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EXPLORING THE CONFORMABLE TIME-FRACTIONAL (3 + 1)-DIMENSIONAL MODIFIED KORTEWEG-DEVRIES-ZAKHAROV- KUZNETSOV EQUATION VIA THREE INTEGRATION SCHEMES

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Abstract In this paper, the nonlinear conformable time-fractional (3 + 1)-dimensional modified KdV-Zakharov-Kuznetsov equation is being explored using three well-established integration schemes named as: the exp_ζ function method, the hyperbolic function and modified Kudryashov schemes. In returns, many new exact solitary wave solutions, including rational, dark, singular and combined dark-singular solitons, are obtained and have been compared with those given in the literature. Moreover, the obtained solutions are demonstrated by $2D$ and $3D$ graphs for suitable values of the parameters to observe the dynamical behavior of the secured solutions.

Keywords Modified Kdv-Zakharov-Kuznetsov equation, conformable derivative, three integration schemes, solitary wave solutions.

MSC(2010) 35Q51, 35Q53, 37K40.

1. Introduction

Fractional derivative is becoming a hot spot of international research these days; it can describe the nonlinear phenomena more precisely. Through fractional derivative, we can adapt the nonlinear partial differential equation into an ordinary differential equation and secure the exact solutions. In recent years, many developments in fractional order derivatives, Like Caputo, Hilfer, Riemann-Liouville, form and so on, have been made in the literature but the well known product, quotient and the chain rules were the setbacks of one definition or another [11, 12, 21, 39]. Therefore the most fascinating definition of the fractional derivative with some of its properties are given in [19, 23].

The theory of solitons has become a dynamical subject that study the nonlinear physical models in vast area of applied fields. The most studied solitons are optical solitons [18, 26, 32, 35]. The adynamic of shallow water waves in various places like sea beaches are governed by the Korteweg-deVries (KdV) and Boussinesq Equations [24, 34, 48]. The Korteweg-deVries (KdV), Boussinesq, Kadomtsev-Petviashvili,

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and Whitham-Broer-Kaup (WBK) equations are the well-known completely integrable models that describe the propagation of shallow water [1, 30, 33, 37, 42, 45]. The KdV equation has an impact in modeling blood pressure pulses [7, 40, 46]. Wazwaz [44] introduced the nonlinear modified KdV $(3+1)$-dimensional equations and investigate their exact soliton and kink solutions. The Zakharov-Kuznetsov (ZK) equation is one of the well-studied canonical two dimensional extension of the KdV equation [22, 36]. Our main objective in this article is to construct exact solitary wave solutions of conformable time-fractional $(3+1)$-dimensional modified KdV-Zakharov-Kuznetsov (mKdV-ZK)equation [13]:

$$D_t^{\zeta} v + dv^2 v_x + ev_{xxx} + fv_{xyy} + gv_{xzz} = 0, \quad t > 0, 0 < \zeta \leq 1, \tag{1.1}$$

where d, e, f, g are non-zero constants. When $d = e \neq 0$, $f = g = 0$, Eq. (1.1) becomes fractional order mKdV equation and when $\alpha = 1$, Eq. (1.1) is known as the modified KdV-ZK equation [17, 29]. The existence of the solutions for the modified KdV-ZK equation has been considered in several papers [9, 10, 20]. By means of a traveling wave transformation and the conformable derivative, the aforementioned model is investigated via three integration schemes.

There are various mathematical approaches to solve important physical models with nonlinear characteristics or fractional derivatives. See for example, [3, 8, 16, 25, 28, 38, 51]. In particular [5, 6, 15, 27, 31, 43], the unified approach, the auxiliary equation, the improved $\tan(\frac{\phi(\eta)}{2})$-expansion, and the extended tanh-function methods have been explored for discrete and fractional order PDEs as well. In particular, the $exp_{\ a}$ function method [2, 49] and the hyperbolic function approach [4, 47] both have been utilized to procure the exact solutions of nonlinear partial differential equations.

The conformable derivative The conformable fractional derivative definition is given as [19, 23]:

$$D^{\zeta}(u(t)) = \lim_{h \to 0} \frac{u(t + ht^{1-\zeta}) - u(t)}{h},$$

where $\zeta \in\ (0; 1]$ and $u : [0; 1) \to \mathbb{R}$ in the half space $t > 0$. This fractional derivative supports plenty of properties given below under the assumptions that the order is $\zeta \in (0; 1]$ and that $u = u(t)$ and $v = v(t)$ are sufficiently ζ-differentiable for all $t > 0$. Then,

$$\begin{aligned}
&D^{\zeta}(a_1 u + a_2 v) = a_1 D^{\zeta}(u) + a_2 D^{\zeta}(v), \\
&D^{\zeta}(t^k) = kt^{k-\zeta}, \quad \forall\ k \epsilon R, \\
&D^{\zeta}(\lambda) = 0, \quad \forall\ constant\ \lambda, \\
&D^{\zeta}(uv) = uD^{\zeta}(v) + vD^{\zeta}(u), \\
&D^{\zeta}(\frac{u}{v}) = \frac{vD^{\zeta}(u) + uD^{\zeta}(v)}{v^2}, \\
&D^{\zeta}(u)(t) = t^{1-\zeta} \frac{du}{dt},
\end{aligned}$$

for $\forall a_1,\ a_2 \in \mathbb{R}$. The conformable derivative gives support to Laplace transformations, exponential function properties, chain rule, Taylor Series expansion etc. [19]. Probably the most useful property indicates the relation between the conformable derivative and common derivative.

Theorem 1.1. *Let u be an ζ-conformable differentiable function, and v is also differentiable function defined in the range of u. Then,*

$$D^{\zeta}(u \circ v)(t) = t^{1-\zeta} v^{'}(t) u^{'}(v(t)).$$

The rest of the paper is arranged as follows: In Sec. 2, we sketch the main frame of proposed methods that are known as the exp_{ζ}, the hyperbolic function method and the modified Kudryashov method. In Sec. 3, as an application, many new exact solitary wave solutions, including dark, singular and combined dark-singular solitons of Eq. (1.1), are obtained and have been compared with those given in the literature. Finally, we give some conclusions.

2. A formal sketch of proposed schemes

We provide, in this section, an outline of schemes and the ordinary formulation of a nonlinear fractional order partial differential equation into an ordinary differential equation. For this purpose, consider a nonlinear conformable time-fractional differential equation as follows:

$$F(u, \frac{\partial^{\zeta} u}{\partial t^{\zeta}}, \frac{\partial u}{\partial x}, \frac{\partial^{2\zeta} u}{\partial t^{2\zeta}}, \frac{\partial^2 u}{\partial x^2}, ...) = 0. \tag{2.1}$$

The wave transformation

$$u(x,t) = V(\epsilon), \qquad \epsilon = \sum_{i=1}^{N} k_i x_i - \lambda \frac{t^{\zeta}}{\zeta}, \quad 0 < \zeta \leq 1 \tag{2.2}$$

reduces Eq. (2.1) to a nonlinear ODE of integer order as given below:

$$G(V, V^{'}, V^{''}, ...) = 0, \tag{2.3}$$

where the prime demonstrates the conformable differentiation with respect to ϵ. To make our calculations easy, we can integrate Eq. (2.3) one or more time. In the next subsections, we find the exact solutions of aforementioned equation by using the following three analytical approaches.

2.1. Steps for exp_{ζ} function scheme

Let us consider a non-trivial solution for the Eq. (2.3) in the following form [14, 49, 50]:

$$U(\epsilon) = \frac{A_0 + A_1 \zeta^{\epsilon} + ... + A_N \zeta^{N\epsilon}}{B_0 + B_1 \zeta^{\epsilon} + ... + B_N \zeta^{N\epsilon}}, \ \zeta \neq 0, \tag{2.4}$$

where A_i and B_i, for $(0 \leq i \leq N)$, are found later and N is a free positive constant to be determined with the help of homogeneous balance principle. Replacing the Eq. (2.4) and its necessary derivatives in the nonlinear Eq. (2.3), yields

$$\wp(\zeta^{\epsilon}) = q_0 + q_1 a^{\epsilon} + ... + q_{\tau} a^{\tau\epsilon} = 0. \tag{2.5}$$

Setting $q_i(0 \leq i \leq \tau)$ in Eq. (2.5) to be zero, results give a set of nonlinear equations as follows:

$$q_i = 0, \quad i = 0, ..., \tau, \tag{2.6}$$

by solving the generated set (2.6), we acquire non-trivial solutions of the nonlinear PDE (1.1).

2.2. Steps for hyperbolic function scheme

Let us consider a non-trivial solution to the Eq. (2.3) in the following form [4,41,47]

$$U(\epsilon) = A_0 + \sum_{i=1}^{N} \sinh^{i-1}(\rho)[B_i \sinh(\rho) + A_i \cosh(\rho)], \tag{2.7}$$

where ρ is some specific functions. The positive integer N will be calculated using the homogeneous balance principle, using Eq. (2.7) in Eq. (2.3), and comparing the coefficients, we will find a set of nonlinear equations. The solution of this set finally provides the exact solutions of Eq. (1.1). It is important to mention that the implementation of separation of variables techniques on $\frac{d\rho}{d\epsilon} = \sinh(\rho)$, we find $\sinh(\rho) = \pm\text{csch}(\epsilon)$, $\cosh(\rho) = -\coth(\epsilon)$ and $\sinh(\rho) = \pm\imath\text{sech}(\epsilon)$, $\cosh(\rho) = -\tanh(\epsilon)$. Accordingly, the solution (2.4) can be rewritten as

$$U(\epsilon) = A_0 + \sum_{i=1}^{N} (\pm\text{csch})^{i-1}(\epsilon))[\pm B_i csch(\epsilon) - A_i \coth(\epsilon)],$$

and

$$U(\epsilon) = A_0 + \sum_{i=1}^{N} (\pm\imath\text{sech})^{i-1}(\epsilon)[\pm\imath B_i\text{sech}(\epsilon) - A_i \tanh(\epsilon)].$$

Equally, it is evident that from $\frac{d\rho}{d\epsilon} = \cosh(\rho)$, we find $\sinh(\rho) = -\cot(\epsilon)$, $\cosh(\rho) = \pm\csc(\epsilon)$ and $\sinh(\rho) = \tan(\epsilon)$, $\cosh(\rho) = \pm\sec(\epsilon)$. Accordingly, the solution (2.4) can be rewritten as

$$U(\epsilon) = A_0 + \sum_{i=1}^{N} (-\cot)^{i-1}(\epsilon)[-B_i \cot(\epsilon) \pm A_i \csc(\epsilon)],$$

and

$$U(\epsilon) = A_0 + \sum_{i=1}^{N} (\tan^{i-1})(\epsilon)[B_i \tan(\epsilon) \pm A_i \sec(\epsilon)].$$

2.3. Steps for modified kudryashov scheme

This section provides a concise report of principles of the Modified Kudryashov Method in inducing the exact solutions of nonlinear conformable differential equations [3,16,41].

Let the solution of Eq. (2.3) can be expressed as a finite series of the form

$$V(\epsilon) = a_0 + \sum_{i=0}^{N} a_i Q^i(\epsilon), \tag{2.8}$$

where a_i, $i = 1, ..., N(a_N \neq 0)$ are unknowns to be calculated, and $Q(\epsilon) = \frac{1}{1+d_0 a^\epsilon}$ satisfies the following first-order nonlinear equation

$$Q^{'}(\epsilon) = Q(\epsilon)(Q(\epsilon) - 1)\ln(a), \quad a \neq 0. \tag{2.9}$$

It should be pointed out that the positive integer N in Eq. (2.8) is computed using homogeneous balance principle. Inserting Eq. (2.8) and its necessary derivatives in Eq. (2.3) gives

$$P(Q(\epsilon)) = 0, \tag{2.10}$$

in which $P(Q(\epsilon))$ is a polynomial in $Q(\epsilon)$. By setting the coefficient of each power of $Q(\epsilon)$ in Eq. (2.10) to zero, we will reach a nonlinear algebraic system whose solution generates new solutions for the original Eq. (1.1).

3. Conformable time-fractional $(3+1)$-dimensional mKdV-ZK equation

By introducing a transformation $\epsilon = kx + py + qz - l\frac{t^\lambda}{\lambda}$, Eq. (1.1) can be turned into an ordinary differential equation:

$$-lV' + kdV^2V' + k^3eV''' + kfp^2V''' + kgq^2V''' = 0. \tag{3.1}$$

Integrate Eq. (3.1) once and taking zero constant of integration.

$$\frac{1}{3}dkV^3 + kV''\left(ek^2 + fp^2 + gq^2\right) - lV = 0. \tag{3.2}$$

The exp_ζ **function method:** After balancing the higher terms in Eq. (3.2) to obtain $N = 1$, then the non-trivial solution (2.4) becomes:

$$V(\epsilon) = \frac{A_0 + A_1\zeta^\epsilon}{B_0 + B_1\zeta^\epsilon}. \tag{3.3}$$

Setting the Eq. (3.3) in Eq. (3.2) and after setting the coefficients of ζ^ϵ equal to zero, the obtained nonlinear algebraic system gives the following sets of solutions:

$$A_0 = \pm\frac{i\sqrt{\frac{3}{2}}B_0\log(\zeta)\sqrt{l_1}}{\sqrt{d}}, \quad A_1 = \mp\frac{i\sqrt{\frac{3}{2}}B_1\log(\zeta)\sqrt{l_1}}{\sqrt{d}},$$
$$l = -\frac{1}{2}k\log^2(\zeta)l_1, \; l_1 = ek^2 + fp^2 + gq^2.$$

Therefore, the explicit exact solutions can be written as

$$V_1(x,y,z,t) = \frac{i\sqrt{\frac{3}{2}}\log(\zeta)\left(B_0 - B_1\zeta^{kx+py+qz+(-\frac{1}{2}k\log^2(\zeta)l_1)\frac{t^\lambda}{\lambda}}\right)\sqrt{l_1}}{\sqrt{d}\left(B_0 + B_1\zeta^{kx+py+qz+(\frac{1}{2}k\log^2(\zeta)l_1)\frac{t^\lambda}{\lambda}}\right)}, \tag{3.4}$$

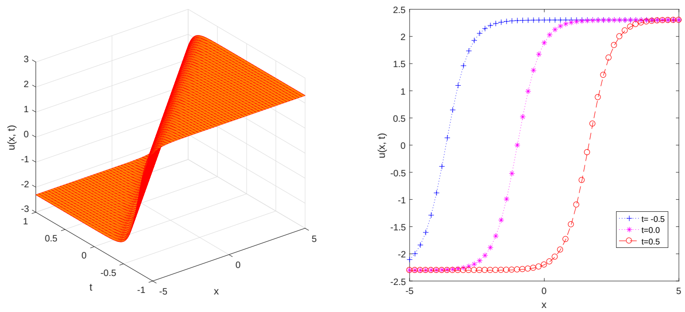
$$V_2(x,y,z,t) = -\frac{i\sqrt{\frac{3}{2}}\log(\zeta)\left(B_0 - B_1\zeta^{kx+py+qz+(-\frac{1}{2}k\log^2(\zeta)l_1)\frac{t^\lambda}{\lambda}}\right)\sqrt{l_1}}{\sqrt{d}\left(B_0 + B_1\zeta^{kx+py+qz+(\frac{1}{2}k\log^2(\zeta)l_1)\frac{t^\lambda}{\lambda}}\right)}. \tag{3.5}$$

The obtained solutions of Eq. (3.2) given in equations (3.4) and (3.5) are graphed here corresponding to the following, for the sake of simplicity, numerical values $e = -1$, $f = -\frac{1}{2} = g$, $k = p = q = 1$, and $d = 3$.

The hyperbolic function method: We again consider the Eq. (3.2) to solve under the implementation of hyperbolic function method.

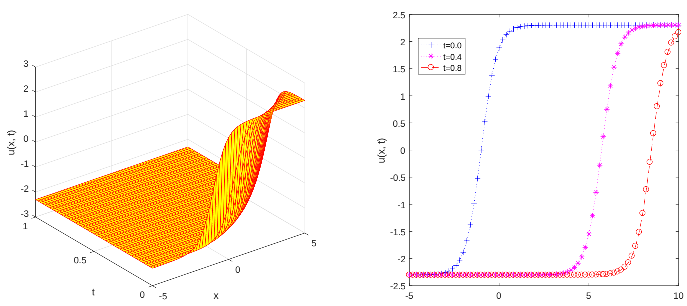
Case-1: $\frac{d\omega}{d\epsilon} = \sinh(\omega)$ and since $N = 1$, the non-trivial solution (2.7) becomes

$$u(\epsilon) = B_1\sinh(\omega) + A_1\cosh(\omega) + A_0. \tag{3.6}$$



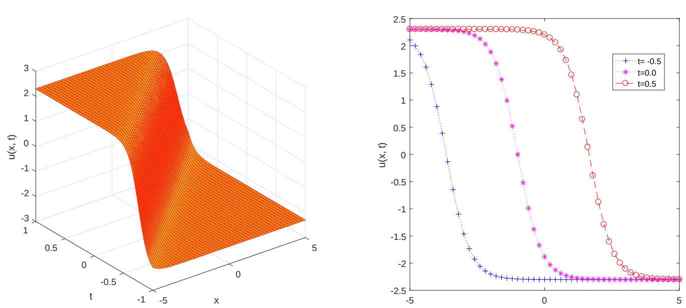
(a) $\lambda = 1$ (b) $\lambda = 1$

Figure 1. $2D$ and 3D plots for the solutions V_1 appear in Eq. (3.4) taking $y = 1$ and $z = 0$.



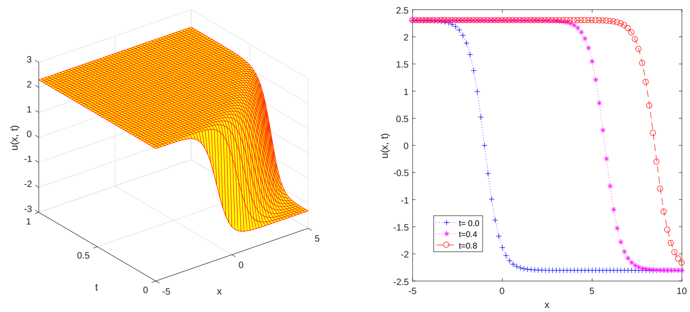
(a) $\lambda = 0.5$ (b) $\lambda = 0.5$

Figure 2. 2D and 3D plots for the solutions V_1 appear in Eq. (3.4) taking $y = 1$ and $z = 0$.



(a) $\lambda = 1$ (b) $\lambda = 1$

Figure 3. 2D and 3D plots for the solutions V_2 appear in Eq. (3.5) taking $y = 1$ and $z = 0$.



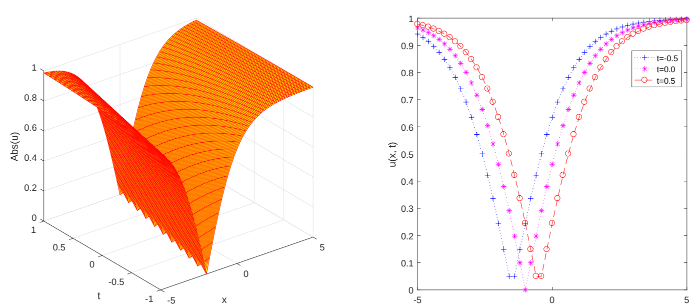
(a) $\lambda = 0.5$ (b) $\lambda = 0.5$

Figure 4. 2D and 3D plots for the solutions V_2 appear in Eq. (3.5) taking $y = 1$ and $z = 0$.

By setting the above non-trivial solution in reduced equation Eq. (3.2) and equating the coefficients of independent functions to zero in the resultant equation, we reach a nonlinear algebraic set of equations which its solution yields

$$A_0 = 0,\ A_1 = \mp\frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2 - fp^2 - gq^2}}{\sqrt{d}},\ B_1 = \pm\frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2 - fp^2 - gq^2}}{\sqrt{d}},$$
$$l = \frac{1}{2}\left(-ek^3 - fkp^2 - gkq^2\right),$$
$$u_{1,2}(x, y, z, t) = \mp\frac{\sqrt{6}\sqrt{-ek^2 - fp^2 - gq^2}}{2\sqrt{d}}\tanh\left(\frac{kx + py + qz - \frac{1}{2}\left(-ek^3 - fkp^2 - gkq^2\right)\frac{t^\lambda}{\lambda}}{2}\right). \tag{3.7}$$

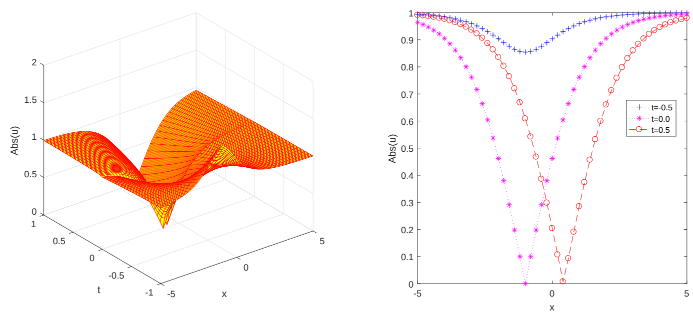
Note that the solutions appear in Eq. (3.7) have been reported in reference [20] and all others are new up to our knowledge.



(a) $\lambda = 1$ (b) $\lambda = 1$

Figure 5. 2D and 3D plots for the solutions $u_{1,2}$ appear in Eq. (3.7) taking $y = 1$ and $z = 0$.

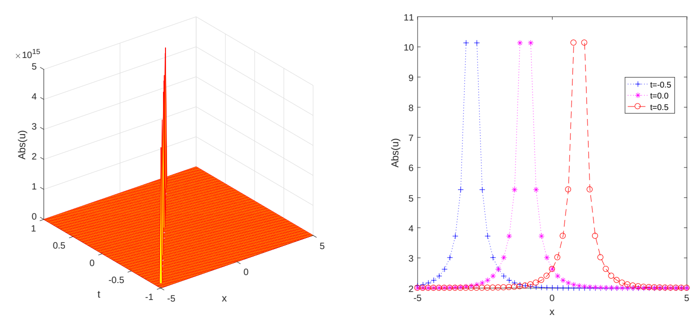
$$A_0 = 0, A_1 = \mp\frac{\sqrt{6}\sqrt{-ek^2 - fp^2 - gq^2}}{\sqrt{d}}, B_1 = 0, l = -2\left(ek^3 + fkp^2 + gkq^2\right),$$



(a) $\lambda = 0.5$ (b) $\lambda = 0.5$

Figure 6. 2D and 3D plots for the solutions $u_{1,2}$ appear in Eq. (3.7) corresponding to $y = 1$, $z = 0$.

$$u_{3,4}(x,y,z,t) = \mp \frac{\sqrt{6}\sqrt{-ek^2-fp^2-gq^2}}{\sqrt{d}} \coth(kx+py+qz+2\left(ek^3+fkp^2+gkq^2\right)\frac{t^\lambda}{\lambda}). \tag{3.8}$$



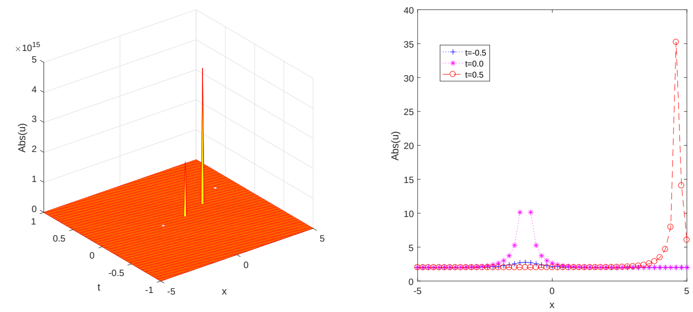
(a) $\lambda = 1$ (b) $\lambda = 1$

Figure 7. 2D and 3D plots for the solutions $u_{3,4}$ appear in Eq. (3.8) taking $y = 1$ and $z = 0$.

$$A_0 = 0, A_1 = 0, B_1 = \mp \frac{\sqrt{6}\sqrt{-ek^2-fp^2-gq^2}}{\sqrt{d}}, l = k\left(ek^2+fp^2+gq^2\right),$$
$$u_{5,6}(x,y,z,t) = \mp \frac{\left(\sqrt{6}\sqrt{-ek^2-fp^2-gq^2}\right)}{\sqrt{d}} \operatorname{csch}(kx+py+qz-k\left(ek^2+fp^2+gq^2\right)\frac{t^\lambda}{\lambda}). \tag{3.9}$$

Thus, the following new explicit exact solutions, for the conformable time-fractional mKdV-ZK equation, can be written as

$$A_0 = 0, A_1 = \mp \frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2-fp^2-gq^2}}{\sqrt{d}}, B_1 = \mp \frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2-fp^2-gq^2}}{\sqrt{d}},$$
$$l = \frac{1}{2}\left(-ek^3-fkp^2-gkq^2\right),$$



(a) $\lambda = 0.5$ (b) $\lambda = 0.5$

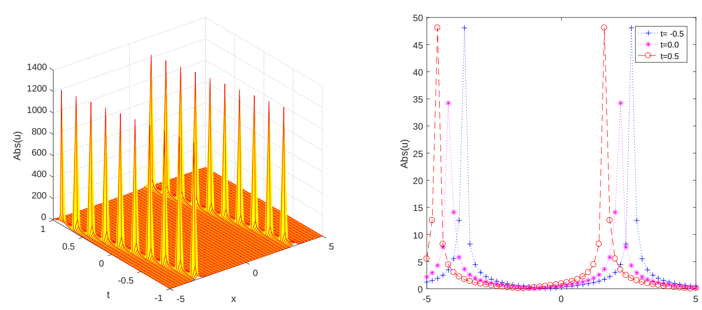
Figure 8. 2D and 3D plots for the solutions $u_{3,4}$ appear in Eq. 3.8 taking $y = 1$ and $z = 0$.

$$u_{7,8}(\epsilon)=\mp\frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2-fp^2-gq^2}}{\sqrt{d}}(\coth(kx+py+qz-l\frac{t^\lambda}{\lambda})+\operatorname{csch}(kx+py+qz-l\frac{t^\lambda}{\lambda})). \tag{3.10}$$

Case-2: $\frac{d\omega}{d\epsilon} = \cosh(\omega)$ and for $N = 1$, we obtain a set of nonlinear equations as which its solution yields

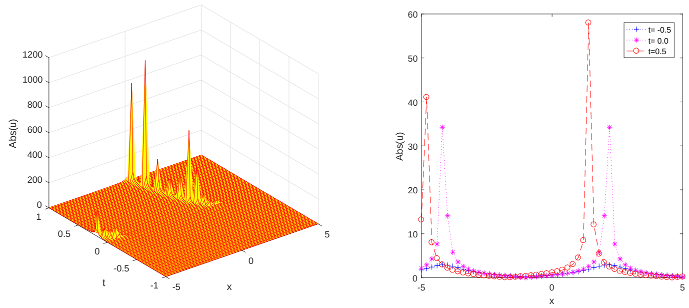
$$A_0 = 0, \quad A_1 = \mp\frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2 - fp^2 - gq^2}}{\sqrt{d}}, B_1 = \mp\frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2 - fp^2 - gq^2}}{\sqrt{d}},$$
$$l = \frac{1}{2}k\left(ek^2 + fp^2 + gq^2\right),$$
$$u_{9,10}(\epsilon) = \pm\frac{\sqrt{6}\sqrt{-ek^2 - fp^2 - gq^2}}{2\sqrt{d}}\tan\left(\frac{kx + py + qz - l\frac{t^\lambda}{\lambda}}{2}\right). \tag{3.11}$$

The following graphical demonstration is of the solution (3.11). For the sake of simplicity, we assume that $e = -1$, $f = -\frac{1}{2} = g$, $k = p = q = 1$, and $d = 3$.



(a) $\lambda = 1$ (b) $\lambda = 1$

Figure 9. 2D and 3D plots for the solutions $V_{9,10}$ appear in Eq. (3.11) corresponding to $y = 1$, $z = 0$.



(a) $\lambda = 0.5$ (b) $\lambda = 0.5$

Figure 10. 2D and 3D plots for the solutions $V_{9,10}$ appear in Eq. (3.11) corresponding to $y = 1$, $z = 0$.

$$A_0 = 0, A_1 = 0, B_1 = \mp\frac{\sqrt{6}\sqrt{-ek^2 - fp^2 - gq^2}}{\sqrt{d}}, l = 2\left(ek^3 + fkp^2 + gkq^2\right),$$

$$u_{11,12}(x,y,z,t) = \mp\frac{\sqrt{6}\sqrt{-ek^2-fp^2-gq^2}}{\sqrt{d}}\cot(kx + py + qz - 2\left(ek^3 + fkp^2 + gkq^2\right)\frac{t^\lambda}{\lambda}). \tag{3.12}$$

Thus, the following new explicit exact solutions of the conformable time-fractional $(3+1)$-dimensional mKdV-ZK equation can be written as

$$A_0 = 0,\ A_1 = \mp\frac{\sqrt{6}\sqrt{-ek^2 - fp^2 - gq^2}}{\sqrt{d}},\ B_1 = 0, l = -ek^3 - fkp^2 - gkq^2,$$

$$u_{13,14}(x,y,z,t) = \pm\frac{\sqrt{6}\sqrt{-ek^2-fp^2-gq^2}}{\sqrt{d}}\csc(kx+py+qz+(ek^3+fkp^2+gkq^2)\frac{t^\lambda}{\lambda}). \tag{3.13}$$

$$A_0 = 0, A_1 = \mp\frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2 - fp^2 - gq^2}}{\sqrt{d}}, B_1 = \pm\frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2 - fp^2 - gq^2}}{\sqrt{d}},$$

$$l = \frac{1}{2}k\left(ek^2 + fp^2 + gq^2\right),$$

$$u_{15,16}(\epsilon) = \pm\frac{\sqrt{\frac{3}{2}}\sqrt{-ek^2-fp^2-gq^2}}{\sqrt{d}}(\cot(kx+py+qz-l\frac{t^\lambda}{\lambda})+\csc(kx+py+qz-l\frac{t^\lambda}{\lambda})). \tag{3.14}$$

The modified kudryashov method: As we know, after balancing the highest derivative term and the highest nonlinear terms appear in Eq. (3.2), we obtain $N = 1$.

Thus, the non trivial solution Eq. (3.2) becomes:

$$V(\epsilon) = a_1 Q(\epsilon) + a_0. \tag{3.15}$$

Inserting the above equation in Eq. (3.2) along with Eq. (2.9) and setting the coefficients of $Q(\epsilon)$ equal to zero, which gives a system of nonlinear algebraic equations.

On solving the obtained system, we find the following two set of solutions.

$$a_0 = \pm \frac{i\sqrt{\frac{3}{2}}\log(a)\sqrt{ek^2+fp^2+gq^2}}{\sqrt{d}}, \quad a_1 = \mp \frac{i\sqrt{6}\log(a)\sqrt{ek^2+fp^2+gq^2}}{\sqrt{d}},$$
$$l = -\frac{1}{2}k\log^2(a)\left(ek^2+fp^2+gq^2\right).$$

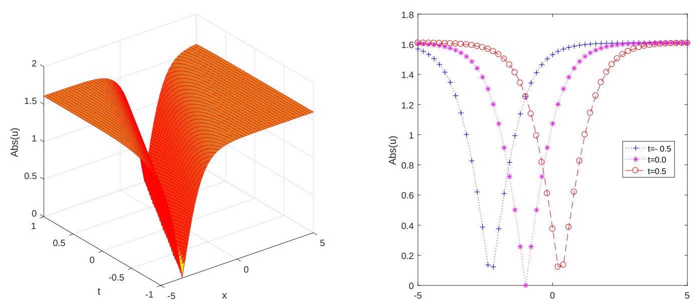
Therefore, the solutions can be written as

$$V_{1,2}(\epsilon) = \mp \frac{i\sqrt{\frac{3}{2}}\log(a)(2Q(\epsilon)-1)\sqrt{ek^2+fp^2+gq^2}}{\sqrt{d}}. \tag{3.16}$$

By replacing the value of $Q(\epsilon) = \frac{1}{d_0 a^{\epsilon}+1}$ in above solutions, which provide the solution in the form:

$$V_{1,2}(x,t) = \mp \frac{i\sqrt{\frac{3}{2}}\log(a)\left(\frac{2}{d_0 a^{\epsilon}+1}-1\right)\sqrt{ek^2+fp^2+gq^2}}{\sqrt{d}}. \tag{3.17}$$

We now present the graphical demonstration of the above solutions. Computer software Matlab 2016 has been used in this work to find solutions and presentation of the graphs of the above-mentioned equations. For the sake of simplicity, we assume that $e=-1$, $f=-\frac{1}{2}=g$, $k=p=q=d_0=1$, $\zeta=5$, and $d=3$.

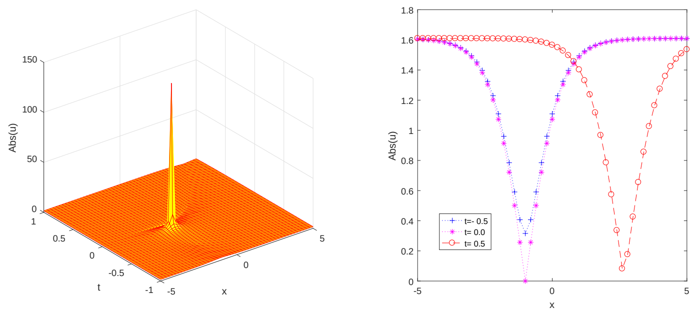


(a) $\lambda = 1$ (b) $\lambda = 1$

Figure 11. 2D and 3D plots for the solutions $V_{1,2}$ appear in Eq.(3.17) corresponding to $y=1$ and $z=0$.

4. Conclusion

In this paper, we have acquired many new wave solutions for the nonlinear conformable fractional $(3+1)$- dimensional modified Kdv-ZK equation. The three prolific integration schemes, namely the exp_{ζ}-function, the hyperbolic function method and the modified Kudryashov method along with the appropriate transformation have been applied to accomplish the objective. Among these solutions, we are with the rational, dark, singular and combined dark-singular solutions and have been compared with those given in the literature. Furthermore, the numerical simulations of some secured solutions have been demonstrated via soft computation



(a) $\lambda = 0.5$ (b) $\lambda = 0.5$

Figure 12. 2D and 3D plots for the solutions $V_{1,2}$ appear in Eq. (3.17) corresponding to $y = 1$ and $z = 0$.

to analyze the dynamical behavior of the waves. Thus we conclude that one can implement the aforesaid approaches to other nonlinear fractional order differential equations.

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