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LUMP SOLUTIONS TO THE GENERALIZED (2+1)-DIMENSIONAL B-TYPE KADOMTSEV-PETVIASHVILI EQUATION*

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Abstract Through symbolic computation with Maple, the (2+1)−dimensional B-type Kadomtsev-Petviashvili(BKP) equation is considered. The generalized bilinear form not the Hirota bilinear method is the starting point in the computation process in this paper. The resulting lump solutions contain six free parameters, four of which satisfy two determinant conditions to guarantee the analyticity and rational localization of the solutions, while the others are arbitrary. Finally, the dynamic properties of these solutions are shown in figures by choosing the values of the parameters.

Keywords Lump solution, generalized bilinear form, B-type Kadomtsev-Petviashvili equation, symbolic computation.

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1. Introduction

Since the Korteweg-de Vries(KdV) equation is solved by the inverse scattering transform [4], there are many studies on other integrable equations [1, 18]. One of the studies about the integrable equations is to discuss its different kinds of solutions, such as solitons [6, 20–23, 32], exact solutions [12, 29–31] and lump solutions [14, 16, 37]. For many years, solitons have been at the forefront of the study of integrable systems, but now other solutions like lump solutions attracte more and more attention. As a kind of rational function solutions, lump solutions have many important applications and localize in many directions in the space, such integrable equations as the KPI equation [8, 28, 33], the three-dimensional three-wave resonant equation [11], Sawada-Kotera equations [35] all have been found to possess lump solutions. In the process of solving the lump solutions, Hirota bilinear forms [7]

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play a crucial role, but in this article, we would like to cite a kind of generalized bilinear differential operators [13], hence we can get a new kind of bilinear differential equations different from Hirota bilinear equations, which share some common features with linear differential equations.

The Kadomtsev-Petviashvili(KP) equation, discovered in 1970 by Kadomtsev and Petviashvili [10], is a two-dimensional extension of the KdV equation. Like the KdV equation, the nonlinear Schrödinger equation and other nonlinear equations, the KP equation possesses soliton solutions via the inverse scattering transform [17]. It can be used to describe various physical phenomena in areas such as water waves, nonlinear optics and plasma physics. For example, in the study of water waves, the KP equation appears in the description of a tsunami wave travelling in the inhomogeneous zone on the bottom of the ocean [36], in plasma physics, it appears in the study of nonlinear ion acoustic waves in magnetized dusty plasma [24,26,27]. Over the past decades, we have constructed and studied many extensions of the KP equation because of the interest about it.

The Kadomtsev-Petviashvili(KP)equation [10]

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{1.1}$$

its lump solutions are as following [14]:

$$\begin{aligned} u &= 2(\ln f)_{xx}, \\ f &= (a_1 x + a_2 y + \frac{a_1 a_2^2 - a_1 a_6^2 + 2a_2 a_5 a_6}{a_1^2 + a_5^2} t + a_4)^2 \\ &\quad + (a_5 x + a_6 y + \frac{2a_1 a_2 a_6 - a_2^2 a_5 + a_5 a_6^2}{a_1^2 + a_5^2} t + a_8) + \frac{3(a_1^2 + a_5^2)^3}{(a_1 a_6 - a_2 a_5)^2}, \end{aligned} \tag{1.2}$$

where the parameters $a_i, i = 1, 2, \ldots, 8$ are arbitrary but $a_1 a_6 - a_2 a_5 \neq 0$, which can ensure that u can be solved.

Recently, another research hot topic is about the KP-type equations, such as the (2+1)-dimensional B-type KP equation [19], the (3+1)-dimensional B-type KP-Boussinesq equation [34] and so on. There are many works to study the multiple wave solutions and lump solutions, Bäcklund transformation and shock wave type solutions for such kind of equations.

In this article, we consider the $(2+1)-$dimensional BKP equation [3,9]

$$P_{BKP}(u) := (u_t + 15uu_{xxx} + 15u_x^3 - 15u_x u_y + u_{5x})_x + 5u_{xxxy} - 5u_{yy} = 0. \tag{1.3}$$

This equation is a member of the BKP soliton hierarchy, and it is also a (2+1)-dimensional generalization of the Caudrey-Dodd-Gibbon-Sawada-Kotera(CDGSK) equation [2,25],

$$v_t + 15vv_{xxx} + 15v_x v_{xx} + 45u^2 v_x + u_{5x} = 0, \tag{1.4}$$

when $v = u_x$ and v is only a function of x and t, Eq.(1.3) becomes Eq.(1.4).The associated spectral problem is of third-order:

$$-\phi_y + \phi_{xxx} + (3v - \lambda)\phi = 0, \tag{1.5}$$

which provides a basis for solving the Cauchy problem of the CDGSK equation(1.4) by the inverse scattering transform [1]. We would like to look for its lump solutions by symbolic computation with Maple.

The layout of this paper are as follows. In Section 2, we introduce the notion of generalized bilinear derivative, then get the generalized bilinear form of the BKP equation. In Section 3, we will begin with the generalized bilinear form which we get from section 2 and do a search for positive quadratic function solutions to the corresponding (2+1)-dimensional bilinear BKP equation. Section 4 is a concluding remarks on this work and presenting some problems to be solved.

2. Generalized Bilinear Derivative

In this seciton, we introduce the generalized bilinear operators. Firstly, let us recall the definition of the Hirota's $D-$operator. Assume that $f(x,y,t,\ldots)$ and $g(x,y,t,\ldots)$ are smooth functions of $x,y,t,\ldots$, for arbitrary nonnegative integers n,m, Hirota [7] defined the differential operators $D_x, D_y, D_t, \ldots$ as follows.

Definition 2.1. Let $S:\mathbb{C}_n\to\mathbb{C}$ be a space of differentiable functions. Then the Hirota $D-$operators $D: S\times S\to S$ are defined to be

$$\begin{aligned} &D_x^n f(x,t)\cdot g(x',t') = (\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})^n f(x,t)g(x',t')|_{x'=x},\\ &D_x^m D_y^n D_t^k f(x,y,t)\cdot g(x',y',t') = (\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})^m(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'})^n(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'})^k \\ &\qquad\qquad \times f(x,y,t)g(x',y',t')|_{x'=x,y'=y,t'=t}, \end{aligned} \tag{2.1}$$

where $n,m,k,\ldots$ are positice integers, and $t,x,k,\ldots$ are independent variables.

We now generalize the Hirota's $D-$operator to more genenral forms, which will be obtained through replacing the Hirota's $D-$operator by a general operator. More precisely, we have the following definition [23].

Definition 2.2. Let $M,p\in\mathbb{N}$be given. We introduce a kind of bilinear differential operator

$$\prod_{i=1}^{M} D_{p,x_i}^{n_i} f\cdot g = \prod_{i=1}^{M}(\frac{\partial}{\partial x_i}+\alpha\frac{\partial}{\partial x_i'})^{n_i} f(x)g(x')\mid_{x_i'=x_i}, \tag{2.2}$$

where $x=(x_1,x_2,\ldots,x_M), x'=(x_1',x_2',\ldots,x_M'), n_1,n_2,\ldots,n_M$ are arbitary non-negative integers, and for an integer m, the mth power of α is defined by

$$\alpha^m = (-1)^{r(m)}, \quad \text{if} \quad m\equiv r(m) mod p. \tag{2.3}$$

with $0\le r(m)\le p$.

Remark 2.1. It is clear that Hirota bilinear operator D_x is just the special case of the generalized bilinear operator $D_{2k,x}$ for $p=2k, (k\in\mathbb{N})$.

But for $p=3$, we have

$$\alpha=-1, \alpha^2=\alpha^3=1, \alpha^4=-1, \alpha^5=\alpha^6=1,\ldots,$$

and for $p=5$, we have

$$\alpha=-1,\alpha^2=1,\alpha^3=-1,\alpha^4=\alpha^5=1,\ldots,$$

$$\alpha^6=-1,\alpha^7=1,\alpha^8=-1,\alpha^9=\alpha^{10}=1,\ldots,$$

in the same way, if $p = 7$, we can get

$$\alpha = -1, \alpha^2 = 1, \alpha^3 = -1, \alpha^4 = 1, \alpha^5 = -1, \alpha^6 = \alpha^7 = 1, \ldots,$$
$$\alpha^8 = -1, \alpha^9 = 1, \alpha^{10} = -1, \alpha^{11} = 1, \alpha^{12} = -1, \alpha^{13} = \alpha^{14} = 1, \ldots,$$

For instance, folliowing those characteristics, we can compute some new bilinear differential operators:

$$\begin{aligned}
&D_{3,x} f \cdot g = f_x g - f g_x,\\
&D_{3,x}^2 f \cdot g = f_{xx} g - 2 f_x g_x + f g_{xx},\\
&D_{3,x}^3 f \cdot g = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} + f g_{xxx},\\
&D_{3,x}^4 f \cdot g = f_{xxxx} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} + 4 f_x g_{xxx} - f g_{xxxx},\\
&D_{3,x}^5 f \cdot g = f_{5x} g - 5 f_{xxxx} g_x + 10 f_{xxx} g_{xx} + 10 f_{xx} g_{xxx} - 5 f_x g_{xxxx} + f g_{5x},\\
&D_{3,x} D_{3,t} f \cdot g = f_{xt} g - f_x g_t - f_t g_x + f g_{xt},\\
&D_{3,x}^2 D_{3,t} f \cdot g = f_{xxt} g - f_{xx} g_t - 2 f_{xt} g_x + 2 f_x g_{xt} + f_t g_{xx} + f g_{xxt},\\
&D_{3,x}^3 D_{3,t} f \cdot g = f_{xxxt} g - f_{xxx} g_t - 3 f_{xxt} g_x + 3 f_{xt} g_{xx} + 3 f_{xx} g_{xt} - 3 f_x g_{xxt}\\
&\qquad\qquad + f_t g_{xxx} - f g_{xxxt};
\end{aligned}$$

and

$$\begin{aligned}
&D_{5,x} f \cdot g = f_x g - f g_x,\\
&D_{5,x}^2 f \cdot g = f_{xx} g - 2 f_x g_x + f g_{xx},\\
&D_{5,x}^3 f \cdot g = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx},\\
&D_{5,x}^4 f \cdot g = f_{xxxx} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} - 4 f_x g_{xxx} + f g_{xxxx},\\
&D_{5,x}^5 f \cdot g = f_{5x} g - 5 f_{xxxx} g_x + 10 f_{xxx} g_{xx} - 10 f_{xx} g_{xxx} + 5 f_x g_{xxxx} + f g_{5x},\\
&D_{5,x} D_{5,t} f \cdot g = f_{xt} g - f_x g_t - f_t g_x + f g_{xt},\\
&D_{5,x}^2 D_{5,t} f \cdot g = f_{xxt} g - f_{xx} g_t - 2 f_{xt} g_x + 2 f_x g_{xt} + f_t g_{xx} - f g_{xxt},\\
&D_{5,x}^3 D_{5,t} f \cdot g = f_{xxxt} g - f_{xxx} g_t - 3 f_{xxt} g_x + 3 f_{xt} g_{xx} + 3 f_{xx} g_{xt} - 3 f_x g_{xxt}\\
&\qquad\qquad - f_t g_{xxx} + f g_{xxxt};
\end{aligned}$$

From above, we can see that

$$D_{5,x}^3 f \cdot f = 0,$$
$$D_{3,x}^3 f \cdot f \neq 0, \quad D_{3,x}^5 f \cdot f \neq 0, \quad D_{5,x}^5 f \cdot f \neq 0,$$

which is different from the Hirota case: $D_x^3 f \cdot f = D_x^5 f \cdot f = 0$.

Particulary, whe $p = 5$, we have the generalize bilinear KdV equation

$$(D_{5,x} D_{5,t} + D_{5,x}^4) f \cdot f = 2(f_{4x} f - 4 f_{xxx} f + 3 f_{xx}^2 + f_{xt} f - f_x f_t) = 0, \tag{2.4}$$

the generalized bilinear Boussinesq equation

$$(D_{5,t}^2 + D_{5,x}^4) f \cdot f = 2(f_{4x} f - 4 f_{xxx} f + 3 f_{xx}^2 + f_{tt} f - f_t^2) = 0, \tag{2.5}$$

the generalized bilinear KP equation

$$(D_{5,x} D_{5,t} + D_{5,x}^4 + D_{5,y}^2) f \cdot f = 2(f_{4x} f - 4 f_{xxx} f + 3 f_{xx}^2 + f_{xt} f - f_x f_t + f_{yy} f - f_y^2) = 0. \tag{2.6}$$

3. Lump solutions to the (2+1) dimensional BKP equation

In this section ,under the first-order logarithmic derivative transformation:

$$u = 2(\ln f)_x, \tag{3.1}$$

where f is positive, with this transformation, the Eq.(1.3) becomes the following generalized bilinear equation:

$$\begin{aligned} B_{BKP}(f) &:= (D_{5,x}^6 - 5D_{5,x}^3 D_{5,y} + D_{5,x}D_{5,t} - 5D_{5,y}^2) f \cdot f \\ &= 30 f_{xxxx} f_{xx} - 20 F_{xxx}^2 - 10 f_{xxxy} f + 30 f_{xxy} f_x - 30 f_{xx} f_{xy} \\ &\quad + 10 f_{xxx} f_y + 2 f_{xt} f - 2 f_x f_t - 10 f_{yy} f + 10 f_y^2 = 0. \end{aligned} \tag{3.2}$$

This is one of the two standard characteristic transformations used in Bell polynomial theories of the integrable equations [5,15], and the other one is $u = 2(lnf)_{xx}$, which can transform the KP equation into a generalized bilinear equation, and it's direct to find the relation between BKP equation and the generalized BKP equaiton:

$$P_{BKP}(u) = [\frac{B_{BKP(f)}}{f^2}]_x. \tag{3.3}$$

Therefore, if f solves the generalized bilinear BKP euqaiton(3.2), then $u = 2(\ln f)_x$ will solve the BKP equation (1.3).

In order to get lump solutions, we first consider positive quadratic solutions f to the generalized bilinear equation (3.2) in the form

$$\begin{aligned} f &= g^2 + h^2 + a_9, \\ g &= a_1 x + a_2 y + a_3 t + a_4, \\ h &= a_5 x + a_6 y + a_7 t + a_8, \end{aligned} \tag{3.4}$$

where $a_i (i = 1, 2, \ldots, 9)$ are real constants which we need to determine later. We can obtain a polynomial of the variables x, y, t by plug the function f of equation (3.4) into the generalized bilinear BKP equation (3.2) and eliminate the coefficients of the polynomial about a_i. Thus, we obtain the following equations:

$$\begin{aligned} & a_1 = a_1, a_2 = a_2, a_3 = \frac{5(a_1 a_2^2 - a_1 a_6^2 + 2a_2 a_5 a_6)}{a_1^2 + a_5^2}, a_4 = a_4, \\ & a_5 = a_5, a_6 = a_6, a_7 = \frac{5(2a_1 a_2 a_6 - a_2^2 a_5 + a_5 a_6^2)}{a_1^2 + a_5^2}, a_8 = a_8, \\ & a_9 = \frac{-3(a_1^2 + a_5^2)^2 (a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2}, \end{aligned} \tag{3.5}$$

which needs to satisfy a determinant condition

$$\begin{aligned} \Delta_1 &:= a_1 a_6 - a_2 a_5 = \begin{vmatrix} a_1 & a_2 \\ a_5 & a_6 \end{vmatrix} \neq 0, \\ \Delta_2 &:= a_1 a_2 + a_5 a_6 = \begin{vmatrix} a_1 & a_2 \\ -a_5 & a_6 \end{vmatrix} < 0. \end{aligned} \tag{3.6}$$

These sets lead to guarantee of the well-defined function f and a class of positive quadratic function solutions to the generalized bilinear BKP equation in Eq.(3.2):

$$\begin{aligned} f = & (a_1 x + a_2 y + \frac{5(a_1 a_2^2 - a_1 a_6^2 + 2a_2 a_5 a_6)}{a_1^2 + a_5^2} t + a_4)^2 \\ & + (a_5 x + a_6 y + \frac{5(2a_1 a_2 a_6 - a_2^2 a_5 + a_5 a_6^2)}{a_1^2 + a_5^2} t + a_8)^2 \\ & + \frac{-3(a_1^2 + a_5^2)^2 (a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2}, \end{aligned} \tag{3.7}$$

and in turn we get the solution of u:

$$u = \frac{4(a_1 g + a_5 h)}{f}, \tag{3.8}$$

where the function f is defined by Eq.(3.7), and the functions of g and h are given as follows:

$$\begin{aligned} g &= a_1 x + a_2 y + \frac{5(a_1 a_2^2 - a_1 a_6^2 + 2a_2 a_5 a_6)}{a_1^2 + a_5^2} t + a_4, \\ h &= a_5 x + a_6 y + \frac{5(2a_1 a_2 a_6 - a_2^2 a_5 + a_5 a_6^2)}{a_1^2 + a_5^2} t + a_8. \end{aligned} \tag{3.9}$$

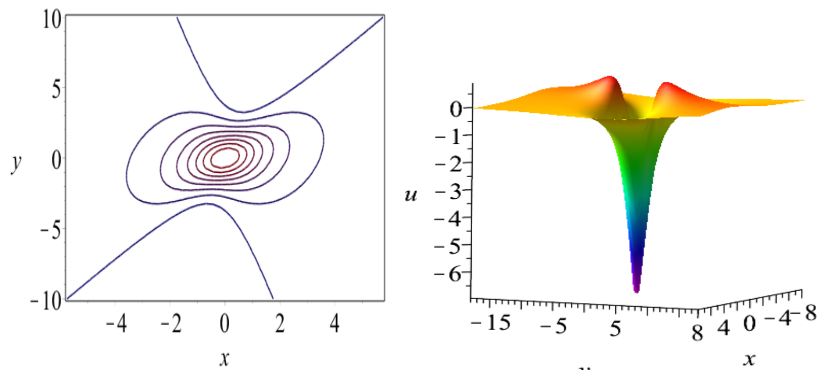

Figure 1. Profiles of Eq.(3.8) with $a_1 = 2, a_2 = 1, a_5 = 1, a_6 = -1, a_4 = 0, a_8 = 0$, when $t = 0$.

In this class of lump solutions, all six involved parameters of $a_1, a_2, a_4, a_5, a_6, a_8$ are arbitrary so that the solutions u are well defined. That is to say, if determinants(3.6) are satisfied, these determinant conditions precisely imply that two direction (a_1, a_2) and (a_5, a_6) are not parallel in the (x, y)-plane.

Note that the solutions defined by Eq.(3.8) are analytic in $\mathbb{R}^3$ if and only if the parameter $a_9 > 0$. The analyticity of the solutions in Eq.(3.8) is guaranteed if the determinant condition(3.6) hold. It is easy to observe that at any given time t, all the above lump solutions $u \to 0$ if and only if the corresponding sum of squares $g^2 + h^2 \to \infty$, or equivalently, $x^2 + y^2 \to \infty$ due to conditions(3.6). Therefore, the nonzero determinant(3.6) condition guarantee both analyticity and localization of the solutions in Eq.(3.8). There are various possibilities to take appropriate parameters to obtain lump solutions.

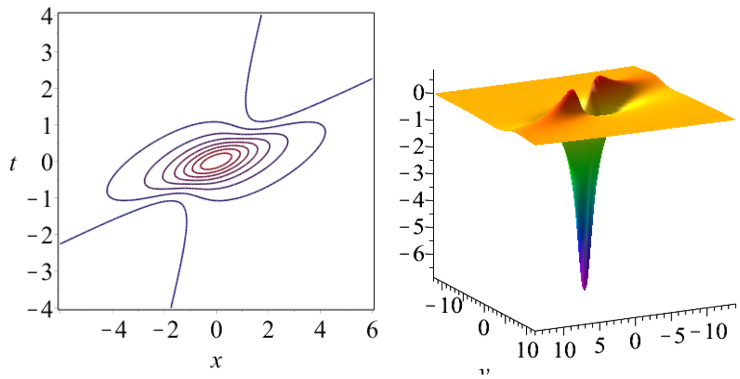

Figure 2. Profiles of Eq.(3.8) with $a_1 = 3, a_2 = 2, a_5 = -1, a_6 = 1, a_4 = 0, a_8 = 0$ when $y = 0$.

4. Concluding remarks

In this article, we studied a new (2+1)-dimensional BKP equation, obtained by using the generalized Hirota bilinear formulation with $p = 5$. Its lump solutions are provided through symbolic computation with Maple, and the analyticity and localization of the resulting solutions are guaranteed by two determinant conditions. In the process of solving the lump solutions, we cite a new kind of generalized bilinear differential operators, hence get a new kind of bilinear differential equations which is different from the known Hirota bilinear equations, as a result we get the different lump solution. These solutions may be of significant importance for explaining special physical phenomena. We hope that these results can be helpful to enrich the dynamic behavior of the KP-type equations in the future research.

Moreover, it is interesting to study the interaction solutions between lump solutions and a pair of resonance stripe solitons by making f as a combination of positive quadratic function and hyperbolic function. Moreover, if we use generalized bilinear derivatives $D_{p,x}, D_{p,y}, D_{p,t}$ with $p = 3$ in section 3, all solutions computed above will be different. In the future work, these problems will be worth discussing.

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