Journal of Applied Analysis and Computation
Volume 11, Number 1, February 2021, 254–270
Website:http://www.jaac-online.com
DOI:10.11948/20190348

ANTI-PERIODIC SOLUTION FOR FOURTH-ORDER IMPULSIVE DIFFERENTIAL EQUATION VIA SADDLE POINT THEOREM*

Suiming Shang[1],Yu Tian[2], Zhanbing Bai[1,†], and Min Zhang[2]

Abstract In this paper, the boundary value problem of fourth-order impulsive differential equation is studied. The solution space is decomposed by Riesz-Frechet theorem and eigenvalue theory. The existence of anti-periodic solution is obtained by saddle point theorem. Furthermore, the results in this paper generalize the existing results in [1] and [23].

Keywords Impulsive differential equation, Riesz-Frechet theorem, Saddle point theorem.

MSC(2010) 34B15, 35A15.

1. Introduction

The following impulsive problem

$$\begin{cases} -u'' = f(x,u), & x \in (0,1) \setminus \{x_1, \dots, x_m\} \\ u(0) = u(1) = 0,\ u(x_j^+) = u(x_j^-), & j = 1, \dots, m \\ u'(x_j^+) = u'(x_j^-) - l_j(u(x_j)), & j = 1, \dots, m \end{cases} \tag{1.1}$$

was considered by Agarwal et al. via Morse theory in [1], where $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Besides, $f(t,u) = \sum_{j=1}^{m+1} a_j(t)\chi_j(t)u + g(t,u)$, where $a_j(t) \in C(\mathbb{R},\mathbb{R})$, $j = 1, 2, \dots, m+1$ and $g(t,u)$ satisfies $|g(t,u)| \le C(|u|^{\gamma-1} + 1)$ for a.e. $t \in [0,T]$ and any $u(t) \in \mathbb{R}, \gamma \in (1,2)$, $C > 0$. $u(x_j^{\pm}) = \lim_{x \to x_j} u(x)$, $u'(x_j^{\pm}) = \lim_{x \to x_j} u'(x)$ and $l_j(u)$ are continuous functions on $\mathbb{R}$. The authors obtained a nontrivial solution for asymptotically piecewise linear problem with impulsive effects that are asymptotically linear at zero and superlinear at infinity.

In 2016, Wang and Wang [23] studied the problem (1.1) by saddle point theorem. The authors considered problem (1.1) under more relaxed assumptions on $f(t,u)$ than [1]. They generalized the non-linear term part $g(t,u)$ of $f(t,u)$ to $|g(t,u)| \le C(|u|^{\gamma-1} + e(t))$, where $e(t) \in L^1(\mathbb{R})$. The existence results generalized

†The corresponding author. Email address:zhanbingbai@163.com.(Z. Bai)
1 College of Mathematics and System Sciencet, Shandong University of Science and Technology, No. 579, Qianwan'gang Road, Qingdao 266590, China
2 School of Science, Beijing University of Posts and Telecommunications, No.10 Xitucheng Road, Beijing 100876, China
*National Natural Science Foundation of China(11571207).

some existing results in [1].

In recent years, the impulsive effect of differential equation has attracted people's great attention ([2, 13, 16–18, 27]). As is known to us, impulsive differential equations better reflect the trajectory of processes which undergo the sudden changes or discontinuous jumps. There are many such processes in nature such as timed fishing, replenishment of population ecosystems, timed dosing in pharmacy, closing of a switch in a circuit system, frequency modulation in communications, some optimal control models in economics, mechanical movements or other vibrations which suddenly suffered from external forces and so on. For the research of impulsive differential equations, one can refer to [6, 7, 9, 11, 15, 24–26].

Anti-periodic boundary value problems have frequently appeared in many scientific fields such as bioengineering, chemical engineering, physics and medicine. There exist various physical phenomena accompanied by anti-periodic behaviors, such as anti-periodic vibration, anti-periodic waves. H. Okochi used the Schauder fixed point theorem to prove the existence of anti-periodic solutions for abstract evolution equations in 1988. Later, a large number of scholars conducted more extensive and in-depth studies on the anti-periodic solution of impulsive differential equation [4, 5, 10, 12, 19–22].

However, to the best of our knowledge, there are few papers concerned with impulsive differential equation involving nonlinear part and anti-periodic boundary condition. Motivated by [1, 3, 18, 23], we study the existence of solution for the following problem

$$\begin{cases} u^{(iv)}(t) = f(t, u(t)), & t \in [0, T]\backslash\{t_1, t_2, \ldots, t_m\}, \\ -\triangle u'''(t_i) = I_{1i}(u(t_i)), & i = 1, 2, \ldots, m, \\ -\triangle u''(t_i) = I_{2i}(u'(t_i)), & i = 1, 2, \ldots, m, \\ u(0) = -u(T), \quad u'(0) = -u'(T), \quad u''(0) = -u''(T), \quad u'''(0) = -u'''(T) \end{cases} \tag{1.2}$$

where $u^{(iv)}(t)$ is the fourth derivative of the function u with respect to the variable t, $f : (0, T) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function on $(0, T) \times \mathbb{R}$, $\Delta u'''(t_i) = u'''(t_i^+) - u'''(t_i^-)$, $\Delta u''(t_i) = u''(t_i^+) - u''(t_i^-)$, $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T$ and I_{1i}, $I_{2i} \in C(\mathbb{R}, \mathbb{R})$.

The fourth-order boundary value problem with impulsive effects is used to describe the localized bending mode on the elastic level, such as the stability of pressure pole in material mechanics, bridge vibration in elastic mechanics. The nonlinear term $f(t, u)$ represents length resistance of the horizontal unit, and the solution $u(t)$ represents the degree of bending deviation or deflection, the variable t represents the space coordinate, the impulsive terms $I_{1i}(u)$, $I_{2i}(u)$ respectively represent the jump produced by the shear force of the beam and the axis of the beam when the sudden force is applied at the u point. Existence of solution means that under certain conditions, the bridge or pole will bend to a certain degree. This kind of bending is related to space coordinates.

We shall apply saddle point theorem [23] to obtain the existence of classical solution for (1.2). With the impulsive effects and boundary value conditions taken into consideration, difficulties are as follows: (1) how to define the new closed subspace; (2) how to give the linear independent basis $w_i(t)$ of finite dimensional subspace of solution space; (3) how to prove the boundness of (PS) sequence $\{u_n\}$ have to be overcome.

This paper is organized as follows. In Section 2, some definitions and lemmas which are critical to main results are given. In Section 3, the existence results of solutions are obtained. In Section 4, an example is presented to affirm Theorem 3.1. In Section 5, the proof of Lemma 2.2 is given.

2. Preliminaries

Let the space

$$X = \{u \in H^2([0,T]) : u(0) = -u(T), u'(0) = -u'(T)\},$$

with the norm

$$\|u\|_X = \left(\int_0^T |u|^2 + |u'|^2 + |u''|^2 dt \right)^{\frac{1}{2}}.$$

Remark 2.1. Let $\|u\| = (\int_0^T |u''|^2 dt)^{\frac{1}{2}}$. We claim $\|u\|_X = \|u\|$(see Lemma 2.3).

Clearly, $(X, \|\cdot\|_X)$ is a reflexive real Banach space. Define the functional $I : X \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_0^T |u''|^2 dt + \sum_{i=1}^m \int_0^{u(t_i)} I_{1i}(s)ds - \sum_{i=1}^m \int_0^{u'(t_i)} I_{2i}(s)ds - \int_0^T F(t,u)dt,$$

where $F(t,u) = \int_0^u f(t,s)ds$. Moreover, we have

$$\langle I'(u), v\rangle = \int_0^T u''v'' - f(t,u)vdt + \sum_{i=1}^m I_{1i}(u(t_i))v(t_i) - \sum_{i=1}^m I_{2i}(u'(t_i))v'(t_i), \quad for\ v \in X.$$

Besides, we define the inner product of X as $\langle u, v\rangle = \int_0^T u''v''dt$ by equivalence of norms.

In order to make it easier to read and understand this paper, we introduce some definitions and lemmas in the following.

2.1. Definitions and Lemmas

Definition 2.1. Let E be a real Banach space. We say that $I \in C^1(E, \mathbb{R})$ satisfies the (PS) condition if any sequence $(u_k) \subset E$ for which $I(u_k)$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence.

Definition 2.2. A function $u \in X$ is said to be a weak solution of problem (1.2) if u satisfies $\langle I'(u), v\rangle = 0$ for all $v \in X$.

Definition 2.3. A function $u \in C^1([0,T])$, $u''(t), u'''(t) \in C([0,T]\backslash\{t_1, t_2, \cdots, t_m\})$ and $u^{(iv)}(t) \in L^1([0,T]\backslash\{t_1, t_2, \cdots, t_m\})$ is said to be a classical solution of problem (1.2) if u satisfies the equation in problem (1.2) for $t \in [0,T]\backslash\{t_1, t_2, \cdots, t_m\}$, and the impulsive conditions and boundary conditions of problem (1.2).

Lemma 2.1 ([14]). *Let $E = V \oplus X$, where E is a real Banach space and $v \neq \{0\}$ and is finite dimensional. Suppose $I \in C^1(E, \mathbb{R})$, satisfies (PS) condition and*
(I_1) there is a constant α and a bounded neighborhood D of 0 in V such that $I|_{\partial D} \leq$

α, and
(I_2) there is a constant $\beta > \alpha$ such that $I|_X \geq \beta$.

Then I possesses a critical value $c \geq \beta$. Moreover c can be characterized as $c = \inf_{h\in\Gamma} \max_{u\in\bar{D}} I(h(u))$, where $\Gamma = \{h \in C(\bar{D}, E)|h = id \text{ on } \partial D\}$.

Lemma 2.2. *If function u is weak solution of problem (1.2), then u is classical solution of problem (1.2).*

Lemma 2.3. *The norm $\|u\|_X^2$ is equivalent to $\int_0^T |u''(t)|^2 dt$.*

Proof. By the boundary condition $u(0) + u(T) = 0$, $u'(0) + u'(T) = 0$, one has

$$u(\xi) = 0,\ u'(\eta) = 0 \text{ for some } \xi \in [0, T],\ \eta \in [0, T].$$

By Hölder's inequality, we have

$$|u(t)| = |\int_\xi^t u'(s)ds| \leq \int_0^T |u'(s)|ds \leq T^{\frac{1}{2}}(\int_0^T |u'(s)|^2 ds)^{\frac{1}{2}}.$$

So $\int_0^T |u(t)|^2 dt \leq T^2 \int_0^T |u'(t)|^2 dt$ holds.

Similarly, we obtain $\int_0^T |u'(t)|^2 dt \leq T^2 \int_0^T |u''(t)|^2 dt$. By the above inequalities, $\|u\|_X^2 \leq C \int_0^T |u''(t)|^2 dt$ holds for some positive constant C. Clearly, $\int_0^T |u''(t)|^2 dt \leq \|u\|_X^2$ is established. The proof is completed. □

Lemma 2.4. *Assume that* $w_j(t) = \begin{cases} \frac{1}{2}t^2 + (\frac{3T}{4} - \frac{t_j}{2})t + \frac{T^2}{8} - \frac{Tt_j}{2}, & t \in [0, t_j], \\ -\frac{1}{2}t^2 + (\frac{t_j}{2} + \frac{T}{4})t + \frac{T^2}{8}, & t \in [t_j, T], \end{cases}$
where $j = 1, 2, \ldots, m$, then $w_j(t)$ are linear independent.

Proof. It is necessary to prove that if $a_1w_1(t) + a_2w_2(t) + \ldots + a_mw_m(t) = 0$, where $t_1 < t_2 < \ldots < t_m$, then $a_1 = a_2 = \ldots = a_m = 0$. By calculating, we obtain

$$\begin{cases} a_1\Big[\frac{t^2}{2} + \Big(\frac{3T}{4} - \frac{t_1}{2}\Big)t + \frac{T^2}{8} - \frac{Tt_1}{2}\Big] + a_2\Big[\frac{t^2}{2} + \Big(\frac{3T}{4} - \frac{t_2}{2}\Big)t + \frac{T^2}{8} - \frac{Tt_2}{2}\Big] + \ldots \\ +a_m\Big[\frac{t^2}{2} + \Big(\frac{3T}{4} - \frac{t_m}{2}\Big)t + \frac{T^2}{8} - \frac{Tt_m}{2}\Big] = 0, \quad t \in [0, t_1], \\ a_1\Big[-\frac{t^2}{2} + \Big(\frac{t_1}{2} + \frac{T}{4}\Big)t + \frac{T^2}{8}\Big] + a_2\Big[\frac{t^2}{2} + \Big(\frac{3T}{4} - \frac{t_2}{2}\Big)t + \frac{T^2}{8} - \frac{Tt_2}{2}\Big] + \ldots \\ +a_m\Big[\frac{t^2}{2} + \Big(\frac{3T}{4} - \frac{t_m}{2}\Big)t + \frac{T^2}{8} - \frac{Tt_m}{2}\Big] = 0, \quad t \in [t_1, t_2], \\ \vdots \\ a_1\Big[-\frac{t^2}{2} + \Big(\frac{t_1}{2} + \frac{T}{4}\Big)t + \frac{T^2}{8}\Big] + a_2\Big[-\frac{t^2}{2} + \Big(\frac{t_2}{2} + \frac{T}{4}\Big)t + \frac{T^2}{8}\Big] + \ldots \\ +a_m\Big[\frac{t^2}{2} + \Big(\frac{3T}{4} - \frac{t_m}{2}\Big)t + \frac{T^2}{8} - \frac{Tt_m}{2}\Big] = 0, \quad t \in [t_{m-1}, t_m], \\ a_1\Big[-\frac{t^2}{2} + \Big(\frac{t_1}{2} + \frac{T}{4}\Big)t + \frac{T^2}{8}\Big] + a_2\Big[-\frac{t^2}{2} + \Big(\frac{t_2}{2} + \frac{T}{4}\Big)t + \frac{T^2}{8}\Big] + \ldots \\ +a_m\Big[-\frac{t^2}{2} + \Big(\frac{t_m}{2} + \frac{T}{4}\Big)t + \frac{T^2}{8}\Big] = 0, \quad t \in [t_m, T]. \end{cases} \tag{2.1}$$

Let $\alpha = (a_1, a_2, \ldots, a_m)$. By the first equation of (2.1),

$$\alpha\Big(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\Big)^{\mathrm{T}} t^2 + \alpha T_1 t + \alpha\Big(\frac{T^2}{8} - \frac{Tt_1}{2}, \frac{T^2}{8} - \frac{Tt_2}{2}, \ldots, \frac{T^2}{8} - \frac{Tt_m}{2}\Big)^{\mathrm{T}} = 0 \tag{2.2}$$

holds for any $t \in [0, t_1]$, where $T_1 = \Big(\frac{3T}{4} - \frac{t_1}{2}, \frac{3T}{4} - \frac{t_2}{2}, \dots, \frac{3T}{4} - \frac{t_m}{2}\Big)^{\mathrm{T}}$, T represents the transpose of a vector. By the second equation of (2.1), we have

$$\alpha\Big(\frac{-1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\Big)^{\mathrm{T}} t^2 + \alpha T_2 t + \alpha\Big(\frac{T^2}{8}, \frac{T^2}{8} - \frac{Tt_2}{2}, \dots, \frac{T^2}{8} - \frac{Tt_m}{2}\Big)^{\mathrm{T}} = 0, \qquad (2.3)$$

for any $t \in [t_1, t_2]$, where $T_2 = \Big(\frac{T}{4} + \frac{t_1}{2}, \frac{3T}{4} - \frac{t_2}{2}, \dots, \frac{3T}{4} - \frac{t_m}{2}\Big)^{\mathrm{T}}$. By analysis, (2.3) implies $\frac{T^2}{8} a_1 + \Big(\frac{T^2}{8} - \frac{Tt_2}{2}\Big) a_2 + \ldots + \Big(\frac{T^2}{8} - \frac{Tt_m}{2}\Big) a_m = 0$, (2.2) implies $\Big(\frac{T^2}{8} - \frac{Tt_1}{2}\Big) a_1 + \Big(\frac{T^2}{8} - \frac{Tt_2}{2}\Big) a_2 + \ldots + \Big(\frac{T^2}{8} - \frac{Tt_m}{2}\Big) a_m = 0$. So we obtain $a_1 = 0$. By the third equation of (2.1), the following equation is established,

$$\alpha\Big(\frac{-1}{2}, \frac{-1}{2}, \dots, \frac{1}{2}\Big)^{\mathrm{T}} t^2 + \alpha T_3 t + \alpha\Big(\frac{T^2}{8}, \frac{T^2}{8}, \dots, \frac{T^2}{8} - \frac{Tt_m}{2}\Big)^{\mathrm{T}} = 0, \qquad (2.4)$$

for any $t \in [t_2, t_3]$, where $T_3 = \Big(\frac{T}{4} + \frac{t_1}{2}, \frac{T}{4} + \frac{t_2}{2}, \dots, \frac{3T}{4} - \frac{t_m}{2}\Big)^{\mathrm{T}}$. Combing (2.4) with (2.3), one has $a_2 = 0$.

Similarly, we get $a_i = 0$, $i = 3, 4, \dots, m-2$. By the m-th equation of (2.1), one has

$$\alpha\Big(\frac{-1}{2}, \dots, \frac{-1}{2}, \frac{1}{2}\Big)^{\mathrm{T}} t^2 + \alpha T_4 t + \alpha\Big(\frac{T^2}{8}, \dots, \frac{T^2}{8}, \frac{T^2}{8} - \frac{Tt_m}{2}\Big)^{\mathrm{T}} = 0 \qquad (2.5)$$

for any $t \in [t_{m-1}, t_m]$, where $T_4 = \Big(\frac{T}{4} + \frac{t_1}{2}, \frac{T}{4} + \frac{t_2}{2}, \dots, \frac{T}{4} + \frac{t_{m-1}}{2}, \frac{3T}{4} - \frac{t_m}{2}\Big)^{\mathrm{T}}$. By the $m+1$-th equation of (2.1), one has

$$\alpha\Big(\frac{-1}{2}, \dots, \frac{-1}{2}\Big)^{\mathrm{T}} t^2 + \alpha\Big(\frac{T}{4} + \frac{t_1}{2}, \frac{T}{4} + \frac{t_2}{2}, \dots, \frac{T}{4} + \frac{t_2}{2}\Big)^{\mathrm{T}} t + \alpha\Big(\frac{T^2}{8}, \dots, \frac{T^2}{8}\Big)^{\mathrm{T}} = 0. \qquad (2.6)$$

By (2.5),(2.6) we obtain $a_m = 0$.

By the above analysis, $w_j(t)$ are linear independent. □

2.2. Preliminary decomposition of solution space

Let the closed linear subspace $N = \{u \in X : u(t_j) = 0, u'(t_j) = 0, j = 1, 2, \dots, m\}$. Since $I_{1i}(0) = 0$ and $I_{2i}(0) = 0$ is necessary for the existence of zero solution of problem (1.2), N is important to solution space. The mapping $u(t) \to u'(t_j)$, $j = 1, 2, \dots, m$, is a bounded linear functional on X. By the Riesz-Frechet representation theorem, there is a unique $w_j(t) \in X$ such that $u'(t_j) = \langle u, w_j \rangle$. By analyzing, we assume

$$w_j(t) = \begin{cases} At^2 + at + a_0, & t \in [0, t_j], \\ Bt^2 + bt + b_0, & t \in [t_j, T]. \end{cases}$$

By Lemma 2.3, we have

$$u'(t_j) = \langle u, w_j \rangle = \int_0^T u''(t) w_j''(t) dt$$

$$
\begin{aligned}
&= \int_0^{t_j} Au''(t)dt + \int_{t_j}^{T} Bu''(t)dt \\
&= (A-B)u'(t_j) + (A+B)u(T).
\end{aligned}
$$

A straight calculation, one has $A = \frac{1}{2}$, $B = -\frac{1}{2}$. Since $w_j(t)$ is continuous at t_j, $w_j(0) = -w_j(T)$ and $w_j'(0) = -w_j'(T)$, we obtain $a = \frac{3T}{4} - \frac{t_j}{2}$, $a_0 = \frac{T^2}{8} - \frac{Tt_j}{2}$, $b = \frac{t_j}{2} + \frac{T}{4}$, $b_0 = \frac{T^2}{8}$. So

$$
w_j = \begin{cases} \frac{1}{2}t^2 + (\frac{3T}{4} - \frac{t_j}{2})t + \frac{T^2}{8} - \frac{Tt_j}{2}, & t \in [0, t_j], \\ -\frac{1}{2}t^2 + (\frac{t_j}{2} + \frac{T}{4})t + \frac{T^2}{8}, & t \in [t_j, T]. \end{cases}
$$

By the Lemma 2.4, $w_j(t)$, $j = 1, 2, \ldots, m$ are linear independent. Considering $u'(t_j) = \langle u, w_j \rangle = 0$ where $u \in N$, we know N is the orthogonal complement of the m-dimensional subspace M which is spanned by $w_j(t)$, $j = 1, 2, \ldots, m$. So we have the orthogonal decomposition $X = N \oplus M$, i.e., $u = v + w$. By this decomposition, we obtain

$$
\begin{aligned}
\int_0^T |u''(t)|^2 dt = \int_0^T |v''(t) + w''(t)|^2 dt &= \int_0^T |v''(t)|^2 + |w''(t)|^2 + 2v''(t)w''(t))dt \\
&= \int_0^T |v''(t)|^2 + |w''(t)|^2 + 2\langle v, w \rangle dt \\
&= \int_0^T |v''(t)|^2 + |w''(t)|^2 dt.
\end{aligned}
$$

Since $v(t) \in N$, we get $v(t_j) = 0$ and $v'(t_j) = 0$. So we have $u(t_j) = w(t_j)$, $u'(t_j) = w'(t_j)$ for any $u \in X$. Moreover, $I(u)$ can be rewritten as

$$
\begin{aligned}
I(u) =& \frac{1}{2} \int_0^T |v''(t)|^2 + |w''(t)|^2 dt + \sum_{i=1}^{m} \int_0^{w(t_i)} I_{1i}(s)ds - \sum_{i=1}^{m} \int_0^{w'(t_i)} I_{2i}(s)ds \\
&- \int_0^T F(t, u)dt. \qquad (2.7)
\end{aligned}
$$

Furthermore, the linear closed subspace N has the decomposition $N = \oplus_{j=1}^{m+1} N_j$, $v = \sum\limits_{j=1}^{m+1} v_j$, where $N_j = X(t_{j-1}, t_j)$, $v_j = \chi_j v$ and

$$
\chi_j(t) = \begin{cases} 1, & t \in (t_{j-1}, t_j), \\ 0, & t \in (0, t_{j-1}] \cup [t_j, T), \end{cases}
$$

is the characteristic function of the interval (t_{j-1}, t_j). Combining the decomposition with (2.7), one has

$$
\begin{aligned}
I(u) =& \frac{1}{2} [\sum_{j=1}^{m+1} \int_{t_{j-1}}^{t_j} |v_j''(t)|^2 + |w''(t)|^2 dt] + \sum_{i=1}^{m} \int_0^{w(t_i)} I_{1i}(s)ds - \sum_{i=1}^{m} \int_0^{w'(t_i)} I_{2i}(s)ds \\
&- \int_0^T F(t, u)dt. \qquad (2.8)
\end{aligned}
$$

3. Main Results

From a mathematical point of view, the case of non-resonance is more conducive to the application of space decomposition techniques and the reduction of energy functional. The proof of main results only be done in non-resonant case. In a practical sense, non-resonant functional differential systems are a common type of systems in nature. They have a wide range of practical applications in the fields of physics, biology, medicine, cybernetics and so on. Combined with the bridge vibration equation we have studied, non-resonance can improve the reliability of the bridge and the safety state pursued in bridge design. Resonance is an important cause of bridge damage. During the erection process, bridges will actively avoid the sway of resonance under the action of high winds, or the impact of dynamic loads such as pedestrian traffic. So we only consider the nonresonant case, where j, a_j is not in the set

$$\sum_j = \{0 < \lambda_1^j \le \lambda_2^j \le \ldots \le \lambda_k^j \to +\infty, k = 1, 2, \ldots\}, j = 1, 2, \ldots, m, m+1,$$

which consists of the eigenvalues of the following problem

$$\begin{cases} u^{(iv)}(t) = \lambda u(t), & t \in [t_{j-1}, t_j], \\ u(t_{j-1}) = 0, & u(t_j) = 0, \\ u'(t_{j-1}) = 0, & u'(t_j) = 0. \end{cases} \tag{3.1}$$

Remark 3.1. By the reference [8], we know the problem (3.1) exists eigenvalues.

To better illustrate main results Theorem 3.1, we give the following assumptions:
(A_1) $f(t,u) = \sum_{j=1}^{m+1} a_j(t)\chi_j(t)u + g(t,u)$, where $a_j(t) \in C(\mathbb{R}, \mathbb{R})$, $j = 1, 2, \ldots, m+1$. Besides $a_j(t)$ is indefinite and $g(t,u)$ satisfies

$$|g(t,u)| \le C(|u|^{\gamma-1} + e(t)) \text{ for } a.e.\ t \in [0,T] \text{ and } \forall u(t) \in \mathbb{R}, \gamma \in (1,2), C > 0, e \in L^2(\mathbb{R}); \tag{3.2}$$

(A_2) The impulsive functions I_{1i} satisfy sublinear growth, that is there exist constants $\alpha_i \ge 0$, $\beta_i > 0$, $\gamma_i > 0$, $i = 1, 2, \ldots, m$ such that $uI_{1i}(u) \ge -\alpha_i + \beta_i|u|^{\gamma_i}$;
(A_3) There exist constants $C' > 0$, $\mu_i > 0$ and $D > 0$, $i = 1, 2, \ldots, m$ such that $uI_{2i}(u) \le C'|u|^{\mu_i} + D$, where $\min\{\gamma_i, \mu_i\} > 2$, $\min\{\beta_i - C'\} > 0$;
(A_4) There exist constants $l_{1i} > 0$, $l_{2i} > 0$, $p_{1i} > 0$, $p_{2i} > 0$, $n_i > 0$, $o_i > 0$ such that $uI_{1i}(u) \le l_{1i} + l_{2i}|u|^{n_i}$ and $uI_{2i}(u) \ge p_{1i} + p_{2i}|u|^{o_i}$, where $\min\{n_i, o_i\} = O > 2$, $\min\{l_{2i} - p_{2i}\} = -L_2 < 0$, $\max \sum_{i=1}^{m} \{l_{1i} - p_{1i}\} = L_1 > 0$.

Theorem 3.1. *Assume that (A_1), (A_2), (A_3), (A_4) hold and $a_j(t) \notin \sum_j$ for $j = 1, 2, \ldots, m, m+1$. Then problem (1.2) has at least one solution.*

Proof. We will apply Lemma 2.1 to complete this section by three steps.

Step 1. The proof that $I(u)$ satisfies (PS) condition is given.

We prove that a sequence $\{u_n\} \in H$ is a (PS) sequence, i.e., considering $I(u_n)$ is bounded, $I'(u_n) \to 0$ as $n \to \infty$, then $\{u_n\}$ has a convergent subsequence.

(1). we prove $\{u_n\}$ is bounded.

Firstly, we rewrite the energy functional with known conditions.

By (A_1), one has

$$F(t,u)=\frac{1}{2}\sum_{j=1}^{m+1}a_{j(t)}\chi_j(t)u^2+G(t,u),$$

where $G(t,u)=\int_0^u g(t,s)ds$. By (3.2), one has

$$|G(t,u)|=|\int_0^u g(t,s)ds|\leq \tilde{C}|u|^\gamma+\int_0^u |e(s)|ds \text{ for } a.e.\ t\in(0,T),\ \forall\ u\in\mathbb{R}, \quad (3.3)$$

where $\tilde{C}=\frac{C}{\gamma}$. By (A_1), (2.8) can be rewritten as

$$\begin{aligned} I(u)=&\frac{1}{2}\Big\{\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}(|v_j''(t)|^2-a_j(t)|v_j(t)|^2)dt+\int_0^T w''(t)dt \\ &-\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}a_j(t)|w(t)|^2dt\Big\} \\ &-\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}a_j(t)v_j(t)w(t)dt-\int_0^T G(t,u)dt+\sum_{i=1}^{m+1}\int_0^{w(t_i)}I_{1i}(s)ds \\ &-\sum_{i=1}^{m+1}\int_0^{w'(t_i)}I_{2i}(s)ds. \end{aligned} \quad (3.4)$$

By (A_2), (A_3), we have

$$-\sum_{i=1}^{m+1}\int_0^{w(t_i)}I_{1i}(s)ds\leq\sum_{i=1}^{m+1}(\alpha_i-\beta_i\|w(t_i)\|_\infty^{\gamma_i})\|w(t_i)\|_\infty, \quad (3.5)$$

$$\sum_{i=1}^{m+1}\int_0^{w'(t_i)}I_{2i}(s)ds\leq\sum_{i=1}^{m+1}(C'\|w'(t_i)\|_\infty^{\mu_i}+D)\|w'(t_i)\|_\infty, \quad (3.6)$$

where $\|u\|_\infty=\max_{t\in[0,T]}|u(t)|$. From the proof of Lemma 2.3, we obtain $\|u\|_\infty\leq\sqrt{T}\|u\|_X$ and $\|u'\|_\infty\leq\sqrt{T}\|u\|_X$.

Secondly, we perform space decomposition to further simplify and shrink the energy functional to obtain the boundedness of the sequence $\{u_n\}$.

Defining J_0 as the set of j which such that $a_j(t)<\lambda_1^j$ and $J_1=\{1,2,\dots,m,m+1\}\backslash J_0$. For any $j\in J_1$, $\lambda_h^j<a_j(t)<\lambda_{h+1}^j$ holds for some $h\geq 1$. Moreover, the decomposition $N_j^+\bigoplus N_j^-$ holds, where N_j^- is the h-dimensional subspace which is expanded by the eigenfunctions of $\lambda_1^j,\lambda_2^j,\dots,\lambda_h^j$ and N_j^+ is it's orthogonal complement. So one has $v_j=v_j^+\bigoplus v_j^-$. Then (3.4) is changed into

$$\begin{aligned} I(u)=&\frac{1}{2}\Big\{\sum_{j\in J_0}\int_{t_{j-1}}^{t_j}(|v_j''(t)|^2-a_j(t)|v_j(t)|^2)dt+\sum_{j\in J_1}\int_{t_{j-1}}^{t_j}(|v_j^{+''}(t)|^2-a_j(t)|v_j^+(t)|^2)dt \\ &+\sum_{j\in J_1}\int_{t_{j-1}}^{t_j}(|v_j^{-''}(t)|^2-a_j(t)|v_j^-(t)|^2)dt \end{aligned}$$

$$+\int_0^T w''(t)dt-\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j} a_j(t)|w(t)|^2dt\Big\}$$
$$-\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j} a_j(t)v_j(t)w(t)dt-\int_0^T G(t,u)dt+\sum_{i=1}^{m+1}\int_0^{w(t_i)} I_{1i}(s)ds$$
$$-\sum_{i=1}^{m+1}\int_0^{w'(t_i)} I_{2i}(s)ds, \tag{3.7}$$

for

$$u=\sum_{j\in J_0} v_j+\sum_{j\in J_1}(v_j^+ + v_j^-)+w\in\bigoplus_{j\in J_0}N_j\oplus\bigoplus_{j\in J_1}(N_j^+\oplus N_j^-)\oplus M. \tag{3.8}$$

By the eigenvalues problem (3.1), we have $\int_{t_{j-1}}^{t_j} u^{(iv)}u-\lambda u^2dt=0$. By partial integration, we obtain

$$\int_{t_{j-1}}^{t_j}|u''|^2dt+u'''(t_j)u(t_j)-u'''(t_{j-1})u(t_{j-1})-(u''(t_j)u'(t_j)-u''(t_{j-1})u'(t_{j-1}))$$
$$-\lambda\int_{t_{j-1}}^{t_j} u^2dt=0.$$

Taking boundary conditions into consideration, $\int_{t_{j-1}}^{t_j}|u''|^2dt=\lambda\int_{t_{j-1}}^{t_j}u^2dt$ is satisfied. So we obtain

$$\begin{aligned}&\int_{t_{j-1}}^{t_j}|v_j''(t)|^2dt\ge\lambda_1^j\int_{t_{j-1}}^{t_j}|v_j(t)|^2dt,\ j\in J_0,\\&\int_{t_{j-1}}^{t_j}|v_j^{+''}(t)|^2dt\ge\lambda_{h+1}^j\int_{t_{j-1}}^{t_j}|v_j^+(t)|^2dt,\ j\in J_1,\\&\int_{t_{j-1}}^{t_j}|v_j^{-''}(t)|^2dt\le\lambda_h^j\int_{t_{j-1}}^{t_j}|v_j^-(t)|^2dt,\ j\in J_1.\end{aligned} \tag{3.9}$$

According to (3.9), we can get the following inequality (3.10), which is important for the amplification of the energy functional I.

$$\int_{t_{j-1}}^{t_j}|v_j''(t)|^2-a_j(t)|v_j(t)|^2dt\ge\int_{t_{j-1}}^{t_j}|v_j''(t)|^2-a_j(t)\frac{1}{\lambda_1^j}(|v_j''(t)|^2dt$$
$$=(1-\frac{a_j(t)}{\lambda_1^j})\int_{t_{j-1}}^{t_j}|v_j''(t)|^2dt\ge b_j\|v_j\|_X^2dt,\ j\in J_0,$$
$$\int_{t_{j-1}}^{t_j}|v_j^{+''}(t)|^2-a_j(t)|v_j^+(t)|^2dt\ge\int_{t_{j-1}}^{t_j}|v_j^{+''}(t)|^2-a_j(t)\frac{1}{\lambda_{h+1}^j}|v_j^{+''}(t)|^2dt$$
$$=(1-\frac{a_j(t)}{\lambda_{h+1}^j})\int_{t_{j-1}}^{t_j}|v_j^{+''}(t)|^2dt\ge b_j^+\|v_j^+\|_X^2,\ j\in J_1,$$
$$\int_{t_{j-1}}^{t_j}|v_j^{-''}(t)|^2-a_j(t)|v_j^-(t)|^2dt\le\int_{t_{j-1}}^{t_j}|v_j^{-''}(t)|^2-a_j(t)\frac{1}{\lambda_h^j}|v_j^{-''}(t)|^2dt,$$
$$=(1-\frac{a_j(t)}{\lambda_h^j})\int_{t_{j-1}}^{t_j}|v_j^{-''}(t)|^2dt\le b_j^-\|v_j^-\|_X^2,\ j\in J_1, \tag{3.10}$$

where $b_j = 1 - \frac{\max\{a_j(t),0\}}{\lambda_1^j}$, $j \in J_0$, $b_j^+ = 1 - \frac{a_j(t)}{\lambda_{h+1}^j}$, $b_j^- = \frac{a_j(t)}{\lambda_h^j} - 1$, $j \in J_1$, are all positive. According to the decomposition (3.8), we choose

$$u_n = \sum_{j\in J_0} v_{nj} + \sum_{j\in J_1}(v_{nj}^+(t)+v_{nj}^-(t))+w_n,\ \bar{u}_n = \sum_{j\in J_0} v_{nj} + \sum_{j\in J_1}(v_{nj}^+(t)-v_{nj}^-(t))-w_n.$$

By calculation, we obtain

$$u_n''\bar{u}_n'' = \sum_{j\in J_0}|v_{nj}''|^2 + \sum_{j\in J_1}(|v_{nj}^{+''}|^2 - |v_{nj}^{-''}|^2) - |w_n''|^2,$$

and $f(t,u_n) = \sum_{j=1}^{m+1} a_j(t)\chi_j(t)u_n + g(t,u_n)$.

Next, we prepare for the boundedness of the sequence $\{u_n\}$, by rewriting and scaling $\langle I'(u_n,\bar{u}_n)\rangle$. A straightforward computation shows that

$$\begin{aligned}\langle I'(u_n,\bar{u}_n)\rangle =& \int_0^T u_n''\bar{u}_n'' - f(t,u_n)\bar{u}_n dt + \sum_{i=1}^m I_{1i}(w_n(t_i))w_n(t_i) - \sum_{i=1}^m I_{2i}(w_n'(t_i))w_n'(t_i)\\ =& \sum_{j\in J_0}\int_{t_{j-1}}^{t_j}(|v_{nj}''|^2 - a_j(t)|v_{nj}|^2)dt + \sum_{j\in J_1}\int_{t_{j-1}}^{t_j}[|v_{nj}^{+''}|^2 - a_j(t)|v_{nj}^+|^2\\ &- (|v_{nj}^{-''}|^2 - a_j(t)|v_{nj}^-|^2)]dt - \int_0^T |w_n''|^2 dt + \sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j} a_j|w_n|^2 dt\\ &+ 2\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j} a_j v_{nj}^- w_n dt - \int_0^T g(t,u_n)\bar{u}_n dt + \sum_{i=1}^m I_{1i}(w_n(t_i))w_n(t_i)\\ &- \sum_{i=1}^m I_{2i}(w_n'(t_i))w_n'(t_i). \end{aligned} \tag{3.11}$$

By $I'(u_n) \to 0$ as $n \to \infty$, one has

$$\begin{aligned} &\sum_{j\in J_0}\int_{t_{j-1}}^{t_j}(|v_{nj}''|^2 - a_j(t)|v_{nj}|^2)dt\\ &+ \sum_{j\in J_1}\int_{t_{j-1}}^{t_j}[|v_{nj}^{+''}|^2 - a_j(t)|v_{nj}^+|^2 - (|v_{nj}^{-''}|^2 - a_j(t)|v_{nj}^-|^2)]dt\\ \le& -\sum_{i=1}^m I_{1i}(w_n(t_i))w_n(t_i) + \sum_{i=1}^m I_{2i}(w_n'(t_i))w_n'(t_i) + \int_0^T |w_n''|^2 dt\\ &- \sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j} a_j|w_n|^2 dt - 2\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j} a_j v_{nj}^- w_n dt + \int_0^T g(t,u_n)\bar{u}_n dt. \end{aligned} \tag{3.12}$$

By (3.10), (A_2), (A_3), (3.3) and Hölder's inequality, (3.12) is changed into

$$\begin{aligned} &\sum_{j\in J_0} b_j\|v_{nj}\|_X^2 + \sum_{j\in J_1}(b_j^+\|v_{nj}^+\|_X^2 + b_j^-\|v_{nj}^-\|_X^2)\\ \le& \sum_{i=1}^m(\alpha_i - \beta_i|w_n|^{\gamma_i})|w_n| + \sum_{i=1}^m(C'|w_n'|^{\mu_i} + D)|w_n'| \end{aligned}$$

$$+\|w_n\|_X^2+2\sum_{j\in J_1}|a_j(t)|\|v_{nj}^-\|_X\|w_n\|_X$$
$$+C\|u_n\|_X^{\gamma-1}\|\bar{u}_n\|_X+\|e\|_{L^2}\|\bar{u}_n\|_X.$$

By simple transposition, we obtain

$$\begin{aligned}&\sum_{j\in J_0}b_j\|v_{nj}\|_X^2+\sum_{j\in J_1}(b_j^+\|v_{nj}^+\|_X^2+b_j^-\|v_{nj}^-\|_X^2)+\sum_{i=1}^m(\beta_i|w_n|^{\gamma_i+1}-C'|w_n|^{\mu_i+1})\\ \le&\|w_n\|_X^2+2\sum_{j\in J_1}|a_j(t)|\|v_{nj}^-\|_X\|w_n\|_X+C\|u_n\|_X^{\gamma-1}\|\bar{u}_n\|_X+\|e\|_{L^2}\|\bar{u}_n\|_X\\ &+\sum_{i=1}^m(\alpha_i+D)\|w_n\|_\infty.\end{aligned}\tag{3.13}$$

Let $E=\min\{\beta_i,C'\}$, $\bar{\mu}=\min\{\gamma_i+1,\mu_i+1\}$, $F=\max\{2|a_j(t)|,C,\|e\|_{L^2},1,\sum_{i=1}^m(\alpha_i+D)\}$, then

$$\begin{aligned}&\sum_{j\in J_0}b_j\|v_{nj}\|_X^2+\sum_{j\in J_1}(b_j^+\|v_{nj}^+\|_X^2+b_j^-\|v_{nj}^-\|_X^2)+E\|w_n\|_X^{\bar{\mu}}\\ \le&F(\|w_n\|_X^2+\sum_{j\in J_1}\|v_{nj}^-\|_X\|w_n\|_X+\|u_n\|_X^{\gamma-1}\|\bar{u}_n\|_X+\|\bar{u}_n\|_X+\|w_n\|_X).\end{aligned}\tag{3.14}$$

Since $\bar{\mu}>2$, $\|\bar{u}_n\|_X=\|u_n\|_X$, $\gamma<2$, we claim the boundness of sequence $\{u_n\}$, i.e. $\|u_n\|_X^2=\sum_{j\in J_0}\|v_{nj}\|_X^2+\sum_{j\in J_1}(\|v_{nj}^+\|_X^2+\|v_{nj}^-\|_X^2)+\|w_n\|_X^2$ is bounded. In fact, let $a=\sum_{j\in J_0}\|v_{nj}\|_X^+\sum_{j\in J_1}(\|v_{nj}^+\|_X^+\|v_{nj}^-\|_X)$, $b=\|w_n\|_X$. So (3.14) can be written as

$$a^2+b^{\bar{\mu}}\le b^2+ab+a^\gamma.\tag{3.15}$$

Clearly, $\|u_n\|_X^2$ is bounded, when a or b is bounded. The following we discussed into two cases. (i)$\frac{a}{b}=c\to\infty$, (3.15) is equality to

$$b^{2-\gamma}(1-\frac{1}{c^2}-\frac{1}{c})+\frac{b^{\bar{\mu}-\gamma}}{c^2}\le c^{\gamma-2}.\tag{3.16}$$

Based on $a\to\infty$, $b\to\infty$, $\bar{\mu}>2$ and $\gamma<2$, (3.16) is contradictory.
(ii)$\frac{b}{a}=c\to\infty$, (3.15) is equality to

$$a^{\bar{\mu}}+a^2c^{-\bar{\mu}}\le a^2c^{2-\bar{\mu}}+a^2c^{1-\bar{\mu}}+a^\gamma c^{-\bar{\mu}}.\tag{3.17}$$

Based on $a\to\infty$, $b\to\infty$, $\bar{\mu}>2$ and $\gamma<2$, (3.17) is contradictory.

So the sequence $\{u_n\}$ is bounded.

(2). the proof that $\{u_n\}$ has convergent subsequence is shown.

For any bounded sequence $\{u_n\}\in X$, it has subsequence $\{u_{n_j}\}$, for the sake of convenience, we still write u_{n_j} as u_n, $u_n\rightharpoonup u$. By calculation, one has

$$\langle I'(u_n)-I'(u),u_n-u\rangle=\int_0^T|(u_n-u)''|^2dt-\int_0^T(f(t,u_n)-f(t,u))(u_n-u)dt$$

$$+\sum_{i=1}^{m}(I_{1i}(u_n)-I_{1i}(u))(u_n-u)$$
$$-\sum_{i=1}^{m}(I_{2i}(u_n')-I_{2i}(u'))(u_n'-u'). \tag{3.18}$$

Since $X\hookrightarrow C[0,T]$, $\{u_n\}$ uniformly converges to u in $C[0,T]$. $\{u_n\}\in X$ implies u_n' converges to u' in $C[0,T]$. So we obtain

$$\langle I'(u_n)-I'(u),u_n-u\rangle\to 0,\quad \int_0^T(f(t,u_n)-f(t,u))(u_n-u)dt\to 0,$$
$$\sum_{i=1}^{m}(I_{1i}(u_n)-I_{1i}(u))(u_n-u)\to 0,\quad \sum_{i=1}^{m}(I_{2i}(u_n')-I_{2i}(u'))(u_n'-u')\to 0.$$

Combining with (3.18), one has $\int_0^T|(u_n-u)''|^2dt=\|u_n-u\|_X\to 0$. So $\{u_n\}$ has a convergent subsequence. I satisfies (PS) condition.

Step 2. The proof that $I(u)$ satisfies the (I_1) condition of Lemma 2.1 is given. We decompose space into

$$X=(\bigoplus_{j\in J_1}N_j^-\oplus M)\oplus(\bigoplus_{j\in J_0}N_j\oplus\bigoplus_{j\in J_1}N_j^+)=X_1\oplus X_2. \tag{3.19}$$

For $u=\sum_{j\in J_1}v_j^-+w\in X_1$, by (3.4) and $\langle v_j^-,w\rangle=0$, we obtain

$$I(u)=\frac{1}{2}\Big\{\sum_{j\in J_1}\int_{t_{j-1}}^{t_j}(|v_j^{-''}(t)|^2-a_j(t)|v_j^-(t)|^2)dt+\int_0^T|w''(t)|^2dt$$
$$-\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}a_j(t)|w(t)|^2dt\Big\}-\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}a_j(t)v_j^-(t)w(t)dt$$
$$-\int_0^T G(t,u)dt+\sum_{i=1}^{m+1}\int_0^{w(t_i)}I_{1i}(s)ds-\sum_{i=1}^{m+1}\int_0^{w'(t_i)}I_{2i}(s)ds.$$

Taking (3.10), (A_2), (A_4) and (3.3), one has

$$I(u)\le-\frac{1}{2}\sum_{j\in J_1}b_j^-\|v_j^-\|_X^2+\|w\|_X^2+\max_{j\in J_1}\{a_j(t)\}\sum_{j\in J_1}\|v_j^-\|_X\|w\|_X+C\|u\|_X^\gamma+\|e\|_{L^2}\|u\|_X$$
$$+\sum_{i=1}^{m}(l_{1i}+l_{2i}|w(t_i)|^{n_i}-p_{1i}-p_{2i}|w'(t_i)|^{o_i})$$
$$\le-\frac{1}{2}\sum_{j\in J_1}b_j^-\|v_j^-\|_X^2-L_2\sum_{i=1}^{m}|w(t_i)|^O+\max\{a_j(t),C,L_1,1\}(\|w\|_X^2$$
$$+\sum_{j\in J_1}\|v_j^-\|_X\|w\|_X+\|u\|_X^\gamma+\|e\|_{L^2}\|u\|_X+1).$$

Since $O>2$, $\gamma>2$, obviously, one has $I(u)\to-\infty$ as $\|u\|_X=\sum_{j\in J_1}\|v_j^-\|_X^2+\|w\|_X^2\to\infty$.

Step 3. The proof that $I(u)$ satisfies the (I_2) condition of Lemma 2.1 is shown. For $u=\sum\limits_{j\in J_0} v_j+\sum\limits_{j\in J_1} v_j^+\in X_2$, by (3.4), $\langle v_j^+,v_j\rangle=0$ and $u(t_i)=u'(t_i)=0$,

$$\begin{aligned}I(u)=&\frac{1}{2}\{\sum_{j\in J_0}\int_{t_{j-1}}^{t_j}(|v_j^{''}(t)|^2-a_j(t)|v_j(t)|^2)dt\\&+\sum_{j\in J_1}\int_{t_{j-1}}^{t_j}(|v_j^{+''}(t)|^2-a_j(t)|v_j^+(t)|^2)dt\}-\int_0^T G(t,u)dt,\end{aligned}$$

holds. Taking (3.10), (3.3) into consideration,

$$I(u)\geq\frac{1}{2}\{\sum_{j\in J_0} b_j\|v_j\|_X^2+\sum_{j\in J_1} b_j^+\|v_j^+\|_X^2\}-\tilde{C}\|u\|_X^{\gamma}-\|e\|_{L^2}\|u\|_X,$$

is established.
Since $\gamma>2$, it is clear I is bounded from below on X_2.

Applying Lemma 2.1, problem (1.2) has at least one solution. □

4. Example

Example 4.1. Set $f(t,u)=\sum\limits_{j=1}^{m} a_j(t)\chi_j(t)u+\frac{u^3-u+1}{u^{2.5}+2}$. By $f(t,u)=\sum\limits_{j=1}^{m+1} a_j(t)\chi_j(t)u+g(t,u)$, we get $g(t,u)=\frac{u^3-u+1}{u^{2.5}+2}$. A straightforward Calculation shows $|g(t,u)|\leq 2|u|^{0.5}+e^{-(t+1)}+1$. Hence (A_1) is satisfied.

Let $I_{1i}(u)=\begin{cases}-3u^3, & |u|<k,\\ 2.5u^6, & |u|\geq k,\end{cases}$ and $I_{2i}(u)=\begin{cases}0, & |u|<k,\\ ku^{4.8}, & |u|\geq k,\end{cases}$
where $k>4$.

Case 1. $|u|<k$:
According to the expression of I_{1i}, I_{2i}, we have $-\alpha_i+2|u|^4\leq uI_{1i}(u)\leq -3|u|^4+1$, $-|u|^4\leq uI_{2i}(u)\leq |u|^4$. In the assumption $uI_{1i}(u)\geq -\alpha_i+\beta_i|u|^{\gamma_i}$, by the corresponding coefficients of the same power, we get $\alpha_i\geq 5|u|^4$, $\beta_i=2$, $\gamma_i=4$. So the conditions (A_2) is satified. In the assumption, $uI_{2i}(u)\leq C'|u|^{\mu_i}+D$, by the corresponding coefficients of the same power, we get $C'=1$, $\mu_i=4$, $D=0$, $\min\{\gamma_i,\mu_i\}=4>2$, $\min\{\beta_i-C'\}=1>0$. So the conditions (A_3) is satified. In the assumption, $uI_{1i}(u)\leq l_{1i}+l_{2i}|u|^{n_i}$ and $uI_{2i}(u)\geq p_{1i}+p_{2i}|u|^{o_i}$, by the corresponding coefficients of the same power, we get $l_{1i}=1$, $l_{2i}=-3$, $n_i=4$, $p_{1i}=0$, $o_i=4$, $p_{2i}=-1$. Besides, one has $\min\{n_i,o_i\}=4>2$, $\min\{\ l_{2i}-p_{2i}\}=-2<0$, $\max\{\ l_{1i}-p_{1i}\}=1>0$. So the conditions (A_4) is satified.

Case 2. $|u|>k$:
According to the expression of I_{1i}, I_{2i}, we have $2|u|^5\leq uI_{1i}(u)\leq 3|u|^8+\varepsilon_1$, $4|u|^{4.5}\leq uI_{2i}(u)\leq |u|^7$. In the assumption $uI_{1i}(u)\geq -\alpha_i+\beta_i|u|^{\gamma_i}$, by the corresponding coefficients of the same power, we get $\alpha_i=0$, $\beta_i=2$, $\gamma_i=5$. So the conditions (A_2) is satified. In the assumption, $uI_{2i}(u)\leq C'|u|^{\mu_i}+D$, by the corresponding coefficients of the same power, we get $C'=1$, $\mu_i=7$, $D=0$, $\min\{\gamma_i,\mu_i\}=5>2$, $\min\{\beta_i-C'\}=1>0$. So the conditions (A_3) is satified. In the assumption, $uI_{1i}(u)\leq l_{1i}+l_{2i}|u|^{n_i}$ and $uI_{2i}(u)\geq p_{1i}+p_{2i}|u|^{o_i}$, by the corresponding coefficients of the same power, we get $l_{1i}=\varepsilon_1$, $0<\varepsilon_1<0.5|u|$, $l_{2i}=3$, $n_i=8$, $p_{1i}=0$,

$p_{2i} = 4$, $o_i = 4.5$. Besides, one has $\min\{n_i, o_i\} = 4.5 > 2$, $\min\{\ l_{2i} - p_{2i}\} = -1 < 0$, $\max\{\ l_{1i} - p_{1i}\} = \varepsilon_1 > 0$. So the conditions (A_4) is satified.

By theorem 3.1, we obtain the problem (1.2) has at least one solution.

5. Conclusion

In this paper, the solution of the fourth-order impulsive differential equations with anti-periodic boundary value conditions is obtained by variational method and saddle point theorem. In terms of mathematics, the concrete form and calculation method of the basis of the finite-dimensional linear subspace of the solution space of the fourth-order differential equation by Riesz-Frechet theorem. In terms of engineering, the studied model can provide guidance for design of curved bridge and stability of the pressure pole.

Acknowledgements

The authors express their sincere thanks to the anonymous reviews for their valuable suggestions and corrections for improving the quality of this paper.

Appendix

The proof of Lemma 2.2

The proof is similar to the reference [18]. From the definition 2.1, we obtain

$$\int_0^T u''v''dt + \sum_{i=1}^{l} I_{1i}(u(t_i))v(t_i) - \sum_{i=1}^{l} I_{2i}(u'(t_i))v'(t_i) - \int_0^T f(t,u)vdt = 0, \quad (5.1)$$

for any $v \in X$. Specially, we take $v \in W_0^{2,2}[t_i, t_{i+1}]$, then $v(t_i) = v(t_{i+1}) = v'(t_i) = v'(t_{i+1}) = 0$. So (5.1) can be rewritten as

$$\int_{t_{i+1}}^{t_i} u''v''dt - \int_{t_{i+1}}^{t_i} f(t,u)vdt = 0, \text{ for any } v \in W_0^{2,2}[t_i, t_{i+1}]. \quad (5.2)$$

We firstly show that

$$\begin{aligned}\int_{t_i}^{t_{i+1}} f(t,u)vdt &= \int_{t_i}^{t_{i+1}} v(t)d[\int_{t_i}^{t} f(s,u(s))ds] \\ &= [\int_{t_i}^{t} f(s,u(s))ds]v(t)|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} v'[\int_{t_i}^{t} f(s,u(s))ds]dt \\ &= -\int_{t_i}^{t_{i+1}} g_{11}(t,u)v'(t)dt = -\int_{t_i}^{t_{i+1}} v'd[\int_{t_i}^{t} g_{11}(s,u)ds] \\ &= \int_{t_i}^{t_{i+1}} g_{12}(t,u)v''(t)dt, \end{aligned} \quad (5.3)$$

where $g_{11}(t,u) = \int_{t_i}^{t} f(s,u(s))ds$, $g_{12}(t,u) = \int_{t_i}^{t} g_{11}(s,u)ds$. By (5.1),(5.2),(5.3), we have

$$\int_{t_i}^{t_{i+1}} [u'' - g_{12}(t,u)]v''(t)dt = 0, \text{ for all } v''(t) \in C_0^{\infty}[t_i, t_{i+1}].$$

By the fundamental lemma, we have

$$u'' - g_{12}(t,u) = ct + b, \tag{5.4}$$

where c, b are arbitrary constants. Taking derivative for two times on (5.4), we have

$$u^{iv}(t) - f(t,u) = 0, \ for \ t \in (t_i, t_{i+1}).$$

So

$$u^{iv}(t) = f(t,u), \tag{5.5}$$

is satisfied for $t \in [0,T] \backslash \{t_1, t_2, \ldots, t_l\}$. Since $u'' = -\int_{t_i}^t g_{11}(s,u)ds + \int_0^t cdt + \int_0^t u'(s)ds + u(0)$, we obtain u'' is absolutely continuous. Clearly, v' is absolutely continuous by $v \in X$. By integrating by parts, we have

$$\int_0^T u''v''dt = \sum_{i=1}^m -\Delta u''(t_i)v'(t_i) + u''(T)v'(T) - u''(0)v'(0) - \int_0^T u'''v'dt.$$

Since $u'''(t) = -\int_{t_i}^t f(u)dt + \int_0^t u''(s)ds - u'(0) - u'''(0)$, we obtain u''' is absolutely continuous. It is obviously v is absolutely continuous by $v \in X$. By integrating by parts, we obtain

$$\int_0^T u'''v'dt = \sum_{i=1}^m -\Delta u'''(t_i)v(t_i) + u'''(T)v(T) - u'''(0)v(0) - \int_0^T u^{iv}vdt.$$

Combining with the above analysis, we have

$$\begin{aligned}\langle I'(u), v\rangle = &\int_0^T (u^{iv} - f(u))vdt + \sum_{i=1}^m (I_{1i} + \Delta u'''(t_i))v(t_i) + \sum_{i=1}^m (I_{2i} - \Delta u''(t_i))v'(t_i) \\ &+ (u''(T) + u''(0))v'(T) + (u'''(0) + u'''(T))v(t) = 0. \end{aligned} \tag{5.6}$$

By (5.5), we obtain (5.6) is equal to

$$\begin{aligned}&\sum_{i=1}^m (I_{1i} + \Delta u'''(t_i))v(t_i) + \sum_{i=1}^m (I_{2i} - \Delta u''(t_i))v'(t_i) + (u''(T) + u''(0))v'(T) \\ &+ (u'''(0) + u'''(T))v(T) = 0. \end{aligned} \tag{5.7}$$

Firstly, we show $I_{1i} + \Delta u'''(t_i) = 0$ for $i = 1, 2, \ldots, m$. Assume that there exists j satisfying

$$I_{1i} + \Delta u'''(t_i) \neq 0. \tag{5.8}$$

Let $v(t) = \prod_{i=0, i\neq j}^{m+1} (t - t_i)^3(3t + t_i - 4t_j)$. Apparently $v(t_i) = 0, i = 0, 1, 2, \ldots, j - 1, j + 1, \ldots$. By calculating we have

$$v'(t_i) = \sum_{k=0, i=0}^{m+1} 12(t - t_k)^2(t - t_j) \prod_{i=0, i\neq j,k} (t - t_i)^3(3t + t_i) - 4t_j$$

and $v'(t_i) = 0, i = 1, 2, \ldots, m, m+1$. Plugging $v(t)$ back into (5.7), we have $I_{1j} + \Delta u'''(t_j) = 0$. This contradicts with (5.8). The first impulsive condition is established. (5.7) is changed into

$$\sum_{i=1}^{m}(I_{2i} - \Delta u''(t_i))v'(t_i) + (u''(T) + u''(0))v'(T) + (u'''(0) + u'''(T))v(T) = 0. \quad (5.9)$$

We take $v_2 \in X$, $v_2'(t) = \prod\limits_{i=0, i\neq j}(t - t_i)$. Similar discuss methods, we obtain $I_{2i} - \Delta u''(t_i) = 0$ for $i = 1, 2, \ldots, m$. Furthermore, (5.9) is changed into $(u''(T) + u''(0))v'(T) + (u'''(0) + u'''(T))v(T) = 0$. So we have $u''(T) = -u''(0)$ and $u'''(0) = -u'''(T)$.

References

[1] R. Agarwal, T. G. Bhaskar and K. Perera, *Some results for impulsive problem via Morse theory*, J. Math. Anal. Appl., 2014, 409, 752–759.

[2] Z. Bai, X. Dong and C. Yin, *Existence results for impulsive nonlinear fractional differential equation with mixed boundary conditions*, Bound. Value. Probl., 2016, 63, 1–11.

[3] Z. Bai, Z. Du and S. Zhang, *Iterative method for a class of fourth-order p-Laplacian beam equation*, J. Appl. Anal. Comput., 2019, 9(4), 1443–1453.

[4] Q. Fan, W. Wang and X. Yi, *Anti-periodie solutions for a class of nonlinear nth-order differential equations with delays*, J. Comput. Appl. Math., 2009, 230(2), 762–769.

[5] D. Faneo and J. J. Nieto, *First-order impulsive ordinary difefrential equations with anti-periodic and nonlinear boundary conditions*, Nonlinear. Anal., 2000, 42(2), 163–173.

[6] D. Grigoriev, *Tropical differential equations*, Adv. Appl. Math., 2017, 82, 120–128.

[7] D. Gao and J. Li, *Applications of variational method to impulsive fractional differential equations with two control parameters*, J. Nonlinear. Model. Anal., 2020, 2(1), 95–110.

[8] Z. Hao and L. Debnath, *On eigenvalue intervals and eigenfunctions of fourth-order singular boundary value problems*, Appl. Math. Lett., 2005, 18(5), 543–553.

[9] J. Li, J. J. Nieto and J. Shen, *Impulsive periodic boundary value problems of first-order differential equations*, J. Math. Anal. Appl., 2007, 325(1), 226–236.

[10] Y. Li and T. Zhang, *Existenee and uniqueness of anti-periodic solution for a kind of forced Raleigh equation with state dependent delay and impulses*, Commun. Nonlinear. Sci., 2010, 15(12), 4076–4083.

[11] A. Lannizzotto, *Three periodic solutions for an ordinary differential inclusion with two parameters*, Ann. Pol. Math., 2011, 103(1), 89–100.

[12] X. Li, Z. Han and S. Shurong, *Anti-periodic boundary value problems for fractional q-difference equations*, J. Appl. Math. Comput., 2016, 50(1), 243–257.

[13] D. Min and F. Chen, *Existence of solutions for a fractional advectional-dispersion equation with impulsive effects via variational approach*, J. Appl. Anal. Comput., 2020, 10(3), 1005–1023.

[14] P. H. Rabinowitz, *Minimal menhods in critical point theory with applications to differential equations*, in: CBMS regional Conference Series of in Mathematics, 65, American Mathematical Society, Providence, Rhode Island, 1986.

[15] J. Sun, H. Chen and J. J. Nieto, *Infinitely many solutions for second-order Hamiltonian system with impulsive effects*, Math. Comput. Model., 2011, 54, 544–555.

[16] H. Shi, H. Chen and H. Liu, *Morse theory and local linking for a class of boundary value problems with impulsive effects*, J. Appl. Math. Comput., 2016, 51(1), 353–365.

[17] Y. Tian and G. Gao, *Variational methods to Sturm-Liouville boundary value problem for impulsive differential equations*, Nonlinear. Anal-Theor., 2010, 72(1), 277–287.

[18] Y. Tian and X. Liu, *Applications of variational methods to Sturm-Liouville boundary value problem for fourth-order impulsive differential equations*, Math. Method. Appl. Sci., 2014, 37, 95–105.

[19] Y. Tian, S. Shang and Q. Huo, *Anti-periodic solutions of fourth-order impulsive differential equation*, Math. Method. Appl. Sci., 2018, 41(2), 769–780.

[20] K. Wang, *A new existence result for nonlinear first-order anti-periodie boundary value problems*, Appl. Math. Lett., 2008, 21(11), 1149–1154.

[21] W. Wang and J. Shen, *Existence of solutions for anti-periodie boundary value problems*, Nonlinear. Anal-Theor., 2009, 70(2), 595–605.

[22] P. Wang and W. Wang, *Anti-periodic boundary value problem for first order impulsive delay difference equations*, Adv. Differ. Equ., 2015, 93, doi:10.1186/s13662–015–0441–7.

[23] Q. Wang and M. Wang, *Existence of solution for impulsive differential equations with indefinite linear part*, Appl. Math. Lett., 2016, 51, 41–47.

[24] X. Zhang and L. Liu, *Initial value problems for nonlinear second order impulsive differential equations of mixed type in Banach spaces*, Nonlinear. Anal-Theor., 2006, 64(11), 2562–2574.

[25] X. Zhang, L. Liu and Y. Wu, *Global solutions of nonlinear second-order impulsive integro-differential equations of mixed type in Banach space*, Nonlinear. Anal-Theor., 2007, 67(8), 2335–2349.

[26] K. V. Zhukovsky, *Operational solution for some types of second order differential equations and for relevant physical problems*, J. Math. Anal. Appl., 2017, 446(1), 628–647.

[27] J. Zhou, H. Chen, O. M. Belal, et al., *Existence and multiplicity of solutions for some damped Dirichlet nonlinear impulsive differential equations*, Dyn. Syst. Differ. Equ., 2016, 24(2), 135–148.