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CONVERGENCE ANALYSIS OF NEW ADDITIVE SCHWARZ METHOD FOR SOLVING NONSELFADJOINT ELLIPTIC PROBLEMS

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Abstract In this paper, we present a two-level additive Schwarz method for solving a system arising from the discretization of the nonselfadjoint elliptic equation. By employing the Cauchy-Schwarz-type inequality and stable decomposition under the energy norm, we obtain the optimal convergence theory for the proposed method. It shows that the convergence rate is bounded and independent of the fine mesh size and the number of subdomains. Some numerical results are reported to verify our theoretical result. Moreover, we demonstrate the benefit compared to the classical two-level additive Schwarz algorithm for solving convection-diffusion equations.

Keywords Additive Schwarz method, AHSS iteration, nonselfadjoint elliptic problems, convergence rate.

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1. Introduction

Domain decomposition methods are some of the most popular methods for the solution of large linear systems arising from partial differential equations (PDEs). For linear problems, domain decomposition methods can often be viewed as preconditioners for Krylov subspace accelerator techniques. The classical two-level additive Schwarz (AS) methods are originally presented for solving selfadjoint positive definite (SPD) problems [12,14,15]. And these methods have been successfully applied to elliptic problems with discontinuous coefficients [13,18,23,30]. Recently, AS methods are also employed to solve the system of equations arising from Discontinuous Galerkin and finite volume element discretizations of selfadjoint elliptic PDEs [1,22,30]. For nonselfadjoint and indefinite linear elliptic problems, two variants of the AS methods are presented in [10]. The analysis shows that the convergence rate is bounded independent of the fine mesh size and the number of subdomains if the coarse mesh size is sufficiently small. Two-level AS methods are also developed for the mortar element and P_1 nonconforming finite element approximation

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of nonselfadjoint and indefinite elliptic problems [11,37]. Additionally, AS methods have been also developed and applied to other problems in [19,26,29,31,32,36,38] and the references therein.

In this paper, we introduce a two-level AS algorithm based on the asymmetric Hermitian/skew-Hermitian splitting (AHSS) iteration proposed in [24]. The AHSS iteration can be viewed as a generalized version of the HSS iteration which is first presented by Bai, Golub and Ng [7]. It has been shown that these methods converge unconditionally to the unique solution of the linear system. Numerical results show that these methods perform very well for convection-diffusion equations. So these methods have been deeply studied and widely developed [3–6,20,24,25,27,28,35]. However, it is very costly to solve the system of equations with shifted skew-Hermitian matrix. Although some techniques, such as the inexact approximations, are used, the difficulty is not easy to overcome since the system of the equations is very large usually. Combining the ideas of the AS method with the AHSS iteration, we present a new two-level AS algorithm. Different from the Schwarz algorithm presented in [10], the Schwarz operator in our algorithm includes both the selfadjoint and skew-selfadjoint parts of the equation. Moreover, since the systems in the subdomains are small and can be easily solved by ILU, the proposed Schwarz algorithm performs very well for convection-diffusion equations. We establish an optimal convergence theory and prove that the convergence rate is bounded and independent of the fine mesh size and the number of subdomains. Further, it shows that the parameters in the Schwarz operators should be chosen as the minimum and the maximum eigenvalues of the selfadjoint part of the coefficient matrix. To confirm the convergence theory and demonstrate the applicability of this method, we show some numerical experiments and compare our approach with the classical AS algorithm.

The rest of this paper is organized as follows. In Section 2, we describe the model problem and introduce the two-level AS algorithm. In Section 3, we present the convergence analysis of the proposed algorithm based on the abstract Schwarz theory in [10]. Some numerical experiments are reported to illustrate the performance of this algorithm in Section 4. Finally, some concluding remarks are given in Section 5.

2. Preliminaries and notations

We consider the following second-order elliptic boundary value problem

$$\begin{cases} -\nabla\cdot(\mathbf{a}(x)\nabla u)+2\mathbf{b}(x)\cdot\nabla u+c(x)u=f(x), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is an open, bounded polygonal domain in R^d and $\mathbf{a}(x)\in C^1(\overline{\Omega},R^{d\times d})$, $\mathbf{b}(x)\in C^1(\overline{\Omega})^d$, $c(x)\in C^1(\overline{\Omega})$ and the right hand side $f(x)\in L^2(\Omega)$. Assume that $\mathbf{a}(x)=(a_{ij}(x))_{d\times d}$ is a symmetric and uniformly positive definite matrix in Ω, i.e., there exists a positive constant m such that $\xi^T\mathbf{a}(x)\xi\geq m|\xi|^2$ for all $\xi\in R^d$ and $x\in\overline{\Omega}$. We assume that $c(x)-\nabla\cdot\mathbf{b}(x)\geq 0$ for any $x\in\overline{\Omega}$, and problem (2.1) has a unique solution in $H_0^1(\Omega)$. For brevity, we omit the variable x in the following discussion.

The weak form of problem (2.1) is: find $u\in H_0^1(\Omega)$ such that

$$A(u,v)=(f,v)\quad\forall\, v\in H_0^1(\Omega), \tag{2.2}$$

where the bilinear form $A(u,v)$ is defined as

$$A(u,v) = \int_\Omega (\mathbf{a}\nabla u \cdot \nabla v + 2\mathbf{b}\cdot\nabla u v + cuv)dx$$
$$= \int_\Omega (\mathbf{a}\nabla u \cdot \nabla v + 2\mathbf{b}\cdot\nabla u v + \nabla\cdot\mathbf{b}uv + \widetilde{c}uv)dx \quad \forall\ u,\ v\in H_0^1(\Omega),$$

where $\widetilde{c} = c - \nabla\cdot\mathbf{b}$. From the assumption for problem (2.1), we see that $\widetilde{c}$ is a nonnegative function, and there exists a constant $C>0$, such that

$$A(u,v) \le C\ |||\ u\ |||_1\ |||v\ |||_1 \quad \forall\ u,\ v\in H_0^1(\Omega), \tag{2.3}$$

Denote

$$\widehat{A}(u,v) = \int_\Omega \mathbf{a}\nabla u\cdot\nabla v dx, \qquad H(u,v) = \int_\Omega (\mathbf{a}\nabla u\cdot\nabla v + \widetilde{c}uv)dx,$$
$$S(u,v) = \int_\Omega (2\mathbf{b}\cdot\nabla u v + \nabla\cdot\mathbf{b}uv)dx = \int_\Omega \mathbf{b}\cdot(\nabla u v - \nabla v u)dx,$$

where $H(u,v)$ and $S(u,v)$ correspond to selfadjoint and skew-selfadjoint parts of $A(u,v)$, respectively. It is clear that $A(u,v)=H(u,v)+S(u,v)$, and there exists a constant $C>0$, such that

$$|S(u,v)| \le C\ |||\ u\ |||_1\ \|v\| \quad \text{and} \quad |S(u,v)| \le C\|u\|\ |||\ v\ |||_1 \quad \forall\ u,\ v\in H_0^1(\Omega). \tag{2.4}$$

Define the norm

$$|||u|||_1 = \sqrt{H(u,u)}.$$

It is easy to see that

$$c\|u\|_1 \le |||u|||_1 \le C\|u\|_1 \quad \forall\ u\in H_0^1(\Omega), \tag{2.5}$$

where c and C are positive constants and $\|\cdot\|_1$ denotes the H^1 norm in Sobolev space. We assume the solution of (2.1) with $\mathbf{a}=\mathbf{I}$, $\mathbf{b}=\mathbf{0}$, and $c=0$ satisfy the following regularity estimate:

$$\|u\|_2 \le C\|f\|, \tag{2.6}$$

where $\|\cdot\|_2$ and $\|\cdot\|$ denote the H^2 norm and L^2 norm in Sobolev space, respectively. The following estimates are straightforward from the assumptions for problem (2.1).

We next introduce the overlapping Schwarz preconditioner for (2.2). Let $\{\Omega_i\}_{1\le i\le N}$ be a set of non-overlapping simplices such that $\overline{\Omega} = \sum_{i=1}^N \overline{\Omega}_i$. Denote the diameter of Ω_i by $\widehat{H}_i$. Let H_0 denote the mesh parameter which is the maximum diameter of all subdomains, i.e., $H_0 = \max\{\widehat{H}_1,\ldots,\widehat{H}_N\}$. Divide each Ω_i into smaller simplices, which denoted as $\tau_i^j(j=1,\ldots)$. Let h_i^j be the diameter of τ_i^j and $h=\max\{h_i^j\}$. By repeatedly adding some layers of fine mesh elements, we extend each subdomain Ω_i to the larger domain Ω_i', such that $\partial\Omega_i'$ does not cut through any fine elements, and denote the corresponding overlap by δ_i. Therefore, $\Omega_i\subset\Omega_i'$, it is enough to assume that every point $x\in\Omega$ belongs to at most N_c overlapping subdomains. The maximum of $\widehat{H}_i/\delta_i$ is defined by $H_0/\delta = \max_{1\le i\le N}\left\{\widehat{H}_i/\delta_i\right\}$. Finally, we introduce a shape-regular coarse mesh on Ω. For simplicity, we assume that the coarse mesh is nested in the fine mesh. Denote the coarse mesh size by H_c and assume that $H_c\le C\widehat{H}_i\le CH_0$, where $C>0$ is a constant.

Define the coarse and fine finite element spaces on Ω by V_H and V_h, which consist of continuous, piecewise linear functions. We introduce the subspaces $V_h^i = V_h \cap H_0^1(\Omega_i')$ $(i = 1, 2, \ldots, N)$. Then the finite element space V_h can be decomposed as

$$V_h = V_H + V_h^1 + V_h^2 + \cdots + V_h^N.$$

Define the operators $A_0 : V_H \to V_H$ and $A : V_h \to V_h$ by

$$\begin{aligned} &(A_0 u_H, v_H) = A(u_H, v_H) \quad \forall\ u_H,\ v_H \in V_H, \\ &(A u_h, v_h) = A(u_h, v_h) \qquad \forall\ u_h, \quad v_h \in V_h. \end{aligned}$$

The operators A_i, H_i and $S_i : V_h^i \to V_h^i$ $(i = 1, 2, \ldots, N)$ are defined by

$$\begin{aligned} &(A_i u_h^i, v_h^i) = A(u_h^i, v_h^i), \qquad (H_i u_h^i, v_h^i) = H(u_h^i, v_h^i), \\ &(S_i u_h^i, v_h^i) = S(u_h^i, v_h^i) \qquad \forall\ u_h^i,\ v_h^i \in V_h^i. \end{aligned}$$

Obviously, $A_i = H_i + S_i$, where $H_i = \frac{1}{2}(A_i + A_i^T)$, $S_i = \frac{1}{2}(A_i - A_i^T)$. Let λ_{max} and λ_{min} be the the maximum and minimum eigenvalues of H_i, respectively. It is well known that $\lambda_{max} = O(h^{-2})$ and λ_{min} is a constant.

Define projection operators P_0, $\widehat{P}_0$ and $Q_0 : V_h \to V_H$ by

$$\begin{aligned} &A(P_0 u_h, v_H) = A(u_h, v_H), \qquad H(\widehat{P}_0 u_h, v_H) = H(u_h, v_H), \\ &(Q_0 u_h, v_H) = (u_h, v_H) \quad \forall\ u_h \in V_h, v_H \in V_H. \end{aligned}$$

The operators P_i, $\widehat{P}_i$ and $Q_i : V_h \to V_h^i$ $(i = 1, 2, \ldots, N)$ are defined by

$$\begin{aligned} &A(P_i u_h, v_h^i) = A(u_h, v_h^i), \qquad H(\widehat{P}_i u_h, v_h^i) = H(u_h, v_h^i), \\ &(Q_i u_h, v_h^i) = (u_h, v_h^i) \quad \forall\ u_h \in V_h, v_h^i \in V_h^i. \end{aligned}$$

From the definitions of P_i $(i = 1, 2, \ldots, N)$, we have

$$\|P_i u_h\|_1 \le C \|u_h\|_{1,\Omega_i'} \quad \text{and} \quad |\!|\!|\ P_i u_h |\!|\!|_1 \le C\ |\!|\!|\ u_h\ |\!|\!|_{1,\Omega_i'} \quad \forall\ u_h \in V_h, \tag{2.7}$$

where $\|\cdot\|_{1,\Omega_i'}$ and $|\!|\!|\cdot|\!|\!|_{1,\Omega_i'}$ are only nonzero on the overlapping subdomain Ω_i'. Define the operator $\mathcal{O} : V_h \to V_h$

$$|\!|\!|\ \mathcal{O} |\!|\!|_1 = \sup_{u_h, v_h \in V_h} \frac{|H(\mathcal{O} u_h, v_h)|}{|\!|\!| u_h\ |\!|\!|_1\ |\!|\!| v_h |\!|\!|_1}. \tag{2.8}$$

The finite element solution of (2.2) is to find $u_h^* \in V_h$ such that

$$A(u_h^*, v_h) = (f, v_h) \quad \forall\ v_h \in V_h. \tag{2.9}$$

From the above analysis, (2.9) can be rewritten as

$$A u_h^* = f, \tag{2.10}$$

where A is nonselfadjoint and positive definite.

Define

$$M_i = \frac{1}{\alpha+\beta}(\alpha I + H_i)(\beta I + S_i),\ N_i = \frac{1}{\alpha+\beta}(\beta I - H_i)(\alpha I - S_i), \tag{2.11}$$

where $\alpha \geq 0$, $\beta > 0$, and I is the identity operator. We observe that $A_i = M_i - N_i$. Based on the AHSS iteration [24], we define the operator $T_i : V_h \to V_h^i$ ($i = 1, 2, \ldots, N$) by

$$M(T_i u_h, v_h^i) = A(u_h, v_h^i) \quad \forall\ u_h \in V_h,\ v_h^i \in V_h^i. \tag{2.12}$$

By the definition of P_i, Q_i and (2.12), we have

$$T_i = M_i^{-1} Q_i A = M_i^{-1} A_i P_i. \tag{2.13}$$

Let $T_0 = P_0$, the two-level AS operator is defined by

$$T = \sum_{i=0}^{N} T_i = P_0 + \sum_{i=1}^{N} M_i^{-1} Q_i A = A_0^{-1} Q_0 A + \sum_{i=1}^{N} M_i^{-1} Q_i A = B^{-1} A, \tag{2.14}$$

where $B^{-1} = A_0^{-1} Q_0 + \sum\limits_{i=1}^{N} M_i^{-1} Q_i$.

Now we present the two-level AS algorithm.

Algorithm 2.1 (Two-level AS algorithm)**.** Find the solution of the problem (2.9) or (2.10) by solving

$$T u_h = g \tag{2.15}$$

with a Krylov subspace method, where $g = A_0^{-1} Q_0 A u_h^* + \sum\limits_{i=1}^{N} M_i^{-1} Q_i A u_h^*$.

To analyze the convergence performance of the AS algorithm, we introduce several important properties.

Lemma 2.1. *There exists a constant $C > 0$, which is independent of H_0 and h, such that for all $u_h \in V_h$,*

$$|||T_0 u_h|||_1 \leq C\ |||\ u_h|||_1$$

and

$$\|T_0 u_h - u_h\| \leq C H_0\ |||\ u_h\ |||_1\ .$$

Proof. It follows from the definition of T_0 and (2.3), we have

$$A(T_0 u_h, T_0 u_h) = A(u_h, T_0 u_h) \leq C\ |||\ u_h\ |||_1\ |||T_0 u_h\ |||_1\ .$$

Since $S(T_0 u_h, T_0 u_h) = 0$, we have $H(T_0 u_h, T_0 u_h) = A(T_0 u_h, T_0 u_h)$. Therefore,

$$|||T_0 u_h|||_1 \leq C\ |||\ u_h\ |||_1\ .$$

Analogously to Lemma 11.3 in [33], by using a "duality" argument and (2.3), we obtain

$$\|T_0 u_h - u_h\| \leq C H_0\ |||\ T_0 u_h - u_h|||_1 \leq C H_0\ |||\ u_h\ |||_1\ .$$

This completes the proof of the lemma. □

Proposition 2.1 (Strengthened Cauchy-Schwarz inequalities)**.** *There exists a constant $0 \leq k_{ij} \leq 1$, for $u_h^i \in V_h^i,\ u_h^j \in V_h^j,\ 1 \leq i, j \leq N$, such that*

$$|H(u_h^i, u_h^j)| \leq k_{ij} H(u_h^i, u_h^i)^{\frac{1}{2}} H(u_h^j, u_h^j)^{\frac{1}{2}}. \tag{2.16}$$

We will denote the spectral radius of $K = \{k_{ij}\}$ by $\rho(K)$.

Proposition 2.2 (Stability of the decomposition)**.** *For any $v_h \in V_h$, there exist $v_h^0 = v_H \in V_H$ and $v_h^i \in V_h^i$ such that $v_h = \sum_{i=0}^{N} v_h^i$ and*

$$\sum_{i=0}^{N} H(v_h^i, v_h^i) \leq C_0^2 H(v_h, v_h), \tag{2.17}$$

where $C_0 = C(1 + H_0/\delta)^{\frac{1}{2}}$ and C is a constant independent of the mesh parameters h, H_0.

Remark 2.1. (i) In Proposition 2.1, we have

$$|H(u_h^i, u_h^j)| = \left| \widehat{A}(u_h^i, u_h^j) + (\widetilde{c}u_h^i, u_h^j) \right| \leq \left| \widehat{A}(u_h^i, u_h^j) \right| + \left| (\widetilde{c}u_h^i, u_h^j) \right|.$$

It follows from Assumption 2.3 in [33], Lemma 3.3 in [9] and Cauchy-Schwarz inequality that

$$\begin{aligned} |H(u_h^i, u_h^j)| &\leq \left| \widehat{A}(u_h^i, u_h^j) \right| + \left| (\widetilde{c}u_h^i, u_h^j) \right| \\ &\leq k_{ij} \widehat{A}(u_h^i, u_h^i)^{\frac{1}{2}} \widehat{A}(u_h^j, u_h^j)^{\frac{1}{2}} + k_{ij} (\widetilde{c}u_h^i, u_h^i)^{\frac{1}{2}} (\widetilde{c}u_h^j, u_h^j)^{\frac{1}{2}} \\ &\leq k_{ij} H(u_h^i, u_h^i)^{\frac{1}{2}} H(u_h^j, u_h^j)^{\frac{1}{2}}. \end{aligned}$$

(ii) Proposition 2.2 can be directly obtained from Lemma 4 in [8] and Theorem 4.1 in [16].

3. Convergence analysis of the two-level additive Schwarz algorithm

In this section, we present the convergence analysis of Algorithm 2.1. Following Eisenstat, Elman and Schultz [17], the convergence rate of AS preconditioned GMRES method can be computed by the two quantities

$$c_T = \inf_{u_h \neq 0} \frac{H(Tu_h, u_h)}{H(u_h, u_h)} \quad \text{and} \quad C_T = \sup_{u_h \neq 0} \frac{|||Tu_h|||_1}{|||u_h|||_1}.$$

Moreover, the residual at the k^{th} iteration is bounded as

$$|||r_k|||_1 \leq \left(1 - \frac{c_T^2}{C_T^2}\right)^{\frac{k}{2}} |||\, r_0|||_1,$$

where $r_k = b - Tu_h^k$.

To estimate the bounds of c_T and C_T and their dependency on h and the number of subdomains, we first present the following two assumptions, and give the main result of this paper based on these assumptions in section 3.1. Then we prove the proposed assumptions in section 3.2.

Assumption 3.1. *Suppose that $\alpha = O(\lambda_{min})$ and $\beta = O(\lambda_{max})$ hold. There exists a constant $C > 0$, independent of H_0 and h, such that for $u_h \in V_h$,*

$$\sum_{i=1}^{N} H(T_i u_h, T_i u_h) \leq C N_c H(u_h, u_h).$$

Assumption 3.2. *If H_0 is sufficiently small, and suppose that $\alpha = O(\lambda_{min})$ and $\beta = O(\lambda_{max})$ hold. There exists a constant $C > 0$, independent of H_0 and h, such that*

$$\sum_{i=0}^{N} H(T_i u_h, T_i u_h) \geq C C_0^{-2} H(u_h, u_h) \quad \forall\ u_h \in V_h,$$

where C_0 is introduced in Proposition 2.2.

3.1. The upper and lower bounds of the operator T

By employing above two assumptions, we provide estimates for the upper and lower bounds of the operator T.

Theorem 3.1. *If Assumptions 3.1 and 3.2 hold and H_0 is sufficiently small, suppose that $\alpha = O(\lambda_{min})$ and $\beta = O(\lambda_{max})$, then*

(1) there exists a constant C_T such that

$$H(Tu_h, Tu_h) \leq C_T^2 H(u_h, u_h) \quad \forall\ u_h \in V_h,$$

where $C_T^2 = C(1 + N_c^2)$ and C is a positive constant independent of H_0 and h.

(2) there exists a constant c_T such that

$$H(Tu_h, u_h) \geq c_T H(u_h, u_h),$$

where $c_T = CC_0^{-2} = C(1 + H_0/\delta)^{-1}$ and C is a positive constant independent of H_0 and h.

Proof. (1) It follows from (2.14) and the mean value inequality, we have

$$\begin{aligned} H(Tu_h, Tu_h) &= H\left(\sum_{i=0}^{N} T_i u_h, \sum_{i=0}^{N} T_i u_h\right) = H\left(T_0 u_h + \sum_{i=1}^{N} T_i u_h, T_0 u_h + \sum_{i=1}^{N} T_i u_h\right) \\ &\leq 2H(T_0 u_h, T_0 u_h) + 2H\left(\sum_{i=1}^{N} T_i u_h, \sum_{i=1}^{N} T_i u_h\right). \end{aligned} \tag{3.1}$$

By Proposition 2.1, we obtain

$$\begin{aligned} H\left(\sum_{i=1}^{N} T_i u_h, \sum_{i=1}^{N} T_i u_h\right) &= \sum_{i=1}^{N}\sum_{j=1}^{N} H(T_i u_h, T_j u_h) \\ &\leq \sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij} H(T_i u_h, T_i u_h)^{\frac{1}{2}} H(T_j u_h, T_j u_h)^{\frac{1}{2}} \\ &\leq \rho(K) \sum_{i=1}^{N} H(T_i u_h, T_i u_h) \leq N_c \sum_{i=1}^{N} H(T_i u_h, T_i u_h). \end{aligned}$$

Combining this inequality with Assumption 3.1 implies

$$H\left(\sum_{i=1}^{N} T_i u_h, \sum_{i=1}^{N} T_i u_h\right) \leq C N_c^2 H(u_h, u_h). \tag{3.2}$$

It follows from (3.1), (3.2) and Lemma 2.1 that

$$H(Tu_h, Tu_h) \le C(1 + N_c^2)H(u_h, u_h) = C_T^2 H(u_h, u_h).$$

(2) From (2.14), we have

$$\begin{aligned} H(Tu_h, u_h) &= H\left(\sum_{i=0}^{N} T_i u_h, u_h\right) \\ &= \sum_{i=0}^{N} H(T_i u_h, T_i u_h) + \sum_{i=0}^{N} \left(H(T_i u_h, u_h) - H(T_i u_h, T_i u_h)\right) \\ &= \sum_{i=0}^{N} H(T_i u_h, T_i u_h) + \sum_{i=0}^{N} H(u_h - T_i u_h, T_i u_h). \end{aligned} \tag{3.3}$$

For $i = 0$, by the definition of T_0, we have

$$\begin{aligned} H(u_h - T_0 u_h, T_0 u_h) &= A(u_h - T_0 u_h, T_0 u_h) - S(u_h - T_0 u_h, T_0 u_h) \\ &= -S(u_h - T_0 u_h, T_0 u_h). \end{aligned}$$

It follows from (2.4) and Lemma 2.1 that

$$|S(u_h - T_0 u_h, T_0 u_h)| \le C\|u_h - T_0 u_h\| \,|\!|\!|\, T_0 u_h |\!|\!|_1 \le CH_0 H(u_h, u_h). \tag{3.4}$$

For $i > 0$, since $S(T_i u_h, T_i u_h) = 0$, we have

$$\begin{aligned} H(u_h - T_i u_h, T_i u_h) &= A(u_h - T_i u_h, T_i u_h) - S(u_h - T_i u_h, T_i u_h) \\ &= A(u_h - T_i u_h, T_i u_h) - S(u_h, T_i u_h). \end{aligned} \tag{3.5}$$

It follows from (2.12) and (3.5) that

$$\begin{aligned} H(u_h - T_i u_h, T_i u_h) &= A(u_h - T_i u_h, T_i u_h) - S(u_h, T_i u_h) \\ &= A(u_h, T_i u_h) - A(T_i u_h, T_i u_h) - S(u_h, T_i u_h) \\ &= M(T_i u_h, T_i u_h) - A(T_i u_h, T_i u_h) - S(u_h, T_i u_h) \\ &= N(T_i u_h, T_i u_h) - S(u_h, T_i u_h). \end{aligned} \tag{3.6}$$

By (2.11), we have

$$\begin{aligned} &N(T_i u_h, T_i u_h) \\ =&\frac{1}{\alpha+\beta}\left(\alpha\beta(T_i u_h, T_i u_h) - \alpha H(T_i u_h, T_i u_h) - \beta S(T_i u_h, T_i u_h) + (H\cdot S)(T_i u_h, T_i u_h)\right) \\ =&\frac{1}{\alpha+\beta}\left(\alpha\beta(T_i u_h, T_i u_h) - \alpha H(T_i u_h, T_i u_h) + (H\cdot S)(T_i u_h, T_i u_h)\right) \\ \le&\frac{1}{\alpha+\beta}\left(\alpha\beta(T_i u_h, T_i u_h) + (H\cdot S)(T_i u_h, T_i u_h)\right). \end{aligned} \tag{3.7}$$

Using Friedrichs' inequality, we obtain

$$(T_i u_h, T_i u_h) \le C\widehat{H}_i^2 \widehat{A}(T_i u_h, T_i u_h) \le C\widehat{H}_i^2 H(T_i u_h, T_i u_h) \le CH_0^2 H(T_i u_h, T_i u_h). \tag{3.8}$$

Combining the definition of S_i and (2.8) implies

$$(H\cdot S)(T_iu_h,T_iu_h)=H(S_iT_iu_h,T_iu_h)\le C\,|||\,S_i\,|||_1\,H(T_iu_h,T_iu_h). \tag{3.9}$$

It follows from the definition of $\widehat{P}_i$ and (2.4) that

$$\begin{aligned}H(S_iu_h,v_h)&=H(S_iu_h,\widehat{P}_iv_h)=(S_iu_h,H_i\widehat{P}_iv_h)\\&=S(u_h,H_i\widehat{P}_iv_h)\le C\,|||\,u_h\,|||_1\,\|H_i\widehat{P}_iv_h\|\quad\forall\,u_h,v_h\in V_h^i.\end{aligned} \tag{3.10}$$

Analogously to the proof of Theorem 2 in [27], if $\beta=O(\lambda_{max})$, we have

$$\begin{aligned}\frac{1}{\beta}\|H_i\widehat{P}_iv_h\|&=\frac{1}{C\lambda_{max}}(H_i\widehat{P}_iv_h,H_i\widehat{P}_iv_h)^{\frac12}=\frac{1}{C\sqrt{\lambda_{max}}}H\left(v_h,\frac{1}{\lambda_{max}}H_i\widehat{P}_iv_h\right)^{\frac12}\\&\le C\frac{1}{\sqrt{\lambda_{max}}}H(v_h,v_h)^{\frac12}=C\frac{1}{\sqrt{\lambda_{max}}}\,|||\,v_h\,|||_1\,.\end{aligned} \tag{3.11}$$

Combining (2.8), (3.10) and (3.11) yields

$$\frac{1}{\beta}\,|||\,S_i|||_1\le C\frac{1}{\sqrt{\lambda_{max}}}. \tag{3.12}$$

Since $\alpha=O(\lambda_{min})$ is a constant, $\beta=O(\lambda_{max})=O(h^{-2})$, it follows from (3.7)-(3.9) and (3.12) that

$$\begin{aligned}N(T_iu_h,T_iu_h)&\le\frac{1}{\alpha+\beta}\left(\alpha\beta(T_iu_h,T_iu_h)+(H\cdot S)(T_iu_h,T_iu_h)\right)\\&\le C(H_0^2+h)H(T_iu_h,T_iu_h).\end{aligned}$$

Summing over i on both sides of the above inequality, we obtain

$$\sum_{i=1}^N N(T_iu_h,T_iu_h)\le C(H_0^2+h)\sum_{i=1}^N H(T_iu_h,T_iu_h). \tag{3.13}$$

It follows from (2.4), Proposition 2.1 and Friedrichs' inequality that

$$\begin{aligned}\left|\sum_{i=1}^N S(u_h-T_iu_h,T_iu_h)\right|&=\left|S\left(u_h,\sum_{i=1}^N T_iu_h\right)\right|\le C\,|||\,u_h\,|||_1\left\|\sum_{i=1}^N T_iu_h\right\|\\&=C\,|||\,u_h\,|||_1\left(\sum_{i=1}^N\sum_{j=1}^N(T_iu_h,T_ju_h)\right)^{\frac12}\\&\le C\,|||\,u_h\,|||_1\left(\sum_{i=1}^N\sum_{j=1}^N k_{ij}(T_iu_h,T_iu_h)^{\frac12}(T_ju_h,T_ju_h)^{\frac12}\right)^{\frac12}\\&\le C\sqrt{\rho(K)}\,|||\,u_h\,|||_1\left(\sum_{i=1}^N\|T_iu_h\|^2\right)^{\frac12}\\&\le C\sqrt{N_c}\,|||\,u_h\,|||_1\left(\sum_{i=1}^N C\widehat{H}_i^2\,|||\,T_iu_h|||_1^2\right)^{\frac12}\end{aligned}$$

$$\leq CH_0\sqrt{N_c} \,|||\, u_h \,|||_1 \left(\sum_{i=1}^{N} H(T_iu_h, T_iu_h)\right)^{\frac{1}{2}}. \tag{3.14}$$

By (3.6), (3.13), (3.14) and Assumption 3.1, we have

$$\begin{aligned}
&\sum_{i=1}^{N} H(u_h - T_iu_h, T_iu_h) \\
\leq &\sum_{i=1}^{N} N(T_iu_h, T_iu_h) + \sum_{i=1}^{N} S(u_h - T_iu_h, T_iu_h) \\
\leq &C(H_0^2 + h)\sum_{i=1}^{N} H(T_iu_h, T_iu_h) + CH_0\sqrt{N_c} \,|||\, u_h \,|||_1 \left(\sum_{i=1}^{N} H(T_iu_h, T_iu_h)\right)^{\frac{1}{2}} \\
\leq &C(H_0^2 + h)N_c \,|||\, u_h \,|||_1^2 + CH_0N_c \,|||\, u_h \,|||_1^2\,.
\end{aligned} \tag{3.15}$$

Combining (3.4) with (3.15) implies

$$\sum_{i=0}^{N} H(u_h - T_iu_h, T_iu_h) \leq C\left((H_0^2 + h + H_0)N_c + H_0\right) |||\, u_h \,|||_1^2\,. \tag{3.16}$$

If H_0 is sufficiently small, from (3.3) and (3.16), we have

$$\begin{aligned}
H(Tu_h, u_h) &= \sum_{i=0}^{N} H(T_iu_h, T_iu_h) + \sum_{i=0}^{N} H(u_h - T_iu_h, T_iu_h) \\
&\geq \sum_{i=0}^{N} H(T_iu_h, T_iu_h) - C\left((H_0^2 + h + H_0)N_c + H_0\right) |||\, u_h |||_1^2 \\
&\geq \sum_{i=0}^{N} H(T_iu_h, T_iu_h).
\end{aligned} \tag{3.17}$$

It follows from Assumption 3.2 and (3.17) that

$$H(Tu_h, u_h) \geq \sum_{i=0}^{N} H(T_iu_h, T_iu_h) \geq CC_0^{-2}H(u_h, u_h).$$

The proof is completed. □

3.2. The verification of Assumptions 3.1 and 3.2

Lemma 3.1. *Suppose that $\alpha = O(\lambda_{min})$ and $\beta = O(\lambda_{max})$ are satisfied. Then Assumption 1 holds.*

Proof. Since $\alpha = O(\lambda_{min})$ and $\beta = O(\lambda_{max})$, from (2.8), (2.11) and (2.13), we have

$$\begin{aligned}
H(T_iu_h, T_iu_h) &= H(M_i^{-1}A_iP_iu_h, T_iu_h) \\
&= \frac{\alpha + \beta}{\beta} H(\beta(\beta I + S_i)^{-1}(\alpha I + H_i)^{-1}A_iP_iu_h, T_iu_h)
\end{aligned}$$

$$\leq C \,|||\, \beta(\beta I + S_i)^{-1} \,|||_1 \,|||(\alpha I + H_i)^{-1} A_i P_i u_h \,|||_1 \,|||T_i u_h \,|||_1 \,. \quad (3.18)$$

Cancelling the common factor and squaring both sides of (3.18), we have

$$H(T_i u_h, T_i u_h) \leq C \,|||\, \beta(\beta I + S_i)^{-1} \,|||_1^2 \,|||(\alpha I + H_i)^{-1} A_i P_i u_h \,|||_1^2 \,. \quad (3.19)$$

It follows from (2.3), (2.7) and (2.8) that

$$\begin{aligned}
|||(\alpha I + H_i)^{-1} A_i P_i u_h|||_1^2 &= H((\alpha I + H_i)^{-1} A_i P_i u_h, (\alpha I + H_i)^{-1} A_i P_i u_h) \\
&= (H_i(\alpha I + H_i)^{-1} A_i P_i u_h, (\alpha I + H_i)^{-1} A_i P_i u_h) \\
&\leq \frac{\lambda_{max}}{\alpha + \lambda_{max}} A(P_i u_h, (\alpha I + H_i)^{-1} A_i P_i u_h) \\
&\leq C \,|||\, P_i u_h \,|||_1 \,|||(\alpha I + H_i)^{-1} A_i P_i u_h|||_1 \\
&\leq C \,|||\, u_h \,|||_{1,\Omega_i'} \,|||(\alpha I + H_i)^{-1} A_i P_i u_h \,|||_1 \,.
\end{aligned}$$

Therefore, we obtain

$$|||\, (\alpha I + H_i)^{-1} A_i P_i u|||_1 \leq C \,|||\, u_h \,|||_{1,\Omega_i'} \,. \quad (3.20)$$

Since $\beta = O(\lambda_{max}) = O(h^{-2})$, there exist $h_0 > 0$ and $0 < q < 1$ such that for $h < h_0$,

$$|||\, \beta^{-1} S_i|||_1 \leq C \frac{1}{\sqrt{\lambda_{max}}} < q < 1. \quad (3.21)$$

From (3.21) and Neumann Lemma, we obtain

$$|||\, \beta(\beta I + S_i)^{-1}|||_1 = |||(I + \beta^{-1} S_i)^{-1}|||_1 \leq C. \quad (3.22)$$

It follows from (3.19), (3.20) and (3.22) that

$$\sum_{i=1}^{N} H(T_i u_h, T_i u_h) \leq C \sum_{i=1}^{N} |||u_h|||_{1,\Omega_i'}^2 \leq C N_c H(u_h, u_h). \quad (3.23)$$

Which completes the proof of Assumption 3.1. □

Lemma 3.2. *Suppose that $\alpha = O(\lambda_{min})$ and $\beta = O(\lambda_{max})$ are satisfied. Then Assumption 2 holds.*

Proof. By the definitions of P_i and T_i, we obtain

$$\begin{aligned}
H(u_h, u_h) = A(u_h, u_h) &= \sum_{i=0}^{N} A(u_h, u_h^i) \\
&= A(P_0 u_h, u_h^0) + \sum_{i=1}^{N} A(P_i u_h, u_h^i) \\
&= A(P_0 u_h, u_h^0) + \sum_{i=1}^{N} (M_i M_i^{-1} A_i P_i u_h, u_h^i) \\
&= A(T_0 u_h, u_h^0) + \frac{1}{\alpha + \beta} \sum_{i=1}^{N} ((\alpha I + H_i)(\beta I + S_i) T_i u_h, u_h^i). \quad (3.24)
\end{aligned}$$

Due to $\alpha = O(\lambda_{min})$, it follows from (2.8) and (3.21) that

$$\begin{aligned}((\alpha I + H_i)(\beta I + S_i)T_i u_h, u_h^i) &= H\left(H_i^{-1}(\alpha I + H_i)(\beta I + S_i)T_i u_h, u_h^i\right) \\ &\le \frac{\alpha + \lambda_{min}}{\lambda_{min}} H((\beta I + S_i)T_i u_h, u_h^i) \\ &\le C\beta \,|||\, I + \beta^{-1} S_i \,|||_1\, H(T_i u_h, T_i u_h)^{\frac{1}{2}} H(u_h^i, u_h^i)^{\frac{1}{2}} \\ &\le C\beta H(T_i u_h, T_i u_h)^{\frac{1}{2}} H(u_h^i, u_h^i)^{\frac{1}{2}}.\end{aligned}$$

Summing over i on both sides of the above inequality and using Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\sum_{i=1}^{N} \frac{1}{\alpha+\beta}((\alpha I + H_i)(\beta I + S_i)T_i u_h, u_h^i) \\ \le &C\frac{\beta}{\alpha+\beta} \sum_{i=1}^{N} H(T_i u_h, T_i u_h)^{\frac{1}{2}} H(u_h^i, u_h^i)^{\frac{1}{2}} \\ \le &C\left(\sum_{i=1}^{N} H(T_i u_h, T_i u_h)\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} H(u_h^i, u_h^i)\right)^{\frac{1}{2}}. \qquad (3.25)\end{aligned}$$

From (2.3), we obtain

$$A(T_0 u_h, u_h^0) \le C \,|||\, T_0 u_h \,|||_1\, |||u_h^0|||_1 = C H(T_0 u_h, T_0 u_h)^{\frac{1}{2}} H(u_h^0, u_h^0)^{\frac{1}{2}}. \qquad (3.26)$$

Using Cauchy-Schwarz inequality and Proposition 2.2, it follows from (3.24)-(3.26) that

$$\begin{aligned}H(u_h, u_h) &\le C\left(\sum_{i=0}^{N} H(T_i u_h, T_i u_h)\right)^{\frac{1}{2}} \left(\sum_{i=0}^{N} H(u_h^i, u_h^i)\right)^{\frac{1}{2}} \\ &\le CC_0 \left(\sum_{i=0}^{N} H(T_i u_h, T_i u_h)\right)^{\frac{1}{2}} H(u_h, u_h)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$H(u_h, u_h) \le CC_0^2 \sum_{i=0}^{N} H(T_i u_h, T_i u_h),$$

which completes the proof of Assumption 3.2. □

Remark 3.1. From Lemmas 3.1 and 3.2, we observe that Theorem 3.1 holds and the Algorithm 2.1 converges for solving nonselfadjoint elliptic equations. Different from the classical Schwarz algorithm, we introduce a new Schwarz operator which includes both the selfadjoint and skew-selfadjoint parts of the equation in Algorithm 2.1, namely, two systems instead of the original system should be solved by ILU in each subdomain. With these techniques, our proposed algorithm performs well for nonselfadjoint elliptic equation when the skew-selfadjoint part is dominant.

4. Numerical examples

In this section, we present some numerical experiments to demonstrate the performance of the proposed AS method. In the experiments, the domain $\Omega = [0,1]^d$ is covered by a uniform coarse mesh of size H_0, and a uniform fine mesh of size h. The fine mesh is decomposed into $N_x \times N_y$ subdomains in $[0,1]^2$ or $N_x \times N_y \times N_z$ subdomains in $[0,1]^3$. All the subdomain problems are solved inexactly by ILU factorization. And the coarse problem is solved exactly. The overlap is denoted by "ovlp", which is chosen as 0, 1 and 2, respectively. Note that AS method with overlap of 0 is equivalent to the block Jacobi method. The linear systems are solved by restarted GMRES(20) and the stopping criterion for GMRES is

$$\frac{\|r_k\|_0}{\|r_0\|_0} \le 10^{-6},$$

where $r_k = b - Tu_h^k$ is the k^{th} step residual. Moreover, the GMRES method is also terminated when the total number of iterations exceeds 1000.

Example 4.1. Consider a two-dimensional convection diffusion equation [2]

$$\begin{cases} -\epsilon\triangle u + \mathbf{b}\cdot\nabla u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{b} = (\cos\pi/8, \sin\pi/8)^T$ and f is chosen that $u = x(1-x)\sin(\pi y)$ is the exact solution.

In the first test, we set $\epsilon = 1$, $h = 1/256$ and vary the subdomain partition, the overlapping size as well as the coarse mesh size H_0. The number of iteration denoted by "IT" is listed in Table 1. It is clear that the number of iteration goes down with the increase of the overlapping size, and is bounded and independent of the number of subdomains, which illustrates that Algorithm 2.1 is optimal. Moreover, the number of iteration also decreases when choosing smaller H_0. Note that it is not easy to choose the parameters α and β to obtain optimal convergence rate for AHSS or HSS iteration, so some techniques have been proposed to compute the optimal parameters [6, 7, 21, 34]. In the implementation, we choose the parameters as $\alpha = h^2/\epsilon$ and $\beta = 1$, respectively. It confirms the theoretical analysis that α and β should be the minimum and maximum eigenvalues, respectively. The numerical results show that it performs very well for this example.

Next, we present some numerical results to compare Algorithm 2.1 and the classical two-level AS algorithm [10, 12]. For brevity, we denote Algorithm 2.1 and the classical two-level AS algorithm by "AHSS-AS"and "AS", respectively. We fix $H_0 = 1/32$, $h = 1/256$, ovlp= 1 and vary the subdomains partition as well as the coefficient ϵ. The numerical results are listed in Table 2, where the symbol "*"indicates that the algorithm failed to converge in 1000 iterations. It shows that the number of iteration for AHSS-AS and AS is almost the same and bounded independent of the number of subdomians when $\epsilon = 1, 0.1, 0.05$. And the iterative steps of AS is less than that of AHSS-AS when $\epsilon = 0.01$. However, AS does not converge in 1000 iterations in the case of $\epsilon = 0.001$, and AHSS-AS works well for this case. If we set smaller coarse mesh size such that $H_0 = 1/64$, the numerical results listed in Table 3 show that AHSS-AS performs as well as AS in the case of $\epsilon = 1, 0.1, 0.05, 0.01$, and it performs better than AS for the case $\epsilon = 0.001$.

Table 1. The numerical results for solving Example 4.1 by Algorithm 2.1 with $\epsilon = 1$ and $h = 1/256$

$N_x \times N_y$	2×2			4×4			8×8			16×16		
ovlp	0	1	2	0	1	2	0	1	2	0	1	2
$H_0 = 1/16$	29	20	17	26	18	16	25	17	15	24	16	13
$H_0 = 1/32$	20	16	16	19	15	14	18	14	13	17	13	12
$H_0 = 1/64$	16	15	15	15	14	14	14	13	14	14	12	13
$H_0 = 1/128$	14	15	14	13	14	15	12	13	14	12	12	13

Table 2. The numerical comparisons between Algorithm 2.1 and the classical two-level AS algorithm for solving Example 4.1 with $H_0 = 1/32$, $h = 1/256$ and ovlp=1

	$N_x \times N_y$	2×2		4×4		8×8		16×16	
ϵ	Algorithm	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT
1	AHSS-AS	4.16e-4	16	4.16e-4	15	4.16e-4	14	4.16e-4	13
	AS	4.16e-4	16	4.16e-4	15	4.16e-4	14	4.16e-4	13
0.1	AHSS-AS	4.50e-4	18	4.50e-4	17	4.50e-4	17	4.50e-4	15
	AS	4.50e-4	19	4.50e-4	17	4.50e-4	16	4.50e-4	15
0.05	AHSS-AS	4.79e-4	19	4.79e-4	18	4.79e-4	17	4.79e-4	16
	AS	4.79e-4	19	4.79e-4	18	4.79e-4	17	4.79e-4	16
0.01	AHSS-AS	5.29e-4	24	5.29e-4	25	5.29e-4	25	5.29e-4	24
	AS	5.29e-4	19	5.29e-4	20	5.29e-4	22	5.29e-4	23
0.001	AHSS-AS	5.52e-4	497	5.52e-4	497	5.52e-4	508	5.52e-4	502
	AS	*	*	*	*	*	*	*	*

Table 3. The numerical comparisons between Algorithm 2.1 and the classical two-level AS algorithm for solving Example 4.1 with $H_0 = 1/64$, $h = 1/256$ and ovlp=1

	$N_x \times N_y$	2×2		4×4		8×8		16×16	
ϵ	Algorithm	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT
1	AHSS-AS	4.16e-4	15	4.16e-4	14	4.16e-4	13	4.16e-4	12
	AS	4.16e-4	15	4.16e-4	14	4.16e-4	13	4.16e-4	12
0.1	AHSS-AS	4.50e-4	17	4.50e-4	16	4.50e-4	15	4.50e-4	14
	AS	4.50e-4	17	4.50e-4	16	4.50e-4	15	4.50e-4	14
0.05	AHSS-AS	4.79e-4	19	4.79e-4	17	4.79e-4	16	4.79e-4	15
	AS	4.79e-4	17	4.79e-4	17	4.79e-4	16	4.79e-4	15
0.01	AHSS-AS	5.29e-4	18	5.29e-4	18	5.29e-4	18	5.29e-4	18
	AS	5.29e-4	18	5.29e-4	19	5.29e-4	20	5.29e-4	21
0.001	AHSS-AS	5.52e-4	236	5.52e-4	238	5.52e-4	240	5.52e-4	244
	AS	*	*	*	*	*	*	*	*

Example 4.2. Consider a three-dimensional convection diffusion equation

$$\begin{cases} -\epsilon \triangle u + \mathbf{b} \cdot \nabla u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{b} = (1, 1, 1)$ and f is chosen that $u(x, y, z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ is the exact solution.

Table 4. The numerical results for solving Example 4.2 by Algorithm 2.1 with $\epsilon = 1$ and $h = 1/64$

$N_x \times N_y \times N_z$	$2 \times 2 \times 2$			$4 \times 4 \times 4$			$8 \times 8 \times 8$		
ovlp	0	1	2	0	1	2	0	1	2
$H_0 = 1/16$	10	9	9	10	9	9	9	8	8
$H_0 = 1/32$	9	10	10	8	9	9	8	8	8

Table 5. The numerical comparisons between Algorithm 2.1 and the classical two-level AS algorithm for solving Example 4.2 with $H_0 = 1/16$, $h = 1/64$ and ovlp=1

	$N_x \times N_y \times N_z$	$2 \times 2 \times 2$		$4 \times 4 \times 4$		$8 \times 8 \times 8$	
ϵ	Algorithm	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT
1	AHSS-AS	1.88e-2	10	1.88e-2	9	1.88e-2	8
	AS	1.88e-2	12	1.88e-2	10	1.88e-2	8
0.1	AHSS-AS	3.64e-2	10	3.64e-2	10	3.64e-2	10
	AS	3.64e-2	13	3.64e-2	11	3.64e-2	10
0.05	AHSS-AS	4.37e-2	12	4.37e-2	12	4.37e-2	12
	AS	4.38e-2	12	4.38e-2	13	4.38e-2	11
0.01	AHSS-AS	5.24e-2	13	5.24e-2	13	5.24e-2	13
	AS	5.20e-2	12	5.20e-2	14	5.20e-2	16
0.005	AHSS-AS	5.94e-2	15	5.94e-2	15	5.94e-2	15
	AS	5.32e-2	49	5.32e-2	45	5.32e-2	40

Table 6. The numerical comparisons between Algorithm 2.1 and the classical two-level AS algorithm for solving Example 4.2 with $H_0 = 1/32$, $h = 1/64$ and ovlp=1

	$N_x \times N_y \times N_z$	$2 \times 2 \times 2$		$4 \times 4 \times 4$		$8 \times 8 \times 8$	
ϵ	Algorithm	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT	$\|u - u_h\|$	IT
1	AHSS-AS	1.88e-2	10	1.88e-2	9	1.88e-2	8
	AS	1.88e-2	12	1.88e-2	10	1.88e-2	8
0.1	AHSS-AS	3.64e-2	8	3.64e-2	8	3.64e-2	8
	AS	3.64e-2	12	3.64e-2	11	3.64e-2	9
0.05	AHSS-AS	4.38e-2	8	4.38e-2	8	4.38e-2	8
	AS	4.38e-2	12	4.38e-2	12	4.38e-2	11
0.01	AHSS-AS	5.59e-2	5	5.59e-2	5	5.59e-2	5
	AS	5.20e-2	10	5.20e-2	13	5.20e-2	14
0.005	AHSS-AS	5.25e-2	5	5.25e-2	5	5.25e-2	5
	AS	5.32e-2	40	5.32e-2	39	5.32e-2	34

We test another example to show the performance of Algorithm 2.1. Similar to Example 4.1, we set $\epsilon = 1$, $h = 1/64$ and vary the coarse mesh size, subdomains partition and overlapping size, respectively. From Table 4, we observe that the number of iteration decreases when we increase the overlap and is bounded independent of the number of subdomains. And the iterative steps decreases when choosing smaller coarse mesh size H_0.

In the next set of experiments, we compare the performance of AHSS-AS and AS for solving this example with different coefficient ε. In the implementation, the parameters α and β are set as $\alpha = h^2/\epsilon$ and $\beta = 1$, respectively. We fix $H_0 = 1/16$, $h = 1/64$, ovlp= 1 and vary the subdomains partition as well as the coefficient ϵ. The numerical results presented in Table 5 show that the iterative steps of both AHSS-AS and AS is bounded independent of the number of subdomains, and AHSS-AS needs less iterative steps than AS for most of the cases. Finally, we change the coarse mesh size $H_0 = 1/32$ and fix other parameters. From Tables 5 and 6, we observe that the iterative steps of both AHSS-AS and AS decrease when choose smaller coarse mesh size. The numerical results in Table 6 also show that the number of iteration of AHSS-AS is much less than that of AS for the cases $\epsilon = 0.01$ and $\epsilon = 0.005$.

5. Concluding remarks

We presented a two-level AS algorithm for solving nonselfadjoint elliptic equations, and established a convergence theory which shows that the convergence rate is bounded independent of the number of subdomains and the fine mesh size. The numerical experiments confirm the convergence analysis, it shows that our algorithm performs very well for solving the convection-diffusion equations by choosing the parameters easily. On the other hand, the numerical comparisons show that our algorithm has a benefit over the classical two-level AS algorithm for the presented numerical examples.

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