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NON-PERIODIC DISCRETE SCHRÖDINGER EQUATIONS WITH SIGN-CHANGING AND SUPER-QUADRATIC TERMS: EXISTENCE OF SOLUTIONS*

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Abstract We study the existence of homoclinic solutions for a class of non-periodic discrete nonlinear Schrödinger equations, where nonlinearities are super-linear at infinity, and primitive functions of nonlinearities are allowed to be sign-changing. By using some weaker conditions, our result extends and improves some results in the literature. Besides, we also give examples to illuminate our results.

Keywords non-periodic discrete Schrödinger equations, super-linear, homoclinic solutions, sign-changing, variational methods.

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1. Introduction and main results

With the development of nonlinear science, many authors focus their attention to the discrete nonlinear Schrödinger equation, which is one of the important nonlinear models in mathematics and physics. Besides, the discrete nonlinear Schrödinger equation has been widely used in many fields, such as biomolecular chains [13], nonlinear optics [8] and Bose-Einstein condensates [18], etc.

We will study the standing waves for the following discrete nonlinear Schrödinger equation

$$-i\dot{\psi}_n - \Delta\psi_n + v_n\psi_n = f_n(\psi_n), \quad n \in Z^m, \tag{1.1}$$

where

$$\begin{aligned}\Delta\psi_n =&\psi_{(n_1+1,n_2,\cdots,n_m)} + \psi_{(n_1,n_2+1,\cdots,n_m)} + \cdots + \psi_{(n_1,n_2,\cdots,n_m+1)}\\ &- 2m\psi_{(n_1,n_2,\cdots,n_m)} + \psi_{(n_1-1,n_2,\cdots,n_m)} + \psi_{(n_1,n_2-1,\cdots,n_m)} + \psi_{(n_1,n_2,\cdots,n_m-1)}\end{aligned}$$

is the discrete Laplace operator in m dimensional space, $V = \{v_n\}_{n\in Z^m}$ is a sequence of real numbers and

$$f_n(e^{i\omega}s) = e^{i\omega}f_n(s), \quad \forall\omega \in R,\ \forall(n, s) \in Z^m \times R.$$

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By the definition of *standing waves* ($\psi_n = u_n e^{-i\omega t}$, $\{u_n\}_{n\in Z^m}$ is a sequence of real numbers, $\lim_{|n|=|n_1|+|n_2|+\cdots+|n_m|\to\infty} u_n = 0$), we get the problem on standing waves of (1.1) reduces to the *homoclinic solutions* of following non-periodic discrete nonlinear Schrödinger equation

$$-\Delta u_n + v_n u_n - \omega u_n = f_n(u_n), \quad n \in Z^m. \tag{1.2}$$

Here, we say that the *homoclinic* solutions of (1.2) are the solutions that satisfy the boundary condition

$$\lim_{|n|=|n_1|+|n_2|+\cdots+|n_m|\to\infty} u_n = 0. \tag{1.3}$$

The study of discrete nonlinear Schrödinger equations can be divided into two cases: periodic and non-periodic cases in n (i.e., the sequences $\{v_n\}$ and f_n are all T-periodic ($T \in N^+$) in n, $\{v_n\}$ and f_n are all non-periodic). For the results of periodic discrete nonlinear Schrödinger equations, we refer readers to see [1–3, 5, 10, 15, 20–23, 26, 27, 30, 35–37], etc. For the results of non-periodic discrete nonlinear Schrödinger equations, we refer readers to see [4, 10–12, 14, 16, 17, 19, 24, 25, 28, 31–34, 38], etc. The authors in [4, 10, 14, 17, 24, 25, 31, 34] studied the case of infinite one dimensional lattice (i.e., $n \in Z$), and the authors in [11, 12, 16, 19, 28, 32, 33, 38] studied the case of infinite m dimensional lattices (i.e., $n \in Z^m$). The authors in [4, 12, 14, 16, 17, 19, 24, 28, 34] obtained the *existence* of homoclinic solutions of (1.2), and the authors in [11, 12, 16, 19, 24, 25, 28, 31–34, 38] obtained the *infinitely many* homoclinic solutions of (1.2) under the odd condition $f_n(-s) = -f_n(s)$ for all n. For discrete nonlinear Schrödinger equations with perturbed terms, we refer the readers to see [6, 7].

In this paper, we will study the existence of homoclinic solutions for the non-periodic equation (1.2) without the odd condition $f_n(-s) = -f_n(s)$. In order to illustrate our result, we will give some comparisons between our result and the existence results of [4, 12, 14, 16, 17, 19, 24, 28, 34].

Now, our result reads as follows.

Theorem 1.1. *Assume that the following assumptions hold.*

($\mathbf{V_1}$) $V = \{v_n\}_{n\in \mathrm{Z}^m}$ *satisfies* $\lim_{|n|\to+\infty} v_n = +\infty$.

($\mathbf{F_1}$) $f_n \in C(\mathrm{R}, \mathrm{R})$*, and there exist constants* $c_1, c_2 > 0$*,* $p > 2$ *and* $c_3 > 1$ *such that*

$$|f_n(s)| \le c_1|s| + c_2|s|^{p-1}, \quad \forall (n, s) \in \mathrm{Z}^m \times \mathrm{R},$$

where $c_1 < \frac{1}{2\gamma_2^2} - c_3|\lambda_1 - \omega|$*,* γ_2 *is the Sobolev embedding constant given in the following Lemma 2.1,* $\lambda_1 := \inf \sigma(-\triangle + V)$ *and* $c_3|\lambda_1 - \omega| < \frac{1}{2\gamma_2^2}$.

($\mathbf{F_2}$) $\lim_{|s|\to+\infty} \frac{F_n(s)}{|s|^2} = +\infty$, $\forall (n, s) \in \mathrm{Z}^m \times \mathrm{R}$*, where* $F_n(s) := \int_0^s f_n(t)dt$.

($\mathbf{F_3}$) $f_n(s)s - 2F_n(s) \ge 0$, $\forall (n, s) \in \mathrm{Z}^m \times \mathrm{R}$*, and there exist* $r_0, c_0 > 0$ *and* $\varrho > 1$ *such that*

$$|F_n(s)|^{\varrho} \le c_0|s|^{2\varrho}(f_n(s)s - 2F_n(s)), \qquad \forall |s| \ge r_0, \quad \forall n \in \mathrm{Z}^m. \tag{1.4}$$

Then (1.2) *has at least a nontrivial homoclinic solution. Here, a solution* u *is nontrivial means* $u_n \not\equiv 0$.

We mention that the authors in [11] have used the assumptions in Theorem 1.1 and the assumptions ($F_n(s) \geq 0$, $\forall |s| \geq r_0$ and $f_n(-s) = -f_n(s)$, $\forall n \in Z^m$) to study the existence of infinitely many homoclinic solutions of (1.2). But we remove the assumptions ($F_n(s) \geq 0$, $\forall |s| \geq r_0$ and $f_n(-s) = -f_n(s)$, $\forall n \in Z^m$) and obtain the *existence* of homoclinic solutions for (1.2) by using the mountain pass theorem in [9].

As usual, the condition (F_2) is called as the super-quadratic condition. In fact, in addition to our result, the above condition (V_1) is also suitable to consider some other wider class of nonlinearities. For example, it could be used to study the case that the primitive functions of the nonlinearities f_n are asymptotically quadratic at infinity (i.e., $\lim_{|s|\to+\infty} \frac{F_n(s)}{|s|^2} = k_n > 0$, $k_n \in l^\infty$, $\forall (n,s) \in Z^m \times R$).

Next, we give an example to illuminate our result.

Example 1.1. Let

$$F_n(s) = a_n\big[|s|^{3.5} - 2s^2\cos(s)\big], \quad \forall (n,s) \in Z^m \times R,$$

where $0 < \inf_{n\in Z^m} a_n < \sup_{n\in Z^m} a_n < +\infty$.

Clearly, F_n is sign-changing. It is easy to show that F_n satisfies our conditions (F_1) (if $\sup_{n\in Z^m} a_n$ is small) and (F_2). Next, we will verify the function F_n also satisfies our condition (F_3). Let

$$\widetilde{F}_n(s) := f_n(s)s - 2F_n(s) = a_n\big[1.5|s|^{3.5} + 2s^3\sin(s)\big],$$

Obviously, $\widetilde{F}_n(s) \geq 0$ in $[-\pi, \pi]$. If $|s| \geq \pi$, then

$$\begin{aligned}\widetilde{F}_n(s) =& a_n\big[1.5|s|^{3.5} + 2s^3\sin(s)\big]\\ \geq& a_n|s|^3\big[1.5|s|^{0.5} - 2\big]\\ \geq& a_n|s|^3\big[1.5|\pi|^{0.5} - 2\big]\\ >& 0.\end{aligned}$$

Thus $\widetilde{F}_n(s) \geq 0$ for all $(n,s) \in Z^m \times R$. Let

$$h_n := c_0|s|^{2\varrho}\big[f_n(s)s - 2F_n(s)\big] - \big|F_n(s)\big|^\varrho,$$

where $\varrho > 1$ and $c_0 > 0$. Clearly, if $|s| \geq r_0$ for some $r_0 > 0$ is large enough, we have

$$\begin{aligned}h_n =& c_0|s|^{2\varrho}\big[f_n(s)s - 2F_n(s)\big] - \big|F_n(s)\big|^\varrho\\ =& a_nc_0\big[|s|^{2\varrho}(1.5|s|^{3.5} + 2s^3\sin(s))\big] - \big|a_n(|s|^{3.5} - 2s^2\cos(s))\big|^\varrho\\ \geq& 1.5a_nc_0|s|^{2\varrho+3.5} - 2a_nc_0|s|^{2\varrho+3} - 2^\varrho a_n^\varrho|s|^{3.5\varrho}\\ >& 0.\end{aligned}$$

So condition (1.4) is satisfied. Therefore, the function F_n satisfies our conditions (F_1)–(F_3).

Remark 1.1 (comparisons)**.** Now we give some detailed comparisons between our result and the existence results ([12,16,19,28] ($n \in Z^m$), [4,14,17,24,34] ($n \in Z$)).

1) The papers [14, 17, 24] are about the asymptotically linear case, the paper [16] is about the sub-linear case, which are all essentially different from the super-linear case in our paper.
2) Our nonlinearities F_n can be sign-changing, which is more general than the case $(F_n(s) \geq 0, \ \forall (n,s) \in Z^m \times R)$ in the most of above papers [12, 19, 24, 28, 34]. The authors in [4] also studied the sign-changing case, but they used the following condition which we do not need:

$$F_n(s) \geq \frac{1}{2}as^2, \quad \text{where } a = \lambda_{k_0} - \omega < 0, \quad \lambda_{k_0} \in \sigma(-\Delta + V), \ k_0 \geq 1.$$

3) The authors in [19, 28, 34] all used the following two conditions:

$$|f_n(s)| \leq a_1(1 + |s|^{\nu-1}), \quad \text{for some } a_1 > 0, \ \nu > 2, \quad \forall (n,s) \in Z^m \times R, \tag{1.5}$$
$$f_n(s) = o(s), \quad s \to 0. \tag{1.6}$$

However, our condition (F_1) is weaker than the conditions (1.5)–(1.6).
4) The authors in [19, 28] used the monotony condition

$$\frac{f_n(s)}{s} \text{ is increasing for } s > 0 \text{ and decreasing for } s < 0, \tag{1.7}$$

and the authors in [34] used the Ambrosetti-Rabinowitz super-linear condition

$$0 < \nu' F_n(s) \leq f_n(s)s \quad \text{for some } \nu' > 2, \quad \forall s \in R\backslash\{0\}. \tag{1.8}$$

It is not hard to check that the function in our Example 1.1 do not satisfy the conditions (1.6)–(1.8), but it satisfies our conditions (F_1)–(F_3). Therefore, our result extends and improves the results in the above papers.

In Section 2, we will give some preliminary lemmas and the detailed proofs of our result.

2. Preliminary lemmas and proof of Theorem 1.1

Let

$$l^p \equiv l^p(Z^m) := \left\{ u = \{u_n\} : \ n \in Z^m, \ u_n \in R, \ \|u\|_p = \left(\sum_{n \in Z^m} |u_n|^p \right)^{1/p} < \infty \right\},$$

are real sequence spaces, where $p \in [1, +\infty)$. Clearly, we have the elementary embedding relations

$$l^p \subset l^q, \quad \|u\|_q \leq \|u\|_p, \quad 1 \leq p \leq q \leq \infty, \quad \text{where } \|u\|_\infty := \max_{n \in Z^m} |u_n|.$$

Let E be the form domain of $-\triangle + V$, i.e., the domain of $(-\triangle + V)^{1/2}$. Under our assumptions, the operator $-\triangle + V$ is an unbounded self-adjoint operator in l^2. Since the operator $-\triangle$ is bounded in l^2, it is easy to see that $E = \{u \in l^2 : \ V^{1/2}u \in l^2\}$, where $V = \{v_n\}_{n \in Z^m}$.

Lemma 2.1 ([33]). *If assumption* (V_1) *holds, then we have:*

(1) the embedding from E into l^q is compact for $2 \leq q < \infty$, and there exist $\gamma_q > 0$ such that

$$\|u\|_q \leq \gamma_q \|u\|, \quad \forall u \in E;$$

(2) the spectrum $\sigma(-\triangle + V)$ is discrete and consists of simple eigenvalues accumulating to $+\infty$.

Remark 2.1. Obviously, there exists a constant $c_3 > 1$ such that

$$\lambda_1 - \omega + c_3|\lambda_1 - \omega| > c > 0,$$

where $\lambda_1 := \inf \sigma(-\triangle + V)$. Let $\overline{f}_n(s) = f_n(s) + c_3|\lambda_1 - \omega|s$, then by (F_1), we have $\overline{f}_n \in C(R, R)$ and

$$\left|\overline{f}_n(s)\right| < c_1'|s| + c_2|s|^{p-1}, \quad \forall (n, s) \in Z^m \times R, \tag{2.1}$$

where $c_1' := \frac{1}{2\gamma_2^2}$, $c_2 > 0$, $p > 2$, γ_2 is the Sobolev embedding constant given in above Lemma 2.1. Besides, it is easy to check that $\overline{f}_n$ also satisfies conditions (F_2) and (F_3), the reasons are as following:

(1) When $|s| \to +\infty$,

$$\frac{\overline{F}_n(s)}{|s|^2} = \frac{F_n(s)}{|s|^2} + \frac{1}{2}c_3|\lambda_1 - \omega| \to +\infty.$$

So $\overline{f}_n$ satisfies condition (F_2).

(2) Clearly, $\overline{f}_n(s)s - 2\overline{F}_n(s) = f_n(s)s - 2F_n(s) \geq 0$. And by (1.4) and (F_2), $f_n(s)s - 2F_n(s) \to +\infty$ when $|s| \to +\infty$, thus there exists $c_0' > 0$ such that

$$c_0' \geq c_0 + \frac{(c_3|\lambda_1 - \omega|)^\varrho}{2[f_n(s)s - 2F_n(s)]}.$$

Then

$$\begin{aligned}
|\overline{F}_n(s)|^\varrho \leq& |F_n(s)|^\varrho + \frac{1}{2}(c_3|\lambda_1 - \omega|)^\varrho |s|^{2\varrho} \\
\leq& c_0|s|^{2\varrho}[f_n(s)s - 2F_n(s)] + \frac{1}{2}(c_3|\lambda_1 - \omega|)^\varrho |s|^{2\varrho} \\
\leq& c_0'|s|^{2\varrho}[f_n(s)s - 2F_n(s)].
\end{aligned}$$

So $\overline{f}_n$ satisfies condition (F_3).

Note that the problem (1.2) is equivalent to the following problem

$$-\Delta u_n + \overline{v}_n u_n = \overline{f}_n(u_n), \quad n \in Z^m, \quad \overline{v}_n := v_n + c_3|\lambda_1 - \omega| - \omega, \tag{$*$}$$

with the boundary condition (1.3). Thus, to prove the Theorems 1.1, we only need to prove the problem $(*)$ has at least a homoclinic solution under the conditions (V_1) and (F_1)–(F_3).

Let L be defined by $Lu_n := -\triangle u_n + \overline{v}_n u_n$. According to Remark 2.1, we can introduce the following inner product and norm on E:

$$(u,\varphi) := (Lu,\varphi)_{l^2} = \sum_{n\in Z^m} \big[(-\Delta u_n + \overline{v}_n u_n)\varphi_n\big],$$
$$\|u\| = (u,u)^{\frac{1}{2}}, \qquad \forall u,\varphi \in E.$$

Therefore, the corresponding functional of $(*)$ can be written as follows

$$\begin{aligned}\Phi(u) =& \frac{1}{2}(Lu,u)_{l^2} - \sum_{n\in Z^m} \overline{F}_n(u_n)\\ =& \frac{1}{2}\|u\|^2 - I(u), \quad \forall u \in E,\end{aligned} \tag{2.2}$$

where $\overline{F}_n(s) := \int_0^s \overline{f}_n(t)dt$ and $I(u) := \sum_{n\in Z^m} \overline{F}_n(u_n)$. Under our assumptions, $I, \Phi \in C^1(E,R)$ and the derivatives are given by

$$\langle \Phi'(u), v\rangle = (u,v) - \langle I'(u), v\rangle, \quad \langle I'(u), v\rangle = \sum_{n\in Z^m} \overline{f}_n(u_n)v_n, \quad \forall u,v \in E.$$

Moreover, the nonzero critical points of Φ are nontrivial solutions of $(*)$.

We shall use the following mountain pass theorem to prove Theorem 1.1.

Lemma 2.2 (Mountain Pass Theorem [9]). *Let E be a real Banach space with its dual space E^*, and suppose that $\Phi \in^1 (E,\mathrm{R})$ satisfies*

$$\max\{\Phi(0), \Phi(e)\} \le \mu \le \eta \le \inf_{\|u\|=\rho} \Phi(u)$$

for some $\mu,\eta,\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{0\le\tau\le1} \Phi(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u^k\} \subset E$ such that

$$\Phi(u^k) \to c \ge \eta, \qquad (1+\|u^k\|)\|\Phi'(u^k)\|_{E^*} \to 0 \quad as \quad n \to \infty. \tag{2.3}$$

Here, we say that $\Phi \in C^1(X,R)$ satisfies $(C)_c$-condition if any sequence $\{u^k\}$ (satisfies the condition (2.3)) has a convergent subsequence. Clearly, (2.1) implies that

$$|\overline{F}_n(s)| \le \frac{c_1'}{2}|s|^2 + \frac{c_2}{p}|s|^p, \qquad c_1' := \frac{1}{2\gamma_2^2}, \quad \forall (n,s) \in Z^m \times R. \tag{2.4}$$

Lemma 2.3. *If assumptions $(V_1), (F_1), (F_2)$ and (F_3) hold, then Φ satisfies $(C)_c$-condition.*

Proof. We assume that for any sequence $\{u^k\} \subset E$, $\Phi(u^k) \to c$ and $\|\Phi^{'}(u^k)\|\big(1+\|u^k\|\big) \to 0$. Then $\Phi'(u^k) \to 0$, and

$$\big\langle \Phi^{'}(u^k), u^k\big\rangle \to 0. \tag{2.5}$$

(i) We prove the boundedness of $\{u^k\}$ by contradiction, if $\|u^k\| \to \infty$, we let $\varsigma^k = \frac{u^k}{\|u^k\|}$, then $\|\varsigma^k\| = 1$. By the definitions of $\Phi(u)$ and $\Phi'(u)$, for k large enough, we have

$$\sum_{n\in Z^m} \Big(\frac{1}{2}\overline{f}_n(u_n^k)u_n^k - \overline{F}_n(u_n^k)\Big) = \Phi(u^k) - \frac{1}{2}\Big\langle \Phi^{'}(u^k), u^k\Big\rangle \le c+1. \tag{2.6}$$

By (2.2), $\Phi(u^k) \to c$ and $\|u^k\| \to \infty$, we have

$$\limsup_{k\to\infty} \sum_{n\in Z^m} \frac{|\overline{F}_n(u_n^k)|}{\|u^k\|^2} \ge \frac{1}{2}. \tag{2.7}$$

Let

$$\Omega_k(a,b) = \{n \in Z^m : a \le |u_n^k| < b\}, \quad 0 \le a < b. \tag{2.8}$$

By $\|\varsigma^k\| = 1$, we could assume that $\varsigma^k \rightharpoonup \varsigma = \{\varsigma_n\}_{n\in Z^m}$ in E passing to a subsequence, which together with Lemma 2.1 implies $\varsigma^k \to \varsigma$ in l^q for $2 \le q < \infty$, and $\varsigma_n^k \to \varsigma_n$ for all $n \in Z^m$.

If $\varsigma = 0$, then $\varsigma^k \to 0$ in l^q, $2 \le q < \infty$, and $\varsigma_n^k \to 0$ for all $n \in Z^m$. It follows from (2.4) that

$$\begin{aligned}\sum_{n\in\Omega_k(0,r_0)} \frac{|\overline{F}_n(u_n^k)|}{|u_n^k|^2}|\varsigma_n^k|^2 \le& (\frac{c_1'}{2} + \frac{c_2}{p}r_0^{p-2}) \sum_{n\in\Omega_k(0,r_0)} |\varsigma_n^k|^2 \\ \le& (\frac{c_1'}{2} + \frac{c_2}{p}r_0^{p-2}) \sum_{n\in Z^m} |\varsigma_n^k|^2 \\ \to& 0.\end{aligned} \tag{2.9}$$

Let $\varrho^{'} = \varrho/(\varrho-1)$. Due to $\varrho > 1$ (see (F_3)), we have that $2\varrho > 2$. So by (F_3), (2.6), the Hölder's inequality and $\varsigma^k \to 0$ in l^q for $2 \le q < \infty$, we have

$$\begin{aligned}&\sum_{n\in\Omega_k(r_0,\infty)} \frac{|\overline{F}_n(u_n^k)|}{|u_n^k|^2}|\varsigma_n^k|^2 \\ &\le \Big[\sum_{n\in\Omega_k(r_0,\infty)} \Big(\frac{|\overline{F}_n(u_n^k)|}{|u_n^k|^2}\Big)^{\varrho}\Big]^{1/\varrho} \Big[\sum_{n\in\Omega_k(r_0,\infty)} |\varsigma_n^k|^{2\varrho^{'}}\Big]^{1/\varrho^{'}} \\ &\le (2c_0)^{1/\varrho}\Big[\sum_{n\in\Omega_k(r_0,\infty)} \Big(\frac{1}{2}\overline{f}_n(u_n^k)u_n^k - \overline{F}_n(u_n^k)\Big)\Big]^{1/\varrho} \Big[\sum_{n\in Z^m} |\varsigma_n^k|^{2\varrho^{'}}\Big]^{1/\varrho^{'}} \\ &\le [2c_0(c+1)]^{1/\varrho}\|\varsigma^k\|_{2\varrho'}^2 \\ &\to 0.\end{aligned} \tag{2.10}$$

Combining (2.9) with (2.10), we have

$$\sum_{n\in Z^m} \frac{|\overline{F}_n(u_n^k)|}{\|u^k\|^2} = \sum_{n\in\Omega_k(0,r_0)} \frac{|\overline{F}_n(u_n^k)|}{|u_n^k|^2}|\varsigma_n^k|^2 + \sum_{n\in\Omega_k(r_0,\infty)} \frac{|\overline{F}_n(u_n^k)|}{|u_n^k|^2}|\varsigma_n^k|^2 \to 0,$$

which contradicts with (2.7).

If $\varsigma \neq 0$, we let $A := \{n \in Z^m : \varsigma_n \neq 0\}$. For all $n \in A$, by $\varsigma_n^k = \frac{u_n^k}{\|u^k\|}$ and $\|u^k\| \to \infty$, we have $\lim_{k\to\infty} |u_n^k| = \infty$. We define

$$\chi_{n,\Omega_k(r_0,\infty)} := \begin{cases} 1, & n \in \Omega_k(r_0,\infty), \quad \forall k \in N, \\ 0, & n \notin \Omega_k(r_0,\infty), \quad \forall k \in N. \end{cases} \tag{2.11}$$

For large $k \in N$, $A \subset \Omega_k(r_0,\infty)$ and $\lim_{k\to\infty} |u_n^k| = \infty$ for all $n \in A$, it follows from (2.2), (2.4), (F_2), the Fadou's Lemma, $\|\varsigma^k\| = 1$, $\|u^k\| \to \infty$, $\Phi(u^k) \to c$ and $\|\varsigma^k\|_2 \le \gamma_2 \|\varsigma^k\|$ (see Lemma 2.1) that

$$\begin{aligned}
0 &= \lim_{k\to\infty} \frac{c + o(1)}{\|u^k\|^2} \\
&= \lim_{k\to\infty} \frac{\Phi(u^k)}{\|u^k\|^2} \\
&= \lim_{k\to\infty} \left[\frac{1}{2} - \sum_{n\in Z^m} \frac{\overline{F}_n(u_n^k)}{(u_n^k)^2} (\varsigma_n^k)^2\right] \\
&= \lim_{k\to\infty} \left[\frac{1}{2} - \sum_{n\in\Omega_k(0,r_0)} \frac{\overline{F}_n(u_n^k)}{(u_n^k)^2} (\varsigma_n^k)^2 - \sum_{n\in\Omega_k(r_0,\infty)} \frac{\overline{F}_n(u_n^k)}{(u_n^k)^2} (\varsigma_n^k)^2\right] \\
&\le \limsup_{k\to\infty} \left[\frac{1}{2} + \left(\frac{c_1'}{2} + \frac{c_2}{p} r_0^{p-2}\right) \sum_{n\in Z^m} |\varsigma_n^k|^2 - \sum_{n\in\Omega_k(r_0,\infty)} \frac{\overline{F}_n(u_n^k)}{(u_n^k)^2} (\varsigma_n^k)^2\right] \\
&\le \frac{1}{2} + \left(\frac{c_1'}{2} + \frac{c_2}{p} r_0^{p-2}\right)\gamma_2^2 - \liminf_{k\to\infty} \sum_{n\in\Omega_k(r_0,\infty)} \frac{\overline{F}_n(u_n^k)}{(u_n^k)^2} (\varsigma_n^k)^2 \\
&= \frac{1}{2} + \left(\frac{c_1'}{2} + \frac{c_2}{p} r_0^{p-2}\right)\gamma_2^2 - \liminf_{k\to\infty} \sum_{n\in Z^m} \frac{|\overline{F}_n(u_n^k)|}{(u_n^k)^2} [\chi_{n,\Omega_k(r_0,\infty)}] (\varsigma_n^k)^2 \\
&\le \frac{1}{2} + \left(\frac{c_1'}{2} + \frac{c_2}{p} r_0^{p-2}\right)\gamma_2^2 - \sum_{n\in Z^m} \liminf_{k\to\infty} \frac{\overline{F}_n(u_n^k)|}{(u_n^k)^2} [\chi_{n,\Omega_k(r_0,\infty)}] (\varsigma_n^k)^2 \\
&= -\infty.
\end{aligned} \tag{2.12}$$

It is a contradiction. So $\{u^k\}$ is bounded in E.

(ii) The boundedness of $\{u^k\}$ implies that $u^k \rightharpoonup u$ in E passing to a subsequence, where $u = \{u_n\}_{n\in Z^m}$. First, we prove

$$\sum_{n\in Z^m} \left[\overline{f}_n(u_n^k)(u_n^k - u_n)\right] \to 0, \quad k \to \infty. \tag{2.13}$$

Note that Lemma 2.1 implies that $u^k \to u$ in l^q for all $2 \le q < \infty$, so we have

$$\|u^k - u\|_2 \to 0, \quad \|u^k - u\|_p \to 0. \tag{2.14}$$

The boundedness of $\{u^k\}$ and Lemma 2.1 imply that $\|u^k\|_q < \infty$ for all $2 \le q < \infty$, it follows from (2.1), (2.14) and the Hölder's inequality that

$$\Big|\sum_{n\in Z^m} \left[\overline{f}_n(u_n^k)(u_n^k - u_n)\right]\Big|$$

$$\begin{aligned}
&\leq \sum_{n\in Z^m} \left|\overline{f}_n(u_n^k)(u_n^k-u_n)\right| \\
&\leq \sum_{n\in Z^m} \left[(c_1'|u_n^k|+c_2|u_n^k|^{p-1})|u_n^k-u_n|\right] \\
&=c_1'\sum_{n\in Z^m} \left[|u_n^k||u_n^k-u_n|\right]+c_2\sum_{n\in Z^m} \left[(|u_n^k|^{p-1}|u_n^k-u_n|\right] \\
&\leq c_1'\|u^k\|_2\|u^k-u\|_2+c_2\|u^k\|_p^{p-1}\|u^k-u\|_p \to 0. \qquad (2.15)
\end{aligned}$$

So (2.13) holds. Therefore, by (2.13), $\Phi'(u^k)\to 0$, $u^k \rightharpoonup u$ in E and the definition of $\Phi^{'}$, we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} \left\langle \Phi^{'}(u^k), u^k-u\right\rangle \\
&= \lim_{k\to\infty}(u^k,u^k-u)-\lim_{k\to\infty}\sum_{n\in Z^m}\left(\overline{f}_n(u_n^k)(u_n^k-u_n)\right) \\
&= \lim_{k\to\infty}\|u^k\|^2-\|u\|^2-0. \qquad (2.16)
\end{aligned}$$

That is,

$$\lim_{k\to\infty}\|u^k\|=\|u\|. \qquad (2.17)$$

It follows from $u^k \rightharpoonup u$ in E that

$$\|u^k-u\|^2=(u^k-u,u^k-u)\to 0,$$

that is, $\{u^k\}$ has a convergent subsequence in E. Thus Φ satisfies $(C)_c$-condition. □

Lemma 2.4. *If assumptions (V_1) and (F_1) hold, then there exist $\rho,\eta>0$ such that*

$$\inf\{\Phi(u)|u\in E, \|u\|=\rho\}>\eta.$$

Proof. By Lemma 2.1 and (2.4), for $u\in E$ we have

$$\begin{aligned}
\Big|\sum_{n\in Z^m}\overline{F}_n(u_n)\Big| &\leq \sum_{n\in Z^m}\Big|\frac{c_1'}{2}|u|^2+\frac{c_2}{p}|u|^p\Big| \\
&=\frac{c_1'}{2}\|u\|_2^2+\frac{c_2}{p}\|u\|_p^p \\
&\leq\frac{\gamma_2^2c_1'}{2}\|u\|^2+\frac{\gamma_p^pc_2}{p}\|u\|^p \\
&=\frac{1}{4}\|u\|^2+\frac{\gamma_p^pc_2}{p}\|u\|^p. \qquad (2.18)
\end{aligned}$$

Then from (2.2) and (2.18) we have

$$\begin{aligned}
\Phi(u) &=\frac{1}{2}\|u\|^2-\sum_{n\in Z^m}\overline{F}_n(u_n) \\
&\geq\frac{1}{2}\|u\|^2-\frac{1}{4}\|u\|^2-\frac{\gamma_p^pc_2}{p}\|u\|^p \\
&=\frac{1}{4}\|u\|^2-\frac{\gamma_p^pc_2}{p}\|u\|^p, \quad \forall u\in E. \qquad (2.19)
\end{aligned}$$

Let $\|u\| = \rho > 0$, then it follows from (2.19) that there exists $\eta > 0$ such that the lemma holds when ρ small enough. The proof is finished. $\square$

Lemma 2.5. *If assumptions* (V_1) *and* (F_2) *hold, then there exists* $\nu \in E$ *with* $\|\nu\| > \rho$ *such that* $\Phi(\nu) < 0$*, where* ρ *is given in lemma 2.4.*

Proof. By (2.2) we have

$$\frac{\Phi(tu)}{t^2} = \frac{1}{2}\|u\|^2 - \frac{1}{t^2}\sum_{n\in Z^m} \overline{F}_n(tu_n).$$

Then it follows from (F_2) and the Fatou's lemma that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\Phi(tu)}{t^2} &= \lim_{t\to\infty} \big[\frac{1}{2}\|u\|^2 - \frac{1}{t^2}\sum_{n\in Z^m} \overline{F}_n(tu_n)\big] \\
&\le \limsup_{t\to\infty} \big[\frac{1}{2}\|u\|^2 - \frac{1}{t^2}\sum_{n\in Z^m} \overline{F}_n(tu_n)\big] \\
&= \frac{1}{2}\|u\|^2 - \liminf_{t\to\infty} \sum_{n\in Z^m} \frac{\overline{F}_n(tu_n)}{t^2u_n^2}u_n^2 dt \\
&\le \frac{1}{2}\|u\|^2 - \int_0^T \liminf_{t\to\infty} \frac{\overline{F}_n(tu_n)}{t^2u_n^2}u_n^2 dt \\
&= -\infty, \quad \text{as } t \to \infty.
\end{aligned}$$

Therefore let $\nu = t_0 u$, the lemma is proved when $t_0 > 0$ large enough. $\square$

Proof of Theorem 1.1. Lemmas 2.4 and 2.5 imply all the conditions of Lemma 2.2 hold. Therefore, Lemmas 2.2 and 2.3 imply there exists $u^0 \in E$ such that $\Phi'(u^0) = 0$ and $\Phi(u^0) = c > 0$, that is, the problem $(*)$ has at least a nontrivial solution $u^0 \in E$. Note that $u_n^0 \in E \subset l^2$, thus by the definition of l^2, we have u_n^0 satisfies the boundary condition (1.3). Therefore, u^0 is a nontrivial *homoclinic* solution of the problem $(*)$. Therefore, the proof of Theorem 1.1 is finished by the Remark 2.1. $\square$

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