig. 1(a), we see that there are two homoclinic orbits defined by $H(\phi,y)=0$ connecting with the saddle point $(0,0)$, passing through the points $(\pm\phi_3,0)$, respectively, and there are two periodic orbits defined by $H(\phi,y)=h\ \ (h\in(h_1,0))$, one of them passing through the points $(\gamma_1,0)$ and $(\gamma_2,0)$, and another passing through the points $(-\gamma_2,0)$ and $(-\gamma_1,0)$. In (ϕ,y)-plane, their expressions are, respectively,

$$y=\pm\phi\sqrt{-\frac{1}{3}\rho\left(\phi_3^2-\phi^2\right)\left(\phi^2+\phi_4^2\right)},\quad 0<\phi\le\phi_3,\tag{3.3}$$

$$y=\pm\phi\sqrt{-\frac{1}{3}\rho\left(\phi_3^2-\phi^2\right)\left(\phi^2+\phi_4^2\right)},\quad -\phi_3\le\phi<0,\tag{3.4}$$

$$y=\pm\sqrt{-\frac{1}{3}\rho\left(\gamma_1^2-\phi^2\right)\left(\phi^2-\gamma_2^2\right)\left(\phi^2+\gamma_3^2\right)},\quad \gamma_2\le\phi\le\gamma_1,\tag{3.5}$$

$$y=\pm\sqrt{-\frac{1}{3}\rho\left(\gamma_1^2-\phi^2\right)\left(\phi^2-\gamma_2^2\right)\left(\phi^2+\gamma_3^2\right)},\quad -\gamma_1\le\phi\le-\gamma_2,\tag{3.6}$$

where $\phi_{3,4}=\frac{1}{2}\sqrt{\mp3\alpha+\sqrt{9\alpha^2-48\beta}}$, $\gamma_1,\gamma_2,\gamma_3$ $(\gamma_3<\gamma_2<\gamma_1)$ satisfy equation

$$(X^2-\gamma_1^2)(X^2-\gamma_2^2)(X^2+\gamma_3^2)=\left(X^4+\frac{3}{2}\alpha X^2+3\beta\right)X^2+\frac{3h}{\rho},h_1<h<0.$$

Substituting (3.3) and (3.4) into the $\frac{d\phi}{d\xi}=y$ and integrating them along the homoclinic orbits, respectively, we have

$$\int_{\phi}^{\phi_3}\frac{ds}{s\sqrt{\left(\phi_3^2-s^2\right)\left(s^2+\phi_4^2\right)}}=\sqrt{-\frac{1}{3}\rho}|\xi|,\tag{3.7}$$

$$\int_{-\phi_3}^{\phi} \frac{ds}{s\sqrt{(\phi_3^2 - s^2)(s^2 + \phi_4^2)}} = -\sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.8}$$

Completing the integrals in (3.7) and (3.8), we obtain two solitary wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm \frac{\sqrt{2}\phi_3\phi_4}{\sqrt{(\phi_3^2 + \phi_4^2)\cosh(\Omega_1\xi) - (\phi_3^2 - \phi_4^2)}}, \tag{3.9}$$

where $\Omega_1 = \frac{2}{3}\phi_3\phi_4\sqrt{-3\rho}$. Substituting (3.9) into (1.5) yields equation

$$\psi(\xi) = \frac{2\sigma\phi_3^2\phi_4^2}{c\Omega_1} \int \frac{d(\Omega_1\xi)}{(\phi_3^2 + \phi_4^2)\cosh(\Omega_1\xi) - (\phi_3^2 - \phi_4^2)} - \frac{c}{2}\xi. \tag{3.10}$$

Completing the integral in (3.10) and replacing $\psi(\xi)$ by $\psi_1(\xi)$, we have

$$\psi_1(x - ct) = \frac{\sigma\phi_3\phi_4}{c\Omega_1} \arcsin\left(\frac{\phi_3^2 + \phi_4^2 - (\phi_3^2 - \phi_4^2)\cosh(\Omega_1(x - ct))}{\phi_3^2 - \phi_4^2 - (\phi_3^2 + \phi_4^2)\cosh(\Omega_1(x - ct))}\right) - \frac{c}{2}(x - ct). \tag{3.11}$$

From (1.2), (1.3), (3.9) and (3.11), we obtain two solitary wave solutions of equation (1.1) as (3.1).

Substituting (3.5) and (3.6) into the $\frac{d\phi}{d\xi} = y$ and integrating them along the periodic orbits, respectively, we have

$$\int_{\phi}^{\gamma_1} \frac{ds}{\sqrt{(\gamma_1^2 - s^2)(s^2 - \gamma_2^2)(s^2 + \gamma_3^2)}} = \sqrt{-\frac{1}{3}\rho}|\xi|, \tag{3.12}$$

$$\int_{-\gamma_1}^{\phi} \frac{ds}{\sqrt{(\gamma_1^2 - s^2)(s^2 - \gamma_2^2)(s^2 + \gamma_3^2)}} = \sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.13}$$

Completing the integrals in (3.12) and (3.13), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\sqrt{\frac{\gamma_1^2(\gamma_2^2 + \gamma_3^2) - \gamma_3^2(\gamma_1^2 - \gamma_2^2)\,\mathrm{sn}^2(\Omega_2\xi, k_1)}{\gamma_2^2 + \gamma_3^2 + (\gamma_1^2 - \gamma_2^2)\,\mathrm{sn}^2(\Omega_2\xi, k_1)}}, \tag{3.14}$$

where $\Omega_2 = \frac{1}{3}\gamma_1\sqrt{-3\rho(\gamma_2^2 + \gamma_3^2)}$, $k_1 = \sqrt{\frac{\gamma_3^2(\gamma_1^2 - \gamma_2^2)}{\gamma_1^2(\gamma_2^2 + \gamma_3^2)}}$. Substituting (3.14) into (1.5) yields equation

$$\psi(\xi) = \frac{\sigma(\gamma_1^2 + \gamma_3^2)}{c\Omega_2} \int \frac{d(\Omega_2\xi)}{1 - \alpha_1^2\mathrm{sn}^2(\Omega_2\xi, k_1)} - \left(\frac{\sigma\gamma_3^2}{c} + \frac{c}{2}\right)\xi, \tag{3.15}$$

where $\alpha_1^2 = \frac{\gamma_2^2 - \gamma_1^2}{\gamma_2^2 + \gamma_3^2}$. Completing the integral in (3.15) and replacing $\psi(\xi)$ by $\psi_2(\xi)$, we have

$$\psi_2(x - ct) = \frac{\sigma(\gamma_1^2 + \gamma_3^2)}{c\Omega_2}\Pi\left(\mathrm{am}(\Omega_2(x - ct), k_1), \alpha_1^2, k_1\right) - \left(\frac{\sigma\gamma_3^2}{c} + \frac{c}{2}\right)(x - ct). \tag{3.16}$$

From (1.2), (1.3), (3.14) and (3.16), we obtain two periodic wave solutions of equation (1.1) as (3.2).

Letting $h \to 0$, we have

$$\gamma_1 \to \phi_3, \gamma_2 \to 0, \gamma_3 \to \phi_4.$$

So that

$$k_1 = \sqrt{\frac{\gamma_3^2\left(\gamma_1^2-\gamma_2^2\right)}{\gamma_1^2\left(\gamma_2^2+\gamma_3^2\right)}} \to 1, \Omega_2 = \frac{1}{3}\gamma_1\sqrt{-3\rho\left(\gamma_2^2+\gamma_3^2\right)} \to \frac{1}{2}\Omega_1,$$

$$\operatorname{sn}\left(\Omega_2\xi, k_1\right) \to \tanh\left(\frac{1}{2}\Omega_1\xi\right),$$

$$\sqrt{\frac{\gamma_1^2\left(\gamma_2^2+\gamma_3^2\right)-\gamma_3^2\left(\gamma_1^2-\gamma_2^2\right)\operatorname{sn}^2\left(\Omega_2\xi,k_1\right)}{\gamma_2^2+\gamma_3^2+\left(\gamma_1^2-\gamma_2^2\right)\operatorname{sn}^2\left(\Omega_2\xi,k_1\right)}} \to \sqrt{\frac{\phi_3^2\phi_4^2-\phi_3^2\phi_4^2\tanh^2\left(\frac{1}{2}\Omega_1\xi\right)}{\phi_4^2+\phi_3^2\tanh^2\left(\frac{1}{2}\Omega_1\xi\right)}}$$

$$= \frac{\sqrt{2}\phi_3\phi_4}{\sqrt{(\phi_3^2+\phi_4^2)\cosh(\Omega_1\xi)-(\phi_3^2-\phi_4^2)}}.$$

Therefore, as $h \to 0$, the periodic wave solutions (3.2) converge to the solitary wave solutions (3.1). □

Proposition 3.2. *Let*

$$\Omega_3 = -\frac{1}{2}\alpha\sqrt{-3\rho},\ k_2 = \sqrt{\frac{\gamma_6^2\left(\gamma_4^2-\gamma_5^2\right)}{\gamma_4^2\left(\gamma_5^2+\gamma_6^2\right)}},$$

$$\Omega_4 = \frac{1}{3}\gamma_4\sqrt{-3\rho\left(\gamma_5^2+\gamma_6^2\right)},\ \alpha_2^2 = \frac{\gamma_5^2-\gamma_4^2}{\gamma_5^2+\gamma_6^2},$$

$$\psi_3(x-ct) = -\frac{3\sigma\alpha}{2c\Omega_3}\arctan\left(\Omega_3(x-ct)\right) - \frac{c}{2}(x-ct),$$

$$\psi_4(x-ct) = \frac{\sigma(\gamma_4^2+\gamma_6^2)}{c\Omega_4}\Pi\left(am\left(\Omega_4(x-ct),k_2\right),\alpha_2^2,k_2\right) - \left(\frac{\sigma\gamma_6^2}{c}+\frac{c}{2}\right)(x-ct),$$

and $\gamma_4, \gamma_5, \gamma_6$ $(\gamma_6 < \gamma_5 < \gamma_4)$ *satisfy equation*

$$(X^2-\gamma_4^2)(X^2-\gamma_5^2)(X^2+\gamma_6^2) = \left(X^2+\frac{3}{2}\alpha\right)X^4 + \frac{3h}{\rho},\ -\frac{\rho\alpha^3}{6} < h < 0,$$

then when $\alpha < 0, \beta = 0$, *equation (1.1) has two solitary wave solutions*

$$\begin{aligned} A(x,t) &= \frac{1}{2}\sqrt{3c^2-4\omega} + \frac{3\sigma\alpha}{c\left(1+\left(\Omega_3(x-ct)\right)^2\right)}, \\ B(x,t) &= \pm\frac{\sqrt{-6\alpha}e^{i(\psi_3(x-ct)-\omega t)}}{2\sqrt{1+\left(\Omega_3(x-ct)\right)^2}},\quad \omega \le \frac{3}{4}c^2, \end{aligned} \tag{3.17}$$

and has two periodic wave solutions

$$\begin{aligned} A(x,t) &= \frac{1}{2}\sqrt{3c^2-4\omega} - \frac{2\sigma\left(\gamma_4^2\left(\gamma_5^2+\gamma_6^2\right)-\gamma_6^2\left(\gamma_4^2-\gamma_5^2\right)sn^2\left(\Omega_4(x-ct),k_2\right)\right)}{c\left(\gamma_5^2+\gamma_6^2+\left(\gamma_4^2-\gamma_5^2\right)sn^2\left(\Omega_4(x-ct),k_2\right)\right)}, \\ B(x,t) &= \pm\sqrt{\frac{\gamma_4^2\left(\gamma_5^2+\gamma_6^2\right)-\gamma_6^2\left(\gamma_4^2-\gamma_5^2\right)sn^2\left(\Omega_4(x-ct),k_2\right)}{\gamma_5^2+\gamma_6^2+\left(\gamma_4^2-\gamma_5^2\right)sn^2\left(\Omega_4(x-ct),k_2\right)}}e^{i(\psi_4(x-ct)-\omega t)},\omega \le \frac{3}{4}c^2. \end{aligned} \tag{3.18}$$

Moreover, as $h \to 0$, the periodic wave solutions (3.18) converge to the solitary wave solutions (3.17).

Proof. When $\alpha < 0, \beta = 0$, from Fig. 1(b), we see that there are two homoclinic orbits defined by $H(\phi, y) = 0$ connecting with the cusp $(0,0)$, passing through the points $\left(\pm\sqrt{-\frac{3}{2}\alpha}, 0\right)$, respectively, and there are two periodic orbits defined by $H(\phi, y) = h \ \left(h \in \left(-\frac{\rho\alpha^3}{6}, 0\right)\right)$, one of them passing through the points $(\gamma_4, 0)$ and $(\gamma_5, 0)$, and another passing through the points $(-\gamma_5, 0)$ and $(-\gamma_4, 0)$. In (ϕ, y)-plane, their expressions are, respectively,

$$y = \pm\phi^2\sqrt{-\frac{1}{3}\rho\left(-\frac{3}{2}\alpha - \phi^2\right)}, \quad 0 < \phi \le \sqrt{-\frac{3}{2}\alpha}, \tag{3.19}$$

$$y = \pm\phi^2\sqrt{-\frac{1}{3}\rho\left(-\frac{3}{2}\alpha - \phi^2\right)}, \quad -\sqrt{-\frac{3}{2}\alpha} \le \phi < 0, \tag{3.20}$$

$$y = \pm\sqrt{-\frac{1}{3}\rho\left(\gamma_4^2 - \phi^2\right)\left(\phi^2 - \gamma_5^2\right)\left(\phi^2 + \gamma_6^2\right)}, \quad \gamma_5 \le \phi \le \gamma_4, \tag{3.21}$$

$$y = \pm\sqrt{-\frac{1}{3}\rho\left(\gamma_4^2 - \phi^2\right)\left(\phi^2 - \gamma_5^2\right)\left(\phi^2 + \gamma_6^2\right)}, \quad -\gamma_4 \le \phi \le -\gamma_5, \tag{3.22}$$

where $\gamma_4, \gamma_5, \gamma_6$ $(\gamma_6 < \gamma_5 < \gamma_4)$ satisfy equation

$$(X^2 - \gamma_4^2)(X^2 - \gamma_5^2)(X^2 + \gamma_6^2) = \left(X^2 + \frac{3}{2}\alpha\right)X^4 + \frac{3h}{\rho}, -\frac{\rho\alpha^3}{6} < h < 0.$$

Substituting (3.19) and (3.20) into the $\frac{d\phi}{d\xi} = y$ and integrating them along the homoclinic orbits, respectively, we have

$$\int_{\phi}^{\sqrt{-\frac{3}{2}\alpha}} \frac{ds}{s^2\sqrt{-\frac{3}{2}\alpha - s^2}} = \sqrt{-\frac{1}{3}\rho}|\xi|, \tag{3.23}$$

$$\int_{-\sqrt{-\frac{3}{2}\alpha}}^{\phi} \frac{ds}{s^2\sqrt{-\frac{3}{2}\alpha - s^2}} = \sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.24}$$

Completing the integrals in (3.23) and (3.24), we obtain two solitary wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\frac{\sqrt{-6\alpha}}{2\sqrt{1 + (\Omega_3\xi)^2}}, \tag{3.25}$$

where $\Omega_3 = -\frac{1}{2}\alpha\sqrt{-3\rho}$. Substituting (3.25) into (1.5) yields equation

$$\psi(\xi) = -\frac{3\sigma\alpha}{2c\Omega_3}\int\frac{d\left(\Omega_3\xi\right)}{1 + \left(\Omega_3\xi\right)^2} - \frac{c}{2}\xi. \tag{3.26}$$

Completing the integral in (3.26) and replacing $\psi(\xi)$ by $\psi_3(\xi)$, we have

$$\psi_3(x - ct) = -\frac{3\sigma\alpha}{2c\Omega_3}\arctan\left(\Omega_3(x - ct)\right) - \frac{c}{2}(x - ct). \tag{3.27}$$

From (1.2), (1.3), (3.25), (3.27) and pay attention to that if $\beta=0$, then

$$g=\frac{1}{2}\sqrt{3c^2-4\omega},$$

we obtain two solitary wave solutions of equation (1.1) as (3.17).

Substituting (3.21) and (3.22) into the $\frac{d\phi}{d\xi}=y$ and integrating them along the periodic orbits, respectively, we have

$$\int_{\phi}^{\gamma_4}\frac{ds}{\sqrt{(\gamma_4^2-s^2)(s^2-\gamma_5^2)(s^2+\gamma_6^2)}}=\sqrt{-\frac{1}{3}\rho}|\xi|, \tag{3.28}$$

$$\int_{-\gamma_4}^{\phi}\frac{ds}{\sqrt{(\gamma_4^2-s^2)(s^2-\gamma_5^2)(s^2+\gamma_6^2)}}=\sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.29}$$

Completing the integrals in (3.28) and (3.29), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi)=\pm\sqrt{\frac{\gamma_4^2(\gamma_5^2+\gamma_6^2)-\gamma_6^2(\gamma_4^2-\gamma_5^2)\operatorname{sn}^2(\Omega_4\xi,k_2)}{\gamma_5^2+\gamma_6^2+(\gamma_4^2-\gamma_5^2)\operatorname{sn}^2(\Omega_4\xi,k_2)}}, \tag{3.30}$$

where $\Omega_4=\frac{1}{3}\gamma_4\sqrt{-3\rho(\gamma_5^2+\gamma_6^2)}, k_2=\sqrt{\frac{\gamma_6^2(\gamma_4^2-\gamma_5^2)}{\gamma_4^2(\gamma_5^2+\gamma_6^2)}}$. Substituting (3.30) into (1.5) yields equation

$$\psi(\xi)=\frac{\sigma\left(\gamma_4^2+\gamma_6^2\right)}{c\Omega_4}\int\frac{d(\Omega_4\xi)}{1-\alpha_2^2\operatorname{sn}^2(\Omega_4\xi,k_2)}-\left(\frac{\sigma\gamma_6^2}{c}+\frac{c}{2}\right)\xi, \tag{3.31}$$

where $\alpha_2^2=\frac{\gamma_5^2-\gamma_4^2}{\gamma_5^2+\gamma_6^2}$. Completing the integral in (3.31) and replacing $\psi(\xi)$ by $\psi_4(\xi)$, we have

$$\psi_4(x-ct)=\frac{\sigma(\gamma_4^2+\gamma_6^2)}{c\Omega_4}\Pi\left(\operatorname{am}(\Omega_4(x-ct),k_2),\alpha_2^2,k_2\right)-\left(\frac{\sigma\gamma_6^2}{c}+\frac{c}{2}\right)(x-ct). \tag{3.32}$$

From (1.2), (1.3), (3.30), (3.32) and pay attention to that if $\beta=0$, then

$$g=\frac{1}{2}\sqrt{3c^2-4\omega},$$

we obtain two solitary wave solutions of equation (1.1) as (3.18).

Letting $h\to 0$, we have

$$\gamma_4\to\sqrt{-\frac{3}{2}\alpha},\gamma_5\to 0,\gamma_6\to 0.$$

So that

$$\int_{\phi}^{\gamma_4} \frac{ds}{\sqrt{(\gamma_4^2 - s^2)(s^2 - \gamma_5^2)(s^2 + \gamma_6^2)}} = \sqrt{-\frac{1}{3}\rho}|\xi| \to \int_{\phi}^{\sqrt{-\frac{3}{2}\alpha}} \frac{ds}{s^2\sqrt{-\frac{3}{2}\alpha - s^2}} = \sqrt{-\frac{1}{3}\rho}|\xi|,$$

$$\int_{-\gamma_4}^{\phi} \frac{ds}{\sqrt{(\gamma_4^2 - s^2)(s^2 - \gamma_5^2)(s^2 + \gamma_6^2)}} = \sqrt{-\frac{1}{3}\rho}|\xi| \to \int_{-\sqrt{-\frac{3}{2}\alpha}}^{\phi} \frac{ds}{s^2\sqrt{-\frac{3}{2}\alpha - s^2}} = \sqrt{-\frac{1}{3}\rho}|\xi|.$$

Therefore, as $h \to 0$, the periodic wave solutions (3.18) converge to the solitary wave solutions (3.17). □

Proposition 3.3. *Let*

$$\Omega_5 = \frac{1}{3}\phi_2\sqrt{-3\rho(\phi_5^2 - \phi_2^2)},\ \Omega_6 = \frac{1}{3}\gamma_8\sqrt{-3\rho(\gamma_7^2 - \gamma_9^2)},$$

$$\Omega_7 = \frac{2}{3}\phi_3\phi_6\sqrt{-3\rho},\ k_3 = \frac{\gamma_9}{\gamma_8}\sqrt{\frac{\gamma_7^2 - \gamma_8^2}{\gamma_7^2 - \gamma_9^2}},\ \alpha_3^2 = \frac{\gamma_8^2 - \gamma_7^2}{\gamma_8^2},\ \alpha_4^2 = \frac{\gamma_9^2}{\gamma_9^2 - \gamma_7^2},$$

$$\phi_2 = \frac{1}{2}\sqrt{-2\alpha - 2\sqrt{\alpha^2 - 4\beta}},\ \phi_5 = \frac{1}{2}\sqrt{-2\alpha + 4\sqrt{\alpha^2 - 4\beta}},$$

$$\phi_{3,6} = \frac{1}{2}\sqrt{-3\alpha \pm \sqrt{9\alpha^2 - 48\beta}},$$

$$\psi_5(x - ct) = -\frac{\sigma\phi_2\sqrt{2(\phi_5^2 - \phi_2^2)}}{2c\Omega_5}\arcsin\left(\frac{\phi_5^2 + (2\phi_2^2 - \phi_5^2)\cosh(2\Omega_5(x - ct))}{2\phi_2^2 - \phi_5^2 + \phi_5^2\cosh(2\Omega_5(x - ct))}\right) + \left(\frac{\sigma}{c}\phi_2^2 - \frac{c}{2}\right)(x - ct),$$

$$\psi_6(x - ct) = \frac{\sigma\phi_2\sqrt{\phi_5^2 - \phi_2^2}}{2c\Omega_5}\arcsin\left(\frac{\phi_5^2 + (\phi_5^2 - 2\phi_2^2)\cosh(2\Omega_5(x - ct))}{\phi_5^2 - 2\phi_2^2 + \phi_5^2\cosh(2\Omega_5(x - ct))}\right) + \left(\frac{\sigma}{c}\phi_2^2 - \frac{c}{2}\right)(x - ct),$$

$$\psi_7(x - ct) = \frac{\sigma\gamma_7^2}{c\Omega_6}\Pi\left(am(\Omega_6(x - ct), k_3), \alpha_3^2, k_3\right) - \frac{c}{2}(x - ct),$$

$$\psi_8(x - ct) = \frac{\sigma\gamma_7^2}{c\Omega_6}\Pi\left(am(\Omega_6(x - ct), k_3), \alpha_4^2, k_3\right) + \left(\frac{\sigma\gamma_7^2}{c} - \frac{c}{2}\right)(x - ct),$$

$$\psi_9(x - ct) = \frac{2\sigma\phi_3\phi_6}{c\Omega_7}\arctan\left(2\phi_3^2\tan\left(\frac{1}{2}\Omega_7(x - ct)\right)\right) - \frac{c}{2}(x - ct),$$

and $\gamma_7, \gamma_8, \gamma_9$ $(\gamma_9 < \gamma_8 < \gamma_7)$ *satisfy equation*

$$(X^2 - \gamma_7^2)(X^2 - \gamma_8^2)(X^2 - \gamma_9^2) = \left(X^4 + \frac{3}{2}\alpha X^2 + 3\beta\right)X^2 + \frac{3h}{\rho},\ 0 < h < h_2,$$

then when $\alpha < 0, 0 < \beta < \frac{3}{16}\alpha^2$, equation (1.1) has two solitary wave solutions

$$\begin{aligned} A(x,t) &= g - \frac{4\sigma\phi_2^2\phi_5^2\cosh^2(\Omega_5(x-ct))}{c\,(\phi_5^2\cosh(2\Omega_5(x-ct))+2\phi_2^2-\phi_5^2)}, \\ B(x,t) &= \pm\frac{\sqrt{2}\phi_2\phi_5\cosh(\Omega_5(x-ct))\,e^{i(\psi_5(x-ct)-\omega t)}}{\sqrt{\phi_5^2\cosh(2\Omega_5(x-ct))+2\phi_2^2-\phi_5^2}}, \end{aligned} \tag{3.33}$$

has two kink and anti-kink wave solutions

$$\begin{aligned} A(x,t) &= g - \frac{4\sigma\phi_2^2\phi_5^2\sinh^2(\Omega_5(x-ct))}{c\,(\phi_5^2\cosh(2\Omega_5(x-ct))+\phi_5^2-2\phi_2^2)}, \\ B(x,t) &= \pm\frac{\sqrt{2}\phi_2\phi_5\sinh(\Omega_5(x-ct))\,e^{i(\psi_6(x-ct)-\omega t)}}{\sqrt{\phi_5^2\cosh(2\Omega_5(x-ct))+\phi_5^2-2\phi_2^2}}, \end{aligned} \tag{3.34}$$

and has some periodic wave solutions

$$\begin{aligned} A(x,t) &= g - \frac{2\sigma\gamma_7^2\gamma_8^2}{c\,(\gamma_8^2+(\gamma_7^2-\gamma_8^2)sn^2(\Omega_6(x-ct),k_3))}, \\ B(x,t) &= \pm\frac{\gamma_7\gamma_8\,e^{i(\psi_7(x-ct)-\omega t)}}{\sqrt{\gamma_8^2+(\gamma_7^2-\gamma_8^2)sn^2(\Omega_6(x-ct),k_3)}}, \end{aligned} \tag{3.35}$$

$$\begin{aligned} A(x,t) &= g - \frac{2\sigma\gamma_7^2\gamma_9^2 sn^2(\Omega_6(x-ct),k_3)}{c\,(\gamma_7^2-\gamma_9^2+\gamma_9^2 sn^2(\Omega_6(x-ct),k_3))}, \\ B(x,t) &= \pm\frac{\gamma_7\gamma_9 sn(\Omega_6(x-ct),k_3)\,e^{i(\psi_8(x-ct)-\omega t)}}{\sqrt{\gamma_7^2-\gamma_9^2+\gamma_9^2 sn^2(\Omega_6(x-ct),k_3)}}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} A(x,t) &= g - \frac{4\sigma\phi_3^2\phi_6^2}{c\,(\phi_3^2+\phi_6^2-(\phi_3^2-\phi_6^2)\cos(\Omega_7(x-ct)))}, \\ B(x,t) &= \pm\frac{\sqrt{2}\phi_3\phi_6\,e^{i(\psi_9(x-ct)-\omega t)}}{\sqrt{\phi_3^2+\phi_6^2-(\phi_3^2-\phi_6^2)\cos(\Omega_7(x-ct))}}. \end{aligned} \tag{3.37}$$

Moreover, as $h \to h_2$, the periodic wave solutions (3.35) converge to the solitary wave solutions (3.33), the periodic wave solutions (3.36) converge to the kink and anti-kink wave solutions (3.34), respectively, and as $h \to 0$, the periodic wave solutions (3.35) converge to the periodic wave solutions (3.37).

Proof. When $\alpha < 0, 0 < \beta < \frac{3}{16}\alpha^2$, from Fig. 1(c), we see that there are two homoclinic orbits and two heteroclinic orbits defined by $H(\phi, y) = h_2$, the homoclinic orbits connecting with the saddle points $(\pm\phi_2, 0)$, respectively, and passing through the points $\pm(\phi_5, 0)$, respectively, the heteroclinic orbits connecting with the saddle points $(\pm\phi_2, 0)$, there are three periodic orbits defined by $H(\phi, y) = h \ (h \in (0, h_2))$, two of them passing through the points $(\gamma_7, 0)$ and $(\gamma_8, 0)$, $(-\gamma_8, 0)$ and $(-\gamma_7, 0)$, respectively, and another passing through the points $(\gamma_9, 0)$ and $(-\gamma_9, 0)$, and there are two periodic orbits defined by $H(\phi, y) = 0$, one of them passing through the points $(\phi_3, 0)$ and $(\phi_6, 0)$, and another passing through the points $(-\phi_6, 0)$ and $(-\phi_3, 0)$, where $\phi_2 = \frac{1}{2}\sqrt{-2\alpha - 2\sqrt{\alpha^2 - 4\beta}}$, $\phi_5 = \frac{1}{2}\sqrt{-2\alpha + 4\sqrt{\alpha^2 - 4\beta}}$, $\phi_{3,6} = \frac{1}{2}\sqrt{-3\alpha \pm \sqrt{9\alpha^2 - 48\beta}}$, and $\gamma_7, \gamma_8, \gamma_9 \ (\gamma_9 < \gamma_8 < \gamma_7)$ satisfy equation

$$(X^2-\gamma_7^2)(X^2-\gamma_8^2)(X^2-\gamma_9^2) = \left(X^4 + \frac{3}{2}\alpha X^2 + 3\beta\right)X^2 + \frac{3h}{\rho}, 0 < h < h_2.$$

In (ϕ, y)-plane, their expressions are, respectively,

$$y = \pm\left(\phi^2 - \phi_2^2\right)\sqrt{-\frac{1}{3}\rho(\phi_5^2 - \phi^2)}, \quad \phi_2 < \phi \le \phi_5, \tag{3.38}$$

$$y = \pm\left(\phi^2 - \phi_2^2\right)\sqrt{-\frac{1}{3}\rho(\phi_5^2 - \phi^2)}, \quad -\phi_5 \le \phi < -\phi_2, \tag{3.39}$$

$$y = \pm\left(\phi_2^2 - \phi^2\right)\sqrt{-\frac{1}{3}\rho(\phi_5^2 - \phi^2)}, \quad -\phi_2 < \phi < \phi_2, \tag{3.40}$$

$$y = \pm\sqrt{-\frac{1}{3}\rho\left(\gamma_7^2 - \phi^2\right)\left(\phi^2 - \gamma_8^2\right)\left(\phi^2 - \gamma_9^2\right)}, \quad \gamma_8 \le \phi \le \gamma_7, \tag{3.41}$$

$$y = \pm\sqrt{-\frac{1}{3}\rho\left(\gamma_7^2 - \phi^2\right)\left(\phi^2 - \gamma_8^2\right)\left(\phi^2 - \gamma_9^2\right)}, \quad -\gamma_7 \le \phi \le -\gamma_8, \tag{3.42}$$

$$y = \pm\sqrt{-\frac{1}{3}\rho\left(\gamma_7^2 - \phi^2\right)\left(\gamma_8^2 - \phi^2\right)\left(\gamma_9^2 - \phi^2\right)}, \quad -\gamma_9 \le \phi \le \gamma_9, \tag{3.43}$$

$$y = \pm\phi\sqrt{-\frac{1}{3}\rho\left(\phi_3^2 - \phi^2\right)\left(\phi^2 - \phi_6^2\right)}, \quad \phi_6 \le \phi \le \phi_3, \tag{3.44}$$

$$y = \pm\phi\sqrt{-\frac{1}{3}\rho\left(\phi_3^2 - \phi^2\right)\left(\phi^2 - \phi_6^2\right)}, \quad -\phi_3 \le \phi \le -\phi_6. \tag{3.45}$$

Substituting (3.38) and (3.39) into the $\frac{d\phi}{d\xi} = y$ and integrating them along the homoclinic orbits, respectively, we have

$$\int_{\phi}^{\phi_5} \frac{ds}{\left(s^2 - \phi_2^2\right)\sqrt{\phi_5^2 - s^2}} = \sqrt{-\frac{1}{3}\rho}|\xi|, \tag{3.46}$$

$$\int_{-\phi_5}^{\phi} \frac{ds}{\left(s^2 - \phi_2^2\right)\sqrt{\phi_5^2 - s^2}} = \sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.47}$$

Completing the integrals in (3.46) and (3.47), we obtain two solitary wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\frac{\sqrt{2}\phi_2\phi_5\cosh(\Omega_5\xi)}{\sqrt{\phi_5^2\cosh(2\Omega_5\xi) + 2\phi_2^2 - \phi_5^2}}, \tag{3.48}$$

where $\Omega_5 = \frac{1}{3}\phi_2\sqrt{-3\rho\left(\phi_5^2 - \phi_2^2\right)}$. Substituting (3.48) into (1.5) yields equation

$$\psi(\xi) = -\frac{\sigma\phi_2^2(\phi_5^2 - \phi_2^2)}{c\Omega_5}\int\frac{d\left(2\Omega_5\xi\right)}{\phi_5^2\cosh(2\Omega_5\xi) + 2\phi_2^2 - \phi_5^2} + \left(\frac{\sigma}{c}\phi_2^2 - \frac{c}{2}\right)\xi. \tag{3.49}$$

Completing the integral in (3.49) and replacing $\psi(\xi)$ by $\psi_5(\xi)$, we have

$$\psi_5(x - ct) = -\frac{\sigma\phi_2\sqrt{2\left(\phi_5^2 - \phi_2^2\right)}}{2c\Omega_5}\arcsin\left(\frac{\phi_5^2 + \left(2\phi_2^2 - \phi_5^2\right)\cosh\left(2\Omega_5(x - ct)\right)}{2\phi_2^2 - \phi_5^2 + \phi_5^2\cosh\left(2\Omega_5(x - ct)\right)}\right) + \left(\frac{\sigma}{c}\phi_2^2 - \frac{c}{2}\right)(x - ct). \tag{3.50}$$

From (1.2), (1.3), (3.48) and (3.50), we obtain two solitary wave solutions of equation (1.1) as (3.33).

Substituting (3.40) into the $\frac{d\phi}{d\xi} = y$ and integrating it along the heteroclinic orbits, we have

$$\int_{\phi}^{0} \frac{ds}{(\phi_2^2 - s^2)\sqrt{\phi_5^2 - s^2}} = \pm\sqrt{-\frac{1}{3}\rho}\xi. \tag{3.51}$$

Completing the integrals in (3.51), we obtain two kink and anti-kink wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\frac{\sqrt{2}\phi_2\phi_5 \sinh(\Omega_5\xi)}{\sqrt{\phi_5^2\cosh(2\Omega_5\xi) + \phi_5^2 - 2\phi_2^2}}. \tag{3.52}$$

Substituting (3.52) into (1.5) yields equation

$$\psi(\xi) = \frac{\sigma\phi_2^2(\phi_2^2 - \phi_5^2)}{c\Omega_5}\int \frac{d\,(2\Omega_5\xi)}{\phi_5^2\cosh(2\Omega_5\xi) + \phi_5^2 - 2\phi_2^2} + \left(\frac{\sigma}{c}\phi_2^2 - \frac{c}{2}\right)\xi. \tag{3.53}$$

Completing the integral in (3.53) and replacing $\psi(\xi)$ by $\psi_6(\xi)$, we have

$$\begin{aligned}\psi_6(x-ct) =& \frac{\sigma\phi_2\sqrt{\phi_5^2 - \phi_2^2}}{2c\Omega_5}\arcsin\left(\frac{\phi_5^2 + \left(\phi_5^2 - 2\phi_2^2\right)\cosh\left(2\Omega_5(x-ct)\right)}{\phi_5^2 - 2\phi_2^2 + \phi_5^2\cosh\left(2\Omega_5(x-ct)\right)}\right)\\ &+ \left(\frac{\sigma}{c}\phi_2^2 - \frac{c}{2}\right)(x-ct).\end{aligned} \tag{3.54}$$

From (1.2), (1.3), (3.52) and (3.54), we obtain two kink and anti-kink wave solutions of equation (1.1) as (3.34).

Substituting (3.41) and (3.42) into the $\frac{d\phi}{d\xi} = y$ and integrating them along the periodic orbits, respectively, we have

$$\int_{\phi}^{\gamma_7} \frac{ds}{\sqrt{(\gamma_7^2 - s^2)(s^2 - \gamma_8^2)(s^2 - \gamma_9^2)}} = \sqrt{-\frac{1}{3}\rho}|\xi|, \tag{3.55}$$

$$\int_{-\gamma_7}^{\phi} \frac{ds}{\sqrt{(\gamma_7^2 - s^2)(s^2 - \gamma_8^2)(s^2 - \gamma_9^2)}} = \sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.56}$$

Completing the integrals in (3.55) and (3.56), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\frac{\gamma_7\gamma_8}{\sqrt{\gamma_8^2 + (\gamma_7^2 - \gamma_8^2)\mathrm{sn}^2(\Omega_6\xi, k_3)}}, \tag{3.57}$$

where $\Omega_6 = \frac{1}{3}\gamma_8\sqrt{-3\rho\left(\gamma_7^2 - \gamma_9^2\right)}, k_3 = \frac{\gamma_9}{\gamma_8}\sqrt{\frac{\gamma_7^2 - \gamma_8^2}{\gamma_7^2 - \gamma_9^2}}$. Substituting (3.57) into (1.5) yields equation

$$\psi(\xi) = \frac{\sigma\gamma_7^2}{c\Omega_6}\int \frac{d\left(\Omega_6\xi\right)}{1 - \alpha_3^2\mathrm{sn}^2\left(\Omega_6\xi, k_3\right)} - \frac{c}{2}\xi, \tag{3.58}$$

where $\alpha_3^2 = \frac{\gamma_8^2 - \gamma_7^2}{\gamma_8^2}$. Completing the integral in (3.58) and replacing $\psi(\xi)$ by $\psi_7(\xi)$, we have

$$\psi_7(x-ct) = \frac{\sigma\gamma_7^2}{c\Omega_6}\Pi\left(\mathrm{am}\left(\Omega_6(x-ct), k_3\right), \alpha_3^2, k_3\right) - \frac{c}{2}(x-ct). \tag{3.59}$$

From (1.2), (1.3), (3.57) and (3.59), we obtain two periodic wave solutions of equation (1.1) as (3.35).

Substituting (3.43) into the $\frac{d\phi}{d\xi}=y$ and integrating it along the periodic orbit, we have

$$\int_{\phi}^{0}\frac{ds}{\sqrt{(\gamma_7^2-s^2)(\gamma_8^2-s^2)(\gamma_9^2-s^2)}}=\pm\sqrt{-\frac{1}{3}\rho}\xi. \tag{3.60}$$

Completing the integral in (3.60), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi)=\pm\frac{\gamma_7\gamma_9\mathrm{sn}(\Omega_6\xi,k_3)}{\sqrt{\gamma_7^2-\gamma_9^2+\gamma_9^2\mathrm{sn}^2(\Omega_6\xi,k_3)}}. \tag{3.61}$$

Substituting (3.61) into (1.5) yields equation

$$\psi(\xi)=\frac{\sigma\gamma_7^2}{c\Omega_6}\int\frac{d(\Omega_6\xi)}{1-\alpha_4^2\mathrm{sn}^2(\Omega_6\xi,k_3)}+\left(\frac{\sigma\gamma_7^2}{c}-\frac{c}{2}\right)\xi, \tag{3.62}$$

where $\alpha_4^2=\frac{\gamma_9^2}{\gamma_9^2-\gamma_7^2}$. Completing the integral in (3.62) and replacing $\psi(\xi)$ by $\psi_8(\xi)$, we have

$$\psi_8(x-ct)=\frac{\sigma\gamma_7^2}{c\Omega_6}\Pi\left(\mathrm{am}(\Omega_6(x-ct),k_3),\alpha_4^2,k_3\right)+\left(\frac{\sigma\gamma_7^2}{c}-\frac{c}{2}\right)(x-ct). \tag{3.63}$$

From (1.2), (1.3), (3.61) and (3.63), we obtain two periodic wave solutions of equation (1.1) as (3.36).

Substituting (3.44) and (3.45) into the $\frac{d\phi}{d\xi}=y$ and integrating them along the periodic orbits, respectively, we have

$$\int_{\phi}^{\phi_3}\frac{ds}{s\sqrt{(\phi_3^2-s^2)(s^2-\phi_6^2)}}=\sqrt{-\frac{1}{3}\rho}|\xi|, \tag{3.64}$$

$$\int_{-\phi_3}^{\phi}\frac{ds}{s\sqrt{(\phi_3^2-s^2)(s^2-\phi_6^2)}}=-\sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.65}$$

Completing the integrals in (3.64) and (3.65), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi)=\pm\frac{\sqrt{2}\phi_3\phi_6}{\sqrt{\phi_3^2+\phi_6^2-(\phi_3^2-\phi_6^2)\cos(\Omega_7\xi)}}, \tag{3.66}$$

where $\Omega_7=\frac{2}{3}\phi_3\phi_6\sqrt{-3\rho}$. Substituting (3.66) into (1.5) yields equation

$$\psi(\xi)=\frac{2\sigma\phi_3^2\phi_6^2}{c\Omega_7}\int\frac{d(\Omega_7\xi)}{\phi_3^2+\phi_6^2+(\phi_6^2-\phi_3^2)\cos(\Omega_7\xi)}-\frac{c}{2}\xi. \tag{3.67}$$

Completing the integral in (3.67) and replacing $\psi(\xi)$ by $\psi_9(\xi)$, we have

$$\psi_9(x-ct)=\frac{2\sigma\phi_3\phi_6}{c\Omega_7}\arctan\left(2\phi_3^2\tan\left(\frac{1}{2}\Omega_7(x-ct)\right)\right)-\frac{c}{2}(x-ct). \tag{3.68}$$

From (1.2), (1.3), (3.66) and (3.68), we obtain two periodic wave solutions of equation (1.1) as (3.37).

Letting $h \to h_2$, we have

$$\gamma_7 \to \phi_5, \gamma_8 \to \phi_2, \gamma_9 \to \phi_2.$$

So that

$$k_3 = \frac{\gamma_9}{\gamma_8}\sqrt{\frac{\gamma_7^2-\gamma_8^2}{\gamma_7^2-\gamma_9^2}} \to 1, \Omega_6 = \frac{1}{3}\gamma_8\sqrt{-3\rho\left(\gamma_7^2-\gamma_9^2\right)} \to \Omega_5, \mathrm{sn}\left(\Omega_6\xi, k_3\right) \to \tanh\left(\Omega_5\xi\right),$$

$$\begin{aligned}
\frac{\gamma_7\gamma_8}{\sqrt{\gamma_8^2+(\gamma_7^2-\gamma_8^2)\mathrm{sn}^2(\Omega_6\xi,k_3)}} &\to \frac{\phi_2\phi_5}{\sqrt{\phi_2^2+(\phi_5^2-\phi_2^2)\tanh^2\left(\Omega_5\xi\right)}} \\
&= \frac{\sqrt{2}\phi_2\phi_5\cosh(\Omega_5\xi)}{\sqrt{\phi_5^2\cosh(2\Omega_5\xi)+2\phi_2^2-\phi_5^2}}, \\
\frac{\gamma_7\gamma_9\mathrm{sn}(\Omega_6\xi,k_3)}{\sqrt{\gamma_7^2-\gamma_9^2+\gamma_9^2\mathrm{sn}^2(\Omega_6\xi,k_3)}} &\to \frac{\phi_2\phi_5\tanh\left(\Omega_5\xi\right)}{\sqrt{\phi_5^2-\phi_2^2+\phi_2^2\tanh^2\left(\Omega_5\xi\right)}} \\
&= \frac{\sqrt{2}\phi_2\phi_5\sinh(\Omega_5\xi)}{\sqrt{\phi_5^2\cosh(2\Omega_5\xi)+\phi_5^2-2\phi_2^2}}.
\end{aligned}$$

Therefore, as $h \to h_2$, the periodic wave solutions (3.35) converge to the solitary wave solutions (3.33), the periodic wave solutions (3.36) converge to the kink and anti-kink wave solutions (3.34), respectively.

Letting $h \to 0$, we have

$$\gamma_7 \to \phi_3, \gamma_8 \to \phi_6, \gamma_9 \to 0.$$

So that

$$\begin{aligned}
&k_3 = \frac{\gamma_9}{\gamma_8}\sqrt{\frac{\gamma_7^2-\gamma_8^2}{\gamma_7^2-\gamma_9^2}} \to 0, \Omega_6 = \frac{1}{3}\gamma_8\sqrt{-3\rho\left(\gamma_7^2-\gamma_9^2\right)} \to \frac{1}{2}\Omega_7, \\
&\mathrm{sn}\left(\Omega_6\xi, k_3\right) \to \sin\left(\frac{1}{2}\Omega_7\xi\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\gamma_7\gamma_8}{\sqrt{\gamma_8^2+(\gamma_7^2-\gamma_8^2)\mathrm{sn}^2(\Omega_6\xi,k_3)}} &\to \frac{\phi_3\phi_6}{\sqrt{\phi_6^2+(\phi_3^2-\phi_6^2)\sin^2\left(\frac{1}{2}\Omega_7\xi\right)}} \\
&= \frac{\sqrt{2}\phi_3\phi_6}{\sqrt{\phi_3^2+\phi_6^2-(\phi_3^2-\phi_6^2)\cos(\Omega_7\xi)}}.
\end{aligned}$$

Therefore, as $h \to 0$, the periodic wave solutions (3.35) converge to the periodic wave solutions (3.37). □

Proposition 3.4. *Let*

$$\Omega_8=-\frac{1}{4}\alpha\sqrt{-\rho},\ \Omega_9=\frac{1}{3}\gamma_{11}\sqrt{-3\rho\left(\gamma_{10}^2-\gamma_{12}^2\right)},$$

$$k_4=\frac{\gamma_{12}}{\gamma_{11}}\sqrt{\frac{\gamma_{10}^2-\gamma_{11}^2}{\gamma_{10}^2-\gamma_{12}^2}},\ \alpha_5^2=\frac{\gamma_{11}^2-\gamma_{10}^2}{\gamma_{11}^2},\ \alpha_6^2=\frac{\gamma_{12}^2}{\gamma_{12}^2-\gamma_{10}^2},$$

$$\psi_{10}(x-ct)=\frac{\sqrt{3}\sigma\alpha}{8c\Omega_8}\arcsin\left(\frac{2-\cosh\left(2\Omega_8(x-ct)\right)}{2\cosh\left(2\Omega_8(x-ct)\right)-1}\right)-\left(\frac{\sigma\alpha}{4c}+\frac{c}{2}\right)(x-ct),$$

$$\psi_{11}(x-ct)=-\frac{\sqrt{3}\sigma\alpha}{8c\Omega_8}\arcsin\left(\frac{\cosh\left(2\Omega_8(x-ct)\right)+2}{2\cosh\left(2\Omega_8(x-ct)\right)+1}\right)-\left(\frac{\sigma\alpha}{4c}+\frac{c}{2}\right)(x-ct),$$

$$\psi_{12}(x-ct)=\frac{\sigma\gamma_{10}^2}{c\Omega_9}\Pi\left(am\left(\Omega_9(x-ct),k_4\right),\alpha_5^2,k_4\right)-\frac{c}{2}(x-ct),$$

$$\psi_{13}(x-ct)=\frac{\sigma\gamma_{10}^2}{c\Omega_9}\Pi\left(am\left(\Omega_9(x-ct),k_4\right),\alpha_6^2,k_4\right)+\left(\frac{\sigma\gamma_{10}^2}{c}-\frac{c}{2}\right)(x-ct),$$

and $\gamma_{10},\gamma_{11},\gamma_{12}$ $(\gamma_{12}<\gamma_{11}<\gamma_{10})$ *satisfy equation*

$$(X^2-\gamma_{10}^2)(X^2-\gamma_{11}^2)(X^2-\gamma_{12}^2)=\left(X^4+\frac{3}{2}\alpha X^2+\frac{9}{16}\alpha^2\right)X^2+\frac{3h}{\rho},\ 0<h<\frac{\rho\alpha^3}{48},$$

then when $\alpha<0,\beta=\frac{3}{16}\alpha^2$, *equation (1.1) has two solitary wave solutions*

$$A(x,t)=\frac{1}{4c}\sqrt{9\alpha^2\sigma^2-16c^2\omega+12c^4}+\frac{2\sigma\alpha\cosh^2(\Omega_8(x-ct))}{c\left(2\cosh(2\Omega_8(x-ct))-1\right)},$$
$$B(x,t)=\pm\frac{\sqrt{-\alpha}\cosh(\Omega_8(x-ct))e^{i(\psi_{10}(x-ct)-\omega t)}}{\sqrt{2\cosh(2\Omega_8(x-ct))-1}},\quad \omega\leq\frac{3}{16c^2}\left(3\alpha^2\sigma+4c^4\right), \tag{3.69}$$

has two kink and anti-kink wave solutions

$$A(x,t)=\frac{1}{4c}\sqrt{9\alpha^2\sigma^2-16c^2\omega+12c^4}+\frac{2\sigma\alpha\sinh^2(\Omega_8(x-ct))}{c\left(2\cosh(2\Omega_8(x-ct))+1\right)},$$
$$B(x,t)=\pm\frac{\sqrt{-\alpha}\sinh(\Omega_8(x-ct))e^{i(\psi_{11}(x-ct)-\omega t)}}{\sqrt{2\cosh(2\Omega_8(x-ct))+1}},\quad \omega\leq\frac{3}{16c^2}\left(3\alpha^2\sigma+4c^4\right), \tag{3.70}$$

and has some periodic wave solutions

$$A(x,t)=\frac{1}{4c}\sqrt{9\alpha^2\sigma^2-16c^2\omega+12c^4}-\frac{2\sigma\gamma_{10}^2\gamma_{11}^2}{c\left(\gamma_{11}^2+(\gamma_{10}^2-\gamma_{11}^2)sn^2(\Omega_9(x-ct),k_4)\right)},$$
$$B(x,t)=\pm\frac{\gamma_{10}\gamma_{11}e^{i(\psi_{12}(x-ct)-\omega t)}}{\sqrt{\gamma_{11}^2+(\gamma_{10}^2-\gamma_{11}^2)sn^2(\Omega_9(x-ct),k_4)}},\quad \omega\leq\frac{3}{16c^2}\left(3\alpha^2\sigma+4c^4\right), \tag{3.71}$$

$$A(x,t)=\frac{1}{4c}\sqrt{9\alpha^2\sigma^2-16c^2\omega+12c^4}-\frac{2\sigma\gamma_{10}^2\gamma_{12}^2sn^2(\Omega_9(x-ct),k_4)}{c\left(\gamma_{10}^2-\gamma_{12}^2+\gamma_{12}^2sn^2(\Omega_9(x-ct),k_4)\right)},$$
$$B(x,t)=\pm\frac{\gamma_{10}\gamma_{12}sn(\Omega_9(x-ct),k_4)e^{i(\psi_{13}(x-ct)-\omega t)}}{\sqrt{\gamma_{10}^2-\gamma_{12}^2+\gamma_{12}^2sn^2(\Omega_9(x-ct),k_4)}},\quad \omega\leq\frac{3}{16c^2}\left(3\alpha^2\sigma+4c^4\right). \tag{3.72}$$

Moreover, as $h \to \frac{\rho\alpha^3}{48}$*, the periodic wave solutions (3.71) converge to the solitary wave solutions (3.69), the periodic wave solutions (3.72) converge to the kink and anti-kink wave solutions (3.70), respectively.*

Proof. When $\alpha < 0, \beta = \frac{3}{16}\alpha^2$, from Fig. 1(d), we see that there are two homoclinic orbits and two heteroclinic orbits defined by $H(\phi, y) = \frac{\rho\alpha^3}{48}$, the homoclinic orbits connecting with the saddle points $\left(\pm\frac{1}{2}\sqrt{-\alpha}, 0\right)$, respectively, and passing through the points $\pm\left(\sqrt{-\alpha}, 0\right)$, respectively, the heteroclinic orbits connecting with the saddle points $\left(\pm\frac{1}{2}\sqrt{-\alpha}, 0\right)$, and there are three periodic orbits defined by $H(\phi, y) = h \ \left(h \in \left(0, \frac{\rho\alpha^3}{48}\right)\right)$, two of them passing through the points $(\gamma_{10}, 0)$ and $(\gamma_{11}, 0)$, $(-\gamma_{11}, 0)$ and $(-\gamma_{10}, 0)$, respectively, and another passing through the points $(\gamma_{12}, 0)$ and $(-\gamma_{12}, 0)$, where $\gamma_{10}, \gamma_{11}, \gamma_{12}$ $(\gamma_{12} < \gamma_{11} < \gamma_{10})$ satisfy equation

$$(X^2 - \gamma_{10}^2)(X^2 - \gamma_{11}^2)(X^2 - \gamma_{12}^2) = \left(X^4 + \frac{3}{2}\alpha X^2 + \frac{9}{16}\alpha^2\right) X^2 + \frac{3h}{\rho}, \ 0 < h < \frac{\rho\alpha^3}{48}.$$

In (ϕ, y)-plane, their expressions are, respectively,

$$y = \pm\left(\phi^2 + \frac{1}{4}\alpha\right)\sqrt{-\frac{1}{3}\rho(-\alpha - \phi^2)}, \quad \frac{1}{2}\sqrt{-\alpha} < \phi \le \sqrt{-\alpha}, \tag{3.73}$$

$$y = \pm\left(\phi^2 + \frac{1}{4}\alpha\right)\sqrt{-\frac{1}{3}\rho(-\alpha - \phi^2)}, \quad -\sqrt{-\alpha} \le \phi < -\frac{1}{2}\sqrt{-\alpha}, \tag{3.74}$$

$$y = \pm\left(-\frac{1}{4}\alpha - \phi^2\right)\sqrt{-\frac{1}{3}\rho(-\alpha - \phi^2)}, \quad -\frac{1}{2}\sqrt{-\alpha} < \phi < \frac{1}{2}\sqrt{-\alpha}, \tag{3.75}$$

$$y = \pm\sqrt{-\frac{1}{3}\rho\left(\gamma_{10}^2 - \phi^2\right)\left(\phi^2 - \gamma_{11}^2\right)\left(\phi^2 - \gamma_{12}^2\right)}, \quad \gamma_{11} \le \phi \le \gamma_{10}, \tag{3.76}$$

$$y = \pm\sqrt{-\frac{1}{3}\rho\left(\gamma_{10}^2 - \phi^2\right)\left(\phi^2 - \gamma_{11}^2\right)\left(\phi^2 - \gamma_{12}^2\right)}, \quad -\gamma_{10} \le \phi \le -\gamma_{11}, \tag{3.77}$$

$$y = \pm\sqrt{-\frac{1}{3}\rho\left(\gamma_{10}^2 - \phi^2\right)\left(\gamma_{11}^2 - \phi^2\right)\left(\gamma_{12}^2 - \phi^2\right)}, \quad -\gamma_{12} \le \phi \le \gamma_{12}. \tag{3.78}$$

Substituting (3.73) and (3.74) into the $\frac{d\phi}{d\xi} = y$ and integrating them along the homoclinic orbits, respectively, we have

$$\int_{\phi}^{\sqrt{-\alpha}} \frac{ds}{\left(s^2 + \frac{1}{4}\alpha\right)\sqrt{-\alpha - s^2}} = \sqrt{-\frac{1}{3}\rho}|\xi|, \tag{3.79}$$

$$\int_{-\sqrt{-\alpha}}^{\phi} \frac{ds}{\left(s^2 + \frac{1}{4}\alpha\right)\sqrt{-\alpha - s^2}} = \sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.80}$$

Completing the integrals in (3.79) and (3.80), we obtain two solitary wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\frac{\sqrt{-\alpha}\cosh(\Omega_8\xi)}{\sqrt{2\cosh(2\Omega_8\xi) - 1}}, \tag{3.81}$$

where $\Omega_8 = -\frac{1}{4}\alpha\sqrt{-\rho}$. Substituting (3.81) into (1.5) yields equation

$$\psi(\xi) = -\frac{3\sigma\alpha}{8c\Omega_8}\int \frac{d\left(2\Omega_8\xi\right)}{2\cosh(2\Omega_8\xi) - 1} - \left(\frac{\sigma\alpha}{4c} + \frac{c}{2}\right)\xi. \tag{3.82}$$

Completing the integral in (3.82) and replacing $\psi(\xi)$ by $\psi_{10}(\xi)$, we have

$$\psi_{10}(x-ct) = \frac{\sqrt{3}\sigma\alpha}{8c\Omega_8} \arcsin\left(\frac{2-\cosh\left(2\Omega_8(x-ct)\right)}{2\cosh\left(2\Omega_8(x-ct)\right)-1}\right) - \left(\frac{\sigma\alpha}{4c}+\frac{c}{2}\right)(x-ct). \tag{3.83}$$

From (1.2), (1.3), (3.81), (3.83) and pay attention to that if $\beta = \frac{3}{16}\alpha^2$, then

$$g = \frac{1}{4c}\sqrt{9\alpha^2\sigma^2 - 16c^2\omega + 12c^4},$$

we obtain two solitary wave solutions of equation (1.1) as (3.69).

Substituting (3.75) into the $\frac{d\phi}{d\xi} = y$ and integrating it along the heteroclinic orbits, we have

$$\int_0^\phi \frac{ds}{\left(-\frac{1}{4}\alpha - s^2\right)\sqrt{-\alpha - s^2}} = \pm\sqrt{-\frac{1}{3}\rho}\xi. \tag{3.84}$$

Completing the integrals in (3.84), we obtain two kink and anti-kink wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\frac{\sqrt{-\alpha}\sinh(\Omega_8\xi)}{\sqrt{2\cosh(2\Omega_8\xi)+1}}. \tag{3.85}$$

Substituting (3.85) into (1.5) yields equation

$$\psi(\xi) = \frac{3\sigma\alpha}{8c\Omega_8}\int\frac{d\left(2\Omega_8\xi\right)}{1+2\cosh(2\Omega_8\xi)} - \left(\frac{\sigma\alpha}{4c}+\frac{c}{2}\right)\xi. \tag{3.86}$$

Completing the integral in (3.86) and replacing $\psi(\xi)$ by $\psi_{11}(\xi)$, we have

$$\psi_{11}(x-ct) = -\frac{\sqrt{3}\sigma\alpha}{8c\Omega_8} \arcsin\left(\frac{\cosh\left(2\Omega_8(x-ct)\right)+2}{2\cosh\left(2\Omega_8(x-ct)\right)+1}\right) - \left(\frac{\sigma\alpha}{4c}+\frac{c}{2}\right)(x-ct). \tag{3.87}$$

From (1.2), (1.3), (3.85), (3.87) and pay attention to that if $\beta = \frac{3}{16}\alpha^2$, then

$$g = \frac{1}{4c}\sqrt{9\alpha^2\sigma^2 - 16c^2\omega + 12c^4},$$

we obtain two kink and anti-kink wave solutions of equation (1.1) as (3.70).

Substituting (3.76) and (3.77) into the $\frac{d\phi}{d\xi} = y$ and integrating them along the periodic orbits, respectively, we have

$$\int_\phi^{\gamma_{10}} \frac{ds}{\sqrt{(\gamma_{10}^2 - s^2)(s^2-\gamma_{11}^2)(s^2-\gamma_{12}^2)}} = \sqrt{-\frac{1}{3}\rho}|\xi|, \tag{3.88}$$

$$\int_{-\gamma_{10}}^{\phi} \frac{ds}{\sqrt{(\gamma_{10}^2 - s^2)(s^2-\gamma_{11}^2)(s^2-\gamma_{12}^2)}} = \sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.89}$$

Completing the integrals in (3.88) and (3.89), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\frac{\gamma_{10}\gamma_{11}}{\sqrt{\gamma_{11}^2 + (\gamma_{10}^2-\gamma_{11}^2)\mathrm{sn}^2(\Omega_9\xi, k_4)}}, \tag{3.90}$$

where $\Omega_9 = \frac{1}{3}\gamma_{11}\sqrt{-3\rho\left(\gamma_{10}^2-\gamma_{12}^2\right)}, k_4 = \frac{\gamma_{12}}{\gamma_{11}}\sqrt{\frac{\gamma_{10}^2-\gamma_{11}^2}{\gamma_{10}^2-\gamma_{12}^2}}$. Substituting (3.90) into (1.5) yields equation

$$\psi(\xi) = \frac{\sigma\gamma_{10}^2}{c\Omega_9}\int \frac{d\left(\Omega_9\xi\right)}{1-\alpha_5^2\mathrm{sn}^2\left(\Omega_9\xi, k_4\right)} - \frac{c}{2}\xi, \tag{3.91}$$

where $\alpha_5^2 = \frac{\gamma_{11}^2-\gamma_{10}^2}{\gamma_{11}^2}$. Completing the integral in (3.91) and replacing $\psi(\xi)$ by $\psi_{12}(\xi)$, we have

$$\psi_{12}(x-ct) = \frac{\sigma\gamma_{10}^2}{c\Omega_9}\Pi\left(\mathrm{am}\left(\Omega_9(x-ct), k_4\right), \alpha_5^2, k_4\right) - \frac{c}{2}(x-ct). \tag{3.92}$$

From (1.2), (1.3), (3.90), (3.92) and pay attention to that if $\beta = \frac{3}{16}\alpha^2$, then

$$g = \frac{1}{4c}\sqrt{9\alpha^2\sigma^2 - 16c^2\omega + 12c^4},$$

we obtain two periodic wave solutions of equation (1.1) as (3.71).

Substituting (3.78) into the $\frac{d\phi}{d\xi} = y$ and integrating it along the periodic orbit, we have

$$\int_\phi^0 \frac{ds}{\sqrt{\left(\gamma_{10}^2-s^2\right)\left(\gamma_{11}^2-s^2\right)\left(\gamma_{12}^2-s^2\right)}} = \pm\sqrt{-\frac{1}{3}\rho}\xi. \tag{3.93}$$

Completing the integral in (3.93), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\frac{\gamma_{10}\gamma_{12}\mathrm{sn}(\Omega_9\xi, k_4)}{\sqrt{\gamma_{10}^2-\gamma_{12}^2+\gamma_{12}^2\mathrm{sn}^2(\Omega_9\xi, k_4)}}. \tag{3.94}$$

Substituting (3.94) into (1.5) yields equation

$$\psi(\xi) = \frac{\sigma\gamma_{10}^2}{c\Omega_9}\int \frac{d\left(\Omega_9\xi\right)}{1-\alpha_6^2\mathrm{sn}^2\left(\Omega_9\xi, k_4\right)} + \left(\frac{\sigma\gamma_{10}^2}{c} - \frac{c}{2}\right)\xi, \tag{3.95}$$

where $\alpha_6^2 = \frac{\gamma_{12}^2}{\gamma_{12}^2-\gamma_{10}^2}$. Completing the integral in (3.95) and replacing $\psi(\xi)$ by $\psi_{13}(\xi)$, we have

$$\psi_{13}(x-ct) = \frac{\sigma\gamma_{10}^2}{c\Omega_9}\Pi\left(\mathrm{am}\left(\Omega_9(x-ct), k_4\right), \alpha_6^2, k_4\right) + \left(\frac{\sigma\gamma_{10}^2}{c} - \frac{c}{2}\right)(x-ct). \tag{3.96}$$

From (1.2), (1.3), (3.94), (3.96) and pay attention to that if $\beta = \frac{3}{16}\alpha^2$, then

$$g = \frac{1}{4c}\sqrt{9\alpha^2\sigma^2 - 16c^2\omega + 12c^4},$$

we obtain two periodic wave solutions of equation (1.1) as (3.72).

Letting $h \to \frac{\rho\alpha^3}{48}$, we have

$$\gamma_{10} \to \sqrt{-\alpha}, \gamma_{11} \to \frac{1}{2}\sqrt{-\alpha}, \gamma_{12} \to \frac{1}{2}\sqrt{-\alpha}.$$

So that

$$k_4 = \frac{\gamma_{12}}{\gamma_{11}}\sqrt{\frac{\gamma_{10}^2-\gamma_{11}^2}{\gamma_{10}^2-\gamma_{12}^2}} \to 1, \quad \Omega_9 = \frac{1}{3}\gamma_{11}\sqrt{-3\rho\left(\gamma_{10}^2-\gamma_{12}^2\right)} \to \Omega_8,$$

$$\operatorname{sn}\left(\Omega_9\xi, k_4\right) \to \tanh\left(\Omega_8\xi\right),$$

$$\frac{\gamma_{10}\gamma_{11}}{\sqrt{\gamma_{11}^2+(\gamma_{10}^2-\gamma_{11}^2)\operatorname{sn}^2(\Omega_9\xi,k_4)}} \to \frac{-\frac{1}{2}\alpha}{\sqrt{-\frac{1}{4}\alpha-\frac{3}{4}\alpha\tanh^2\left(\Omega_8\xi\right)}} = \frac{\sqrt{-\alpha}\cosh(\Omega_8\xi)}{\sqrt{2\cosh(2\Omega_8\xi)-1}},$$

$$\frac{\gamma_{10}\gamma_{12}\operatorname{sn}(\Omega_9\xi,k_4)}{\sqrt{\gamma_{10}^2-\gamma_{12}^2+\gamma_{12}^2\operatorname{sn}^2(\Omega_9\xi,k_4)}} \to \frac{-\frac{1}{2}\alpha\tanh\left(\Omega_8\xi\right)}{\sqrt{-\frac{3}{4}\alpha-\frac{1}{4}\alpha\tanh^2\left(\Omega_8\xi\right)}} = \frac{\sqrt{-\alpha}\sinh(\Omega_8\xi)}{\sqrt{2\cosh(2\Omega_8\xi)+1}}.$$

Therefore, as $h \to \frac{\rho\alpha^3}{48}$, the periodic wave solutions (3.71) converge to the solitary wave solutions (3.69), the periodic wave solutions (3.72) converge to the kink and anti-kink wave solutions (3.70), respectively. □

Proposition 3.5. *Let*

$$\Omega_{10} = \frac{1}{3}\gamma_{14}\sqrt{-3\rho\left(\gamma_{13}^2-\gamma_{15}^2\right)}, \ \Omega_{11} = \frac{2}{3}\phi_1\sqrt{3\rho\left(\phi_7^2-\phi_1^2\right)},$$

$$k_5 = \frac{\gamma_{15}}{\gamma_{14}}\sqrt{\frac{\gamma_{13}^2-\gamma_{14}^2}{\gamma_{13}^2-\gamma_{15}^2}}, \ \alpha_7^2 = \frac{\gamma_{14}^2-\gamma_{13}^2}{\gamma_{14}^2}, \ \alpha_8^2 = \frac{\gamma_{15}^2}{\gamma_{15}^2-\gamma_{13}^2},$$

$$\phi_1 = \frac{1}{2}\sqrt{-2\alpha+2\sqrt{\alpha^2-4\beta}}, \ \phi_7 = \frac{1}{2}\sqrt{-2\alpha-4\sqrt{\alpha^2-4\beta}},$$

$$\psi_{14}(x-ct) = \frac{\sigma\gamma_{13}^2}{c\Omega_{10}}\Pi\left(am\left(\Omega_{10}(x-ct),k_5\right),\alpha_7^2,k_5\right) - \frac{c}{2}(x-ct),$$

$$\psi_{15}(x-ct) = \frac{\sigma\gamma_{13}^2}{c\Omega_{10}}\Pi\left(am\left(\Omega_{10}(x-ct),k_5\right),\alpha_8^2,k_5\right) + \left(\frac{\sigma\gamma_{13}^2}{c}-\frac{c}{2}\right)(x-ct),$$

$$\psi_{16}(x-ct) = -\frac{2\sigma\phi_1\sqrt{\phi_1^2-\phi_7^2}}{c\Omega_{11}}\arctan\left(\frac{\phi_1}{\sqrt{\phi_1^2-\phi_7^2}}\tan\left(\frac{1}{2}\Omega_{11}(x-ct)\right)\right) + \left(\frac{\sigma\phi_1^2}{c}-\frac{c}{2}\right)(x-ct),$$

and $\gamma_{13}, \gamma_{14}, \gamma_{15}$ $(\gamma_{15} < \gamma_{14} < \gamma_{13})$ *satisfy equation*

$$(X^2-\gamma_{13}^2)(X^2-\gamma_{14}^2)(X^2-\gamma_{15}^2) = \left(X^4+\frac{3}{2}\alpha X^2+3\beta\right)X^2+\frac{3h}{\rho}, \ h_1 < h < h_2,$$

then when $\alpha < 0$, $\frac{3}{16}\alpha^2 < \beta < \frac{1}{4}\alpha^2$, *equation (1.1) has two solitary wave solutions as (3.33), has two kink and anti-kink wave solutions as (3.34), and has some periodic* *wave solutions*

$$
\begin{aligned}
A(x,t) &= g - \frac{2\sigma\gamma_{13}^2\gamma_{14}^2}{c\,(\gamma_{14}^2 + (\gamma_{13}^2 - \gamma_{14}^2)sn^2(\Omega_{10}(x-ct),k_5))},\\
B(x,t) &= \pm\frac{\gamma_{13}\gamma_{14}\,e^{i(\psi_{14}(x-ct)-\omega t)}}{\sqrt{\gamma_{14}^2 + (\gamma_{13}^2 - \gamma_{14}^2)sn^2(\Omega_{10}(x-ct),k_5)}},
\end{aligned}
\tag{3.97}
$$

$$
\begin{aligned}
A(x,t) &= g - \frac{2\sigma\gamma_{13}^2\gamma_{15}^2 sn^2(\Omega_{10}(x-ct),k_5)}{c\,(\gamma_{13}^2 - \gamma_{15}^2 + \gamma_{15}^2 sn^2(\Omega_{10}(x-ct),k_5))},\\
B(x,t) &= \pm\frac{\gamma_{13}\gamma_{15} sn(\Omega_{10}(x-ct),k_5)\,e^{i(\psi_{15}(x-ct)-\omega t)}}{\sqrt{\gamma_{13}^2 - \gamma_{15}^2 + \gamma_{15}^2 sn^2(\Omega_{10}(x-ct),k_5)}},
\end{aligned}
\tag{3.98}
$$

$$
\begin{aligned}
A(x,t) &= g - \frac{2\sigma\phi_1^2\phi_7^2\,(\cos(\Omega_{11}(x-ct)) - 1)}{c\,(\phi_7^2 - 2\phi_1^2 + \phi_7^2\cos(\Omega_{11}(x-ct)))},\\
B(x,t) &= \pm\frac{\phi_1\phi_7\sqrt{\cos(\Omega_{11}(x-ct)) - 1}\,e^{i(\psi_{16}(x-ct)-\omega t)}}{\sqrt{\phi_7^2 - 2\phi_1^2 + \phi_7^2\cos(\Omega_{11}(x-ct))}}.
\end{aligned}
\tag{3.99}
$$

Moreover, as $h \to h_2$, the periodic wave solutions (3.97) converge to the solitary wave solutions (3.33), the periodic wave solutions (3.98) converge to the kink and anti-kink wave solutions (3.34), respectively, and as $h \to h_1$, the periodic wave solutions (3.98) converge to the periodic wave solutions (3.99).

Proof. We only proof the results about the periodic wave solutions, the proof for other results are similar to the Proof for Proposition 3.3, we omit them here. When $\alpha < 0$, $\frac{3}{16}\alpha^2 < \beta < \frac{1}{4}\alpha^2$, from Fig. 1(e), we see that there are three periodic orbits defined by $H(\phi,y) = h\ \ (h \in (h_1,h_2))$, two of them passing through the points $(\gamma_{13},0)$ and $(\gamma_{14},0)$, $(-\gamma_{14},0)$ and $(-\gamma_{13},0)$, respectively, and another passing through the points $(-\gamma_{15},0)$ and $(\gamma_{15},0)$, and there is one periodic orbits defined by $H(\phi,y) = h_1$ passing through the points $(-\phi_7,0)$ and $(\phi_7,0)$, where $\phi_7 = \frac{1}{2}\sqrt{-2\alpha - 4\sqrt{\alpha^2-4\beta}}$, and $\gamma_{13},\gamma_{14},\gamma_{15}$ $(\gamma_{15} < \gamma_{14} < \gamma_{13})$ satisfy equation

$$
(X^2-\gamma_{13}^2)(X^2-\gamma_{14}^2)(X^2-\gamma_{15}^2) = \left(X^4 + \frac{3}{2}\alpha X^2 + 3\beta\right)X^2 + \frac{3h}{\rho},\ \ h_1 < h < h_2.
$$

In (ϕ,y)-plane, their expressions are, respectively,

$$
y = \pm\sqrt{-\frac{1}{3}\rho\,(\gamma_{13}^2-\phi^2)\,(\phi^2-\gamma_{14}^2)\,(\phi^2-\gamma_{15}^2)},\quad \gamma_{14} \le \phi \le \gamma_{13}, \tag{3.100}
$$

$$
y = \pm\sqrt{-\frac{1}{3}\rho\,(\gamma_{13}^2-\phi^2)\,(\phi^2-\gamma_{14}^2)\,(\phi^2-\gamma_{15}^2)},\quad -\gamma_{13} \le \phi \le -\gamma_{14}, \tag{3.101}
$$

$$
y = \pm\sqrt{-\frac{1}{3}\rho\,(\gamma_{13}^2-\phi^2)\,(\gamma_{14}^2-\phi^2)\,(\gamma_{15}^2-\phi^2)},\quad -\gamma_{15} \le \phi \le \gamma_{15}, \tag{3.102}
$$

$$
y = \pm\left(\phi_1^2-\phi^2\right)\sqrt{-\frac{1}{3}\rho\,(\phi_7^2-\phi^2)},\quad -\phi_7 \le \phi \le \phi_7, \tag{3.103}
$$

where $\phi_1 = \frac{1}{2}\sqrt{-2\alpha + 2\sqrt{\alpha^2-4\beta}}$.

Substituting (3.100) and (3.101) into the $\frac{d\phi}{d\xi}=y$ and integrating them along the periodic orbits, respectively, we have

$$\int_{\phi}^{\gamma_{13}}\frac{ds}{\sqrt{\left(\gamma_{13}^2-s^2\right)\left(s^2-\gamma_{14}^2\right)\left(s^2-\gamma_{15}^2\right)}}=\sqrt{-\frac{1}{3}\rho}|\xi|, \tag{3.104}$$

$$\int_{-\gamma_{13}}^{\phi}\frac{ds}{\sqrt{\left(\gamma_{13}^2-s^2\right)\left(s^2-\gamma_{14}^2\right)\left(s^2-\gamma_{15}^2\right)}}=\sqrt{-\frac{1}{3}\rho}|\xi|. \tag{3.105}$$

Completing the integrals in (3.104) and (3.105), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi)=\pm\frac{\gamma_{13}\gamma_{14}}{\sqrt{\gamma_{14}^2+(\gamma_{13}^2-\gamma_{14}^2)\mathrm{sn}^2(\Omega_{10}\xi,k_5)}}, \tag{3.106}$$

where $\Omega_{10}=\frac{1}{3}\gamma_{14}\sqrt{-3\rho\left(\gamma_{13}^2-\gamma_{15}^2\right)}, k_5=\frac{\gamma_{15}}{\gamma_{14}}\sqrt{\frac{\gamma_{13}^2-\gamma_{14}^2}{\gamma_{13}^2-\gamma_{15}^2}}$. Substituting (3.106) into (1.5) yields equation

$$\psi(\xi)=\frac{\sigma\gamma_{13}^2}{c\Omega_{10}}\int\frac{d\left(\Omega_{10}\xi\right)}{1-\alpha_7^2\mathrm{sn}^2\left(\Omega_{10}\xi,k_5\right)}-\frac{c}{2}\xi, \tag{3.107}$$

where $\alpha_7^2=\frac{\gamma_{14}^2-\gamma_{13}^2}{\gamma_{14}^2}$. Completing the integral in (3.107) and replacing $\psi(\xi)$ by $\psi_{14}(\xi)$, we have

$$\psi_{14}(x-ct)=\frac{\sigma\gamma_{13}^2}{c\Omega_{10}}\Pi\left(\mathrm{am}\left(\Omega_{10}(x-ct),k_5\right),\alpha_7^2,k_5\right)-\frac{c}{2}(x-ct). \tag{3.108}$$

From (1.2), (1.3), (3.106) and (3.108), we obtain two periodic wave solutions of equation (1.1) as (3.97).

Substituting (3.102) into the $\frac{d\phi}{d\xi}=y$ and integrating it along the periodic orbit, we have

$$\int_{\phi}^{0}\frac{ds}{\sqrt{\left(\gamma_{13}^2-s^2\right)\left(\gamma_{14}^2-s^2\right)\left(\gamma_{15}^2-s^2\right)}}=\pm\sqrt{-\frac{1}{3}\rho}\xi. \tag{3.109}$$

Completing the integral in (3.109), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi)=\pm\frac{\gamma_{13}\gamma_{15}\mathrm{sn}(\Omega_{10}\xi,k_5)}{\sqrt{\gamma_{13}^2-\gamma_{15}^2+\gamma_{15}^2\mathrm{sn}^2(\Omega_{10}\xi,k_5)}}. \tag{3.110}$$

Substituting (3.110) into (1.5) yields equation

$$\psi(\xi)=\frac{\sigma\gamma_{13}^2}{c\Omega_{10}}\int\frac{d\left(\Omega_{10}\xi\right)}{1-\alpha_8^2\mathrm{sn}^2\left(\Omega_{10}\xi,k_5\right)}+\left(\frac{\sigma\gamma_{13}^2}{c}-\frac{c}{2}\right)\xi, \tag{3.111}$$

where $\alpha_8^2=\frac{\gamma_{15}^2}{\gamma_{15}^2-\gamma_{13}^2}$. Completing the integral in (3.111) and replacing $\psi(\xi)$ by $\psi_{15}(\xi)$, we have

$$\psi_{15}(x-ct)=\frac{\sigma\gamma_{13}^2}{c\Omega_{10}}\Pi\left(\mathrm{am}\left(\Omega_{10}(x-ct),k_5\right),\alpha_8^2,k_5\right)+\left(\frac{\sigma\gamma_{13}^2}{c}-\frac{c}{2}\right)(x-ct). \tag{3.112}$$

From (1.2), (1.3), (3.110) and (3.112), we obtain two periodic wave solutions of equation (1.1) as (3.98).

Substituting (3.103) into the $\frac{d\phi}{d\xi} = y$ and integrating it along the periodic orbit, we have

$$\int_{\phi}^{0} \frac{ds}{(\phi_1^2 - s^2)\sqrt{\phi_7^2 - s^2}} = \pm\sqrt{-\frac{1}{3}}\rho\xi. \tag{3.113}$$

Completing the integrals in (3.113), we obtain two periodic wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\frac{\phi_1\phi_7\sqrt{\cos(\Omega_{11}\xi) - 1}}{\sqrt{\phi_7^2 - 2\phi_1^2 + \phi_7^2\cos(\Omega_{11}\xi)}}, \tag{3.114}$$

where $\Omega_{11} = \frac{2}{3}\phi_1\sqrt{3\rho\left(\phi_7^2 - \phi_1^2\right)}$. Substituting (3.114) into (1.5) yields equation

$$\psi(\xi) = \frac{2\sigma\phi_1^2\left(\phi_1^2 - \phi_7^2\right)}{c\Omega_{11}}\int \frac{d\left(\Omega_{11}\xi\right)}{\phi_7^2 - 2\phi_1^2 + \phi_7^2\cos\left(\Omega_{11}\xi\right)} + \left(\frac{\sigma\phi_1^2}{c} - \frac{c}{2}\right)\xi. \tag{3.115}$$

Completing the integral in (3.115) and replacing $\psi(\xi)$ by $\psi_{16}(\xi)$, we have

$$\begin{aligned}\psi_{16}(x - ct) = &-\frac{2\sigma\phi_1\sqrt{\phi_1^2 - \phi_7^2}}{c\Omega_{11}}\arctan\left(\frac{\phi_1}{\sqrt{\phi_1^2 - \phi_7^2}}\tan\left(\frac{1}{2}\Omega_{11}(x - ct)\right)\right)\\ &+\left(\frac{\sigma\phi_1^2}{c} - \frac{c}{2}\right)(x - ct).\end{aligned} \tag{3.116}$$

From (1.2), (1.3), (3.114) and (3.116), we obtain two periodic wave solutions of equation (1.1) as (3.99).

Letting $h \to h_2$, we have

$$\gamma_{13} \to \phi_5, \gamma_{14} \to \phi_2, \gamma_{15} \to \phi_2.$$

So that

$$\begin{aligned}
&k_5 = \frac{\gamma_{15}}{\gamma_{14}}\sqrt{\frac{\gamma_{13}^2 - \gamma_{14}^2}{\gamma_{13}^2 - \gamma_{15}^2}} \to 1, \Omega_{10} = \frac{1}{3}\gamma_{14}\sqrt{-3\rho\left(\gamma_{13}^2 - \gamma_{15}^2\right)} \to \Omega_5,\\
&\operatorname{sn}\left(\Omega_{10}\xi, k_5\right) \to \tanh\left(\Omega_5\xi\right),\\
&\frac{\gamma_{13}\gamma_{14}}{\sqrt{\gamma_{14}^2 + (\gamma_{13}^2 - \gamma_{14}^2)\operatorname{sn}^2(\Omega_{10}\xi, k_5)}} \to \frac{\phi_2\phi_5}{\sqrt{\phi_2^2 + \left(\phi_5^2 - \phi_2^2\right)\tanh^2\left(\Omega_5\xi\right)}}\\
&\qquad = \frac{\sqrt{2}\phi_2\phi_5\cosh(\Omega_5\xi)}{\sqrt{\phi_5^2\cosh(2\Omega_5\xi) + 2\phi_2^2 - \phi_5^2}},\\
&\frac{\gamma_{13}\gamma_{15}\operatorname{sn}(\Omega_{10}\xi, k_5)}{\sqrt{\gamma_{13}^2 - \gamma_{15}^2 + \gamma_{15}^2\operatorname{sn}^2(\Omega_{10}\xi, k_5)}} \to \frac{\phi_2\phi_5\tanh\left(\Omega_5\xi\right)}{\sqrt{\phi_5^2 - \phi_2^2 + \phi_2^2\tanh^2\left(\Omega_5\xi\right)}}\\
&\qquad = \frac{\sqrt{2}\phi_2\phi_5\sinh(\Omega_5\xi)}{\sqrt{\phi_5^2\cosh(2\Omega_5\xi) + \phi_5^2 - 2\phi_2^2}}.
\end{aligned}$$

Therefore, as $h \to h_2$, the periodic wave solutions (3.97) converge to the solitary wave solutions (3.33), the periodic wave solutions (3.98) converge to the kink and anti-kink wave solutions (3.34), respectively.

Letting $h \to h_1$, we have

$$\gamma_{13} \to \phi_1, \gamma_{14} \to \phi_1, \gamma_{15} \to \phi_7.$$

So that

$$\begin{aligned}
&k_5 = \frac{\gamma_{15}}{\gamma_{14}}\sqrt{\frac{\gamma_{13}^2-\gamma_{14}^2}{\gamma_{13}^2-\gamma_{15}^2}} \to 0, \Omega_{10} = \frac{1}{3}\gamma_{14}\sqrt{-3\rho\left(\gamma_{13}^2-\gamma_{15}^2\right)} \to \frac{1}{2}\Omega_{11},\\
&\mathrm{sn}\left(\Omega_{10}\xi, k_5\right) \to \sin\left(\frac{1}{2}\Omega_{11}\xi\right),\\
&\frac{\gamma_{13}\gamma_{15}\mathrm{sn}(\Omega_{10}\xi,k_5)}{\sqrt{\gamma_{13}^2-\gamma_{15}^2+\gamma_{15}^2\mathrm{sn}^2(\Omega_{10}\xi,k_5)}} \to \frac{\phi_1\phi_7\sin\left(\frac{1}{2}\Omega_{11}\xi\right)}{\sqrt{\phi_1^2-\phi_7^2+\phi_7^2\sin^2\left(\frac{1}{2}\Omega_{11}\xi\right)}}\\
&\qquad\qquad = \frac{\phi_1\phi_7\sqrt{\cos\left(\Omega_{11}\xi\right)-1}}{\sqrt{\phi_7^2-2\phi_1^2+\phi_7^2\cos(\Omega_{11}\xi)}}.
\end{aligned}$$

Therefore, as $h \to h_1$, the periodic wave solutions (3.98) converge to the periodic wave solutions (3.99). □

Proposition 3.6. *When $\alpha < 0, \beta = \frac{1}{4}\alpha^2$, equation (1.1) has two kink and anti-kink wave solutions*

$$\begin{aligned}
&A(x,t) = \frac{1}{2c}\sqrt{3\alpha^2\sigma^2-4c^2\omega+3c^4} + \frac{\sigma\alpha\left(\Omega_{12}(x-ct)\right)^2}{c\left(1+\left(\Omega_{12}(x-ct)\right)^2\right)},\\
&B(x,t) = \pm\Omega_{12}(x-ct)\sqrt{\frac{-\alpha}{2\left(1+\left(\Omega_{12}(x-ct)\right)^2\right)}}e^{i(\psi_{17}(x-ct)-\omega t)}, \qquad (3.117)\\
&\omega \le \frac{1}{4c}\left(3\alpha^2\sigma^2+3c^4\right),
\end{aligned}$$

where $\Omega_{12} = -\frac{1}{6}\alpha\sqrt{-3\rho}$, $\psi_{17}(x-ct) = \frac{\sigma\alpha}{2c\Omega_{12}}\arctan\left(\Omega_{12}(x-ct)\right)-\left(\frac{\sigma\alpha}{2c}+\frac{c}{2}\right)(x-ct)$.

Proof. When $\alpha < 0, \beta = \frac{1}{4}\alpha^2$, from Fig. 1(f), we see that there are two heteroclinic orbits defined by $H(\phi,y) = \frac{\rho\alpha^3}{24}$ connecting with the cusps $\left(\pm\frac{1}{2}\sqrt{-2\alpha}, 0\right)$. In (ϕ,y)-plane, their expressions are

$$y = \pm\left(-\frac{1}{2}\alpha-\phi^2\right)\sqrt{-\frac{1}{3}\rho\left(-\frac{1}{2}\alpha-\phi^2\right)}, \quad -\frac{1}{2}\sqrt{-2\alpha} < \phi < \frac{1}{2}\sqrt{-2\alpha}. \qquad (3.118)$$

Substituting (3.118) into the $\frac{d\phi}{d\xi} = y$ and integrating it along the heteroclinic orbits, we have

$$\int_\phi^0 \frac{ds}{\left(-\frac{1}{2}\alpha-s^2\right)\sqrt{-\frac{1}{2}\alpha-s^2}} = \pm\sqrt{-\frac{1}{3}\rho}\xi. \qquad (3.119)$$

Completing the integrals in (3.119), we obtain two kink and anti-kink wave solutions of (1.6) as follows:

$$\phi(\xi) = \pm\Omega_{12}\xi\sqrt{\frac{-\alpha}{2\left(1+(\Omega_{12}\xi)^2\right)}}, \tag{3.120}$$

where $\Omega_{12} = -\frac{1}{6}\alpha\sqrt{-3\rho}$. Substituting (3.120) into (1.5) yields equation

$$\psi(\xi) = \frac{\sigma\alpha}{2c\Omega_{12}}\int\frac{d\left(\Omega_{12}\xi\right)}{1+(\Omega_{12}\xi)^2} - \left(\frac{\sigma\alpha}{2c}+\frac{c}{2}\right)\xi. \tag{3.121}$$

Completing the integral in (3.121) and replacing $\psi(\xi)$ by $\psi_{17}(\xi)$, we have

$$\psi_{17}(x-ct) = \frac{\sigma\alpha}{2c\Omega_{12}}\arctan\left(\Omega_{12}(x-ct)\right) - \left(\frac{\sigma\alpha}{2c}+\frac{c}{2}\right)(x-ct). \tag{3.122}$$

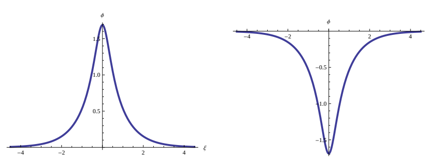
From (1.2), (1.3), (3.120), (3.122) and pay attention to that if $\beta = \frac{1}{4}\alpha^2$, then

$$g = \frac{1}{2c}\sqrt{3\alpha^2\sigma^2 - 4c^2\omega + 3c^4},$$

we obtain two kink and anti-kink wave solutions of equation (1.1) as (3.117). □

4. Numerical simulations

Example 4.1. If $\rho = -1.5, \alpha = -1.2, \beta = -1.0$, then $h_1 \approx -2.702018916$. Taking $h = -2.0$, we have $\gamma_1 \approx 1.533774952, \gamma_2 \approx 1.027947900, \gamma_3 \approx 1.268519802$. Taking $h = -0.1$, we have $\gamma_1 \approx 1.683377144, \gamma_2 \approx 0.2535277454, \gamma_3 \approx 1.047871618$. Taking $h = -0.000001$, we have $\gamma_1 \approx 1.688763438, \gamma_2 \approx 0.0008164964176, \gamma_3 \approx 1.025632788$. Taking $h = -0.000000001$, we have $\gamma_1 \approx 1.688763491, \gamma_2 \approx 0.000025819$, $\gamma_3 \approx 1.025632551$. The profiles of (3.9) are shown in Fig. 2(a) and (b), the limiting process of (3.14) are similar to that in Fig. 3(a)-(h).



(a) (b)

Figure 2. The profiles of (3.9).

Example 4.2. If $\rho = -2.0, \alpha = -1.5$, then $-\frac{\rho\alpha^3}{6} = -1.125$. Taking $h = -1.0$, we have $\gamma_4 \approx 1.331401565, \gamma_5 \approx 1.090430135, \gamma_6 \approx 0.8436041767$. Taking $h = -0.05$, we have $\gamma_4 \approx 1.494986778, \gamma_5 \approx 0.4368609095, \gamma_6 \approx 0.4193243616$. Taking $h = -0.005$, we have $\gamma_4 \approx 1.499505440, \gamma_5 \approx 0.2418688693, \gamma_6 \approx 0.2387825643$. Taking $h = -0.001$, we have $\gamma_4 \approx 1.499901205, \gamma_5 \approx 0.1611527290, \gamma_6 \approx 0.1602305454$. The profiles of (3.25) are shown in Fig. 4(a) and (b), the limiting process of (3.30) are similar to that in Fig. 5(a)-(h).

Example 4.3. If $\rho = -1.5, \alpha = -2.0, \beta = 0.5$, then $h_2 \approx 0.1035533906$. Taking $h = 0.001$, we have $\gamma_7 \approx 1.538347555, \gamma_8 \approx 0.7950785458, \gamma_9 \approx 0.03656373036$. Taking

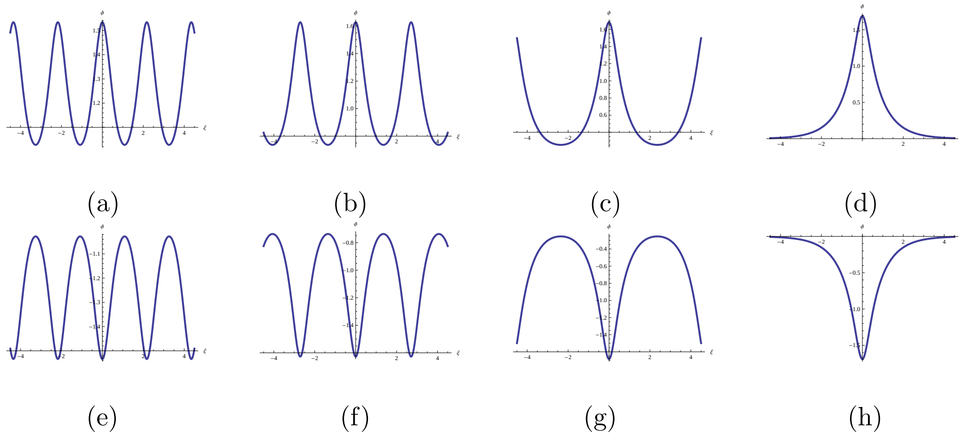


Figure 3. The limiting precess of (3.14) tends to (3.9) when $h \to 0$.

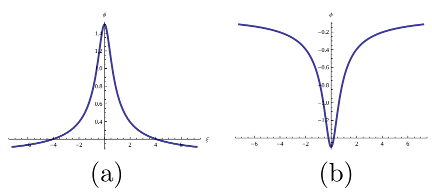


Figure 4. The profiles of (3.25).

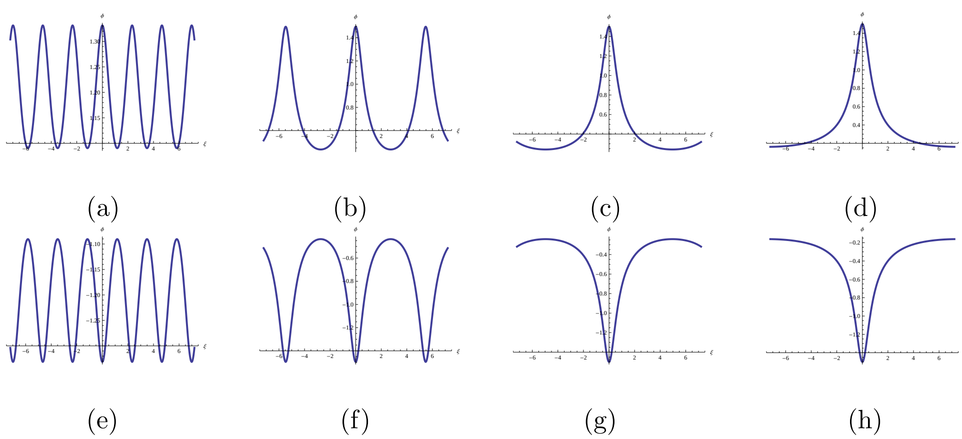


Figure 5. The limiting precess of (3.30) tends to (3.25) when $h \to 0$.

$h = 0.07$, we have $\gamma_7 \approx 1.548898304, \gamma_8 \approx 0.6921817194, \gamma_9 \approx 0.3489964330$. Taking $h = 0.1$, we have $\gamma_7 \approx 1.553264923, \gamma_8 \approx 0.5929516612, \gamma_9 \approx 0.4855681285$. Taking $h = 0.1035533$, we have $\gamma_7 \approx 1.553773961, \gamma_8 \approx 0.5414660585, \gamma_9 \approx 0.5409260443$. The profiles of (3.48) are shown in Fig. 6(a) and (b), the limiting process of (3.57) are similar to that in Fig. 7(a)-(h). The profiles of (3.52) are shown in Fig. 8(a) and (b), the limiting process of (3.61) are similar to that in Fig. 9(a)-(h).

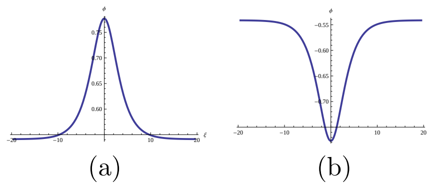


Figure 6. The profiles of (3.48).

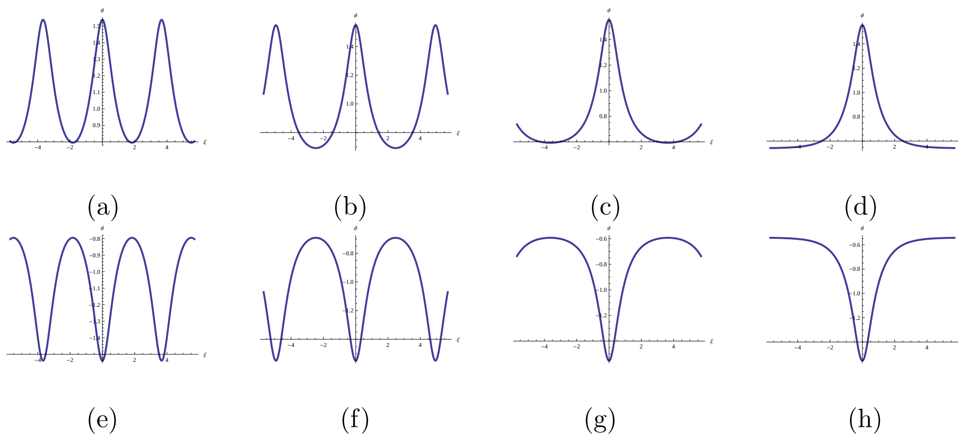


Figure 7. The limiting precess of (3.57) tends to (3.48) when $h \to h_2$.

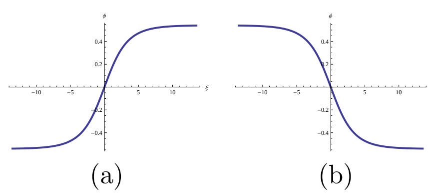


Figure 8. The profiles of (3.52).

Example 4.4. If $\rho = -1.5, \alpha = -2.0, \beta = 0.5$, then $h_2 \approx 0.1035533906$. Taking $h = 0.07$, we have $\gamma_7 \approx 1.548898304, \gamma_8 \approx 0.6921817194, \gamma_9 \approx 0.3489964330$. Taking $h = 0.03$, we have $\gamma_7 \approx 1.542873100, \gamma_8 \approx 0.7587922583, \gamma_9 \approx 0.2092293118$. Taking $h = 0.01$, we have $\gamma_7 \approx 1.539766907, \gamma_8 \approx 0.7844810203, \gamma_9 \approx 0.1170786149$. Taking $h = 0.0001$, we have $\gamma_7 \approx 1.538204864, \gamma_8 \approx 0.7961108128, \gamma_9 \approx 0.01154854563$. The profiles of (3.66) are shown in Fig. 10(a) and (b), the limiting process of (3.57) are similar to that in Fig. 11(a)-(h).

Example 4.5. If $\rho = -1.5, \alpha = -1.2, \beta = 0.31$, then $h_1 \approx 0.05181966008, h_2 \approx 0.07418033992$. Taking $h = 0.072$, we have $\gamma_{13} \approx 1.018443976, \gamma_{14} \approx 0.6802733772$, $\gamma_{15} \approx 0.5477225575$. Taking $h = 0.069$, we have $\gamma_{13} \approx 1.011131104, \gamma_{14} \approx 0.7183708135$, $\gamma_{15} \approx 0.5114266959$. Taking $h = 0.065$, we have $\gamma_{13} \approx 1.0, \gamma_{14} \approx 0.7571030846$, $\gamma_{15} \approx 0.4762299017$. Taking $h = 0.05182$, we have $\gamma_{13} \approx 0.9080818774, \gamma_{14} \approx 0.9069726505$, $\gamma_{15} \approx 0.3908809475$. The profiles of (3.114) are shown in Fig. 12(a) and (b), the limiting process of (3.110) are similar to that in Fig. 13(a)-(h).

Example 4.6. Taking $\rho = -1.5, \alpha = -1.2$, the profiles of (3.120) are shown in Fig.

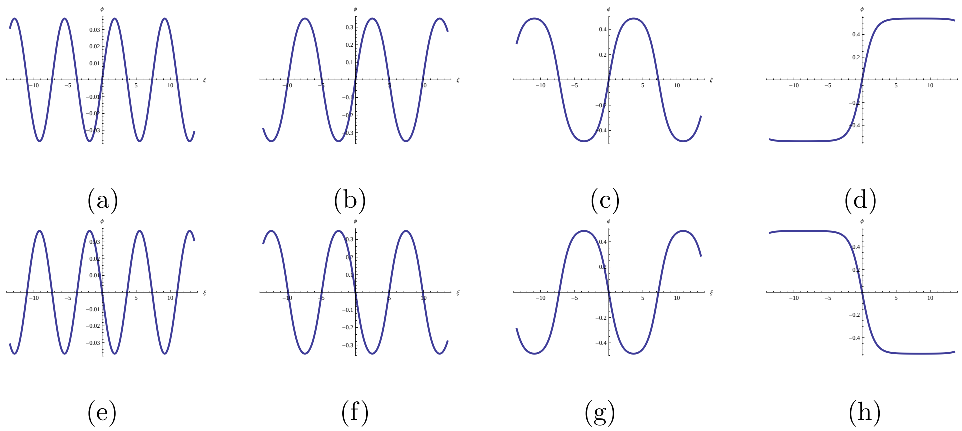


Figure 9. The limiting precess of (3.61) tends to (3.52) when $h \to h_2$.

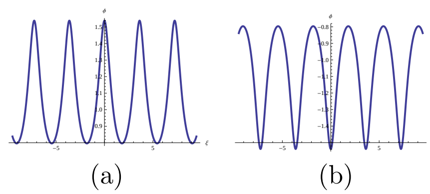


Figure 10. The profiles of (3.66).

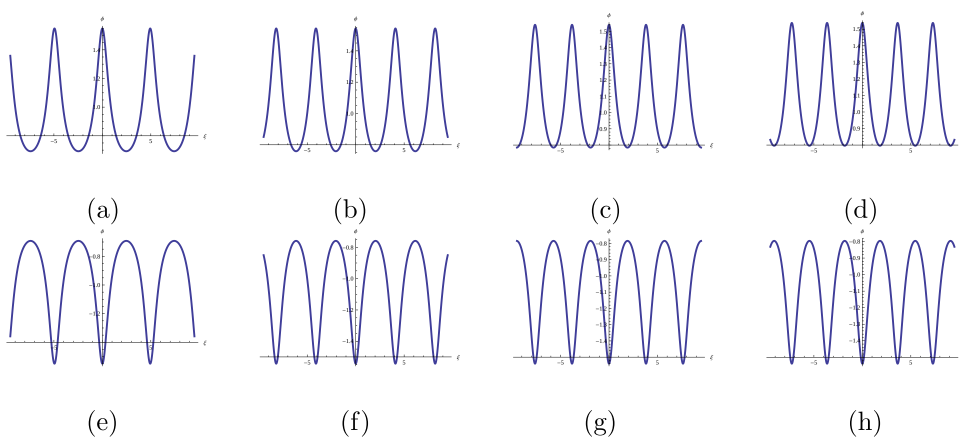


Figure 11. The limiting precess of (3.57) tends to (3.66) when $h \to 0$.

14(a) and (b).

5. Conclusion

In this paper, we investigate the dynamic properties and present some explicit exact traveling wave solutions of the long waves-short waves model (1.1) using

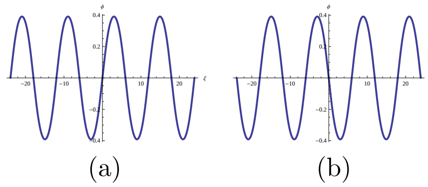


Figure 12. The profiles of (3.114).

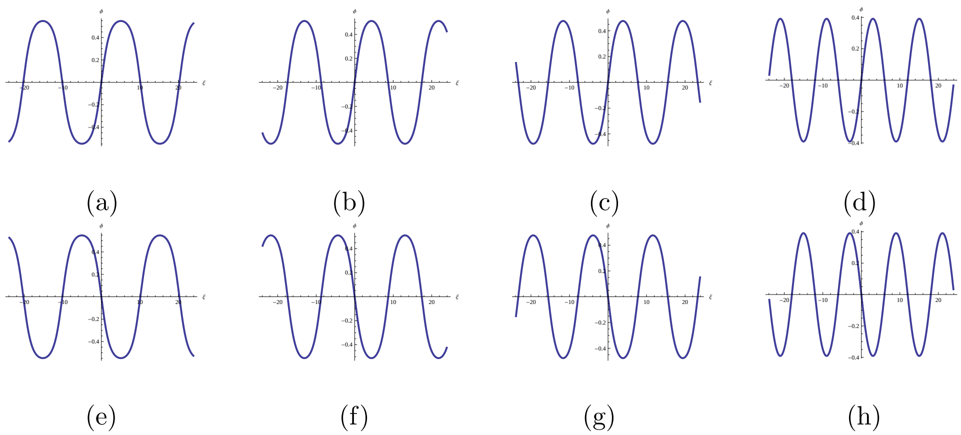


Figure 13. The limiting precess of (3.110) tends to (3.114) when $h \to h_1$.

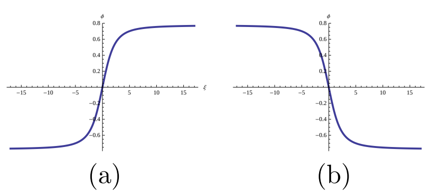


Figure 14. The profiles of (3.120).

the bifurcation method of dynamical system. Also, the relations of the periodic wave solutions and other traveling wave solutions are stated. Our main results for equation (1.1) are given in Propositions 3.1-3.6. Comparing with the before references, we obtained some new results. Actually, there are some interesting and important problems of equation (1.1) to be further studied. For examples, how to search for more new explicit exact non-smooth traveling wave solutions and non-traveling wave solutions of equation (1.1)? We will study equation (1.1) further.

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