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THE BIFURCATION TRAVELLING WAVES OF A GENERALIZED BROER-KAUP EQUATION*

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Abstract In this paper, by using the dynamical system theory and simulation method, we study a generalized Broer-Kaup (gBK) equation. Under different parameter conditions, the travelling wave solutions such as kink wave, periodic blow-up wave, periodic wave and solitary wave are given, and the dynamic characters of these solutions are investigated.

Keywords Broer-Kaup equation, periodic wave, periodic blow-up wave, solitary wave, kink wave.

MSC(2010) 35B10, 35C07, 35C08.

1. Introduction

Now, periodic waves, periodic blow-up waves, solitary waves and kink waves have always been hot research issues, such as, in [1–3, 5, 16] the authors gave detailed studies. Tu [17] once employed Lie algebras as a tool for generating nonlinear-equation hierarchies and proposed a powerful approach for producing Hamiltonian structures of soliton hierarchies. Ma [13] called the method the Tu-Ma scheme. By following the way one has obtained some interesting integrable hierarchies of evolution type and their Hamiltonian structures, Darboux transformations, and some other properties [6,7,9,10,13–15,24]. Zhang et al. [23] adopt the Tu-Ma scheme to derive a integrable soliton hierarchy which can reduce to a coupled generalized Broer-Kaup (gBK) equation

$$\begin{cases} v_t = v_{xx} - 2vv_x - 2w_x, \\ w_t = -w_{xx} - 2(wv)_x - 2v_x, \end{cases} \tag{1.1}$$

whose two kinds of Darboux transformations, the bilinear representation, the bilinear Bäcklund transformation and a Lax pair equation are given, respectively.

In this paper, using dynamical system theory and simulation method (see [11,12, 18–22] and the references therein), the travelling wave solutions of equation (1.1) are studied. The expressions of solutions of kink wave, periodic blow-up wave, periodic wave and solitary wave are obtained.

The rest of this paper is organized as follows. In Section 2, we derive travelling wave solutions. In Section 3, we give classifications of travelling wave solutions, and

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their numerical simulations are made using mathematical software *Mathematica 7.0*. In Section 4, we give a short conclusion.

2. Travelling wave solutions of equation (1.1)

Making the transformation $v(x,t) = \varphi(\xi)$, $w(x,t) = \psi(\xi)$, with $\xi = x - ct$ in equation (1.1), we have the following ordinary differential equations.

$$\begin{cases} 2\psi' = \varphi'' - 2\varphi\varphi' + c\varphi', \\ c\psi' = \psi'' + 2(\psi\varphi)' + 2\varphi', \end{cases} \tag{2.1}$$

where c is the wave speed, and the symbol $'$ indicates derivative with respect to ξ.

Integrating (2.1) once with respect to ξ, we have the following equations.

$$\begin{cases} 2\psi = \varphi' - \varphi^2 + c\varphi + A, \\ c\psi = \psi' + 2\psi\varphi + 2\varphi + B, \end{cases} \tag{2.2}$$

where A and B are the integration constants. Substituting the first equation of (2.2) into the second equation of (2.2), we get

$$\varphi'' = 2\varphi^3 - 3c\varphi^2 + (c^2 - 2A - 4)\varphi + Ac - 2B. \tag{2.3}$$

For simplicity, we denote $w = \psi(\xi) = \frac{1}{2}(\varphi' - \varphi^2 + c\varphi + A)$ in the entire process. In (2.3) multiplied by $2\varphi'$ both side at the same time and integrating it, we have

$$(\varphi')^2 = \varphi^4 - 2c\varphi^3 + (c^2 - 2A - 4)\varphi^2 + 2(Ac - 2B)\varphi + h, \tag{2.4}$$

where h is any integration constant.

Let

$$f(\varphi) = \varphi^4 - 2c\varphi^3 + (c^2 - 2A - 4)\varphi^2 + 2(Ac - 2B)\varphi + h, \tag{2.5}$$

then (2.4) becomes

$$(\varphi')^2 = f(\varphi). \tag{2.6}$$

2.1. Cases of $f(\varphi)$ has four conjugate complex roots

When $f(\varphi)$ has four conjugate complex roots $\varphi_1, \overline{\varphi_1}, \varphi_2, \overline{\varphi_2}$, then

$$f(\varphi) = (\varphi - \varphi_1)(\varphi - \overline{\varphi_1})(\varphi - \varphi_2)(\varphi - \overline{\varphi_2}). \tag{2.7}$$

Taking $\varphi_0 = b_1 - a_1 g$ is an original value, substituting (2.7) into (2.6) and integrating it, we have

$$\int_{\varphi_0}^{\varphi} \frac{1}{\sqrt{(t-\varphi_1)(t-\overline{\varphi_1})(t-\varphi_2)(t-\overline{\varphi_2})}}\,dt = \pm \int_0^{\xi} dt. \tag{2.8}$$

From (2.8), using formula 267.00 in [4], we get two travelling wave solutions as follows:

$$\varphi = b_1 + \frac{a_1}{g} - a_1\left(g + \frac{1}{g}\right)\frac{1}{1 + \operatorname{sc}(\frac{P+Q}{2}\xi, k)}, \tag{2.9}$$

and

$$\varphi = b_1 + \frac{a_1}{g} - a_1(g + \frac{1}{g})\frac{1}{1 - \mathrm{sc}(\frac{P+Q}{2}\xi, k)}, \tag{2.10}$$

where $b_1 = \frac{\varphi_1+\overline{\varphi_1}}{2}$, $a_1 = \sqrt{-\frac{(\varphi_1-\overline{\varphi_1})^2}{4}}$, $b_2 = \frac{\varphi_2+\overline{\varphi_2}}{2}$, $a_2 = \sqrt{-\frac{(\varphi_2-\overline{\varphi_2})^2}{4}}$, $P = \sqrt{(b_1-b_2)^2+(a_1+a_2)^2}$, $Q = \sqrt{(b_1-b_2)^2+(a_1-a_2)^2}$, $k = \sqrt{\frac{4PQ}{(P+Q)^2}}$ and $g = \sqrt{\frac{4a_1^2-(P-Q)^2}{-4a_1^2+(P+Q)^2}}$.

2.2. Cases of $f(\varphi)$ has one double real root and two conjugate complex roots

When $f(\varphi)$ has one double real root φ_1 and two conjugate complex roots $\varphi_2, \overline{\varphi_2}$, then

$$f(\varphi) = (\varphi - \varphi_1)^2(\varphi - \varphi_2)(\varphi - \overline{\varphi_2}). \tag{2.11}$$

(1) When $\varphi < \varphi_1$, taking an original value φ_0, and it satisfies $\varphi_0 < \varphi_1$, substituting (2.11) into (2.6) and integrating it, we have

$$\int_{\varphi_0}^{\varphi} \frac{1}{(\varphi_1 - t)\sqrt{(t-\varphi_2)(t-\overline{\varphi_2})}} dt = \pm \int_0^{\xi} dt. \tag{2.12}$$

From (2.12), we get two travelling wave solutions as follows:

$$\varphi = \frac{\varphi_1(P_0e^{\tau})^2 + 2(\varphi_1\varphi_2 + \varphi_1\overline{\varphi_2} - 2\varphi_2\overline{\varphi_2})P_0e^{\tau} + \varphi_1(\overline{\varphi_2} - \varphi_2)^2}{(P_0e^{\tau})^2 - 2(\varphi_2 + \overline{\varphi_2} - 2\varphi_1)P_0e^{\tau} + (\overline{\varphi_2} - \varphi_2)^2}, \tag{2.13}$$

and

$$\varphi = \frac{\varphi_1(P_0e^{-\tau})^2 + 2(\varphi_1\varphi_2 + \varphi_1\overline{\varphi_2} - 2\varphi_2\overline{\varphi_2})P_0e^{-\tau} + \varphi_1(\overline{\varphi_2} - \varphi_2)^2}{(P_0e^{-\tau})^2 - 2(\varphi_2 + \overline{\varphi_2} - 2\varphi_1)P_0e^{-\tau} + (\overline{\varphi_2} - \varphi_2)^2}, \tag{2.14}$$

where $P_0 = \varphi_2 + \overline{\varphi_2} - 2\varphi_1 + \frac{2\sqrt{(\overline{\varphi_2}-\varphi_1)(\varphi_2-\varphi_1)}\left(\sqrt{(\overline{\varphi_2}-\varphi_1)(\varphi_2-\varphi_1)}+\sqrt{(\overline{\varphi_2}-\varphi_0)(\varphi_2-\varphi_0)}\right)}{\varphi_1-\varphi_0}$, and $\tau = \sqrt{(\overline{\varphi_2}-\varphi_1)(\varphi_2-\varphi_1)}\xi$.

(2)When $\varphi > \varphi_1$, taking an original value φ_0, and it satisfies $\varphi_1 < \varphi_0$, substituting (2.11) into (2.6) and integrating it, we have

$$\int_{\varphi_0}^{\varphi} \frac{1}{(t - \varphi_1)\sqrt{(t-\varphi_2)(t-\overline{\varphi_2})}} dt = \pm \int_0^{\xi} dt. \tag{2.15}$$

From (2.15), we get two travelling wave solutions as follows:

$$\varphi = \frac{\varphi_1(P_1e^{\tau})^2 - 2(\overline{\varphi_2}\varphi_1 + \varphi_2\varphi_1 - 2\overline{\varphi_2}\varphi_2)P_1e^{\tau} + \varphi_1(\varphi_2 - \overline{\varphi_2})^2}{(P_1e^{\tau})^2 + 2(\overline{\varphi_2} + \varphi_2 - 2\varphi_1)P_1e^{\tau} + (\varphi_2 - \overline{\varphi_2})^2}, \tag{2.16}$$

and

$$\varphi = \frac{\varphi_1(P_1e^{-\tau})^2 - 2(\overline{\varphi_2}\varphi_1 + \varphi_2\varphi_1 - 2\overline{\varphi_2}\varphi_2)P_1e^{-\tau} + \varphi_1(\varphi_2 - \overline{\varphi_2})^2}{(P_1e^{-\tau})^2 + 2(\overline{\varphi_2} + \varphi_2 - 2\varphi_1)P_1e^{-\tau} + (\varphi_2 - \overline{\varphi_2})^2}, \tag{2.17}$$

where $P_1 = 2\varphi_1 - \overline{\varphi_2} - \varphi_2 + \frac{2\sqrt{(\varphi_1-\overline{\varphi_2})(\varphi_1-\varphi_2)}\left(\sqrt{(\varphi_1-\overline{\varphi_2})(\varphi_1-\varphi_2)}+\sqrt{(\varphi_0-\overline{\varphi_2})(\varphi_0-\varphi_2)}\right)}{\varphi_0-\varphi_1}$, and $\tau = \sqrt{(\varphi_1-\overline{\varphi_2})(\varphi_1-\varphi_2)}\xi$.

2.3. Cases of $f(\varphi)$ has two simple real roots and two conjugate complex roots

When φ_1 and φ_2 are two simple real roots, $\varphi_3, \overline{\varphi_3}$ are two conjugate complex roots of $f(\varphi)$, where $\varphi_1 < \varphi_2$, then

$$f(\varphi) = (\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3)(\varphi - \overline{\varphi_3}). \tag{2.18}$$

(1) When $\varphi < \varphi_1$, substituting (2.18) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_1} \frac{1}{\sqrt{(t-\varphi_1)(t-\varphi_2)(t-\varphi_3)(t-\overline{\varphi_3})}} dt = \pm \int_{\xi}^{0} dt. \tag{2.19}$$

From (2.19), using formula $3.145-3$ in [8], we get a travelling wave solution as follows:

$$\varphi = \frac{m_2\varphi_1 - m_1\varphi_2 + (m_1\varphi_2 + m_2\varphi_1)\mathrm{cn}(\tau, k)}{m_2 - m_1 + (m_1 + m_2)\mathrm{cn}(\tau, k)}, \tag{2.20}$$

where $\tau = \sqrt{m_1 m_2}\xi$, $m_1 = \sqrt{(\varphi_2 - \varphi_3)(\varphi_2 - \overline{\varphi_3})}$, $m_2 = \sqrt{(\varphi_1 - \varphi_3)(\varphi_1 - \overline{\varphi_3})}$, and $k = \sqrt{\frac{(m_1+m_2)^2 + (\varphi_2 - \varphi_1)^2}{4m_1 m_2}}$.

(2)When $\varphi_2 < \varphi$, substituting (2.18) into (2.6) and integrating it, we have

$$\int_{\varphi_2}^{\varphi} \frac{1}{\sqrt{(t-\varphi_1)(t-\varphi_2)(t-\varphi_3)(t-\overline{\varphi_3})}} dt = \pm \int_{0}^{\xi} dt. \tag{2.21}$$

From (2.21), using formula $3.145-1$ in [8], we get a travelling wave solution as follows:

$$\varphi = \frac{m_2\varphi_2 - m_1\varphi_1 + (m_1\varphi_1 + m_2\varphi_2)\mathrm{cn}(\tau, k)}{m_2 - m_1 + (m_1 + m_2)\mathrm{cn}(\tau, k)}. \tag{2.22}$$

2.4. Cases of $f(\varphi)$ has one double root and two simple real roots

2.4.1. Cases of φ_1 is a double root

When φ_1 is a double root, φ_2 and φ_3 are two simple real roots of $f(\varphi)$, where $\varphi_1 < \varphi_2 < \varphi_3$, then

$$f(\varphi) = (\varphi - \varphi_1)^2(\varphi - \varphi_2)(\varphi - \varphi_3). \tag{2.23}$$

(1) If $\varphi \le \varphi_1$, taking an original value φ_0, and it satisfies $\varphi_0 < \varphi_1$, substituting (2.23) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_0} \frac{1}{(\varphi_1 - t)\sqrt{(\varphi_3 - t)(\varphi_2 - t)}} dt = \pm \int_{\xi}^{0} dt. \tag{2.24}$$

From (2.24), we get two travelling wave solutions as follows:

$$\varphi = \frac{\varphi_1(P_2e^{\tau})^2 + 2(\varphi_1\varphi_2 + \varphi_1\varphi_3 - 2\varphi_2\varphi_3)P_2e^{\tau} + \varphi_1(\varphi_3 - \varphi_2)^2}{(P_2e^{\tau})^2 - 2(\varphi_2 + \varphi_3 - 2\varphi_1)P_2e^{\tau} + (\varphi_3 - \varphi_2)^2}, \tag{2.25}$$

and

$$\varphi = \frac{\varphi_1(P_2e^{-\tau})^2 + 2(\varphi_1\varphi_2 + \varphi_1\varphi_3 - 2\varphi_2\varphi_3)P_2e^{-\tau} + \varphi_1(\varphi_3 - \varphi_2)^2}{(P_2e^{-\tau})^2 - 2(\varphi_2 + \varphi_3 - 2\varphi_1)P_2e^{-\tau} + (\varphi_3 - \varphi_2)^2}, \tag{2.26}$$

where $P_2 = \varphi_2 + \varphi_3 - 2\varphi_1 + \frac{2\sqrt{(\varphi_3-\varphi_1)(\varphi_2-\varphi_1)}\left(\sqrt{(\varphi_3-\varphi_1)(\varphi_2-\varphi_1)}+\sqrt{(\varphi_3-\varphi_0)(\varphi_2-\varphi_0)}\right)}{\varphi_1-\varphi_0}$, and $\tau = \sqrt{(\varphi_3-\varphi_1)(\varphi_2-\varphi_1)}\xi$.

(2) If $\varphi_1 < \varphi \le \varphi_2$, substituting (2.23) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_2} \frac{1}{(t-\varphi_1)\sqrt{(\varphi_3-t)(\varphi_2-t)}}dt = \pm\int_{\xi}^{0} dt. \tag{2.27}$$

From (2.27), we get a travelling wave solution as follows:

$$\varphi = \varphi_1 + \frac{2(\varphi_3-\varphi_1)(\varphi_2-\varphi_1)}{\varphi_2+\varphi_3-2\varphi_1+(\varphi_3-\varphi_2)\cosh\tau}. \tag{2.28}$$

(3) If $\varphi_3 \le \varphi$ substituting (2.23) into (2.6) and integrating it, we have

$$\int_{\varphi_3}^{\varphi} \frac{1}{(t-\varphi_1)\sqrt{(t-\varphi_3)(t-\varphi_2)}}dt = \pm\int_{0}^{\xi} dt. \tag{2.29}$$

From (2.29), we get a travelling wave solution as follows:

$$\varphi = \varphi_1 + \frac{2(\varphi_3-\varphi_1)(\varphi_2-\varphi_1)}{\varphi_2+\varphi_3-2\varphi_1-(\varphi_3-\varphi_2)\cosh(\tau-2nT)}, \tag{2.30}$$

where $T = \operatorname{arccosh}[1+\frac{2(\varphi_2-\varphi_1)}{\varphi_3-\varphi_1}]$, $(2n-1)T < \tau < (2n+1)T$ and $n = 0, \pm1, \pm2, \cdots$.

2.4.2. Cases of φ_2 is a double root

When φ_2 is a double root, φ_1 and φ_3 are two simple real roots of $f(\varphi)$, where $\varphi_1 < \varphi_2 < \varphi_3$, then

$$f(\varphi) = (\varphi-\varphi_1)(\varphi-\varphi_2)^2(\varphi-\varphi_3). \tag{2.31}$$

(1) If $\varphi \le \varphi_1$ substituting (2.31) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_1} \frac{1}{(\varphi_2-t)\sqrt{(\varphi_3-t)(\varphi_1-t)}}dt = \pm\int_{\xi}^{0} dt. \tag{2.32}$$

From (2.32), we get a travelling wave solution as follows:

$$4d_1(\varphi_3-\varphi)(\varphi_1-\varphi)\cot^2(\sqrt{d_1}\xi) = [-2d_1 + d_2(\varphi_2-\varphi)]^2, \tag{2.33}$$

where $d_1 = (\varphi_3-\varphi_2)(\varphi_2-\varphi_1), d_2 = \varphi_1+\varphi_3-2\varphi_2$.

(2) If $\varphi_3 \le \varphi$ substituting (2.31) into (2.6) and integrating it, we have

$$\int_{\varphi_3}^{\varphi} \frac{1}{(t-\varphi_2)\sqrt{(t-\varphi_3)(t-\varphi_1)}}dt = \pm\int_{0}^{\xi} dt. \tag{2.34}$$

From (2.34), we get a travelling wave solution as follows:

$$4d_1(\varphi-\varphi_3)(\varphi-\varphi_1)\cot^2(\sqrt{d_1}\xi) = [-2d_1 - d_2(\varphi-\varphi_2)]^2. \tag{2.35}$$

2.4.3. Cases of φ_3 is a double root

When φ_3 is a double root, φ_1 and φ_2 are two simple real roots of $f(\varphi)$, where $\varphi_1 < \varphi_2 < \varphi_3$, then

$$f(\varphi) = (\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3)^2. \tag{2.36}$$

(1) If $\varphi \le \varphi_1$, substituting (2.36) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_1} \frac{1}{(\varphi_3 - t)\sqrt{(\varphi_2 - t)(\varphi_1 - t)}} dt = \pm \int_{\xi}^{0} dt. \tag{2.37}$$

From (2.37), we get a travelling wave solution as follows:

$$\varphi = \varphi_3 + \frac{2(\varphi_3 - \varphi_1)(\varphi_3 - \varphi_2)}{\varphi_1 + \varphi_2 - 2\varphi_3 + (\varphi_2 - \varphi_1)\cosh(\tau - 2nT)}, \tag{2.38}$$

where $\tau = \sqrt{(\varphi_3 - \varphi_1)(\varphi_3 - \varphi_2)}\xi$, $T = \operatorname{arccosh}[\frac{2(\varphi_3 - \varphi_1)}{\varphi_2 - \varphi_1} - 1]$, $(2n-1)T < \tau < (2n+1)T$, $n = 0, \pm 1, \pm 2, \pm 3, \cdots$.

(2) If $\varphi_2 \le \varphi < \varphi_3$, substituting (2.36) into (2.6) and integrating it, we have

$$\int_{\varphi_2}^{\varphi} \frac{1}{(\varphi_3 - t)\sqrt{(t - \varphi_2)(t - \varphi_1)}} dt = \pm \int_{0}^{\xi} dt. \tag{2.39}$$

From (2.39), we get a travelling wave solution as follows:

$$\varphi = \varphi_3 - \frac{2(\varphi_3 - \varphi_1)(\varphi_3 - \varphi_2)}{2\varphi_3 - \varphi_1 - \varphi_2 + (\varphi_2 - \varphi_1)\cosh\tau}. \tag{2.40}$$

(3) If $\varphi_3 \le \varphi$, taking an original value φ_0, and it satisfies $\varphi_3 < \varphi_0$, substituting (2.36) into (2.6) and integrating it, we have

$$\int_{\varphi_0}^{\varphi} \frac{1}{(t - \varphi_3)\sqrt{(t - \varphi_1)(t - \varphi_2)}} dt = \pm \int_{0}^{\xi} dt. \tag{2.41}$$

From (2.41), we get two travelling wave solutions as follows:

$$\varphi = \frac{\varphi_3(P_3 e^{\tau})^2 - 2(\varphi_1\varphi_3 + \varphi_2\varphi_3 - 2\varphi_1\varphi_2)P_3 e^{\tau} + \varphi_3(\varphi_2 - \varphi_1)^2}{(P_3 e^{\tau})^2 + 2(\varphi_1 + \varphi_2 - 2\varphi_3)P_3 e^{\tau} + (\varphi_2 - \varphi_1)^2}, \tag{2.42}$$

and

$$\varphi = \frac{\varphi_3(P_3 e^{-\tau})^2 - 2(\varphi_1\varphi_3 + \varphi_2\varphi_3 - 2\varphi_1\varphi_2)P_3 e^{-\tau} + \varphi_3(\varphi_2 - \varphi_1)^2}{(P_3 e^{-\tau})^2 + 2(\varphi_1 + \varphi_2 - 2\varphi_3)P_3 e^{-\tau} + (\varphi_2 - \varphi_1)^2}, \tag{2.43}$$

where $P_3 = 2\varphi_3 - \varphi_1 - \varphi_2 + \frac{2\sqrt{(\varphi_3-\varphi_1)(\varphi_3-\varphi_2)}\left(\sqrt{(\varphi_3-\varphi_1)(\varphi_3-\varphi_2)}+\sqrt{(\varphi_0-\varphi_1)(\varphi_0-\varphi_2)}\right)}{\varphi_0-\varphi_3}$, and $\tau = \sqrt{(\varphi_3 - \varphi_1)(\varphi_3 - \varphi_2)}\xi$.

2.5. Cases of $f(\varphi)$ has two double real roots

When φ_1 and φ_2 are two double real roots of $f(\varphi)$, where $\varphi_1 < \varphi_2$, then

$$f(\varphi) = (\varphi - \varphi_1)^2(\varphi - \varphi_2)^2. \tag{2.44}$$

(1) If $\varphi \leq \varphi_1$, taking an original value φ_0, and it satisfies $\varphi_0 < \varphi_1$, substituting (2.44) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_0} \frac{1}{(\varphi_2 - t)(\varphi_1 - t)} dt = \pm \int_{\xi}^{0} dt. \tag{2.45}$$

From (2.45), we get two travelling wave solutions as follows:

$$\varphi = \varphi_1 + \frac{(\varphi_1 - \varphi_0)(\varphi_2 - \varphi_1)}{\varphi_1 - \varphi_0 - (\varphi_2 - \varphi_0)\exp[(\varphi_2 - \varphi_1)\xi]}, \tag{2.46}$$

and

$$\varphi = \varphi_1 + \frac{(\varphi_1 - \varphi_0)(\varphi_2 - \varphi_1)}{\varphi_1 - \varphi_0 - (\varphi_2 - \varphi_0)\exp[-(\varphi_2 - \varphi_1)\xi]}. \tag{2.47}$$

(2) If $\varphi_1 < \varphi \leq \varphi_2$, taking an original value φ_0, and it satisfies $\varphi_1 < \varphi_0 < \varphi_2$, substituting (2.44) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_0} \frac{1}{(\varphi_2 - t)(t - \varphi_1)} dt = \pm \int_{\xi}^{0} dt. \tag{2.48}$$

From (2.48), we get two travelling wave solutions as follows:

$$\varphi = \varphi_1 + \frac{(\varphi_0 - \varphi_1)(\varphi_2 - \varphi_1)}{\varphi_0 - \varphi_1 + (\varphi_2 - \varphi_0)\exp[-(\varphi_2 - \varphi_1)\xi]}, \tag{2.49}$$

and

$$\varphi = \varphi_1 + \frac{(\varphi_0 - \varphi_1)(\varphi_2 - \varphi_1)}{\varphi_0 - \varphi_1 + (\varphi_2 - \varphi_0)\exp[(\varphi_2 - \varphi_1)\xi]}. \tag{2.50}$$

(3) If $\varphi_2 \leq \varphi$, taking an original value φ_0, and it satisfies $\varphi_2 < \varphi_0$, substituting (2.44) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_0} \frac{1}{(t - \varphi_2)(t - \varphi_1)} dt = \pm \int_{\xi}^{0} dt. \tag{2.51}$$

From (2.51), we get two travelling wave solutions as follows:

$$\varphi = \varphi_1 + \frac{(\varphi_0 - \varphi_1)(\varphi_2 - \varphi_1)}{\varphi_0 - \varphi_1 - (\varphi_0 - \varphi_2)\exp[(\varphi_2 - \varphi_1)\xi]}, \tag{2.52}$$

and

$$\varphi = \varphi_1 + \frac{(\varphi_0 - \varphi_1)(\varphi_2 - \varphi_1)}{\varphi_0 - \varphi_1 - (\varphi_0 - \varphi_2)\exp[-(\varphi_2 - \varphi_1)\xi]}. \tag{2.53}$$

2.6. Cases of $f(\varphi)$ has a triple root and a simple real root

2.6.1. Cases of φ_1 is a triple root

When φ_1 is a triple root, and φ_2 is a simple real root of $f(\varphi)$, where $\varphi_1 < \varphi_2$, then

$$f(\varphi) = (\varphi - \varphi_1)^3(\varphi - \varphi_2). \tag{2.54}$$

(1) If $\varphi \leq \varphi_1$, taking an original value φ_0, and it satisfies $\varphi_0 < \varphi_1$, substituting (2.54) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_0} \frac{1}{(\varphi_1 - t)\sqrt{(\varphi_2 - t)(\varphi_1 - t)}} dt = \pm \int_{\xi}^{0} dt. \tag{2.55}$$

From (2.55), we get two travelling wave solutions as follows:

$$\varphi = \varphi_1 - \frac{4(\varphi_1 - \varphi_0)}{(\varphi_2 - \varphi_1)(\varphi_1 - \varphi_0)\xi^2 + 4\sqrt{(\varphi_2 - \varphi_0)(\varphi_1 - \varphi_0)}\xi + 4}, \tag{2.56}$$

and

$$\varphi = \varphi_1 - \frac{4(\varphi_1 - \varphi_0)}{(\varphi_2 - \varphi_1)(\varphi_1 - \varphi_0)\xi^2 - 4\sqrt{(\varphi_2 - \varphi_0)(\varphi_1 - \varphi_0)}\xi + 4}. \tag{2.57}$$

(2) If $\varphi_2 \leq \varphi$, substituting (2.54) into (2.6) and integrating it, we have

$$\int_{\varphi_2}^{\varphi} \frac{1}{(t - \varphi_1)\sqrt{(t - \varphi_2)(t - \varphi_1)}} dt = \pm \int_{0}^{\xi} dt. \tag{2.58}$$

From (2.58), we get a travelling wave solution as follows:

$$\varphi = \varphi_1 - \frac{4(\varphi_2 - \varphi_1)}{(\varphi_2 - \varphi_1)^2(\xi - 2nT)^2 - 4}, \tag{2.59}$$

where $T = \frac{2}{\varphi_2 - \varphi_1}$, $(2n-1)T < \xi < (2n+1)T$, $n = 0, \pm 1, \pm 2, \pm 3, \cdots$.

2.6.2. Cases of φ_1 is a simple real root

When φ_1 is a simple real root, and φ_2 is a triple root of $f(\varphi)$, where $\varphi_1 < \varphi_2$, then

$$f(\varphi) = (\varphi - \varphi_1)(\varphi - \varphi_2)^3. \tag{2.60}$$

(1) If $\varphi \leq \varphi_1$, substituting (2.60) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_1} \frac{1}{(\varphi_2 - t)\sqrt{(\varphi_2 - t)(\varphi_1 - t)}} dt = \pm \int_{\xi}^{0} dt. \tag{2.61}$$

From (2.58), we get a travelling wave solution as follows:

$$\varphi = \varphi_2 + \frac{4(\varphi_2 - \varphi_1)}{(\varphi_2 - \varphi_1)^2(\xi - 2nT)^2 - 4}, \tag{2.62}$$

where $T = \frac{2}{\varphi_2 - \varphi_1}$, $(2n-1)T < \xi < (2n+1)T$, $n = 0, \pm 1, \pm 2, \pm 3, \cdots$.

(2) If $\varphi_2 \le \varphi$, taking an original value φ_0, and it satisfies $\varphi_2 < \varphi_0$, substituting (2.60) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_0} \frac{1}{(t-\varphi_2)\sqrt{(t-\varphi_1)(t-\varphi_2)}}dt = \pm \int_{\xi}^{0} dt. \tag{2.63}$$

From (2.63), we get two travelling wave solutions as follows:

$$\varphi = \varphi_2 + \frac{4(\varphi_2-\varphi_0)}{(\varphi_2-\varphi_1)(\varphi_2-\varphi_0)\xi^2 + 4\sqrt{(\varphi_2-\varphi_0)(\varphi_1-\varphi_0)}\xi - 4}, \tag{2.64}$$

and

$$\varphi = \varphi_2 + \frac{4(\varphi_2-\varphi_0)}{(\varphi_2-\varphi_1)(\varphi_2-\varphi_0)\xi^2 - 4\sqrt{(\varphi_2-\varphi_0)(\varphi_1-\varphi_0)}\xi - 4}. \tag{2.65}$$

2.7. Cases of $f(\varphi)$ has a quadruple real root

From (2.5), $f(\varphi)$ has a quadruple real root $\frac{c}{2}$, so

$$f(\varphi) = (\varphi - \frac{c}{2})^4. \tag{2.66}$$

(1) If $\varphi < \frac{c}{2}$, taking an original value φ_0, and it satisfies $\varphi_0 < \frac{c}{2}$, substituting (2.66) into (2.6) and integrating it, we have

$$\int_{\varphi_0}^{\varphi} \frac{1}{(\frac{c}{2}-t)^2}dt = \pm \int_{0}^{\xi} dt. \tag{2.67}$$

From (2.67), we get two travelling wave solutions as follows:

$$\varphi = \frac{c}{2} - \frac{c-2\varphi_0}{2+(c-2\varphi_0)\xi}, \tag{2.68}$$

and

$$\varphi = \frac{c}{2} - \frac{c-2\varphi_0}{2-(c-2\varphi_0)\xi}. \tag{2.69}$$

(2) If $\varphi > \frac{c}{2}$, taking an original value φ_0, and it satisfies $\varphi_0 > \frac{c}{2}$, substituting (2.66) into (2.6) and integrating it, we have

$$\int_{\varphi_0}^{\varphi} \frac{1}{(t-\frac{c}{2})^2}dt = \pm \int_{0}^{\xi} dt. \tag{2.70}$$

From (2.70), we get two travelling wave solutions as follows:

$$\varphi = \frac{c}{2} + \frac{2\varphi_0-c}{2-(2\varphi_0-c)\xi}, \tag{2.71}$$

and

$$\varphi = \frac{c}{2} + \frac{2\varphi_0-c}{2+(2\varphi_0-c)\xi}. \tag{2.72}$$

2.8. Cases of $f(\varphi)$ has four simple real roots

When $\varphi_1, \varphi_2, \varphi_3$ and φ_4 are four simple real roots of $f(\varphi)$, where $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4$, then

$$f(\varphi) = (\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3)(\varphi - \varphi_4). \tag{2.73}$$

(1) If $\varphi \leq \varphi_1$, substituting (2.73) into (2.6) and integrating it, we have

$$\int_{\varphi}^{\varphi_1} \frac{1}{\sqrt{(\varphi_4 - t)(\varphi_3 - t)(\varphi_2 - t)(\varphi_1 - t)}} dt = \pm \int_{\xi}^{0} dt. \tag{2.74}$$

From (2.74), using formula 251.00 in [4], we get a travelling wave solution as follows:

$$\varphi = \varphi_2 - \frac{(\varphi_4 - \varphi_2)(\varphi_2 - \varphi_1)}{\varphi_4 - \varphi_2 - (\varphi_4 - \varphi_1)\mathrm{sn}^2(\tau, k)}, \tag{2.75}$$

where $k = \sqrt{\frac{(\varphi_3 - \varphi_2)(\varphi_4 - \varphi_1)}{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}}$ and $\tau = \frac{\sqrt{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}}{2}\xi$.

(2) If $\varphi_2 \leq \varphi \leq \varphi_3$, substituting (2.73) into (2.6) and integrating it, we have

$$\int_{\varphi_2}^{\varphi} \frac{1}{\sqrt{(\varphi_4 - t)(\varphi_3 - t)(t - \varphi_2)(t - \varphi_1)}} dt = \pm \int_{0}^{\xi} dt. \tag{2.76}$$

From (2.76), using formula 254.00 in [4], we get a travelling wave solution as follows:

$$\varphi = \varphi_1 + \frac{(\varphi_3 - \varphi_1)(\varphi_2 - \varphi_1)}{\varphi_3 - \varphi_1 - (\varphi_3 - \varphi_2)\mathrm{sn}^2(\tau, k)}. \tag{2.77}$$

(3) If $\varphi_4 \leq \varphi$, substituting (2.73) into (2.6) and integrating it, we have

$$\int_{\varphi_4}^{\varphi} \frac{1}{\sqrt{(t - \varphi_4)(t - \varphi_3)(t - \varphi_2)(t - \varphi_1)}} dt = \pm \int_{0}^{\xi} dt. \tag{2.78}$$

From (2.78), using formula 258.00 in [4], we get a travelling wave solution as follows:

$$\varphi = \varphi_3 + \frac{(\varphi_3 - \varphi_1)(\varphi_4 - \varphi_3)}{\varphi_3 - \varphi_1 - (\varphi_4 - \varphi_1)\mathrm{sn}^2(\tau, k)}. \tag{2.79}$$

3. Classifications of travelling wave solutions

Let

$$f_0(\varphi) = \varphi^4 - 2c\varphi^3 + (c^2 - 2A - 4)\varphi^2 + 2(Ac - 2B)\varphi, \tag{3.1}$$

then $f(\varphi) = f_0(\varphi) + h$ and

$$f_0'(\varphi) = 4\varphi^3 - 6c\varphi^2 + 2(c^2 - 2A - 4)\varphi + 2(Ac - 2B). \tag{3.2}$$

Let $p = -\frac{1}{4}(c^2 + 2A + 8)$, $q = -(c + B)$, and $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$, then

(1) If $\Delta > 0$, $f_0'(\varphi) = 0$ there will be one real root and two conjugate complex roots.

(2) If $\Delta = 0$, $f_0'(\varphi) = 0$ there will be three real roots of which at least two are equal.

(3) If $\Delta < 0$, $f_0'(\varphi) = 0$ there will be three real and unequal roots.

Clearly, we have the following results.

(1) If $\Delta \geq 0$, then $f_0(\varphi)$ has an unequal minimum point φ_0^*, $D_0 = f_0(\varphi_0^*)$ is minimal value.

(2) If $\Delta = 0$, and $f_0'(\varphi) = 0$ has a double real root φ_{01}^*, then φ_{01}^* is inflection point of $f_0(\varphi)$, let $D_{01} = f_0(\varphi_{01}^*)$.

(3) If $\Delta < 0$, then $f_0(\varphi)$ has three extreme points φ_1^*, φ_2^* and φ_3^*, where $\varphi_1^* < \varphi_2^* < \varphi_3^*$, φ_1^* and φ_3^* are two minimum points, φ_2^* is a maximum point. Let $D_1 = \frac{f_0(\varphi_1^*)+f_0(\varphi_3^*)}{2} - \frac{|f_0(\varphi_1^*)-f_0(\varphi_3^*)|}{2}$, $D_2 = \frac{f_0(\varphi_1^*)+f_0(\varphi_3^*)}{2} + \frac{|f_0(\varphi_1^*)-f_0(\varphi_3^*)|}{2}$ and $D_3 = f_0(\varphi_2^*)$, then D_1 and D_2 are two minimum values, D_3 is a maximum value, and $D_1 \leq D_2 < D_3$.

Thus, we have the following theorem.

Theorem 3.1. *(1) Under any one of the following two conditions, $f(\varphi)$ has four conjugate complex roots.*
(a) $\Delta \geq 0$ and $h > -D_0$. (b) $\Delta < 0$ and $h > -D_1$.

(2) Under any one of the following three conditions, $f(\varphi)$ has a double real root and two conjugate complex roots.
(a) $\Delta > 0$ and $h = -D_0$. (b) $\Delta = 0$, $p \neq 0$, $q \neq 0$ and $h = -D_0$. (c) $\Delta < 0$ and $h = -D_1 > -D_2$.

(3) If $\Delta = 0$, $p = q = 0$ and $h = -D_0$, then $\frac{c}{2}$ is a quadruple real root of $f(\varphi)$.

(4) Under any one of the following four conditions, $f(\varphi)$ has two simple real roots and two conjugate complex roots.
(a) $\Delta > 0$ and $h < -D_0$. (b) $\Delta = 0$, $h < -D_0$ and $h \neq -D_{01}$. (c) $\Delta < 0$ and $-D_2 < h < -D_1$, (d) $\Delta < 0$ and $h < -D_3$.

(5) If $\Delta = 0$ and $h = -D_{01}$, then $f(\varphi)$ has a simple real root and a triple real root.

(6) Under any one of the following two conditions, $f(\varphi)$ has two simple real roots and a double real root.
(a) $\Delta < 0$ and $h = -D_2 < -D_1$. (b) $\Delta < 0$ and $h = -D_3$.

(7) If $\Delta < 0$ and $h = -D_2 = -D_1$, then $f(\varphi)$ has two double real roots.

(8) If $\Delta < 0$ and $-D_3 < h < -D_2$, then $f(\varphi)$ has four simple real roots.

Let $\frac{d\varphi}{d\xi} = y$, then equation (2.3) becomes the following two dimensional system.

$$\begin{cases} \dfrac{d\varphi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{1}{2} f_0'(\varphi). \end{cases} \tag{3.3}$$

Using the dynamical system theory of planar systems, we know that the singular points of system (3.3) have the following properties.

(1) $(\varphi_0^*, 0)$ is a saddle point.

(2) $(\varphi_{01}^*, 0)$ is a degenerate saddle point.

(3) $(\varphi_1^*, 0)$ and $(\varphi_3^*, 0)$ are two saddle points.

(4) $(\varphi_2^*, 0)$ is a centre point.

Based on the above analysis, we obtain the classification of the travelling wave solutions of equation (1.1).

3.1. Bounded smooth periodic wave

When $\Delta < 0$ and $-D_3 < h < -D_2$, the travelling wave solution (2.77) is a bounded smooth periodic wave.

3.2. Bounded smooth solitary waves

When $\Delta < 0$ and $h = -D_2 < -D_1$, the travelling wave solutions (2.28) and (2.40) are two bounded smooth solitary waves.

3.3. Bounded kink waves

When $\Delta < 0$ and $h = -D_2 = -D_1$, the travelling wave solutions (2.49) and (2.50) are two bounded kink waves.

3.4. Unbounded periodic blow-up waves

(1) Under any one of the following four conditions, the travelling wave solutions (2.20) and (2.22) are two periodic blow-up waves.
(a) $\Delta > 0$ and $h < -D_0$. (b) $\Delta = 0$, $h < -D_0$ and $h \neq -D_{01}$. (c) $\Delta < 0$ and $-D_2 < h < -D_1$. (d) $\Delta < 0$ and $h < -D_3$.

(2) When $\Delta < 0$ and $h = -D_2 < -D_1$, the travelling wave solutions (2.30) and (2.38) are two periodic blow-up waves.

(3) When $\Delta < 0$ and $h = -D_3$, the travelling wave solutions (2.33) and (2.35) are two periodic blow-up waves.

(4) When $\Delta = 0$ and $h = -D_{01}$, the travelling wave solutions (2.59) and (2.62) are two periodic blow-up waves.

(5) When $\Delta < 0$ and $-D_3 < h < -D_2$, the travelling wave solutions (2.75) and (2.79) are two periodic blow-up waves.

3.5. Unbounded periodic waves

Under any one of the following two conditions, the travelling wave solutions (2.9) and (2.10) are two unbounded periodic waves.
(a) $\Delta \geq 0$ and $h > -D_0$. (b) $\Delta < 0$ and $h > -D_1$.

3.6. Unbounded kink waves

(1) Under any one of the following three conditions, the travelling wave solutions (2.13), (2.14), (2.16) and (2.17) are four unbounded kink waves.
(a) $\Delta > 0$ and $h = -D_0$. (b) $\Delta = 0$, $p \neq 0, q \neq 0$ and $h = -D_0$. (c) $\Delta < 0$ and $h = -D_1 > -D_2$.

(2) When $\Delta < 0$ and $h = -D_2 < -D_1$, the travelling wave solutions (2.25), (2.26), (2.42) and (2.43) are four unbounded kink waves.

(3) When $\Delta < 0$ and $h = -D_2 = -D_1$, the travelling wave solutions (2.46), (2.47), (2.52) and (2.53) are four unbounded kink waves.

(4) When $\Delta = 0$ and $h = -D_{01}$, the travelling wave solutions (2.56), (2.57), (2.64) and (2.65) are four unbounded kink waves.

(5) When $\Delta = 0$, $p = q = 0$ and $h = -D_0$, the travelling wave solutions (2.68), (2.69), (2.71) and (2.72) are four unbounded kink waves.

Based on the above analysis, on $\xi - v$ plane, we will simulate the travelling waves using mathematical software *Mathematica 7.0*.

Example 3.1. Choosing $c = -2$, $A = 4$ and $B = 4$, then $\varphi_1^* \doteq -3.48929$, $\varphi_2^* \doteq -1.28917$ and $\varphi_3^* \doteq 1.77846$, so $D_1 \doteq -49.7095$, $D_2 \doteq -7.44017$ and $D_3 \doteq 22.1497$.

(1) Choosing $h = 5$, we get $\varphi_1 \doteq -3.79386$, $\varphi_2 \doteq -3.13829$, $\varphi_3 \doteq 0.150996$ and $\varphi_4 \doteq 2.78116$. Substituting these data into (2.75), (2.79) and (2.77), we draw two unbounded periodic blow-up wave graphs as Fig. 1(a) and Fig. 1(b), and a bounded smooth periodic wave graph as Fig. 2, respectively.

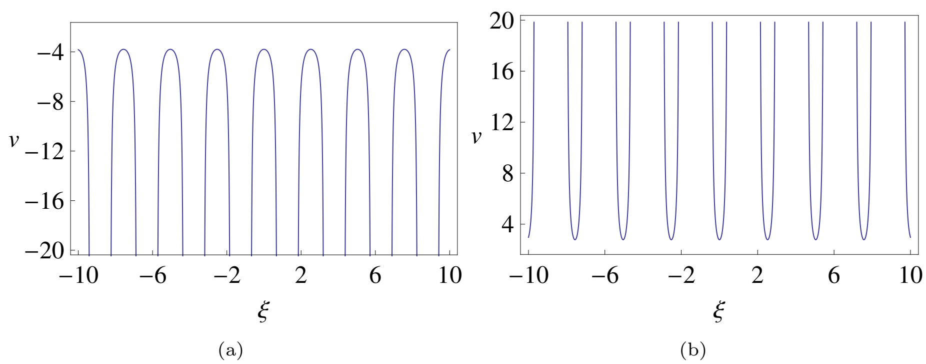


Figure 1. The two unbounded periodic blow-up waves with $c = -2$, $A = 4$, $B = 4$ and $h = 5$.

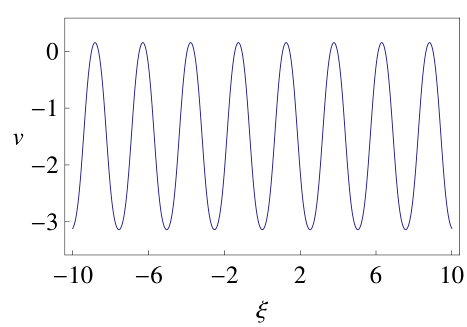


Figure 2. The bounded smooth periodic wave with $c = -2$, $A = 4$, $B = 4$ and $h = 5$.

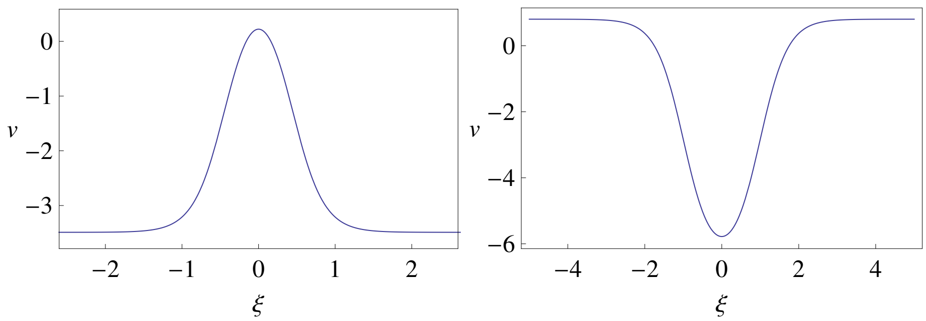
(2) Choosing $h = 7.44017$, we get $\varphi_1 \doteq -3.48929$, $\varphi_2 \doteq 0.221659$ and $\varphi_3 \doteq 2.75692$. Substituting these data into (2.28), we draw a bounded smooth solitary wave graph as Fig. 3(a).

Example 3.2. Choosing $c = -6$, $A = 4$ and $B = 4$, then $\varphi_1^* \doteq -6.938$, $\varphi_2^* \doteq -2.86651$ and $\varphi_3^* \doteq 0.804512$, so $D_1 \doteq -91.2461$, $D_2 \doteq -29.2876$ and $D_3 \doteq 165.534$.

Choosing $h = 29.2876$, we get $\varphi_1 \doteq -7.82988$, $\varphi_2 \doteq -5.77914$ and $\varphi_3 \doteq 0.804512$. Substituting these data into (2.40), we draw a bounded smooth solitary wave graph as Fig. 3(b).

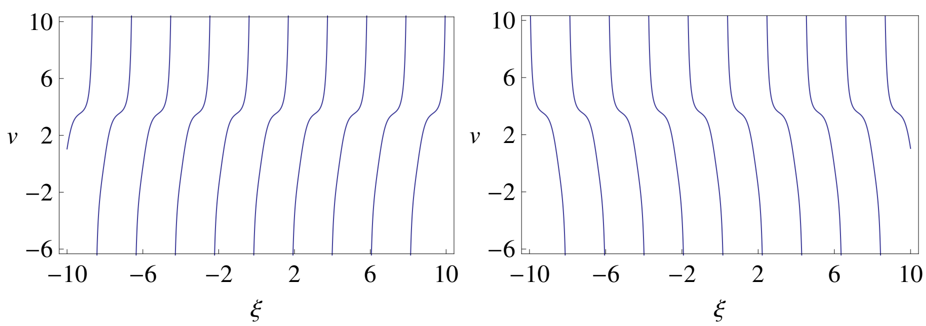
Example 3.3. Choosing $c = 2$, $A = 1$ and $B = 4$, then $\varphi_0^* \doteq 3.5251$, so $D_0 \doteq -87.9564$.

Choosing $h = 90$, we get $\varphi_1 \doteq -1.53628 - 2.19052i$, $\overline{\varphi_1} \doteq -1.53628 + 2.19052i$, $\varphi_2 \doteq 3.53628 - 0.258965i$ and $\overline{\varphi_2} \doteq 3.53628 + 0.258965i$. Substituting these data into (2.9) and (2.10), we draw two unbounded periodic wave graphs as Fig. 4(a) and Fig. 4(b) respectively.



(a) $c = -2$ and $h = 7.44017$ (b) $c = -6$ and $h = 29.2876$

Figure 3. The two bounded smooth solitary waves with $A = 4$ and $B = 4$.



(a) (b)

Figure 4. The two unbounded periodic waves with $c = 2$, $A = 1$, $B = 4$ and $h = 90$.

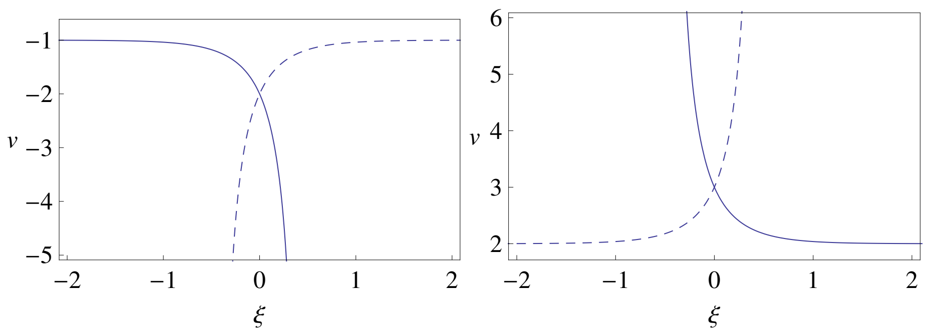
Example 3.4. Choosing $c = 1$, $A = 0$ and $B = -1$, then $\varphi_1^* = -1$, $\varphi_2^* = 0.5$ and $\varphi_3^* = 2$, so $D_1 = D_2 = -4$ and $D_3 = 1.0625$.

Choosing $h = 4$, we get $\varphi_1 = -1$ and $\varphi_2 = 2$.

(1) We take $\varphi_0 = -2$, substituting these data into (2.46) and (2.47), we draw two unbounded kink wave graphs as Fig. 5(a).

(2) We take $\varphi_0 = 3$, substituting these data into (2.52) and (2.53), we draw two unbounded kink wave graphs as Fig. 5(b).

(3) We take $\varphi_0 = 1$, substituting these data into (2.49) and (2.50), we draw two bounded kink wave graphs as Fig. 6.



(a) $\varphi_0 = -2$ (b) $\varphi_0 = 3$

Figure 5. The four unbounded kink waves with $c = 1$, $A = 0$, $B = -1$ and $h = 4$.

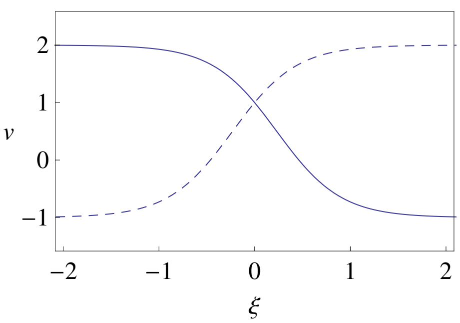


Figure 6. The two bounded kink waves with $c=1$, $A=0$, $B=-1$, $h=4$ and $\varphi_0=1$.

4. Conclusion

In this paper, we studied the bifurcation and global behavior of the gBK equation. We gave the conditions for periodic waves, solitary waves, blow-up waves and kink waves existing and we obtained representations of all the waves. On $\xi - v$ plane, their planar graphs are simulated under some parameters (see Figs. 1–6). The unbounded periodic blow-up wave solutions and unbounded kink wave solutions in this paper are new results to the gBK equation.

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