Journal of Applied Analysis and Computation
Volume 10, Number 6, December 2020, 2659–2668
Website:http://www.jaac-online.com
DOI:10.11948/20200014

POSITIVE PERIODIC SOLUTIONS FOR A NONLINEAR DIFFERENTIAL SYSTEM WITH TWO PARAMETERS*

Ruixiong Fan[1] and Chengbo Zhai[1,†]

Abstract In this article, we investigate a nonlinear system of differential equations with two parameters

$$\begin{cases} x'(t) = a(t)x(t) - \lambda f(t, x(t), y(t)), \\ y'(t) = -b(t)y(t) + \mu g(t, x(t), y(t)), \end{cases}$$

where $a, b \in C(\mathbf{R}, \mathbf{R}_+)$ are $\omega-$periodic for some period $\omega > 0$, $a, b \not\equiv 0$, $f, g \in C(\mathbf{R}\times\mathbf{R}_+ \times\mathbf{R}_+, \mathbf{R}_+)$ are $\omega-$periodic functions in t, λ and μ are positive parameters. Based upon a new fixed point theorem, we establish sufficient conditions for the existence and uniqueness of positive periodic solutions to this system for any fixed $\lambda, \mu > 0$. Finally, we give a simple example to illustrate our main result.

Keywords Positive periodic solutions, differential system, existence and uniqueness.

MSC(2010) 34C25, 34K13, 34B27, 93C15.

1. Introduction

During the past decades, there are many people paying much attention on the study of periodic solutions for differential equations, see the papers [1–4,7,8,12,13] and the references therein. In [12], the authors discussed a logistic system with impulsive perturbations

$$\begin{cases} \frac{\partial}{\partial t}x(t, y) = A(y, t, D)x(t, y) + f(t, y),\ y \in \Omega,\ t > 0,\ t \neq \tau_k,\ k \in Z_0^+, \\ x(t, y) = 0,\ y \in \partial\Omega,\ t > 0, \\ \Delta x(t, y) = B_k x(t, y) + c_k,\ y \in \Omega,\ t = \pi_k,\ k \in Z_0^+, \end{cases}$$

where Ω is an open-bounded domain in $\mathbf{R}^2$ and $\partial\Omega$ is smooth enough, $x(t, y)$ is the population number of isolated species at time t and location y, $f(t, y) = f(t+T, y)$ for $t \geq 0$. $A(y, t, D)$ is a operator with $A(y, t, D) = A(y, t+T, D)$. By discussing

[†]The corresponding author. Email address:cbzhai@sxu.edu.cn(C. Zhai)
[1]School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, China
[*]This paper was supported financially by Shanxi Province Science Foundation (201901D111020) and Graduate Science and Technology Innovation Project of Shanxi (2019BY014).

the exponential stability of the impulsive evolution operator $\Phi(t,\theta), t \geq \theta \geq 0$, the authors gave the existence of periodic mild solution for the $T-$periodic logistic system and T_0-periodic impulsive perturbations in a special Banach $PC([0,qT],X)$ with X is a Banach space. In [7], by using Schauder's fixed point theorem, Ma et al. established the existence of positive periodic solutions for the following second-order differential equation

$$u'' + a(t)u = f(t,u) + c(t),$$

where $a \in L^1(\mathbf{R}/\mathbf{TZ};\mathbf{R}_+)$, $c \in L^1(\mathbf{R}/\mathbf{TZ};\mathbf{R})$ with $L^1(\mathbf{R}/\mathbf{TZ})$ is composed by the integrable $T-$periodic functions, f is a Carathéodory function which may be singular at $u = 0$. Also, in [13], in order to obtain the uniqueness of positive periodic solutions, the authors employed some fixed point theorems for mixed monotone operators to investigate positive periodic solutions for the first-order functional equation

$$y'(t) = -\delta(t)y(t) + f(t, y(t-\tau(t)), y(t-\tau(t))) + g(t, y(t-\tau(t))),$$

where $T > 0$, $\delta, \tau : \mathbf{R} \to \mathbf{R}$ are continuous $T-$periodic functions and $\delta(t) > 0$ for $t \in \mathbf{R}$, $f : \mathbf{R}^3 \to \mathbf{R}$ and $g : \mathbf{R}^2 \to \mathbf{R}$. Likewise, in [2], Kang employed the similar method to study the following integral equation

$$\phi(x) = \int_{[x,x+\omega]\cap G} K(x,y)[f_1(y,\phi(y-\tau(y))) + f_2(y,\phi(y-\tau(y)))]dy,\ x \in G,$$

where G is a closed subset in $\mathbf{R}^N$ and has periodic structure. The existence and uniqueness of positive periodic solutions for this integral equation was given.

Recently, there are several articles reported on the existence of periodic solutions for some systems of differential equations, see [5,6,9–11] for example. In [11], Radu Precup gave the existence of multiple positive periodic solutions for the following differential system

$$\begin{cases} u_1'(t) = -a_1(t)u_1(t) + \epsilon_1 f_1(t,u_1(t),u_2(t)), \\ u_2'(t) = -a_2(t)u_2(t) + \epsilon_2 f_2(t,u_1(t),u_2(t)), \end{cases}$$

where for $i \in \{1,2\} : a_i \in C(\mathbf{R},\mathbf{R}), \int_0^\omega a_i dt \neq 0, \epsilon_i = \operatorname{sign}\int_0^\omega a_i(t)dt, f_i \in C(\mathbf{R} \times \mathbf{R}_+^2, \mathbf{R}_+)$, and $a_i, f_i(\cdot,u_1,u_2)$ are $\omega-$periodic functions for some $\omega > 0$. The method is a different version of Krasnosel'skii's fixed point theorem in cones.

Very recently, in [9], the authors discussed the following system of differential equations

$$\begin{cases} u_i'(t) = u_i(t)[a_i(t) - f_i(t,u(t),v(t))], i = 1,2,\ldots,n, \\ v_j'(t) = v_j(t)[-b_j(t) + g_j(t,u(t),v(t))], j = 1,2,\ldots,m, \end{cases}$$

where $u(t) = (u_1(t),u_2(t),\ldots,u_n(t))^T, v(t) = (v_1(t),v_2(t),\ldots,v_m(t))^T$, and f_i, g_j for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, are $\omega-$periodic functions in t. By using a fixed point theorem, they gave the existence of positive periodic solutions.

As we know, there are not so many papers reporting on the uniqueness of positive periodic solutions. Motivated by some articles [14,15], we will study the uniqueness of positive periodic solutions for the following differential system

$$\begin{cases} x'(t) = a(t)x(t) - \lambda f(t, x(t), y(t)), \\ y'(t) = -b(t)y(t) + \mu g(t, x(t), y(t)), \end{cases} \tag{1.1}$$

where $a, b \in C(\mathbf{R}, \mathbf{R}_+)$ are $\omega-$periodic for the same period $\omega > 0$, $a, b \not\equiv 0$, $f, g \in C([0, \omega] \times \mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}_+)$ are $\omega-$periodic functions in their first variable with $\omega > 0$, λ and μ are two positive parameters. A pair of functions (x, y) is called a positive $\omega-$periodic solution of system (1.1) if $x(t), y(t)$ are $\omega-$periodic functions and they satisfy (1.1). In this article, by using a fixed point theorem, we intend to give the existence and uniqueness of positive periodic solutions for system (1.1) for any fixed positive constants λ and μ. Our results show that the unique positive periodic solution exists in a product set and can be approximated by constructing an iterative sequence for any initial point in the product set. Moreover, our result is an extension and improvement of the previous works. In the last, a simple example is presented to illustrate the feasibility of our proposed theoretical result.

2. Preliminaries

Let $a, b \in C(\mathbf{R}, \mathbf{R}_+)$ with not identically zero, and $f_1, f_2 \in C(\mathbf{R}, \mathbf{R})$ be $\omega-$periodic functions. From [11], the following differential system

$$\begin{cases} x'(t) = a(t)x(t) - f_1(t), \\ y'(t) = -b(t)y(t) + f_2(t), \end{cases} \tag{2.1}$$

has a unique $\omega-$periodic solution (x, y) given by

$$\begin{cases} x(t) = \int_t^{t+\omega} H_1(t, s) f_1(s) ds, \\ y(t) = \int_t^{t+\omega} H_2(t, s) f_2(s) ds, \end{cases} \tag{2.2}$$

where

$$H_1(t, s) = \frac{e^{-\int_t^s a(\xi)d\xi}}{1 - e^{-\int_0^\omega a(\xi)d\xi}}, \ H_2(t, s) = \frac{e^{\int_t^s b(\xi)d\xi}}{e^{\int_0^\omega b(\xi)d\xi} - 1}, (t, s) \in (\mathbf{R}, \mathbf{R}). \tag{2.3}$$

Denote

$$l_1 = \min_{t\in[0,\omega]} \int_t^{t+\omega} H_1(t, s) ds, \ l_2 = \min_{t\in[0,\omega]} \int_t^{t+\omega} H_2(t, s) ds,$$
$$L_1 = \max_{t\in[0,\omega]} \int_t^{t+\omega} H_1(t, s) ds, \ L_2 = \max_{t\in[0,\omega]} \int_t^{t+\omega} H_2(t, s) ds.$$

If $a(t) \geq 0, b(t) \geq 0$ for $t \in \mathbf{R}$, then $L_1 \geq l_1 \geq 0, L_2 \geq l_2 \geq 0$. Therefore, (x, y) is a periodic solution of system (1.1) if and only if (x, y) is a solution of the following integral equation system

$$\begin{cases} x(t) = \lambda \int_t^{t+\omega} H_1(t, s) f(s, x(s), y(s)) ds, \\ y(t) = \mu \int_t^{t+\omega} H_2(t, s) g(s, x(s), y(s)) ds, \end{cases} \tag{2.4}$$

which can be considered as an operator equation.

In the following, we state some notations, concepts and lemmas, see [14,15] and references therein.

Let $(E, \|\cdot\|)$ be a real Banach space with a partial order induced by a cone $P \in E$. For $x, y \in E$, the notation $x \sim y$ means that exist $\alpha > 0$ and $\beta > 0$ such that $\alpha x \leq y \leq \beta x$. For fixed $h > 0$ (*i.e.*, $h \geq \theta$ and $h \neq \theta$), we define a set $P_h = \{x \in E \mid x \sim h\}$ and thus $P_h \subset P$. For $h_1, h_2 \in P$ with $h_1, h_2 \neq \theta$, let $h = (h_1, h_2)$, then $h \in \overline{P} := P \times P$. If P is normal, then $\overline{P} = (P, P)$ is normal(See [15]).

Let $\mathbf{\Phi}$ denote the class of those functions $\varphi : (0,1) \to (0,1)$ which satisfies the condition $\varphi(r) > r$ for $r \in (0,1)$.

Lemma 2.1 (see [14,15]). $\overline{P}_h = \{(x,y) : x \in P_{h_1}, y \in P_{h_2}\} = P_{h_1} \times P_{h_2}$.

Lemma 2.2 (see [15]). *Let P be a normal cone in a Banach space E and $h = (h_1, h_2) \in P \times P$ with $h_1, h_2 \neq \theta$. Let the operators $A, B : P \times P \to P$ be increasing and satisfy the following conditions:*

(C_1) *there exist $\varphi_1, \varphi_2 \in \mathbf{\Phi}$ such that*

$$A(rx, ry) \geq \varphi_1(r) A(x,y), B(rx, ry) \geq \varphi_2(r) B(x,y), \;\; r \in (0,1), x, y \in P;$$

(C_2) *there exist $(e_1, e_2) \in \overline{P}_h$ such that $A(e_1, e_2) \in \overline{P}_{h_1}$, $B(e_1, e_2) \in \overline{P}_{h_2}$.*

Then:

(a) $A : P_{h_1} \times P_{h_2} \to P_{h_1}$, $B : P_{h_1} \times P_{h_2} \to P_{h_2}$, and exist $x_1, y_1 \in P_{h_1}$, $x_2, y_2 \in P_{h_2}$, $\gamma \in (0,1)$ such that $\gamma(y_1, y_2) \leq (x_1, x_2) \leq (y_1, y_2)$ and

$$x_1 \leq A(x_1, x_2) \leq y_1, x_2 \leq B(x_1, x_2) \leq y_2;$$

(b) for any fixed $\lambda, \mu > 0$, the operator equation $(x,y) = (\lambda A(x,y), \mu B(x,y))$ has a unique solution $(x^*_{\lambda,\mu}, y^*_{\lambda,\mu})$ in $\overline{P}_h$. In addition, for any initial point $(x_0, y_0) \in \overline{P}_h$, the sequence

$$(x_n, y_n) = (\lambda A(x_{n-1}, y_{n-1}), \mu B(x_{n-1}, y_{n-1})), n = 1, 2, \ldots,$$

satisfies $\|x_n - x^*_{\lambda,\mu}\| \to 0$, $\|y_n - y^*_{\lambda,\mu}\| \to 0$, as $n \to \infty$.

3. Main results

In this section, let $E = \{x \in C(\mathbf{R}, \mathbf{R}) : x(t) = (t + \omega) \;\text{ for every } t \in \mathbf{R}\}$, then E is a Banach space equipped with the norm $\|x\|_\infty = \max_{t \in [0,\omega]} |x(t)|$. We consider the system (1.1) in product space $E \times E$. For $(x,y) \in E \times E$, define $\|(x,y)\| = \|x\|_\infty + \|y\|_\infty$. Then $(E \times E, \|(\cdot,\cdot)\|)$ is also a Banach space. Let

$$\overline{P} = \{(x,y) \in E \times E : x(t) \geq 0, y(t) \geq 0, t \in \mathbf{R}\}, \;\; P = \{x \in E : x(t) \geq 0, t \in \mathbf{R}\},$$

then $\overline{P} \subset E \times E$ and $\overline{P} = P \times P$ is normal, and thus $E \times E$ has a partial order: $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1(t) \leq x_2(t)$, $y_1(t) \leq y_2(t)$, $t \in \mathbf{R}$. Let

$$h_1(t) = \int_t^{t+\omega} H_1(t,s)ds, \;\; h_2(t) = \int_t^{t+\omega} H_2(t,s)ds, \;\; t \in \mathbf{R}. \tag{3.1}$$

Remark 3.1. From (2.3), we have that $h_1(t)$ and $h_2(t)$ are $\omega-$periodic functions. Further, it is clear that $h_1, h_2 \in P$.

Theorem 3.1. *Let h_1, h_2 be given by* (3.1). *Assume that:*

(M_1) $f, g \in C(\boldsymbol{R} \times \boldsymbol{R}_+ \times \boldsymbol{R}_+, \boldsymbol{R}_+)$ *and* $f(t, l_1, l_2) > 0$, $g(t, l_1, l_2) > 0$ *for* $t \in [0, \omega]$;

(M_2) f, g *are increasing with respect to the second and the third variables, i.e.,* $f(t, x_1, y_1) \le f(t, x_2, y_2)$, $g(t, x_1, y_1) \le g(t, x_2, y_2)$ *for any* $t \in [0, \omega]$, $0 \le x_1 \le x_2$, $0 \le y_1 \le y_2$;

(M_3) *there exist* $\varphi_1, \varphi_2 \in \boldsymbol{\Phi}$ *such that*

$$f(t, rx, ry) \ge \varphi_1(r) f(t, x, y), \ \ g(t, rx, ry) \ge \varphi_2(r) g(t, x, y)$$

for $t \in \boldsymbol{R}$, $x, y \in \boldsymbol{R}_+$, *where* $r \in (0, 1)$.

Then:

(a) there exist $x_1, y_1 \in P_{h_1}$, $x_2, y_2 \in P_{h_2}$, $\gamma \in (0, 1)$ *such that* $\gamma(y_1, y_2) \le (x_1, x_2) \le (y_1, y_2)$ *and*

$$x_1(t) \le \int_t^{t+\omega} H_1(t, s) f(s, x(s), y(s)) ds \le y_1(t), \ \ t \in [0, \omega],$$
$$x_2(t) \le \int_t^{t+\omega} H_2(t, s) g(s, x(s), y(s)) ds \le y_2(t), \ \ t \in [0, \omega],$$

where $H_i(t, s)$, $i = 1, 2$, *are given by* (2.3);

(b) for any fixed λ, $\mu > 0$, *the system* (1.1) *has a unique positive periodic solution* $(x^*_{\lambda,\mu}, y^*_{\lambda,\mu})$ *in* $\overline{P}_h$, *where* $h(t) = (h_1(t), h_2(t))$, $t \in [0, \omega]$;

(c) for any initial point $(x_0, y_0) \in \overline{P}_h$, *if*

$$x_{n+1} = \lambda \int_t^{t+\omega} H_1(t, s) f(s, x_n(s), y_n(s)) ds, \ \ n = 1, 2, \ldots,$$
$$y_{n+1} = \mu \int_t^{t+\omega} H_2(t, s) g(s, x_n(s), y_n(s)) ds, \ \ n = 1, 2, \ldots,$$

then $x_n(t) \to x^*_{\lambda,\mu}(t)$, $y_n(t) \to y^*_{\lambda,\mu}(t)$ *as* $n \to \infty$.

Proof. We define three operators $A, B : P \times P \to E$ and $N : P \times P \to E \times E$ by

$$A(x, y) = \int_t^{t+\omega} H_1(t, s) f(s, x(s), y(s)) ds,$$
$$B(x, y) = \int_t^{t+\omega} H_2(t, s) g(s, x(s), y(s)) ds,$$
$$N(x, y)(t) = (\lambda A(x, y), \mu B(x, y)),$$

where $H_1(t, s), H_2(t, s)$ are given by (2.3). From the condition (M_1), we can easily obtain $A, B : \overline{P} \to P$ and $N : \overline{P} \to \overline{P}$. By the above discussion, we know that $(x, y) \in \overline{P}$ is a solution of the system (1.1) if and only if $(x, y) \in \overline{P}$ is a fixed point of operator N. By Lemma 2.1, we only need to show that A, B satisfy all assumptions of Lemma 2.2.

First we prove that A, B are increasing. In fact, for $x_i, y_i \in P$, $i = 1, 2$, with $x_1 \le x_2$, $y_1 \le y_2$, we know that $x_1(t) \le x_2(t)$, $y_1(t) \le y_2(t)$, by employing (M_2),

$$\begin{aligned}
A(x_1, y_1)(t) &= \int_t^{t+\omega} H_1(t,s) f(s, x_1(s), y_1(s)) ds \\
&\le \int_t^{t+\omega} H_1(t,s) f(s, x_2(s), y_2(s)) ds \\
&= A(x_2, y_2)(t), \\
B(x_1, y_1)(t) &= \int_t^{t+\omega} H_2(t,s) g(s, x_1(s), y_1(s)) ds \\
&\le \int_t^{t+\omega} H_2(t,s) g(s, x_2(s), y_2(s)) ds \\
&= B(x_2, y_2)(t).
\end{aligned}$$

So we have $A(x_1, y_1) \le A(x_2, y_2)$, $B(x_1, y_1) \le B(x_2, y_2)$.

In the sequel, for any $r \in (0, 1)$ and $x, y \in P$, by (M_3) we get

$$\begin{aligned}
A(rx, ry)(t) &= \int_t^{t+\omega} H_1(t,s) f(s, rx(s), ry(s)) ds \\
&\ge \varphi_1(r) \int_t^{t+\omega} H_1(t,s) f(s, x(s), y(s)) ds \\
&= \varphi_1(r) A(x, y)(t), \\
B(rx, ry)(t) &= \int_t^{t+\omega} H_2(t,s) g(s, rx(s), ry(s)) ds \\
&\ge \varphi_2(r) \int_t^{t+\omega} H_2(t,s) g(s, x(s), y(s)) ds \\
&= \varphi_2(r) B(x, y)(t).
\end{aligned}$$

That is, $A(rx, ry) \ge \varphi_1(r) A(x, y)$, $B(rx, ry) \ge \varphi_2(r) B(x, y)$ for any $r \in (0, 1)$, $x, y \in P$.

Now we prove that $A(h_1, h_2) \in P_{h_1}$,$B(h_1, h_2) \in P_{h_2}$. Let

$$\begin{aligned}
r_1 &= \min_{t\in[0,\omega]} \{f(t, l_1, l_2)\},\ R_1 = \max_{t\in[0,\omega]} \{f(t, L_1, L_2)\}, \\
r_2 &= \min_{t\in[0,\omega]} \{g(t, l_1, l_2)\},\ R_2 = \max_{t\in[0,\omega]} \{g(t, L_1, L_2)\}.
\end{aligned}$$

From (M_1) and (M_2), we have $R_1 \ge r_1 > 0$, $R_2 \ge r_2 > 0$. Noting that $l_1 \le h_1(t) \le L_1$ and $l_2 \le h_2(t) \le L_2$, from (M_2), we have

$$\begin{aligned}
A(h_1, h_2)(t) &= \int_t^{t+\omega} H_1(t,s) f(s, h_1(s), h_2(s)) ds \\
&\ge \int_t^{t+\omega} H_1(t,s) f(s, l_1, l_2) ds \\
&= r_1 \int_t^{t+\omega} H_1(t,s) ds = r_1 h_1,
\end{aligned}$$

$$\begin{aligned} A(h_1,h_2)(t) &= \int_t^{t+\omega} H_1(t,s)f(s,h_1(s),h_2(s))ds \\ &\le \int_t^{t+\omega} H_1(t,s)f(s,L_1,L_2)ds \\ &= R_1\int_t^{t+\omega} H_1(t,s)ds = R_1h_1. \end{aligned}$$

That is, $r_1h_1 \le A(h_1,h_2) \le R_1h_1$ and thus $A(h_1,h_2) \in P_{h_1}$. Similarly, we can prove $B(h_1,h_2) \in P_{h_2}$.

Finally, by Lemma 2.2, we have the following conclusions:

(1) there exist $x_1, y_1 \in P_{h_1}$, $x_2, y_2 \in P_{h_2}$, $\gamma \in (0,1)$ such that $\gamma(y_1,y_2) \le (x_1,x_2) \le (y_1,y_2)$ and

$$x_1 \le A(x_1,y_1) \le y_1,\ x_2 \le B(x_1,y_1) \le y_2,$$

that is,

$$\begin{aligned} x_1(t) &\le \int_t^{t+\omega} H_1(t,s)f(s,x(s),y(s))ds \le y_1(t),\ t\in[0,\omega], \\ x_2(t) &\le \int_t^{t+\omega} H_2(t,s)g(s,x(s),y(s))ds \le y_2(t),\ t\in[0,\omega]; \end{aligned}$$

(2) for any fixed $\lambda, \mu > 0$, the operator equation $(x,y) = (\lambda A(x,y), \mu B(x,y))$ has a unique solution $(x^*_{\lambda,\mu}, y^*_{\lambda,\mu})$ in $\overline{P}_h$. That is $(x^*_{\lambda,\mu}, y^*_{\lambda,\mu}) = N(x^*_{\lambda,\mu}, y^*_{\lambda,\mu})$. So the system (1.1) has a unique positive periodic solution $(x^*_{\lambda,\mu}, y^*_{\lambda,\mu})$ in $\overline{P}_h$;

(3) for any given point $(x_0,y_0) \in \overline{P}_h$, defining

$$\begin{aligned} x_{n+1} &= \lambda A(x_n,y_n)(t) = \lambda\int_t^{t+\omega} H_1(t,s)f(s,x_n(s),y_n(s))ds,\ n=1,2,\dots, \\ y_{n+1} &= \mu B((x_n,y_n)(t) = \mu\int_t^{t+\omega} H_2(t,s)g(s,x_n(s),y_n(s))ds,\ n=1,2,\dots, \end{aligned}$$

one has $x_n(t) \to x^*_{\lambda,\mu}(t)$, $y_n(t) \to y^*_{\lambda,\mu}(t)$ as $n \to \infty$.

□

Remark 3.2. The study of positive periodic solutions for differential systems is still few and the unique results are also not so many. The method is new to investigate nonlinear systems of differential equations, which presents the existence and uniqueness of positive periodic solutions. Moreover, the unique periodic solution can be approximated by an iteration.

Remark 3.3. By applying the same discussion with Theorem 3.1, we can study the following differential equation

$$x'(t) = a(t)x(t) - \lambda f(t,x(t)),$$

where $a \in C(\mathbf{R},\mathbf{R}_+)$ is $\omega-$periodic for some $\omega > 0, f(t,x) \in C(\mathbf{R}\times\mathbf{R}_+,\mathbf{R}_+)$ is an $\omega-$periodic function in t. Also, for any fixed $\lambda > 0$, we can get the existence and uniqueness of positive periodic solutions and the unique periodic solution can be also approximated by giving an iterative sequence.

4. An example

Example 4.1. We consider the following differential system:

$$\begin{cases} x'(t) = 4\cos^2 t \cdot x(t) - \sin^2 t \cdot [x^{\frac{1}{2}}(t) + y^{\frac{1}{3}}(t)] - 1, \\ y'(t) = -4\sin^2 t \cdot y(t) + \cos^2 t \cdot [x^{\frac{1}{4}}(t) + y^{\frac{1}{5}}(t)] + 2, \end{cases} \tag{4.1}$$

where $a(t) = 4\cos^2 t,\ b(t) = 4\sin^2 t \geq 0$ and $a(t), b(t)$ are π-periodic in t. In this example, we let

$$f(t,x,y) = \sin^2 t[x^{\frac{1}{2}} + y^{\frac{1}{3}}] + 1,\ g(t,x,y) = \cos^2 t[x^{\frac{1}{4}} + y^{\frac{1}{5}}] + 2$$

and they are π-periodic functions in t. Obviously, $f, g \in C(\mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}_+)$ are increasing with respect to second and third variables for any $t \in \mathbf{R}$. Also, let $\varphi_1(r) = r^{\frac{1}{2}},\ \varphi_2(r) = r^{\frac{1}{4}}$ for $r \in (0,1)$. We can see that $\varphi_1(r), \varphi_2(r) \in (0,1)$ and $\varphi_1(r) = r^{\frac{1}{2}} > r,\ \varphi_2(r) = r^{\frac{1}{4}} > r$, and thus $\varphi_1, \varphi_2 \in \mathbf{\Phi}$. Further, for $r \in (0,1), t \in \mathbf{R}, x, y \in \mathbf{R}_+$,

$$\begin{aligned} f(t,rx,ry) &= \sin^2 t[(rx)^{\frac{1}{2}} + (ry)^{\frac{1}{3}}] + 1 \\ &\geq r^{\frac{1}{2}}[\sin^2 t(x^{\frac{1}{2}} + y^{\frac{1}{3}}) + 1] \\ &= \varphi_1(r) f(t,x,y), \\ g(t,rx,ry) &= \cos^2 t[(rx)^{\frac{1}{4}} + (ry)^{\frac{1}{5}}] + 2 \\ &\geq r^{\frac{1}{4}}[\cos^2 t(x^{\frac{1}{4}} + y^{\frac{1}{5}}) + 2] \\ &= \varphi_2(r) g(t,x,y). \end{aligned}$$

In addition,

$$\begin{aligned} H_1(t,s) &= \frac{e^{-\int_t^s 4\cos^2 \xi d\xi}}{1 - e^{-\int_0^\pi 4\cos^2 \xi d\xi}} = \frac{e^{-[2(s-t)+\sin 2s - \sin 2t]}}{1 - e^{-2\pi}}, \\ H_2(t,s) &= \frac{e^{\int_t^s 4\sin^2 \xi d\xi}}{e^{\int_0^\pi 4\sin^2 \xi d\xi} - 1} = \frac{e^{[2(s-t)-\sin 2s+\sin 2t]}}{e^{2\pi} - 1} \end{aligned}$$

and l_1, l_2 are given as in Section 2. Hence,

$$f(t, l_1, l_2) \geq f(t,0,0) = 1 > 0,\ g(t, l_1, l_2) \geq g(t,0,0) = 2 > 0.$$

Hence, all the conditions of Theorem 3.1 are satisfied. Then, by employing Theorem 3.1, the system (4.1) has a unique positive periodic solution (x^*, y^*) in $\overline{P_h}$, where $h(t) = (h_1(t), h_2(t)),\ h_1(t) = \int_t^{t+\pi} H_1(t,s)ds,\ h_2(t) = \int_t^{t+\pi} H_2(t,s)ds$, and for any given point $(x_0, y_0) \in \overline{P_h}$, defining

$$\begin{aligned} x_{n+1}(t) &= \int_t^{t+\pi} \frac{e^{-[2(s-t)+\sin 2s-\sin 2t]}}{1 - e^{-2\pi}} \{\sin^2 s[x_n^{\frac{1}{2}}(s) + y_n^{\frac{1}{3}}(s)] + 1\} ds, n = 1, 2, \ldots, \\ y_{n+1}(t) &= \int_t^{t+\pi} \frac{e^{[2(s-t)-\sin 2s+\sin 2t]}}{e^{2\pi} - 1} \{\cos^2 s[x_n^{\frac{1}{4}}(s) + y_n^{\frac{1}{5}}(s)] + 2\} ds, n = 1, 2, \ldots \end{aligned}$$

we have $x_n(t) \to x^*(t),\ y_n(t) \to y^*(t)$ as $n \to \infty$.

5. Conclusion

Periodic differential systems are increasingly being used to describe many problems in economical, population dynamics, control, ecology and epidemiology. Due to its deep realistic background and important role, people are paying more and more attention. For nonlinear systems of differential equations, there are still few results reported on positive periodic solutions and the uniqueness of positive periodic solutions is seldom seen in literature. In this paper, we study the system (1.1) for any fixed positive parameters λ and μ. By using a fixed point theorem, we give some new existence and uniqueness of positive periodic solutions for (1.1). Our results show that the unique positive periodic solution exists in a product set $\bar{P}_h = P_{h_1} \times P_{h_2}$ and can be approximated by constructing an iterative sequence for any initial point in the product set $\bar{P}_h$. In the last, a simple example is presented to illustrate the feasibility of our proposed theoretical result.

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