Journal of Applied Analysis and Computation
Volume 10, Number 6, December 2020, 2669–2681

Website:http://www.jaac-online.com
DOI:10.11948/20200022

NONLOCAL SYMMETRIES AND EXACT SOLUTIONS OF A VARIABLE COEFFICIENT AKNS SYSTEM*

Xiangpeng Xin[1,†], Lihua Zhang[2], Yarong Xia[3] and Hanze Liu[1]

Abstract In this paper, nonlocal symmetries of variable coefficient Ablowitz-Kaup-Newell-Segur(AKNS) system are studied for the first time. In order to construct some new analytic solutions, a new variable is introduced, which can transform nonlocal symmetries into Lie point symmetries. Furthermore, using the Lie point symmetries of closed system, we give out two types of symmetry reductions and some analytic solutions. For some interesting solutions, such as interaction solutions among solitons and other complicated waves, we give corresponding images to describe their dynamic behavior.

Keywords Nonlocal symmetry, variable coefficient equations, analytic solution, Lie point symmetry.

MSC(2010) 76M60,17B80,83C15.

1. Introduction

Lie group [20] was proposed by Norwegian mathematician Sophus Lie in the 19th century. With the development of nonlinear equations [4, 11, 12, 18, 28, 37] and integrable system [5,8,13,26,27,34] theory, finding solutions to nonlinear equations has become an important research problem in the mathematical physics field. In the 20th century, the Lie group theory developed rapidly, it was not only used to solve differential equations, but also established the relationships with many disciplines, such as, mathematics [1, 24], physics [7, 17], fractional derivative problem [25, 35], bifurcation theory [9], etc.

With the development of symmetry theory, a lot of studies have been devoted to seeking the generalized Lie point symmetry. P.J.Olver [23] has construct a new type of symmetry by using recursion operator which was called nonlocal symmetry. Compared with the local symmetry, nonlocal symmetry was not easy to construct the solutions of differential equations [6, 15], because the nonlocal symmetries contain some auxiliary variables. G.W. Bluman et al. [2, 3] have presented many

[†]The corresponding author. Email address:xinxiangpeng@lcu.edu.cn.(X. Xin)
[1]School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, China
[2]School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, China
[3]School of Information and Engineering, Xi'an University, Xi'an 710065, China
*This work is supported by the National Natural Science Foundation of China (Nos.11505090,1171041), the doctorial foundation of Liaocheng University under Grant No.318051413, the Natural Science Foundation of Shaanxi Province under Grant No.2014JM2-1009, and the Science and Technology Innovation Foundation of Xi'an under Grant No.2017CGWL06.

methods to find nonlocal symmetries of the partial differential equations (PDEs)by using potential systems. F. Galas [10]obtained the nonlocal symmetries by using the pseudo-potentials of PDEs, and construct exact solutions by the obtained nonlocal symmetries. Recently, Lou et al. [14, 21, 29–33] found that Painlevé analysis can also be used to construct the nonlocal symmetries which was called residual symmetries.

In this paper, we consider the variable coefficient AKNS system. With the help of lax pair, the nonlocal symmetries of this system are obtained which be transformed into local symmetries by introducing new variables variables, and analytic solutions are constructed by using the Lie group theory. In [36], the Lie group method is used to study the AKNS system with constant coefficients and it can be found that the results in this paper are special cases of our article. In [19], the constant coefficients AKNS system is studied by using the residue symmetry method, because the nonlocal symmetries obtained in this article are different from our article, so the results are also very different. By comparing with the conclusions of the two articles, we can see that our results are new.

This paper is arranged as follows: In Sec.2, the nonlocal symmetries were constructed by using the Lax pair of variable coefficient AKNS system. In Sec.3, the process of transforming from nonlocal symmetries to local symmetries was introduced in detail. In Sec.4 some symmetry reductions and analytic solutions were obtained by using the Lie point symmetry of closed system. Finally, some conclusions and discussions are given in Sec.6.

2. Nonlocal symmetries of variable coefficient AKNS system

The time-dependent coefficient AKNS system [16] reads

$$\begin{cases} u_t + \delta(2\alpha v u^2 - \alpha u_{xx}) = 0, \\ v_t - \delta(2\alpha v^2 u - \alpha v_{xx}) = 0, \end{cases} \tag{2.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are the real functions, $\delta = \delta(t)$ is a real function of t. The system(2.1) was obtained via the variable transformation from time-dependent Whitham-Broer-Kaup equations, which is used for the shallow water under the Boussinesq approximation. Lax pair,infinitely-many conservation laws and soliton solutions are given in [16]. When $\delta = 1, \alpha = i/2$ Eq.(2.1) reduces to the well-known AKNS system, where $i^2 = -1$, nonlocal symmetries and exact solutions for the constant coefficient AKNS system have been obtained [22]. To our knowledge, nonlocal symmetries for Eq.(2.1) have not been obtained and discussed, which will be the goal of this paper.

The corresponding Lax pair has been obtained in [16],

$$\begin{aligned} \varphi_x &= U\varphi, \\ \varphi_t &= V\varphi, \end{aligned} \tag{2.2}$$

where

$$\varphi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, U = \begin{pmatrix} \lambda & v \\ u & -\lambda \end{pmatrix}, V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$

and $A = \alpha\delta uv - 2\lambda^2\alpha\delta, B = -\alpha\delta v_x - 2\lambda\alpha\delta v, C = \alpha\delta u_x - 2\lambda\alpha\delta u$.

To seek the nonlocal symmetries of variable coefficient AKNS system(2.1), one must solve the following linearized equations,

$$\begin{aligned} &\sigma_t^1 + 2\alpha v u^2\sigma^3 - \alpha u_{xx}\sigma^3 + 2\alpha\delta u^2\sigma^2 + 4\alpha uv\delta\sigma^1 - \alpha\delta\sigma_{xx}^1 = 0, \\ &\sigma_t^2 - 2\alpha v^2 u\sigma^3 + \alpha v_{xx}\sigma^3 - 2\alpha\delta v^2\sigma^1 - 4\alpha uv\delta\sigma^2 + \alpha\delta\sigma_{xx}^2 = 0, \end{aligned} \tag{2.3}$$

$\sigma^1, \sigma^2, \sigma^3$ are symmetries of u, v, δ, which means Eqs.(2.1) is form invariant under the transformations

$$\begin{aligned} &u \to u + \epsilon\sigma^1, \\ &v \to v + \epsilon\sigma^2, \\ &\delta \to \delta + \epsilon\sigma^3, \end{aligned} \tag{2.4}$$

with the infinitesimal parameter ϵ.

Be different from Lie point symmetries, we assume nonlocal symmetries of the system(2.1)have the following form,

$$\begin{aligned} \sigma^1 &= \bar{X}(x, t, u, v, \delta, \phi_1, \phi_2)u_x + \bar{T}(x, t, u, v, \delta, \phi_1, \phi_2)u_t - \bar{U}(x, t, u, v, \delta, \phi_1, \phi_2), \\ \sigma^2 &= \bar{X}(x, t, u, v, \delta, \phi_1, \phi_2)v_x + \bar{T}(x, t, u, v, \delta, \phi_1, \phi_2)v_t - \bar{V}(x, t, u, v, \delta, \phi_1, \phi_2), \\ \sigma^3 &= \bar{T}(x, t, u, v, \delta, \phi_1, \phi_2)\delta_t - \bar{\Delta}(x, t, u, v, \delta, \phi_1, \phi_2). \end{aligned} \tag{2.5}$$

Then, one can using Lie group method to find their solutions of $\sigma^1, \sigma^2, \sigma^3$. By substituting Eq.(2.5) into Eq.(2.3) and eliminating $u_t, v_t, \phi_{1x}, \phi_{1t}, \phi_{2x}, \phi_{2t}$ in terms of the lax pair(2.3), it yields a system of determining equations for the functions $\bar{X}, \bar{T}, \bar{U}, \bar{V}, \bar{\Delta}$, solving these determining equations can obtain,

$$\begin{aligned} &\bar{X}(x, t, u, v, \delta, \phi_1, \phi_2) = c_1 x + c_2, \\ &\bar{T}(x, t, u, v, \delta, \phi_1, \phi_2) = F_1(t), \\ &\bar{U}(x, t, u, v, \delta, \phi_1, \phi_2) = (-2c_1 - c_3)u + c_4\phi_2^2, \\ &\bar{V}(x, t, u, v, \delta, \phi_1, \phi_2) = c_3 v + c_4\phi_1^2, \\ &\bar{\Delta}(x, t, u, v, \delta, \phi_1, \phi_2) = \delta(2c_1 - \tfrac{dF_1(t)}{dt}), \end{aligned} \tag{2.6}$$

where $c_i (i = 1, ..., 4)$ are four arbitrary constants and $F_1(t)$ is arbitrary function of t.

Remark 2.1. It is show that the results(2.6) are local symmetries of variable coefficient AKNS system when $c_4 = 0$, and they are nonlocal symmetries when $c_4 \neq 0$.

Nonlocal symmetries need to be transformed into local ones [14, 21] before construct analytic solutions, so we construct a closed system whose Lie symmetries contain above nonlocal symmetries.

3. Localization of the nonlocal symmetry

For simplicity, let $c_1 = c_2 = c_3 = 0, c_4 = 1, F_1(t) = 0$ in formula (2.6) i.e.,

$$\begin{aligned} \sigma^1 &= -\phi_2^2, \\ \sigma^2 &= -\phi_1^2, \\ \sigma^3 &= 0. \end{aligned} \tag{3.1}$$

To localize the nonlocal symmetry (3.1), we have to solve the following linearized equations,

$$\begin{aligned} &\sigma_x^4 - \sigma^2\phi_2 - v\sigma^5 - \lambda\sigma^4 = 0, \\ &\sigma_x^5 - \sigma^1\phi_1 - u\sigma^4 + \lambda\sigma^5 = 0, \\ &\sigma_t^4 - \alpha uv\phi_1\sigma^3 - \alpha\delta v\phi_1\sigma^1 - \alpha\delta u\phi_1\sigma^2 - \alpha\delta uv\sigma^4 + 2\lambda\alpha v\phi_2\sigma^3 + 2\lambda\alpha\delta\phi_2\sigma^2 \\ &+2\lambda\alpha\delta v\sigma^5 + 2\lambda^2\alpha\phi_1\sigma^3 + 2\lambda^2\alpha\delta\sigma^4 + \alpha\sigma^3\phi_2 v_x + \alpha\delta\phi_2\sigma_x^2 + \alpha\delta\sigma^5 v_x = 0, \\ &\sigma_t^5 + \alpha\delta u\phi_2\sigma^2 + \alpha\delta uv\sigma^5 + \alpha uv\phi_2\sigma^3 + \alpha\delta v\sigma^1\phi_2 + 2\lambda\alpha u\phi_1\sigma^3 + 2\lambda\alpha\delta\phi_1\sigma^1 \\ &+2\lambda\alpha\delta u\sigma^4 - 2\lambda^2\alpha\phi_2\sigma^3 - 2\lambda^2\alpha\delta\sigma^5 - \alpha\sigma^3\phi_1 u_x - \alpha\delta\phi_1\sigma_x^1 - \alpha\delta\sigma^4 u_x = 0, \end{aligned} \tag{3.2}$$

which is form invariant under the following transformation,

$$\begin{aligned} \phi_1 &\to \phi_1 + \epsilon\sigma^4, \\ \phi_2 &\to \phi_2 + \epsilon\sigma^5, \\ f &\to f + \epsilon\sigma^6, \end{aligned} \tag{3.3}$$

with the infinitesimal parameter ϵ, and $\sigma^1, \sigma^2, \sigma^3$ given by (3.1). It is not difficult to verify that the solutions of (3.2) have the following forms,

$$\sigma^4 = \phi_1 f, \qquad \sigma^5 = \phi_2 f, \tag{3.4}$$

where f satisfies the following equations,

$$\begin{aligned} f_x &= -\phi_1\phi_2, \\ f_t &= \alpha\delta(v\phi_2^2 + 4\lambda\phi_1\phi_2 - u\phi_1^2), \end{aligned} \tag{3.5}$$

it is easy to obtain the following result,

$$\sigma^6 = f^2. \tag{3.6}$$

One can see that the nonlocal symmetry (3.1) in the original space $\{x, t, u, v, \delta\}$ has been successfully localized to a Lie point symmetry in the enlarged space $\{x, t, u, v, \delta, \phi 1, \phi 2, f\}$. It is not difficult to verify that the auxiliary dependent variable f just satisfies the Schwartzian form of the variable coefficient AKNS system

$$\delta\frac{\partial C}{\partial t} - \alpha^2\delta^3\frac{\partial S}{\partial x} - (8\lambda\alpha\delta^2 + 3\delta C)\frac{\partial C}{\partial x} - C\frac{\partial \delta}{\partial t} = 0, \tag{3.7}$$

where $C = \frac{\frac{\partial \phi}{\partial t}}{\frac{\partial \phi}{\partial x}}$,and $S = \frac{\frac{\partial^3 \phi}{\partial x^3}}{\frac{\partial \phi}{\partial x}} - \frac{3\left(\frac{\partial^2 \phi}{\partial x^2}\right)^2}{2\left(\frac{\partial \phi}{\partial x}\right)^2}$ is the Schwartzian derivative.

After we successfully transform the nonlocal symmetries(3.1) into local symmetries. New analytic solutions can be constructed naturally by Lie group theory. With the Lie point symmetry(3.1),(3.4),(3.6), by solving the following initial value problem,

$$\begin{aligned}
&\tfrac{d\bar{u}(\epsilon)}{d\epsilon} = -\phi_2^2, && \bar{u}\,|_{\epsilon=0} = u,\\
&\tfrac{d\bar{v}(\epsilon)}{d\epsilon} = -\phi_1^2, && \bar{v}\,|_{\epsilon=0} = v,\\
&\tfrac{d\bar{\delta}(\epsilon)}{d\epsilon} = 0, && \bar{\delta}\,|_{\epsilon=0} = \delta,\\
&\tfrac{d\bar{\phi}_1(\epsilon)}{d\epsilon} = \phi_1 f, && \bar{\phi}_1\,|_{\epsilon=0} = \phi_1,\\
&\tfrac{d\bar{\phi}_2(\epsilon)}{d\epsilon} = \phi_2 f, && \bar{\phi}_2\,|_{\epsilon=0} = \phi_2,\\
&\tfrac{d\bar{f}(\epsilon)}{d\epsilon} = f^2, && \bar{f}\,|_{\epsilon=0} = f,
\end{aligned} \tag{3.8}$$

where ϵ is the group parameter, we arrive at the symmetry group theorem as follows:

Theorem 3.1. *If $\{u, v, \delta, \phi_1, \phi_2, f\}$ is the solution of the prolonged system (2.1)(2.2)and (3.5),with $\lambda = 0$, so is $\{\bar{u}, \bar{v}, \bar{\delta}, \bar{\phi}_1, \bar{\phi}_2, \bar{f}\}$*

$$\begin{aligned}
&\bar{u} = u + \tfrac{\epsilon\phi_2^2}{1+\epsilon f}, && \bar{v} = v + \tfrac{\epsilon\phi_1^2}{1+\epsilon f}, && \bar{\delta} = \delta,\\
&\bar{\phi}_1 = \tfrac{\epsilon\phi_1}{1+\epsilon f}, && \bar{\phi}_2 = \tfrac{\epsilon\phi_2}{1+\epsilon f}, && \bar{f} = \tfrac{\epsilon f}{1+\epsilon f},
\end{aligned} \tag{3.9}$$

with ϵ is an arbitrary group parameter.

Here we give a simple example, starting from a soliton solution of (2.1)

$$u = -\tanh(2\alpha t - x) - 1, v = -\tanh(2\alpha t - x) + 1, \delta = 1, \tag{3.10}$$

it's not difficult to derive the special solutions for the variables ϕ_1, ϕ_2, f from(2.2)and (3.5),

$$\phi_1 = 1 - \tanh(2\alpha t - x), \phi_2 = 1 + \tanh(2\alpha t - x), f = -\frac{2}{1 + e^{4\alpha t - 2x}}. \tag{3.11}$$

Using theorem 1, it's not hard to verify

$$\begin{aligned}
&u = \tfrac{2(2\varepsilon-1)e^{4\alpha t-2x}}{1-2\varepsilon+e^{4\alpha t-2x}}, \quad v = \tfrac{2}{1-2\varepsilon+e^{4\alpha t-2x}}, \quad \phi_1 = \tfrac{2e^{2x}}{(2\varepsilon-1)e^{2x}-e^{4\alpha t}},\\
&\phi_2 = -\tfrac{2e^{4t\alpha}}{(2\varepsilon-1)e^{2x}-e^{4\alpha t}}, \quad f = -\tfrac{2}{1-2\varepsilon+e^{4\alpha t-2x}}, \quad \delta = 1,
\end{aligned}$$

are still the solutions to the system(2.1),(2.2)and (3.5).

Remark 3.1. One can see from the results, the form of the solutions of u, v from the soliton solutions become non-soliton solutions. We can get more solutions by repeating the theorem 3.1. These solutions can not be obtained by traditional Lie group methods, so they are new analytic solutions of system(2.1).

To search for more similarity reductions and analytic solutions of Eq.(2.1), we use classical Lie group method. Assume the symmetries of whole prolonged system have the vector form,

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + \Delta\frac{\partial}{\partial \delta} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q} + F\frac{\partial}{\partial f}, \tag{3.12}$$

where X, T, U, Δ, P, Q, F are the functions with respect to $x, t, u, \delta, \phi_1, \phi_2, f$, which means that the closed system is invariant under the transformations

$$(x, t, u, v, \delta, \phi_1, \phi_2, f) \rightarrow (x+\epsilon X, t+\epsilon T, u+\epsilon U, v+\epsilon V, \delta+\epsilon\Delta, \phi_1+\epsilon P, \phi_2+\epsilon Q, f+\epsilon F), \tag{3.13}$$

with a small parameter ϵ. Symmetries in the vector form (3.12) can be assumed as

$$\begin{aligned}
\sigma^1 &= X u_x + T u_t - U,\\
\sigma^2 &= X v_x + T v_t - V,\\
\sigma^3 &= T\delta_t - \Delta,\\
\sigma^4 &= X\phi_{1x} + T\phi_{1t} - P,\\
\sigma^5 &= X\phi_{2x} + T\phi_{2t} - Q,\\
\sigma^6 &= X f_x + T f_t - F,
\end{aligned} \tag{3.14}$$

where X, T, U, Δ, P, Q, F are the functions with respect to $\{x, t, u, \delta, \phi_1, \phi_2, f\}$. And $\sigma^i, (i = 1, ..., 6)$ satisfy the linearized equations of the prolonged system,
i.e., (2.3),(3.2),and

$$\begin{aligned}
&\sigma^6_x + \sigma^4\phi_2 + \sigma^5\phi_1 = 0,\\
&\sigma^6_t - 4\alpha\lambda\sigma^3\phi_1\phi_2 - 4\alpha\lambda\delta\sigma^4\phi_2 + 2\alpha\delta\sigma^4\varphi_1 u - 4\alpha\lambda\delta\sigma^5\phi_1\\
&-2\alpha\delta\sigma^5\phi_2 v + \alpha\sigma^3\phi_1^2 u - \alpha\sigma^3\phi_2^2 v + \alpha\delta\sigma^1\phi_1^2 - \alpha\delta\sigma^2\phi_2^2 = 0.
\end{aligned} \tag{3.15}$$

Substituting Eqs.(3.14) into Eqs.(2.3),(3.2),(3.15) and eliminating $u_t, v_t, \phi_{1x}, \phi_{1t}, \phi_{2x}, \phi_{2t}, f_x, f_t$ in terms of the closed system, determining equations for the functions $X, T, U, V, \Delta, P, Q, F$ can be obtained, by solving these equations, one can get

$$\begin{aligned}
&X = c_1, \quad T = F_2(t), \quad U = c_2 u + c_3\phi_2^2, \quad V = -c_2 v + c_3\phi_1^2,\\
&\Delta = -\delta\tfrac{dF(t)}{dt}, \quad P = -\tfrac{\phi_1}{2}(c_2 - c_4 + 2c_3 f),\\
&Q = \tfrac{\phi_2}{2}(c_2 + c_4 - 2c_3 f), \quad F = -c_3 f^2 + c_4 f + c_5,
\end{aligned} \tag{3.16}$$

where $c_i, (i = 1, 2, ..., 5)$ are arbitrary constants, $F_2(t)$ is arbitrary function of t.

Remark 3.2. When $c_1 = c_2 = c_4 = c_5 = F_2(t) = 0, c_3 = -1$, the nonlocal symmetry is just the one expressed by (2.6), and when $c_3 = c_4 = c_5 = 0$ the related symmetries only Lie point symmetry of variable coefficient AKNS system.

4. Symmetry reduction and analytic solutions of variable coefficient AKNS system

In this section, we will give two nontrivial similarity reductions and group invariant solutions of variable coefficient AKNS system(2.1)under consideration $c_3 \neq 0$.

case 1: $c_5 \neq 0$.

Without loss of generality, let $c_2 = c_4 = 0, c_1 = c_3 = 1, c_5 = k_1, F_2(t) = k_2$, with k_1, k_2 are two arbitrary constants. By solving the following characteristic equations,

$$\frac{dx}{1} = \frac{dt}{k_2} = \frac{du}{\phi_2^2} = \frac{dv}{\phi_1^2} = \frac{d\delta}{0} = \frac{d\phi_1}{-\phi_1 f} = \frac{d\phi_2}{-\phi_2 f} = \frac{df}{-f^2 + k_1}, \tag{4.1}$$

one can obtain

$$\begin{aligned} &u = \frac{\sqrt{k_1}F_4(\xi) - F_3^2(\xi)\tanh(\Theta)}{\sqrt{k_1}}, \quad v = \frac{\sqrt{k_1}F_5(\xi) - F_2^2(\xi)\tanh(\Theta)}{\sqrt{k_1}} \\ &\phi_1 = F_2(\xi)\sqrt{\tanh^2(\Theta) - 1}, \quad \phi_2 = F_3(\xi)\sqrt{\tanh^2(\Theta) - 1}, \\ &f = \sqrt{k_1}\tanh(\Theta), \delta = k_3, \end{aligned} \tag{4.2}$$

where $\Theta = \sqrt{k_1}(F_1(\xi) + x), \xi = t - k_2 x$.

Substituting Eqs.(4.2)into the prolonged system yields,

$$\begin{aligned} &F_2 = Ce^{\int \frac{k_3\alpha k_2^2 F_{1\xi\xi} + 2k_3\alpha\lambda k_2 F_{1\xi} - F_{1\xi} - 2k_3\alpha\lambda}{2k_3\alpha k_2(k_2 F_{1\xi} - 1)} d\xi}, \quad F_3 = \frac{k_1 - k_1 k_2 F_{1\xi}}{F_2}, \\ &F_4 = -\frac{-k_2^2 k_3\alpha F_{1\xi\xi} + 4k_1 k_2\lambda k_3\alpha F_{1\xi} - k_1 F_{1\xi} - 4k_1\lambda k_3\alpha}{2k_3\alpha F_2^2}, \\ &F_5 = \frac{k_2^2 k_3\alpha F_2^2 F_{1\xi\xi} + 4k_1\lambda k_3\alpha F_2^2 F_{1\xi} - 4\lambda k_3\alpha F_2^2 + F_2^2 F_{1\xi}}{2k_3\alpha k_2^2 F_{1\xi}^2 - 4k_3\alpha k_1 k_2 F_{1\xi} + 2k_3\alpha k_1}, \end{aligned} \tag{4.3}$$

where C is arbitrary constant. One can see that through the Eqs.(4.2)and (4.3), if we know the form of $F_1(\xi)$, then u, v can be obtained directly. We known that auxiliary dependent variable f satisfies the Schwartzian form, by substituting $f = \sqrt{k_1}\tanh(\Theta)$ into (3.7), one can get,

$$\begin{aligned} &\alpha^2 k_3^2 k_2^4(2k_2 F - k_2^2 F^2 - 1)F_{\xi\xi\xi} - 3k_3^2\alpha^2 k_2^6 F_\xi^3 + [4\alpha^2 k_3^2 k_2^5(k_2 F - 1)F_{\xi\xi} \\ &+4k_1 k_3^2\alpha^2 k_2^4 F^2(k_2^2 F^2 - 4k_2 F + 6) + (2k_2 - 16k_1 k_2^3 k_3^2\alpha^2 - 4k_3\alpha\lambda k_2^2)F \\ &+4k_1 k_3^2\alpha^2 + 4k_3\lambda\alpha k_2 + 1]F_\xi = 0, \end{aligned} \tag{4.4}$$

where $F = F(\xi) = F_{1\xi}$.

It is not difficult to verify that the above equation is equivalent to the following elliptic equation,

$$F_\xi = \frac{1}{k_3\alpha k_2^3}\sqrt{A_0 + A_1 F + A_2 F^2 + A_3 F^3 + A_4 F^4} \tag{4.5}$$

where

$$\begin{aligned} &A_0 = 2k_3 C_1\alpha^2 k_2^5 + 2k_3^2 C_2\alpha^2 k_2^5 + 4C\alpha\lambda k_2 - 1, \\ &A_1 = -(4k_3^2 C_1\alpha^2 k_2^6 + 6k_3^2 C_2\alpha^2 k_2^6 + 4k_3\alpha\lambda k_2^2 - 2k_2), \\ &A_2 = 2k_3^2 C_1\alpha^2 k_2^7 + 6k_3^2 C_2\alpha^2 k_2^7 + 4k_3^2\alpha^2 k_1 k_2^4, \\ &A_3 = -2k_3^2 C_2\alpha^2 k_2^8 - 8k_3^2\alpha^2 k_1 k_2^5, \\ &A_4 = 4k_3^2\alpha^2 k_1 k_2^6. \end{aligned}$$

C_1, C_2 are arbitrary constants.

It is know that the general solution of Eq.(4.5) can be written in terms of Jacobi elliptic functions. Hence, expression of solution (4.2) reflects the wave interaction between the soliton and the Elliptic function periodic wave. A simple solution of Eq.(4.5) is given as,

$$F = b_0 + b_1 sn(\xi, n), \tag{4.6}$$

substituting Eq.(4.6) into Eq.(4.5) yields

$$b_0 = 2\alpha\lambda k_3, b_1 = 8k_3^2\alpha^2\lambda^3, k_1 = \frac{n^2}{256k_3^4\alpha^4\lambda^6}, k_2 = \frac{1}{2\lambda\alpha k_3}, \tag{4.7}$$

with $k_3, \lambda, \alpha \in R, 0 \leq n \leq 1$.

Substituting Eqs.(4.7),(4.6) and $F_{1\xi} = F$ into Eq.(4.3), one can obtain the solutions of u, v. Because the expression is too prolix, it is omitted here. In order to study the properties of these solutions of AKNS system, we give some pictures of u, v as following,

In Fig.1, we plot the interaction solutions between solitary waves and elliptic function waves expressed by (4.2) with parameters $C = 5, C_1 = 2, k_1 = 0.18, k_2 = 10, \lambda = 0.1, \alpha = 1, n = 0.1$.

We can see that the component u exhibits a soliton propagates on a Jacobi elliptic sine function wave background. In Fig.1, the first picture(a) shows that the height of the soliton is approximately 0.03 at $t = -10$. With the development of time, soliton produces elastic collisions with other waves, and the height increases continuously. Picture(e) shows that soliton is roughly in line with its adjacent wave at $t = 14$. After the collision, the soliton reverts to the original height and continues to collide with the adjacent waves see the pictures($f \rightarrow i$). The corresponding 3d image is given below, exhibits a soliton propagating on period waves background. As one can see from the expression(4.2), u, v possess similar form, so there is no more detailed discussion here.

In order to study the properties of the solutions, we draw the corresponding 3-D images using Maple software,(see Fig.2) and the parameters used in the figures are selected same as Fig.1.

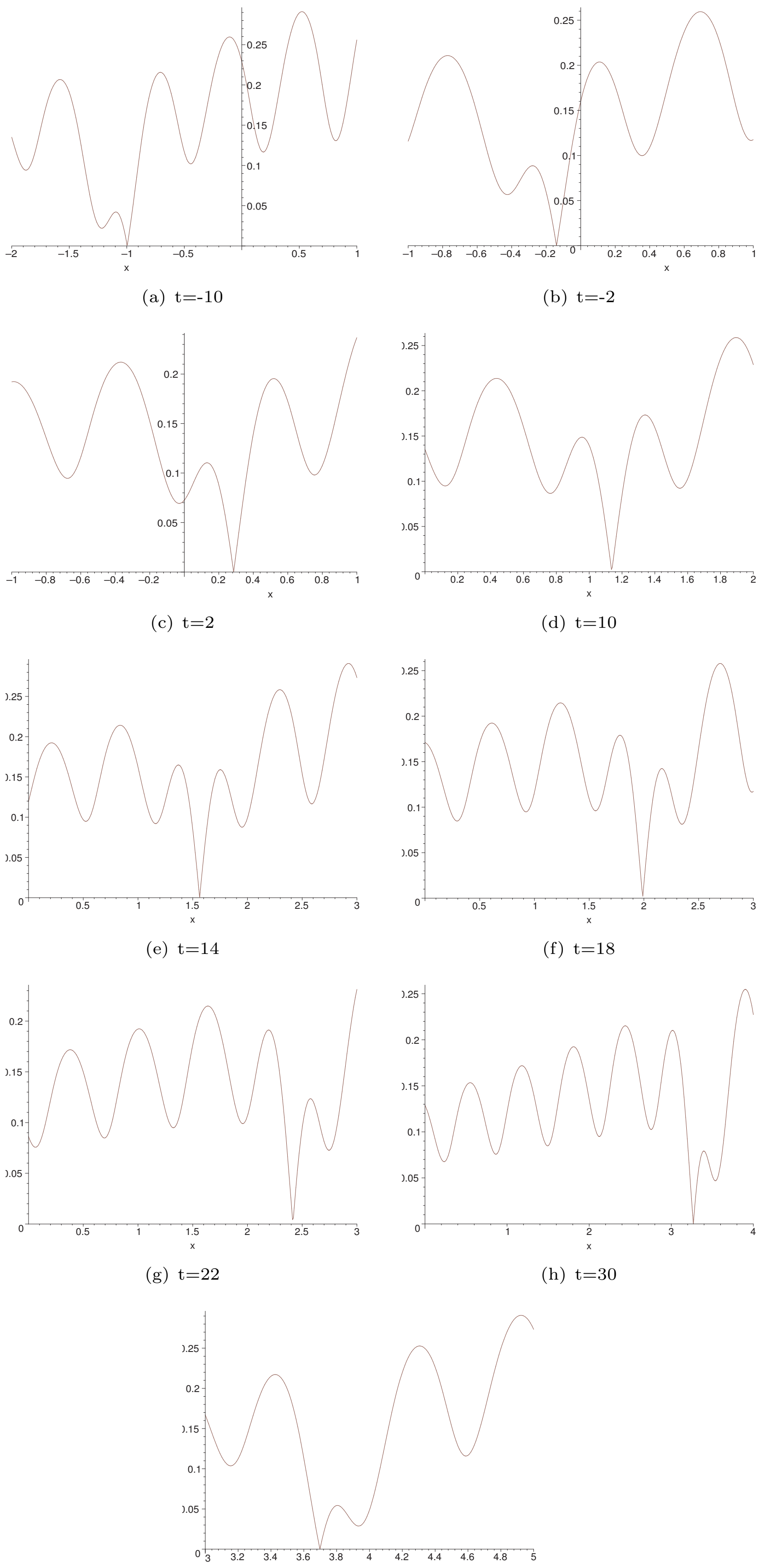
In fact, it is of interest to study these types of solutions, for example, in describing localized states in optically refractive index gratings. In the ocean, there are some typical nonlinear waves such as the solitary waves and the cnoidal periodic waves.

case 2: $c_5 = 0$.

We let $c_1 = k_1, c_2 = 2k_2, c_3 = k_3, c_4 = c_5 = 0, F_1(t) = 1$, with k_1, k_2, k_3 are arbitrary constants. By solving the following characteristic equation,

$$\begin{aligned} \frac{dx}{k_1} = \frac{dt}{1} &= \frac{du}{k_3\phi_2^2 + 2k_2 u} = \frac{dv}{k_3\phi_1^2 + 2k_2 v} = \frac{d\delta}{0} \\ &= \frac{d\phi_1}{-\phi_1(k_3 f + k_2)} = \frac{d\phi_2}{-\phi_2(k_3 f - k_2)} = \frac{df}{-k_3 f^2}, \end{aligned} \tag{4.8}$$

one can obtain the following results,



(a) t=-10 (b) t=-2

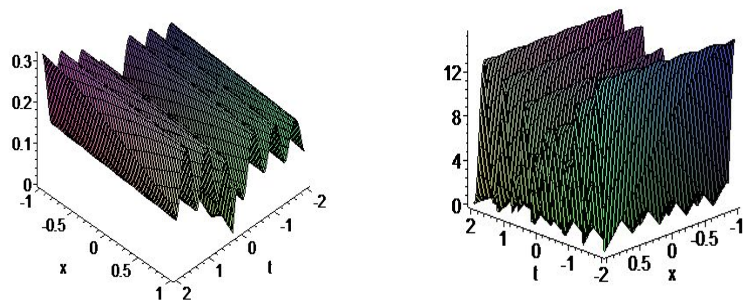
(c) t=2 (d) t=10

(e) t=14 (f) t=18

(g) t=22 (h) t=30

(i) t=34

Figure 1. Interaction solutions to the variable coefficient AKNS system



(a) $u(x,t)$ (b) $v(x,t)$

Figure 2. 3-D Interaction solutions to the variable coefficient AKNS system

$$
\begin{aligned}
&u = e^{2k_2t}(\tilde{F}_4(\varsigma) - \frac{\tilde{F}_3^2(\varsigma)}{\tilde{F}_1(\varsigma)+k_3t}), v = e^{-2k_2t}(\tilde{F}_5(\varsigma) - \frac{\tilde{F}_2^2(\varsigma)}{\tilde{F}_1(\varsigma)+k_3t}),\\
&\phi_1 = \frac{e^{-k_2t}\tilde{F}_2(\varsigma)}{\tilde{F}_1(\varsigma)+k_3t}, \phi_2 = \frac{e^{k_2t}\tilde{F}_3(\varsigma)}{\tilde{F}_1(\varsigma)+k_3t}, f = \frac{1}{\tilde{F}_1(\varsigma)+k_3t}, \delta = \tilde{C}.
\end{aligned}
\tag{4.9}
$$

where $\varsigma = x - k_1t$, $\tilde{C}$ is a arbitrary constant.

Substituting Eqs.(4.9)into the prolonged system yields,

$$
\begin{aligned}
&\tilde{F}_2 = \tilde{C}_1 e^{\int -\lambda + \frac{\tilde{F}_{1\varsigma\varsigma}}{\tilde{F}_{1\varsigma}} + \frac{k_1}{2\tilde{C}\alpha} - \frac{k_3}{2\tilde{C}\alpha_2\tilde{F}_{1\varsigma}} d\varsigma}, \tilde{F}_3 = \frac{\tilde{F}_{1\varsigma}}{\tilde{F}_2},\\
&\tilde{F}_4 = -\frac{-\tilde{C}\alpha\tilde{F}_{1\varsigma\varsigma} - 4\lambda\tilde{C}\alpha\tilde{F}_{1\varsigma} + k_1\tilde{F}_{1\varsigma} - k_3}{2C\alpha\tilde{F}_2^2},\\
&\tilde{F}_5 = \frac{\tilde{C}\alpha\tilde{F}_2^2F_{1\varsigma\varsigma} - 4\lambda\tilde{C}\alpha\tilde{F}_2^2\tilde{F}_{1\varsigma} + k_1\tilde{F}_2^2\tilde{F}_{1\varsigma} - k_3\tilde{F}_2^2}{2\tilde{C}\alpha\tilde{F}_{1\varsigma}^2},
\end{aligned}
\tag{4.10}
$$

where $\tilde{C}_1$ is a arbitrary constant and $F = F(\varsigma) = F_{1\varsigma}$ satisfies the following equation

$$
\tilde{C}^2\alpha^2(\tilde{F}^2\tilde{F}_{\varsigma\varsigma\varsigma} + 3\tilde{F}_\varsigma^3) - (4\tilde{C}^2\alpha^2\tilde{F}\tilde{F}_{\varsigma\varsigma} + 4\tilde{C}\alpha\lambda k_3\tilde{F} - 2k_1k_3\tilde{F} + 3k_3^2)\tilde{F}_\varsigma = 0, \tag{4.11}
$$

the equation(4.11) is equivalent to the following elliptic equation,

$$
\tilde{F}_\varsigma = \frac{\sqrt{-2\tilde{C}^2\alpha^2\tilde{C}_2\tilde{F}^3 + 2\tilde{C}^2\alpha^2\tilde{C}_1\tilde{F}^2 + (4\tilde{C}k_3\alpha\lambda - 2k_1k_3)\tilde{F} + k_3^2}}{\tilde{C}\alpha}. \tag{4.12}
$$

To solve the equation(4.12), we assume a solution with the following form,

$$
\tilde{F} = \frac{1}{l_0 + l_1 sn(\varsigma, m)}, \tag{4.13}
$$

substituting Eq.(4.13) into Eq.(4.11) yields the following eight sets of solutions,

$$
\begin{aligned}
&\{k_1 = \pm 2\tilde{C}m\alpha + 2\tilde{C}\alpha\lambda, k_3 = \tfrac{\tilde{C}m\alpha}{l_1}, l_0 = l_1\},\\
&\{k_1 = 2\tilde{C}\alpha\lambda \pm 2\tilde{C}\alpha, k_3 = \pm\tfrac{\tilde{C}\alpha}{l_0}, l_1 = \pm l_0 m\},\\
&\{k_1 = \pm 2\tilde{C}m\alpha + 2\tilde{C}\alpha\lambda, k_3 = -\tfrac{\tilde{C}m\alpha}{l_1}, l_0 = \mp l_1\},\\
&\{k_1 = 2\tilde{C}\alpha\lambda \pm 2\tilde{C}\alpha, k_3 = \pm\tfrac{\tilde{C}\alpha}{l_0}, l_1 = \mp l_0 m\},
\end{aligned}
\tag{4.14}
$$

Remark 4.1. Substituting Eqs.(4.14), (4.13) and (4.10) into Eq.(4.9) yields the analytic solutions of variable coefficient AKNS system(2.1). It can be known from the expression (4.9) that u, v are rational function form solutions. If take $k_2 = 0$, then solutions are transformed into elliptic function solutions.

5. Summary and Discussion

In this paper, we have studied nonlocal symmetries and analytic solutions of the variable coefficient AKNS system for the first time. First of all, starting from the known Lax pairs of the variable coefficient AKNS system, nonlocal symmetries are derived directly through a particular assumption. To take advantage of nonlocal symmetries, an auxiliary variable is introduced. Then, the primary nonlocal symmetries are equivalent to a Lie point symmetries of a prolonged system. Applying the Lie group theorem to these local symmetries, the corresponding group invariant solutions are derived.

Secondly, several classes of analytic solutions are provided in this paper, including some special forms of analytic solutions. For example, analytic interaction solutions among solitons and other complicated waves, exponential solution, etc., These forms of solutions display solitons fission and fusions which can be easily applicable to the analysis of physically interesting processes for example the generation process of Rogue waves of variable coefficient AKNS system.

It is very meaningful to study the nonlocal symmetries and analytic solutions of variable coefficient integrable models. However, there is still a lot of work to be done. For example, in a large number of nonlocal symmetries of an integrable model Which one can be localized. Is it possible to apply the nonlocal symmetry theory of constant coefficient differential equation to the variable coefficient differential equation? Above topics will be discussed in the future series research works.

References

[1] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989.

[2] G. W. Bluman, A. F. Cheviakov and S.C. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Springer, New York, 2010.

[3] G. W. Bluman and A. F. Cheviakov, *Framework for potential systems and nonlocal symmetries: Algorithmic approach*, J. Math. Phys., 2005, 46, 123506.

[4] C. Chun and B. Neta, *Comparative study of methods of various orders for finding simple roots of nonlinear equations,* J. Appl. Anal. Comput., 2019, 9(2), 400–427.

[5] S. Chen, B. Tian, Y. Sun and C. Zhang, *Generalized Darboux Transformations, Rogue Waves, and Modulation Instability for the Coherently Coupled Nonlinear Schrödinger Equations in Nonlinear Optics*, Ann. Phys. (Berlin), 2019, 531(8), 1900011.

[6] Z. Cheng and Z. Bi, *Study on a kind of p-Laplacian neutral differential equation with multiple variable coefficients,* J. Appl. Anal. Comput., 2019, 9(2), 501–525.

[7] X. Du, B. Tian, Q. Qu, Y. Yuan and X. Zhao, *Lie group analysis, solitons, self-adjointness and conservation laws of the modified Zakharov-Kuznetsov equation in an electron-positron-ion magnetoplasma*, Chaos Solitons Fract., 2020, 134, 109709.

[8] Z. Du, B. Tian, H. Chai and X. Zhao, *Dark-bright semi-rational solitons and breathers for a higher-order coupled nonlinear Schrödinger system in an optical fiber*, Appl. Math. Lett., 2020, 102, 106110.

[9] A. Deliceoglu and D. Bozkurt, *Structural bifurcation of divergence-free vector fields near non-simple degenerate points with symmetry*, J. Appl. Anal. Comput., 2019, 9(2), 718–738.

[10] F. Galas, *New nonlocal symmetries with pseudopotentials*, J. Phys. A: Math. Gen., 1992, 25, L981.

[11] X. Gao, *Mathematical view with observational/experimental consideration on certain (2+1)-dimensional waves in the cosmic/laboratory dusty plasmas*, Appl. Math. Lett., 2019, 91, 165–172.

[12] X. Gao, Y. Guo and W. Shan, *Water-wave symbolic computation for the Earth, Enceladus and Titan: The higher-order Boussinesq-Burgers system, auto- and non-auto-Bäcklund transformations*, Appl. Math. Lett., 2020, 104, 106170.

[13] C. Hu, B. Tian, H. Yin, C. Zhang and Z. Zhang, *Dark breather waves, dark lump waves and lump wave-soliton interactions for a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation in a fluid*, Comput. Math. Appl., 2019, 78, 166–177.

[14] X. Hu, S. Lou and Y. Chen, *Explicit solutions from eigenfunction symmetry of the Korteweg-de Vries equation*, Phys. Rev. E, 2012, 85, 85056607–1.

[15] J. Han and L. Yan, *A time fractional functional differential equation driven by the fractional Brownian motion,* J. Appl. Anal. Comput., 2019, 9(2), 547–567.

[16] Q. Huang, Y. Gao, and Y. Feng, *Lax pair, infinitely-many conservation laws and soliton solutions for a set of the time-dependent Whitham-Broer-Kaup equations for the shallow water*, Waves in Random and Complex Media 2019, 29(1), 19–33.

[17] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Boston, MA: Reidel, 1985.

[18] H. Khan, C. Tunc and A. Khan, *Stability results and existence theorems for nonlinear delay-fractional differential equations with ϕ_p^*-operator,* J. Appl. Anal. Comput., 2020, 10(2), 584–597.

[19] P, Liu, B. Zeng, J. Yang and B. Ren, *Exact solutions and residual symmetries of the Ablowitz-Kaup-Newell-Segur system*, Chin. Phys. B, 2015, 24(1), 010202.

[20] S. Lie, *über die Integration durch bestimmte Integrale von einer Classe linearer partieller Differentialgleichungen,* Arch. Math. 1881, 6, 328–368.

[21] S. Lou, X. Hu and Y. Chen, *Nonlocal symmetries related to Bäcklund transformation and their applications*, J. Phys. A: Math. Theor., 2012, 45, 155209.

[22] Q. Miao, X. Xin and Y. Chen, *Nonlocal symmetries and explicit solutions of the AKNS system*, Appl. Math. Lett., 2014, 28, 7–13.

[23] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Berlin: Springer, 1986.

[24] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, New York: Academic, 1982.

[25] C. Qin, S. Tian, L. Zou, et al, *Lie symmetry analysis, conservation laws and exact solutions of fourth-order time fractional Burgers equation*, J. Appl. Anal. Comput., 2018, 8(6), 1727–1746.

[26] W. Qian, Y. Li and X. Yang, *The Isoenergetic KAM-Type Theorem at Resonant Case for Nearly Integrable Hamiltonian Systems,* J. Appl. Anal. Comput., 2019, 9(5), 1616–1638.

[27] S. Sui and B. Li, *Bifurcation of limit cycles from the global center of a class of integrable non-Hamilton systems,* J. Appl. Anal. Comput., 2018,8(5), 1441–1451.

[28] M. Wang, B. Tian, Y. Sun and Z. Zhang, *Lump, mixed lump-stripe and rogue wave-stripe solutions of a (3+1)-dimensional nonlinear wave equation for a liquid with gas bubbles*, Comput. Math. Appl., 2020,79, 576–587.

[29] X. Xin and X. Liu, *Interaction Solutions for (1+1)-Dimensional Higher-Order Broer-Kaup System*, Commun. Theor. Phys., 2016, 66(5), 479–482.

[30] X. Xin, Y. Liu and X. Liu, *Nonlocal symmetries, exact solutions and conservation laws of the coupled Hirota equations*, Appl. Math. Lett., 2016, 55, 63–71.

[31] Y. Xia, X. Xin and S. Zhang, *Residual symmetry, interaction solutions, and conservation laws of the (2+1)-dimensional dispersive long-wave system*, Chin. Phys. B, 2017, 26(3), 030202.

[32] X. Xin, L. Zhang, Y. Xia, et al. *Nonlocal symmetries and exact solutions of the (2+1)-dimensional generalized variable coefficient shallow water wave equation*, Appl. Math. Lett., 2019, 94, 112–119.

[33] X. Xin, H. Liu, L. Zhang, et al. *High order nonlocal symmetries and exact interaction solutions of the variable coefficient KdV equation*, Appl. Math. Lett., 2019, 88, 132–140.

[34] H. Yin, B. Tian and X. Zhao, *Chaotic breathers and breather fission/fusion for a vector nonlinear Schrödinger equation in a birefringent optical fiber or wavelength division multiplexed system*, Appl. Math. Comput., 2020, 368, 124768.

[35] X. Zheng and L. Wei, *Symmetry analysis conservation laws of a time fractional fifth-order Sawada-Kotera equation*, J. Appl. Anal. Comput., 2017, 7(4), 1275–1284.

[36] Z. Zhao and B. Han, *On Symmetry Analysis and Conservation Laws of the AKNS System*, Z. Naturforsch., 2016, 71(8)a, 741–750.

[37] C. Zhang, B. Tian, Q. Qu, L. Liu and H. Tian, *Vector bright solitons and their interactions of the couple Fokas-Lenells system in a birefringent optical fiber*, Z. Angew. Math. Phys., 2020, 71(1), 1–19.