Journal of Applied Analysis and Computation
Volume 11, Number 1, February 2021, 309–332
Website:http://www.jaac-online.com
DOI:10.11948/20190412

FINITE ITERATIVE (R,S)-CONJUGATE SOLUTIONS OF THE GENERALIZED COMPLEX COUPLED SYLVESTER-TRANSPOSE EQUATIONS

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Abstract The iterative method of generalized complex coupled Sylvester-transpose equations $AXB + CY^TD = E,\ MX^TN + GYH = F$ over (R,S)-conjugate matrix solution (X, Y) is proposed. Usually, the type of matrix arises from some physical problems with some form of generalized reflexive symmetry. On the condition that the coupled matrix equations are consistent, we show the solution pair (X^*, Y^*) can be obtained by generalization of CG iterative method within finite iterative steps in the absence of roundoff-error for any initial guess chosen by the (R,S)-conjugate matrix. Moreover, the optimal approximation (R,S)-conjugate matrix solutions can be derived by searching the least Frobenius norm solution of the novel generalized complex coupled Sylvester-transpose matrix equations. Finally, some numerical examples are given to illustrate the presented iterative algorithm is efficient.

Keywords Generalized complex coupled Sylvester-transpose equations, (R,S)-conjugate, least Frobenius norm solution, optimal approximation solution, numerical test.

MSC(2010) 65H10, 65K05, 49M15.

1. Introduction

The Sylvester matrix equations have numerous applications in control theory, model reduction, system stability, image restoration, pole assignment, decoupling techniques for ordinary and partial differential equations, filtering, etc. [1, 6, 7, 10, 11, 15, 16, 26]. Particularly, Benner discussed two applications to solve Sylvester equations with factorized right-hand side which come from systems and control theory, such as, observer design and model reduction using the cross-Gramian [4, 5]. It is worthy of mention that, recently, Wei et al. investigated robust fast fusion of multi-band images, i.e., hyperspectral data, based on solving a Sylvester matrix equation [27, 28].

By extending the well-known Jacobi and Gauss-Seidel iterations for $Ax = b$, Ding et al. gained the GI method and least squares method for the Sylvester matrix equation and some generalized forms [17–20]. Liang et al. given the mod-

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ified conjugate gradient iterative approach for the matrix equations $A_1XB_1 + C_1X^TD_1 = F_1,\ A_2XB_2 + C_2X^TD_2 = F_2$ [24]. Moreover, Masoud extended the GP-BiCG method for solving the Sylvester transpose matrix equations $\sum_{i=1}^{r}(A_iXB_i + C_iX^TD_i) = E$, see [21] for more details. In recent years, the coupled matrix equations have received considerable attentions on account of wide applications in some practical problems, for instance, the neural network and nonlinear programming, dynamic analysis etc. Beik et al. presented the global Krylov subspace methods for solving general coupled matrix equations in [2]. Furthermore, they also studied the coupled Sylvester-transpose matrix equations over generalized centro-symmetric matrices solution [3]. Dehghan et al. have obtained some achievements about various coupled Sylvester matrix equations, for more details, see [12–14]. Hajarian considered the iterative approach for the periodic solution of the Sylvester matrix equations $\widehat{A}_j\widehat{X}_j\widehat{B}_j + \widehat{C}_j\widehat{X}_{j+1}\widehat{D}_j = \widehat{E}_j$ $(j = 1, 2, \cdots)$, see [22]. Xie et al. proposed the iterative method to solve the generalized coupled Sylvester-transpose linear matrix equations over reflexive or antireflexive matrix [29, 30], however they didn't further study the novel appearance of (R,S)-conjugate matrix solutions and discuss their optimal solution forms which tend to be importantly for real computation.

If a linear differential operator contains no odd derivative terms with domain and boundary condition symmetry, the corresponding matrix, say A, occurred either from finite difference, boundary element, or finite element discretization, can often be coped with in such a way that A possesses the SAS property [8], namely, $A = PAP$, where P is a permutation matrix. As the extensions of reflexive matrix, Trench presented the so-called (R,S)-conjugate matrix and investigated its characterizations and properties in detail [25]. For the specific type of matrix, Chen provided three examples that were obtained from physical problems in three different application fields: one deals with the altitude estimation of a level network which yields a linear least-squares problem, the second is an electric network resulting in a linear system, and the third problem arises from structural analysis of trusses, for more details see [9]. Considering the practical applications, in the present paper, we conceive and analyze an efficient algorithm for solving the following generalized coupled complex Sylvester-transpose equations over (R,S)-conjugate matrix solutions:

$$\begin{cases} AXB + CY^TD = E, \\ MX^TN + GYH = F, \end{cases} \tag{1.1}$$

where $A,\ G \in \mathbb{C}^{m\times p}$, $C,\ M \in \mathbb{C}^{m\times q}$, $B,\ H \in \mathbb{C}^{q\times n}$, $D,\ N \in \mathbb{C}^{p\times n}$, $E,\ F \in \mathbb{C}^{m\times n}$ are given constant matrices, $X \in \mathbb{C}^{p\times q}$, $Y \in \mathbb{C}^{p\times q}$ are unknown matrices to be determined.

As is known that the CG method is the popular and efficient iterative method for the symmetric and positive definite linear system. By the Kronecker product and Vec operator, matrix equation can be transformed into the linear systems. Then the CG method can be applied to various of linear matrix equations on certain condition. We have to consider some more cheap methods because of the high computational expense once the dimensions of the matrices increasing. Based on the above analysis, in this paper, we propose a modified conjugate gradient method to solve the system (1.1) when the system is consistent and verify that the (R,S)-conjugate matrix solution pair (X^*, Y^*) can be obtained within finite iterative steps in the absence of roundoff-error for any initial value given (R,S)-conjugate matrix. Moreover, we prove that the optimal (R,S)-conjugate matrix can be derived by searching the least Frobenius norm solution of the new generalized complex coupled Sylvester-transpose linear matrix equations.

As a matter of convenience, we use the following notation throughout this paper: Let $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$ and $\mathbb{SOR}^{m\times m}$ denote the set of $m\times n$ real, complex matrix and m-order symmetric orthogonal matrix, respectively. For $A\in\mathbb{C}^{m\times n}$, it writes A^{+}, A^{H}, $\overline{A}$, A^{T}, $\mathcal{R}(A)$ and $\|A\|$ to denote Moore-Penrose generalized inverse, the conjugate transpose, conjugate, transpose, the column space and the Frobenius norm of the matrix A, respectively. For any matrices $A=(a_{ij}),\ B=(b_{ij})$, matrix $A\otimes B$ denotes the Kronecker product defined as $A\otimes B=(a_{ij}B)$. For the matrix $X=(x_1,x_2,\cdots,x_n)\in\mathbb{C}^{m\times n}$, $\mathrm{vec}(X)$ denotes the vec operator defined as $\mathrm{vec}(X)=(x_1^T,x_2^T,\cdots,x_n^T)^T\in\mathbb{C}^{mn}$. $\mathrm{invvec}(X)$ denotes the inverse operation of $\mathrm{vec}(X)$. The inner product in space $\mathbb{C}^{m\times n}$ is defined as

$$\langle A,B\rangle=\mathbf{Re}\mathrm{tr}(B^HA),\quad\forall A,\quad B\in\mathbb{C}^{m\times n},\tag{1.2}$$

The inner product space is denoted as $(\mathbb{C}^{m\times n},\mathbb{R},\langle\cdot,\cdot\rangle)$.

Definition 1.1. Let $R\in\mathbb{SOR}^{n\times n}$, $S\in\mathbb{SOR}^{m\times m}$, namely, $R=R^T=R^{-1}$, $S=S^T=S^{-1}$. An $m\times n$ complex matrix X is said to be a (R,S)-conjugate matrix, if $RXS=\overline{X}$, and denote $X\in\mathbb{RSC}^{m\times n}$.

Definition 1.2. Let $R\in\mathbb{SOR}^{n\times n}$, $S\in\mathbb{SOR}^{m\times m}$, namely, $R=R^T=R^{-1}$, $S=S^T=S^{-1}$. An $m\times n$ complex matrix X is said to be a shew (R,S)-conjugate matrix, if $RXS=-\overline{X}$, and denote $X\in\mathbb{SRSC}^{m\times n}$.

The rest of this paper is organized as follows. In section 2, we construct the modified conjugate gradient (MCG) method for solving the system (1.1) and show that a (R,S)-conjugate solution pair (X^*,Y^*) of (1.1) can be obtained by the MCG method within finite iterative steps in the absence of roundoff-error for any given initial point of (R,S)-conjugate matrix. Furthermore, we demonstrate the least Frobenius norm solution can be obtained by setting a kind of special initial matrix. In section 3, it obtains the optimal approximation (R,S)-conjugate matrix solution by searching the least Frobenius norm solution of the new generalized coupled complex Sylvester matrix equations. In section 4, some numerical examples are given to illustrate the introduced iterative algorithm is efficient. Conclusions are arranged in section 5.

2. The iterative method

First of all, in this section, some basic properties of the permutation matrix will be recalled. Then the problem (1.1) will be gradually transformed into its equivalent form by the Kronecker product and relevant knowledge. In addition, the necessary and sufficient conditions of the consistency of the linear matrix equations (1.1) will be given further. Finally, the modified conjugate gradient method (MCG) for solving (1.1) based on the classical conjugate gradient method is proposed and discussed in detail.

According to Theorem 4.3.8 and Corollary 4.3.10 in [23], for any $X\in\mathbb{C}^{p\times q}$, there exists a permutation matrix $P_{pq}\in\mathbb{R}^{pq\times pq}$ such that

$$\mathrm{vec}(X^T)=P_{pq}\mathrm{vec}(X),\tag{2.1}$$

where

$$P_{pq} = \sum_{i=1}^{p}\sum_{j=1}^{q} E_{ij} \otimes E_{ij}^T, \quad E_{ij} = e_i e_j^T,$$

and $e_i \in \mathbb{R}^p$ ($e_j \in \mathbb{R}^q$) is a column vector with a unity in the i-th (j-th) entry and zeros elsewhere. Then it can be easily shown that

$$P_{pq}P_{qp} = I_{pq}, \, P_{pq}^T = P_{pq}^{-1} = P_{pq}, \, P_{qp}\mathrm{vec}(X^T) = \mathrm{vec}(X),$$

and

$$B \otimes A = P_{mq}^T (A \otimes B) P_{pn}, \quad (A \otimes B) P_{pn} = P_{qm}^T (B \otimes A), \tag{2.2}$$

where $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times n}$.

From above discussion and the definition of Kronecker product, if the (R-S)-conjugate matrix solutions exist, then Eqs. (1.1) are equivalent to

$$\begin{pmatrix} B^T \otimes A & (D^T \otimes C)P_{pq} \\ (N^T \otimes M)P_{pq} & H^T \otimes G \end{pmatrix} \begin{pmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}(E) \\ \mathrm{vec}(F) \end{pmatrix}. \tag{2.3}$$

Let

$$T := \begin{pmatrix} B^T \otimes A & (D^T \otimes C)P_{pq} \\ (N^T \otimes M)P_{pq} & H^T \otimes G \end{pmatrix}, \tag{2.4}$$

$$z := \begin{pmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{pmatrix}, \qquad f := \begin{pmatrix} \mathrm{vec}(E) \\ \mathrm{vec}(F) \end{pmatrix}. \tag{2.5}$$

As a result, (2.3) can be written as

$$Tz = f. \tag{2.6}$$

Lemma 2.1. *Eqs.* (1.1) *are consistent for the* (R,S)*-conjugate matrices if and only if*

$$rank(T) = rank(T, f).$$

Moreover, if

$$rank(T) = rank(T, f) = 2mn,$$

then, Eqs. (1.1) *have a unique* (R,S)*-conjugate solution.*

Lemma 2.2. *Eqs.* (1.1) *are consistent for the* (R,S)*-conjugate matrices if and only if the following matrix equations*

$$\begin{cases} AXB + CY^TD = E, \\ MX^TN + GYH = F, \\ \overline{A}RXS\overline{B} + \overline{C}SY^TR\overline{D} = \overline{E}, \\ \overline{M}SX^TR\overline{N} + \overline{G}RYS\overline{H} = \overline{F}, \end{cases} \tag{2.7}$$

are consistent.

Proof. If Eqs. (1.1) exist the solutions $X^*,\ Y^* \in \mathbb{RSC}^{p\times q}$, namely, $\overline{X^*} = RX^*S,\ \overline{Y^*} = RY^*S$, it is evident that $X^*,\ Y^*$ satisfy Eqs. (2.7). In fact,

$$\begin{aligned}\overline{A}RX^*S\overline{B} + \overline{C}S(Y^*)^T R\overline{D} &= \overline{AR\overline{X^*}SB} + \overline{CS\overline{(Y^*)^T}RD}\\ &= \overline{AX^*B + C(Y^*)^T D} = \overline{E},\end{aligned}$$

and

$$\begin{aligned}\overline{M}S(X^*)^T R\overline{N} + \overline{G}RY^*S\overline{H} &= \overline{MS\overline{(X^*)^T}RN} + \overline{GR\overline{(Y^*)}SH}\\ &= \overline{M(X^*)^T N + GY^*H} = \overline{F}.\end{aligned}$$

On the other hand, suppose that the solutions $\widehat{X},\ \widehat{Y} \in \mathbb{C}^{p\times q}$ of Eqs. (2.7) exist, by choosing $X^* = \frac{1}{2}(\widehat{X} + R\overline{\widehat{X}}S),\ Y^* = \frac{1}{2}(\widehat{Y} + R\overline{\widehat{Y}}S)$, as well as notice that $X^*,\ Y^* \in \mathbb{RSC}^{p\times q}$, we have

$$\begin{aligned}AX^*B + C(Y^*)^T D &= \frac{1}{2}\big(A(\widehat{X} + R\overline{\widehat{X}}S)B + C(\widehat{Y} + R\overline{\widehat{Y}}S)^T D\big)\\ &= \frac{1}{2}\big(A\widehat{X}B + C\widehat{Y}^T D + AR\overline{\widehat{X}}SB + C(R\overline{\widehat{Y}}S)^T D\big)\\ &= \frac{1}{2}\big(A\widehat{X}B + C\widehat{Y}^T D + \overline{\overline{A}R\widehat{X}S\overline{B} + \overline{C}S\widehat{Y}^T R\overline{D}}\ \big)\\ &= \frac{1}{2}\big(E + \overline{\overline{E}}\big) = E\end{aligned}$$

and

$$\begin{aligned}M(X^*)^T N + GY^*H &= \frac{1}{2}\big(M(\widehat{X} + R\overline{\widehat{X}}S)^T N + G(\widehat{Y} + R\overline{\widehat{Y}}S)H\big)\\ &= \frac{1}{2}\big(M\widehat{X}^T N + G\widehat{Y}H + M(R\overline{\widehat{X}}S)^T N + GR\overline{\widehat{Y}}SH\big)\\ &= \frac{1}{2}\big(M\widehat{X}^T N + G\widehat{Y}H + \overline{\overline{M}S\widehat{X}^T R\overline{N} + \overline{G}R\widehat{Y}S\overline{H}}\big)\\ &= \frac{1}{2}(F + \overline{\overline{F}}) = F.\end{aligned}$$

Hence, $X^*,\ Y^*$ are the solution of Eqs. (1.1). □

Now, the following so-called modified conjugate gradient (MCG)method to solve the (R-S)-conjugate solution of (1.1) will be proposed exactly.

(MCG method for (R-S)-conjugate solution)

Step 1 Input $A,\ G \in \mathbb{C}^{m\times p},\ C,\ M \in \mathbb{C}^{m\times q},\ B,\ H \in \mathbb{C}^{q\times n},\ D,\ N \in \mathbb{C}^{p\times n}$, $E,\ F \in \mathbb{C}^{m\times n}$, choose $R \in \mathbb{SOR}^{p\times p}$, $S \in \mathbb{SOR}^{q\times q}$ and initial matrix $X_1 \in \mathbb{RSC}^{p\times q}$, $Y_1 \in \mathbb{RSC}^{p\times q}$ in Definition 1.1. Compute

$$R_1 := \begin{pmatrix} R_1^{(1)} & 0 \\ 0 & R_1^{(2)} \end{pmatrix},$$

$$R_1^{(1)} = E - AX_1B - CY_1^T D,\ R_1^{(2)} = F - MX_1^T N - GY_1H,$$

$$\widetilde{R}_1 := \begin{pmatrix} \widetilde{R}_1^{(1)} & 0 \\ 0 & \widetilde{R}_1^{(2)} \end{pmatrix},$$

$$\widetilde{R}_1^{(1)} = A^H R_1^{(1)} B^H + \overline{N} {R_1^{(2)}}^T \overline{M},\ \widetilde{R}_1^{(2)} = \overline{D} {R_1^{(1)}}^T \overline{C} + G^H R_1^{(2)} H^H,$$
$$M_1 = \frac{1}{2}(\widetilde{R}_1^{(1)} + R\overline{\widetilde{R}_1^{(1)}}S), \quad N_1 = \frac{1}{2}(\widetilde{R}_1^{(2)} + R\overline{\widetilde{R}_1^{(2)}}S).$$

Set $k := 1$.

Step 2 If $R_k = 0$ or $R_k \neq 0$ and $M_k = N_k = 0$, stop; Otherwise, go to Step 3.

Step 3 Update the sequences

$$X_{k+1} = X_k + \alpha_k M_k,\ Y_{k+1} = Y_k + \alpha_k N_k,$$
$$R_{k+1} := \begin{pmatrix} R_{k+1}^{(1)} & 0 \\ 0 & R_{k+1}^{(2)} \end{pmatrix},$$
$$R_{k+1}^{(1)} = E - AX_{k+1}B - CY_{k+1}^T D,\ R_{k+1}^{(2)} = F - MX_{k+1}^T N - GY_{k+1}H,$$
$$\widetilde{R}_{k+1} := \begin{pmatrix} \widetilde{R}_{k+1}^{(1)} & 0 \\ 0 & \widetilde{R}_{k+1}^{(2)} \end{pmatrix},$$
$$\widetilde{R}_{k+1}^{(1)} = A^H R_{k+1}^{(1)} B^H + \overline{N} {R_{k+1}^{(2)}}^T \overline{M}, \quad \widetilde{R}_{k+1}^{(2)} = \overline{D} {R_{k+1}^{(1)}}^T \overline{C} + G^H R_{k+1}^{(2)} H^H,$$
$$M_{k+1} = \frac{1}{2}(\widetilde{R}_{k+1}^{(1)} + R\overline{\widetilde{R}_{k+1}^{(1)}}S) + \beta_k M_k,\ N_{k+1} = \frac{1}{2}(\widetilde{R}_{k+1}^{(2)} + R\overline{\widetilde{R}_{k+1}^{(2)}}S) + \beta_k N_k, \tag{2.8}$$

where
$$\alpha_k := \frac{\|R_k\|^2}{\|M_k\|^2 + \|N_k\|^2}, \quad \beta_k := \frac{\|R_{k+1}\|^2}{\|R_k\|^2}. \tag{2.9}$$

Step 4 Set $k := k + 1$, return to Step 2.

(MCG method for skew (R-S)-conjugate solution)

Step 1 Input $A,\ G \in \mathbb{C}^{m\times p}$, $C,\ M \in \mathbb{C}^{m\times q}$, $B,\ H \in \mathbb{C}^{q\times n}$, $D,\ N \in \mathbb{C}^{p\times n}$, $E,\ F \in \mathbb{C}^{m\times n}$, choose $R \in \mathbb{SOR}^{p\times p}$, $S \in \mathbb{SOR}^{q\times q}$ and initial matrix $X_1 \in \mathbb{SRSC}^{p\times q}$, $Y_1 \in \mathbb{SRSC}^{p\times q}$ in Definition 1.2. Compute

$$R_1 := \begin{pmatrix} R_1^{(1)} & 0 \\ 0 & R_1^{(2)} \end{pmatrix},$$
$$R_1^{(1)} = E - AX_1B - CY_1^T D,\ R_1^{(2)} = F - MX_1^T N - GY_1H,$$
$$\widetilde{R}_1 := \begin{pmatrix} \widetilde{R}_1^{(1)} & 0 \\ 0 & \widetilde{R}_1^{(2)} \end{pmatrix},$$
$$\widetilde{R}_1^{(1)} = A^H R_1^{(1)} B^H + \overline{N} {R_1^{(2)}}^T \overline{M},\ \widetilde{R}_1^{(2)} = \overline{D} {R_1^{(1)}}^T \overline{C} + G^H R_1^{(2)} H^H,$$
$$M_1 = \frac{1}{2}(\widetilde{R}_1^{(1)} - R\overline{\widetilde{R}_1^{(1)}}S), \quad N_1 = \frac{1}{2}(\widetilde{R}_1^{(2)} - R\overline{\widetilde{R}_1^{(2)}}S).$$

Set $k := 1$.

Step 2 If $R_k = 0$ or $R_k \neq 0$ and $M_k = N_k = 0$, stop; Otherwise, go to Step 3.
Step 3 Update the sequences

$$
\begin{aligned}
&X_{k+1} = X_k + \alpha_k M_k,\, Y_{k+1} = Y_k + \alpha_k N_k,\\
&R_{k+1} := \begin{pmatrix} R_{k+1}^{(1)} & 0 \\ 0 & R_{k+1}^{(2)} \end{pmatrix},\\
&R_{k+1}^{(1)} = E - AX_{k+1}B - CY_{k+1}^T D,\, R_{k+1}^{(2)} = F - MX_{k+1}^T N - GY_{k+1}H,\\
&\widetilde{R}_{k+1} := \begin{pmatrix} \widetilde{R}_{k+1}^{(1)} & 0 \\ 0 & \widetilde{R}_{k+1}^{(2)} \end{pmatrix},\\
&\widetilde{R}_{k+1}^{(1)} = A^H R_{k+1}^{(1)} B^H + \overline{N} {R_{k+1}^{(2)}}^T \overline{M}, \quad \widetilde{R}_{k+1}^{(2)} = \overline{D} {R_{k+1}^{(1)}}^T \overline{C} + G^H R_{k+1}^{(2)} H^H,\\
&M_{k+1} = \frac{1}{2}(\widetilde{R}_{k+1}^{(1)} - R\overline{\widetilde{R}_{k+1}^{(1)}}S) + \beta_k M_k, \quad N_{k+1} = \frac{1}{2}(\widetilde{R}_{k+1}^{(2)} - R\overline{\widetilde{R}_{k+1}^{(2)}}S) + \beta_k N_k,
\end{aligned}
\tag{2.10}
$$

where

$$
\alpha_k := \frac{\|R_k\|^2}{\|M_k\|^2 + \|N_k\|^2}, \quad \beta_k := \frac{\|R_{k+1}\|^2}{\|R_k\|^2}. \tag{2.11}
$$

Step 4 Set $k := k+1$, return to Step 2.

Now, it will show that the sequence pair $\{(X_k, Y_k)\}$ generated by Algorithm 1 converges to the solution (X^*, Y^*) of (1.1) within finite iterative steps in the absence of roundoff-error for any initial value over (R,S)-conjugate matrix.

For matrices A, B, C, D with suitable dimensions, some vital and useful results will be presented explictly by the definition of the inner product space $(\mathbb{C}^{m\times n}, \mathbb{R}, \langle\cdot,\cdot\rangle)$ and properties of matrix trace:

$$
\begin{aligned}
\langle A, B\rangle = \mathbf{Re}[\mathrm{tr}(B^H A)] &= \mathbf{Re}[\mathrm{tr}(AB^H)] = \mathbf{Re}[\mathrm{tr}(\overline{AB^H})\\
&= \mathbf{Re}[\mathrm{tr}(A^H B)] = \langle B, A\rangle = \langle B^T, A^T\rangle,
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
\langle ABC, D\rangle &= \mathbf{Re}[\mathrm{tr}((ABC)^H D)]\\
&= \mathbf{Re}[\mathrm{tr}(C^H B^H A^H D)] = \mathbf{Re}[\mathrm{tr}(B^H A^H D C^H)]\\
&= \langle B, A^H D C^H\rangle
\end{aligned}
\tag{2.13}
$$

and

$$
\begin{aligned}
\langle \overline{A}, \overline{B}\rangle &= \mathbf{Re}[\mathrm{tr}(\overline{A}^H \overline{B})]\\
&= \mathbf{Re}[\mathrm{tr}(\overline{A^H B})] = \mathbf{Re}[\mathrm{tr}(A^H B)]\\
&= \langle A, B\rangle.
\end{aligned}
\tag{2.14}
$$

Moreover, for arbitrary complex matrix $Z \in \mathbb{C}^{p\times q}$ if $R \in \mathbb{SOR}^{p\times p}$, $S \in \mathbb{SOR}^{q\times q}$, and $X \in \mathbb{RSC}^{p\times q}$, then

$$
\langle X, \frac{Z + R\overline{Z}S}{2}\rangle = \frac{1}{2}\Big(\langle X, Z\rangle + \langle X, R\overline{Z}S\rangle\Big) \tag{2.15}
$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\langle X, Z\rangle + \langle RXS, \overline{Z}\rangle\Big) \\
&= \frac{1}{2}\Big(\langle X, Z\rangle + \langle \overline{X}, \overline{Z}\rangle\Big) \\
&= \langle X, Z\rangle.
\end{aligned}
$$

Lemma 2.3. *Let the sequences* $\{R_k\}, \{M_k\}, \{N_k\}, \{\widetilde{R}_j^{(1)}\}, \{\widetilde{R}_j^{(2)}\}, \{\alpha_k\}$ *be generated by Algorithm* 1*, then we have*

$$\langle R_{k+1}, R_j\rangle = \langle R_k, R_j\rangle - \alpha_k\Big(\langle M_k, \widetilde{R}_j^{(1)}\rangle + \langle N_k, \widetilde{R}_j^{(2)}\rangle\Big), \quad k, j = 1, 2, \cdots .$$

Proof. By Algorithm 1 and formulas (2.11)-(2.14), it gets

$$
\begin{aligned}
\langle R_{k+1}, R_j\rangle &= \langle R_{k+1}^{(1)}, R_j^{(1)}\rangle + \langle R_{k+1}^{(2)}, R_j^{(2)}\rangle \\
&= \langle E - AX_{k+1}B - CY_{k+1}^T D, R_j^{(1)}\rangle + \langle F - MX_{k+1}^T N - GY_{k+1}H, R_j^{(2)}\rangle \\
&= \langle E - A(X_k + \alpha_k M_k)B - C(Y_k + \alpha_k N_k)^T D, R_j^{(1)}\rangle + \\
&\quad \langle F - M(X_k + \alpha_k M_k)^T N - G(Y_k + \alpha_k N_k)H, R_j^{(2)}\rangle \\
&= \langle E - AX_k B - CY_k^T D, R_j^{(1)}\rangle + \langle F - MX_k^T N - GY_k H, R_j^{(2)}\rangle \\
&\quad -\alpha_k\Big(\langle AM_k B + CN_k^T D, R_j^{(1)}\rangle + \langle MM_k^T N + GN_k H, R_j^{(2)}\rangle\Big) \\
&= \langle R_k, R_j\rangle - \alpha_k\Big(\langle M_k, A^H R_j^{(1)} B^H + \overline{N} {R_j^{(2)}}^T \overline{M}\rangle \\
&\quad + \langle N_k, \overline{D} {R_j^{(1)}}^T C + G^H R_j^{(2)} H^H\rangle\Big) \\
&= \langle R_k, R_j\rangle - \alpha_k\Big(\langle M_k, \widetilde{R}_j^{(1)}\rangle + \langle N_k, \widetilde{R}_j^{(2)}\rangle\Big),
\end{aligned}
$$

which completes the proof. □

Lemma 2.4. *Let the sequences* $\{R_k\}, \{M_k\}, \{N_k\}$ *be generated by Algorithm* 1*, then we have*

$$\langle R_i, R_j\rangle = 0, \quad \langle M_i, M_j\rangle + \langle N_i, N_j\rangle = 0, \quad i, j = 1, 2, \cdots, k, \ i \neq j.$$

Proof. Firstly, the objective is to find the following fact

$$\langle R_i, R_j\rangle = 0, \quad \langle M_i, M_j\rangle + \langle N_i, N_j\rangle = 0, \quad 1 \le j < i \le k. \tag{2.16}$$

By mathematical induction. For $k = 2$, by Lemma 2.3, relation (2.15) and notice that $M_j \in \mathbb{RSC}^{p\times q}$, $N_j \in \mathbb{RSC}^{p\times q}$, $j = 1, 2, \cdots, k$, generated by Algorithm 1, we get

$$
\begin{aligned}
\langle R_2, R_1\rangle &= \langle R_1, R_1\rangle - \alpha_1(\langle M_1, \widetilde{R}_1^{(1)}\rangle + \langle N_1, \widetilde{R}_1^{(2)}\rangle) \qquad (2.17) \\
&= \|R_1\|^2 - \alpha_1\Big(\langle M_1, \frac{\widetilde{R}_1^{(1)} + R\overline{\widetilde{R}_1^{(1)}}S}{2}\rangle + \langle N_1, \frac{\widetilde{R}_1^{(2)} + R\overline{\widetilde{R}_1^{(2)}}S}{2}\rangle\Big) \\
&= \|R_1\|^2 - \frac{\|R_1\|^2}{\|M_1\|^2 + \|N_1\|^2}\Big(\langle M_1, M_1\rangle + \langle N_1, N_1\rangle\Big) \\
&= \|R_1\|^2 - \|R_1\|^2 = 0.
\end{aligned}
$$

In addition, by formulas (2.10), (2.11), (2.15), (2.17) and Lemma 2.3, it gives rise to

$$\begin{aligned}\langle M_2,M_1\rangle+\langle N_2,N_1\rangle &= \Big\langle \frac{\widetilde{R}_2^{(1)}+R\overline{\widetilde{R}_2^{(1)}}S}{2}+\beta_1 M_1, M_1\Big\rangle+\Big\langle \frac{\widetilde{R}_2^{(2)}+R\overline{\widetilde{R}_2^{(2)}}S}{2}+\beta_1 N_1, N_1\Big\rangle \\ &= \langle \widetilde{R}_2^{(1)}, M_1\rangle + \langle \widetilde{R}_2^{(2)}, N_1\rangle + \beta_1(\|M_1\|^2+\|N_1\|^2) \\ &= \frac{1}{\alpha_1}\Big(\langle R_1,R_2\rangle - \langle R_2,R_2\rangle\Big) + \beta_1(\|M_1\|^2+\|N_1\|^2) \\ &= 0.\end{aligned}$$

Therefore, (2.16) holds for $k=2$.

Assume that (2.16) holds for $k=l$ ($l>2$). For $k=l+1$, it follows from Lemma 2.3, Algorithm 1 and (2.15) that

$$\begin{aligned}\langle R_{l+1},R_l\rangle &= \langle R_l,R_l\rangle - \alpha_l\Big(\langle M_l,\widetilde{R}_l^{(1)}\rangle + \langle N_l,\widetilde{R}_l^{(2)}\rangle\Big) \\ &= \|R_l\|^2-\alpha_l\Big(\langle M_l,\frac{\widetilde{R}_l^{(1)}+R\overline{\widetilde{R}_l^{(1)}}S}{2}\rangle + \langle N_l,\frac{\widetilde{R}_l^{(2)}+R\overline{\widetilde{R}_l^{(2)}}S}{2}\rangle\Big) \\ &= \|R_l\|^2-\alpha_l\Big(\langle M_l, M_l-\beta_{l-1}M_{l-1}\rangle + \langle N_l, N_l-\beta_{l-1}N_{l-1}\rangle\Big) \\ &= \|R_l\|^2-\alpha_l\Big(\|M_l\|^2+\|N_l\|^2\Big)=0,\end{aligned}$$

where the fourth equality holds by the induction assumption.

By Algorithm 1, (2.10), (2.11), (2.15) and induction, it generates

$$\begin{aligned}\langle M_{l+1},M_l\rangle+\langle N_{l+1},N_l\rangle &= \Big\langle \frac{\widetilde{R}_{l+1}^{(1)}+R\overline{\widetilde{R}_{l+1}^{(1)}}S}{2}+\beta_l M_l, M_l\Big\rangle \\ &\quad +\Big\langle \frac{\widetilde{R}_{l+1}^{(2)}+R\overline{\widetilde{R}_{l+1}^{(2)}}S}{2}+\beta_l N_l, N_l\Big\rangle \\ &= \langle \widetilde{R}_{l+1}^{(1)}, M_1\rangle + \langle \widetilde{R}_{l+1}^{(2)}, N_l\rangle + \beta_l(\|M_l\|^2+\|N_l\|^2) \\ &= \frac{1}{\alpha_l}\Big(\langle R_l,R_{l+1}\rangle - \langle R_{l+1},R_{l+1}\rangle\Big) + \beta_l(\|M_l\|^2+\|N_l\|^2) \\ &= 0.\end{aligned}$$

On the other hand, suppose that

$$\langle R_l,R_j\rangle=0,\quad \langle M_l,M_j\rangle+\langle N_l,N_j\rangle=0,\quad j=1,2,\cdots,l-2.$$

In fact, for $j=l-1$, the conclusions have been obtained by the above analysis.

Next, the following results will be definitely certified

$$\langle R_{l+1},R_j\rangle=0,\quad \langle M_{l+1},M_j\rangle+\langle N_{l+1},N_j\rangle=0,\quad j=1,2,\cdots,l.$$

For $j=1$, by Lemma 2.3, relation (2.15) and the induction, it deduces

$$\begin{aligned}\langle R_{l+1},R_1\rangle &= \langle R_l,R_1\rangle - \alpha_l\Big(\langle M_l,\widetilde{R}_1^{(1)}\rangle + \langle N_l,\widetilde{R}_1^{(2)}\rangle\Big) \\ &= -\alpha_l\Big(\langle M_l,\frac{\widetilde{R}_1^{(1)}+R\overline{\widetilde{R}_1^{(1)}}S}{2}\rangle + \langle N_l,\frac{\widetilde{R}_1^{(2)}+R\overline{\widetilde{R}_1^{(2)}}S}{2}\rangle\Big)\end{aligned}$$

$$
\begin{aligned}
&= -\alpha_l\Big(\langle M_l, M_1\rangle + \langle N_l, N_1\rangle\Big)\\
&= 0.
\end{aligned}
$$

Then, for $j = 2, 3, \cdots, l-1$, one obtains

$$
\begin{aligned}
\langle R_{l+1}, R_j\rangle &= \langle R_l, R_j\rangle - \alpha_l\Big(\langle M_l, \widetilde{R}_j^{(1)}\rangle + \langle N_l, \widetilde{R}_j^{(2)}\rangle\Big)\\
&= -\alpha_l\Big(\langle M_l, \frac{\widetilde{R}_j^{(1)} + R\overline{\widetilde{R}_j^{(1)}}S}{2}\rangle + \langle N_l, \frac{\widetilde{R}_j^{(2)} + R\overline{\widetilde{R}_j^{(2)}}S}{2}\rangle\Big)\\
&= -\alpha_l\Big(\langle M_l, M_j - \beta_{j-1}M_{j-1}\rangle + \langle N_l, N_j - \beta_{j-1}N_{j-1}\rangle\Big)\\
&= -\alpha_l\Big(\langle M_l, M_j\rangle + \langle N_l, N_j\rangle - \beta_{j-1}\big(\langle M_l, M_{j-1}\rangle + \langle N_l, N_{j-1}\rangle\big)\Big)\\
&= 0.
\end{aligned}
$$

Moreover, by applying Lemma 2.3, (2.15) and the induction, for $j = 1, 2, \cdots, l-1$, it yields that

$$
\begin{aligned}
\langle M_{l+1}, M_j\rangle + \langle N_{l+1}, N_j\rangle &= \langle \frac{\widetilde{R}_{l+1}^{(1)} + R\overline{\widetilde{R}_{l+1}^{(1)}}S}{2}, M_j\rangle + \langle \frac{\widetilde{R}_{l+1}^{(2)} + R\overline{\widetilde{R}_{l+1}^{(2)}}S}{2}, N_j\rangle\\
&\quad + \beta_l\Big(\langle M_l, M_j\rangle + \langle N_l, N_j\rangle\Big)\\
&= \langle \widetilde{R}_{l+1}^{(1)}, M_j\rangle + \langle \widetilde{R}_{l+1}^{(2)}, N_j\rangle\\
&= \frac{1}{\alpha_l}\Big(\langle R_j, R_{l+1}\rangle - \langle R_{j+1}, R_{l+1}\rangle\Big)\\
&= 0.
\end{aligned}
$$

Therefore, (2.16) holds for $k = l+1$. When $j < i$, by the symmetric property of the inner product space, it will gets (2.16). This completes the proof. □

Lemma 2.5. *Suppose that* (1.1) *is consistent, i.e.* (2.7) *holds. Let* (X^*, Y^*) *be an arbitrary solution pair of* (1.1)*. Then for any initial guess* $X_1,\ Y_1 \in \mathbb{RSC}^{p\times q}$*, the sequences* $\{X_k\}$*,* $\{Y_k\}$*,* $\{R_k\}$*,* $\{M_k\}$*,* $\{N_k\}$ *generated by Algorithm* 1 *satisfy*

$$
\langle X^* - X_k, M_k\rangle + \langle Y^* - Y_k, N_k\rangle = \|R_k\|^2 \quad k = 1, 2, \cdots. \tag{2.18}
$$

Proof. The conclusion will be accomplished by mathematical induction. Notice that the sequences pair $\{(X_k, Y_k)\}, (k = 1, 2, \cdots)$ generated by Algorithm 1 are all (R,S)-conjugate matrix, since initial matrix (X_1, Y_1) is chosen with (R,S)-conjugate matrix. Then for $k = 1$, it follows from Algorithm 1 and formulas (2.13)-(2.15) that

$$
\begin{aligned}
&\langle X^* - X_1, M_1\rangle + \langle Y^* - Y_1, N_1\rangle\\
=&\langle X^* - X_1, \frac{\widetilde{R}_1^{(1)} + R\overline{\widetilde{R}_1^{(1)}}S)}{2}\rangle + \langle Y^* - Y_1, \frac{\widetilde{R}_1^{(2)} + R\overline{\widetilde{R}_1^{(2)}}S)}{2}\rangle\\
=&\langle X^* - X_1, \widetilde{R}_1^{(1)}\rangle + \langle Y^* - Y_1, \widetilde{R}_1^{(2)}\rangle\\
=&\langle X^* - X_1, A^H R_1^{(1)} B^H + \overline{N} {R_1^{(2)}}^T \overline{M}\rangle\\
&+ \langle Y^* - Y_1, \overline{D} {R_1^{(1)}}^T \overline{C} + G^H R_1^{(2)} H^H\rangle\\
=&\langle X^* - X_1, A^H R_1^{(1)} B^H\rangle + \langle Y^* - Y_1, \overline{D} {R_1^{(1)}}^T \overline{C}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\langle X^*-X_1, \overline{N}{R_1^{(2)}}^T\overline{M}\rangle + \langle Y^*-Y_1, G^H R_1^{(2)} H^H\rangle \\
=&\langle A(X^*-X_1)B, R_1^{(1)}\rangle + \langle C(Y^*-Y_1)^T D, R_1^{(1)}\rangle \\
&\times \langle M(X^*-X_1)^T N, R_1^{(2)}\rangle + \langle G(Y^*-Y_1)H, R_1^{(2)}\rangle \\
=&\langle AX^*B + C{Y^*}^T D - AX_1B - CY_1^T D, R_1^{(1)}\rangle \\
&\times \langle M{X^*}^T N + GY^*H - MX_1^T N - GY_1H, R_1^{(2)}\rangle \\
=&\langle R_1^{(1)}, R_1^{(1)}\rangle + \langle R_1^{(2)}, R_1^{(2)}\rangle \\
=&\|R_1\|^2.
\end{aligned}
$$

The seventh equality holds since (X^*, Y^*) is a solution of the system (1.1). Hence, (2.18) holds for $k = 1$.

Assume that (2.18) holds for $k = l,\ (l \geq 1)$. For $k = l + 1$, it follows from the update formulas of X_{l+1}, Y_{l+1} that

$$
\begin{aligned}
\langle X^*-X_{l+1}, M_l\rangle + \langle Y^*-Y_{l+1}, N_l\rangle &= \langle X^*-X_l-\alpha_l M_l, M_l\rangle \\
&\quad +\langle Y^*-Y_l-\alpha_l N_l, N_l\rangle \\
&= \langle X^*-X_l, M_l\rangle + \langle Y^*-Y_l, N_l\rangle \\
&\quad -\alpha_l\Big(\|M_l\|^2 + \|N_l\|^2\rangle\Big) \\
&= \|R_l\|^2 - \|R_l\|^2 = 0.
\end{aligned}
\tag{2.19}
$$

In a similar way as above discussion, one immediately has

$$
\begin{aligned}
&\langle X^*-X_{l+1}, \widetilde{R}_{l+1}^{(1)}\rangle + \langle Y^*-Y_{l+1}, \widetilde{R}_{l+1}^{(2)}\rangle \\
=&\ \langle X^*-X_{l+1}, A^H R_{l+1}^{(1)} B^H + \overline{N}{R_{l+1}^{(2)}}^T\overline{M}\rangle \\
&+\langle Y^*-Y_{l+1}, \overline{D}{R_{l+1}^{(1)}}^T\overline{C} + G^H R_{l+1}^{(2)} H^H\rangle \\
=&\ \langle A(X^*-X_{l+1})B, R_{l+1}^{(1)}\rangle + \langle M(X^*-X_{l+1})^T N, R_{l+1}^{(2)}\rangle \\
&+\langle C(Y^*-Y_{l+1})^T D, R_{l+1}^{(1)}\rangle + \langle G(Y^*-Y_{l+1})H, R_{l+1}^{(2)}\rangle \\
=&\ \langle AX^*B + C{Y^*}^T D - AX_{l+1}B - C{Y_{l+1}}^T D, R_{l+1}^{(1)}\rangle \\
&+\langle M{X^*}^T N + GY^*H - M{X_{l+1}}^T N - GY_{l+1}H, R_{l+1}^{(2)}\rangle \\
=&\ \langle R_{l+1}^{(1)}, R_{l+1}^{(1)}\rangle + \langle R_{l+1}^{(2)}, R_{l+1}^{(2)}\rangle \\
=&\ \|R_{l+1}\|^2.
\end{aligned}
\tag{2.20}
$$

From, (2.19), (2.20), it generates

$$
\begin{aligned}
\langle X^*-X_{l+1}, M_{l+1}\rangle + \langle Y^*-Y_{l+1}, N_{l+1}\rangle &= \langle X^*-X_{l+1}, \frac{\widetilde{R}_{l+1}^{(1)} + R\overline{\widetilde{R}_{l+1}^{(1)}}S}{2} + \beta_l M_l\rangle \\
&\quad +\langle Y^*-Y_{l+1}, \frac{\widetilde{R}_{l+1}^{(2)} + R\overline{\widetilde{R}_{l+1}^{(2)}}S}{2} + \beta_l N_l\rangle \\
&= \langle X^*-X_{l+1}, \widetilde{R}_{l+1}^{(1)}\rangle + \langle Y^*-Y_{l+1}, \widetilde{R}_{l+1}^{(2)}\rangle \\
&= \|R_{l+1}\|^2.
\end{aligned}
$$

This completes the proof. □

Remark 2.1. The above lemmas are achieved with the assumption that initial guess is (R,S)-conjugate matrix. Similarly, if the initial guess is shew (R,S)-conjugate matrix, the same results can be obtained easily. So, it doesn't need to show these results in details.

Theorem 2.1. *Assume that the system* (1.1) *is consistent. Then, for any initial matrices* $X_1,\ Y_1 \in \mathbb{RSC}^{p\times q}$*, an exact solution of the system* (1.1) *can be derived at most* $2mn+1$ *iteration steps by Algorithm* 1.

Proof. Assume $R_k \neq 0$ for $k = 1, 2, \cdots, 2mn$. It follows from Lemma 2.5 that $\|M_k\|^2 + \|N_k\|^2 \neq 0$ for $k = 1, 2, \cdots, 2mn$. Then R_{2mn+1} will be derived by Algorithm 1. According to Lemma 2.4, we know $\langle R_i, R_j\rangle = 0$ for $i, j = 1, 2, \cdots, 2mn + 1,\ i \neq j$. Then the matrix sequence of $R_1, R_2, \cdots, R_{2pq}$ is the orthogonal basis of the linear space

$$\mathcal{H} = \left\{ H | H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \right\},$$

where $H_1, H_2 \in R^{m\times n}$. Since $R_{2mn+1} \in \mathcal{H}$ and $\langle R_{2mn+1}, R_k\rangle = 0$ for $k = 1, 2, \cdots, 2mn$, hence $R_{2mn+1} = 0$, which completes the proof. □

When system (1.1) is consistent, the solution may not be unique. However, it needs to find the unique least Frobenius norm solution of the system (1.1) for some practical problems. To this end, the following important lemma will be introduced.

Lemma 2.6. *Suppose that linear matrix equation* $Ax = b$ *has a solution* $x^* \in \mathcal{R}(A^H)$*. Then* x^* *is the unique least Frobenius norm solution of* $Ax = b$.

Proof. By singular value decomposition of matrix A, it gives rise to

$$A = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^H = U_1 \Sigma V_1^H,$$

where $U = (U_1,\ U_2)$, $V = (V_1,\ V_2)$ are unitary matrices.

Obviously, the unique least norm solution of $Ax = b$ is

$$x^* = A^+ b = V_1 \Sigma^{-1} U_1^H \in \mathcal{R}(V_1).$$

Note that

$$x^* \in \mathcal{R}(A^H) = \mathcal{R}(V_1 \Sigma U_1^H) \in \mathcal{R}(V_1).$$

Therefore, x^* is the unique least Frobenius norm solution of $Ax = b$. □

Theorem 2.2. *Suppose the system* (1.1) *is consistent. By choosing the initial matrix pair*

$$X_1 = A^H U_1 B^H + \overline{N} V_1^T \overline{M} + RA^T \overline{U_1} B^T S + RNV_1^H MS, \tag{2.21}$$

$$Y_1 = \overline{D} U_1^T \overline{C} + G^H V_1 H^H + RDU_1^H CS + RG^T \overline{V_1} H^T S, \tag{2.22}$$

where $U_1, V_1 \in \mathbb{C}^{m\times n}$ *are two arbitrary matrices,* (*especially, take* $X_1 = Y_1 = 0 \in \mathbb{C}^{p\times q}$)*, then the solution* (X^*, Y^*) *given by Algorithm* 1 *is the unique least Frobenius norm solution of system* (1.1).

Proof. From Lemma 2.2, (2.7) is equivalent to

$$\widehat{T}z = \widehat{f}, \tag{2.23}$$

where

$$\widehat{T} := \begin{pmatrix} B^T \otimes A & (D^T \otimes C)P_{pq} \\ (N^T \otimes M)P_{pq} & H^T \otimes G \\ B^H S \otimes \overline{A}R & (D^H R \otimes \overline{C}S)P_{pq} \\ (N^H R \otimes \overline{M}S)P_{pq} & H^H S \otimes \overline{G}R \end{pmatrix}, \tag{2.24}$$

$$z := \begin{pmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{pmatrix}, \quad \widehat{f} := \begin{pmatrix} \mathrm{vec}(E) \\ \mathrm{vec}(F) \\ \mathrm{vec}(\overline{E}) \\ \mathrm{vec}(\overline{F}) \end{pmatrix}.$$

For $U_1, V_1 \in \mathbb{C}^{m\times n}$, it follows from the formulas (2.21), (2.22) and (2.2) that

$$\begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(Y_1) \end{pmatrix} = \begin{pmatrix} \overline{B} \otimes A^H & (M^H \otimes \overline{N})P_{mn} & (SB) \otimes (RA^T) & \big((SM^T) \otimes (RN)\big)P_{mn} \\ (C^H \otimes \overline{D})P_{mn} & \overline{H} \otimes G^H & \big((SC^T) \otimes (RD)\big)P_{mn} & (SH) \otimes (RG^T) \end{pmatrix}$$
$$\times \begin{pmatrix} \mathrm{vec}(U_1) \\ \mathrm{vec}(V_1) \\ \mathrm{vec}(\overline{U_1}) \\ \mathrm{vec}(\overline{V_1}) \end{pmatrix} \in \mathcal{R}(\widehat{T}^H). \tag{2.25}$$

Clearly, if we select X_1, Y_1 according to the formulas (2.21) and (2.22), respectively, then matrices X_k, Y_k generated by Algorithm 1 will satisfy

$$\begin{pmatrix} \mathrm{vec}(X_k) \\ \mathrm{vec}(Y_k) \end{pmatrix} \in \mathcal{R}(\widehat{T}^H).$$

Hence, from Lemma 2.6, the solution (X^*, Y^*) generated by Algorithm 1 is the unique least Frobenius norm solution of system (1.1). □

3. The optimal (R,S)-conjugate solution

In this section, the optimal approximate solution problem of the system (1.1) will be reconsidered as follows:

$$\|\widetilde{X} - \widehat{X}\| + \|\widetilde{Y} - \widehat{Y}\| = \min_{X,\ Y\in\mathcal{S}} \|X - \widehat{X}\| + \|Y - \widehat{Y}\|, \tag{3.1}$$

where, $\widehat{X},\ \widehat{Y} \in \mathbb{C}^{p\times q}$ are given matrices, $\mathcal{S}$ denotes non-empty solution set of system (1.1). $\widetilde{X},\ \widetilde{Y}$ are the matrices to be determined.

Firstly, for $P \in \mathbb{RSC}^{p\times q},\ Q \in \mathbb{SRSC}^{p\times q}$, notice that the relation

$$\begin{aligned}\langle P, Q\rangle &= \mathbf{Re}[\mathrm{tr}(P^H Q)] = \mathbf{Re}[\mathrm{tr}(SP^T RQ)]\\ &= \mathbf{Re}[\mathrm{tr}(P^T RQS)] = -\mathbf{Re}[\mathrm{tr}(P^T \overline{Q})]\\ &= -\mathbf{Re}[\mathrm{tr}((P^T\overline{Q})^T)] = -\mathbf{Re}[\mathrm{tr}(Q^H P)]\\ &= -\langle P, Q\rangle\end{aligned} \tag{3.2}$$

holds.

Next, an approach of solving the optimal approximate solution of the problem (3.1) will be provided. Since the system (1.1) is consistent, note that $X,\ Y \in \mathbb{RSC}^{p\times q}$ as well as (3.2), one gets

$$\begin{aligned}&\|X-\widehat{X}\|^2+\|Y-\widehat{Y}\|^2\\ =&\|X-\frac{\widehat{X}+R\overline{\widehat{X}}S}{2}-\frac{\widehat{X}-R\overline{\widehat{X}}S}{2}\|^2+\|Y-\frac{\widehat{Y}+R\overline{\widehat{Y}}S}{2}-\frac{\widehat{Y}-R\overline{\widehat{Y}}S}{2}\|^2\\ =&\|X-\frac{\widehat{X}+R\overline{\widehat{X}}S}{2}\|^2+\|Y-\frac{\widehat{Y}+R\overline{\widehat{Y}}S}{2}\|^2+\|\frac{\widehat{X}-R\overline{\widehat{X}}S}{2}\|^2+\|\frac{\widehat{Y}-R\overline{\widehat{Y}}S}{2}\|^2\\ :=&\|X'\|^2+\|Y'\|^2+C_0,\end{aligned} \tag{3.3}$$

where

$$X'=X-\frac{\widehat{X}+R\overline{\widehat{X}}S}{2},\ Y'=Y-\frac{\widehat{Y}+R\overline{\widehat{Y}}S}{2},\ C_0=\|\frac{\widehat{X}-R\overline{\widehat{X}}S}{2}\|^2+\|\frac{\widehat{Y}-R\overline{\widehat{Y}}S}{2}\|^2.$$

In addition, system (1.1) can be written as

$$AX'B+C(Y')^TD=E-A\frac{\widehat{X}+R\overline{\widehat{X}}S}{2}B-C(\frac{\widehat{Y}+R\overline{\widehat{Y}}S}{2})^TD:=E',$$

and

$$M(X')^TN+GY'H=F-M(\frac{\widehat{X}+R\overline{\widehat{X}}S}{2})^TN-G\frac{\widehat{Y}+R\overline{\widehat{Y}}S}{2}H:=F'.$$

Therefore, the optimal approximate (R,S)-conjugate solution of the system (1.1) will be obtained by solving the least Frobenius norm solution pairs $(X')^*,\ (Y')^*$ of the following generalized complex coupled Sylvester-transpose matrix equations

$$AX'B+C(Y')^TD=E',\ M(X')^TN+GY'H=F'.$$

Finally, the optimal approximate (R,S)-conjugate solution is normally expressed as

$$\widetilde{X}=(X')^*+\frac{\widehat{X}+R\overline{\widehat{X}}S}{2},\ \widetilde{Y}=(Y')^*+\frac{\widehat{Y}+R\overline{\widehat{Y}}S}{2}. \tag{3.4}$$

4. Numerical experiments

In this section, we report some numerical results to illustrate the efficiency of Algorithm 1. The tests have been carried out by MATLAB R2011b (7.13). The relevant parameters of convergence performance including iteration step (denoted as 'IT'), elapsed CPU time (denoted as 'CPU'), as well as the residual error (denoted as 'RES') defined by

$$\mathrm{RES} := \|E - AX_kB - CY_k^TD\| + \|F - MX_k^TN - GY_kH\|,$$

the relative error (denoted as 'δ_k') defined by

$$\delta_k := \frac{\|X_k - X^*\| + \|Y_k - Y^*\|}{\|X^*\| + \|Y^*\|}.$$

The stop criterion is set with RES $< 10^{-10}$.

Example 4.1. Firstly, consider Eqs. (1.1) with the following matrices:

$$A = \begin{pmatrix} 8-i & 4+i & 3-i \\ 1-2i & 1+2i & 1+3i \\ 2-2i & 1-6i & 1-5i \\ 5+i & 4-i & 5+2i \\ 1+i & 3-i & 1+i \end{pmatrix}, B = \begin{pmatrix} 7+2i & 1-i & 1-2i & 1+3i & 1-2i \\ 1+6i & 1-5i & 1+4i & 1-4i & 9+3i \\ 1-2i & 1-3i & 2i & 2-i & 1+i \\ 1-2i & 1+3i & 2-2i & 1+4i & 1-i \end{pmatrix},$$

$$C = \begin{pmatrix} 1+i & 1-i & 2+2i & -3-i \\ 3+2i & 4+i & 2-i & 2+i \\ 1-i & 4-i & 15+i & 1-i \\ -3-i & -1+i & -2+i & 2-i \\ 2-i & 4+i & 1-3i & 2+2i \end{pmatrix}, D = \begin{pmatrix} 3+2i & 11-2i & 3-i & 16+i & 1-i \\ 1+3i & 5-i & 4i & -2i & 2-2i \\ 2-2i & 1-3i & 2-i & 1+i & 1-i \end{pmatrix},$$

$$M = \begin{pmatrix} 3+i & 3-2i & 6+2i & 3-i \\ 2+2i & 3+i & 2-i & 2+i \\ 1+i & 2-i & 3+i & 4-i \\ 1-i & -1+i & 1+i & 3+i \\ 4-2i & 2+i & 1-3i & 1-i \end{pmatrix}, N = \begin{pmatrix} 2-2i & 2+2i & 3+5i & 1-i & 2+i \\ 2-i & 5+i & 1-4i & 2i & 1+2i \\ 2-2i & 1-3i & 2-i & 1-i & 1+i \end{pmatrix},$$

$$G = \begin{pmatrix} 12-i & 1+i & 3-i \\ 1-i & 1+2i & 1-2i \\ 3-2i & -3i & 2-5i \\ 3+i & 2-2i & 1-2i \\ 1-i & 3-i & 1-i \end{pmatrix}, \ H = \begin{pmatrix} 1-3i & 3+i & 1-2i & 4+3i & 1-2i \\ 1+3i & 4-2i & 2-2i & 3-3i & 2-3i \\ 2i & 3-3i & 3-2i & 1+i & 1-i \\ 1+i & 1-i & 2+i & 1-3i & 2+2i \end{pmatrix},$$

where

$$R = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix}, \quad S = \begin{pmatrix} 0\,1\,0\,0 \\ 1\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,1\,0 \end{pmatrix},$$

and $E,\ F$ are generated by the given exact solution (X^*, Y^*).

We choose the initial matrix pair $X_1 = 0,\ Y_1 = 0$. By implementing Algorithm 1, we obtain the exact solution just need to 34 steps of iteration numbers:

$$X_{34} = \begin{pmatrix} 2+2i & 2-2i & 1+i & 1-i \\ 1-3i & 1+2i & 3-i & 4+i \\ 1-2i & 1+3i & 4-i & 3+i \end{pmatrix},$$

$$Y_{34} = \begin{pmatrix} 3+3i & 3-3i & 2+i & 2-i \\ 2-3i & 1+3i & 3-2i & 4+2i \\ 1-3i & 2+3i & 4-2i & 3+2i \end{pmatrix}.$$

The elapsed CPU time is 0.0098 (t), the corresponding residual error, relative error and least Frobenius norms are

$$\begin{aligned} &\mathrm{RES} = 2.0703e-011, \quad \delta_k = 9.1735e-015, \\ &\|X_{34}\| = 10.1980, \quad \|Y_{34}\| = 12.5698, \end{aligned}$$

respectively. The above results are presented in Fig. 1.

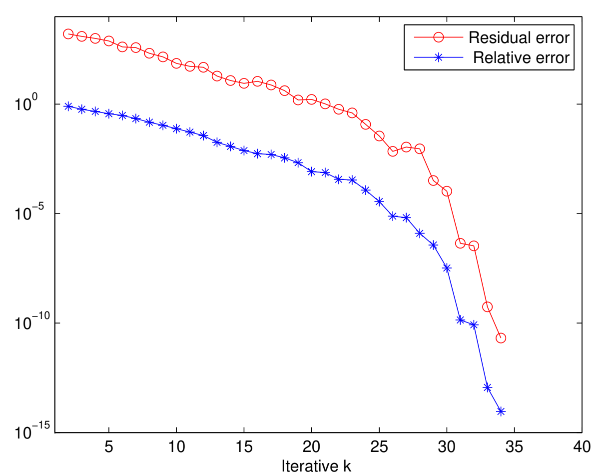

Figure 1. The Relative error and Residual error for Example 4.1.

Example 4.2. Consider Eqs. (1.1) with the following matrices:

$$A=\begin{pmatrix} 1+i & 1-i & 2+2i & -3-i \\ 3+2i & 4+i & 2-i & 2+i \\ 1-i & 4-i & 7+i & 2-i \\ -1-i & -1+i & -1+i & 2+i \\ 4-i & 4+i & 2-3i & 1+i \end{pmatrix}, B=\begin{pmatrix} 2i & 2i & 3-i & 1+i & 1-i \\ 0 & 5-i & 4i & -2i & 1-2i \\ -2i & 3i & 4-i & 1+i & 1-i \end{pmatrix},$$

$$C=\begin{pmatrix} 4-i & 3+i & 4-i \\ -2i & 1-2i & 3i \\ 5-2i & -6i & 1-5i \\ 5+i & 4-i & 2i \\ 1+i & 2-i & 1+i \end{pmatrix}, D=\begin{pmatrix} -2i & -i & -2i & 3i & 2i \\ 6i & 5i & 4i & 4i & 3i \\ 2i & 3i & 2i & i & i \\ i & i & 2i & 4i & i \end{pmatrix},$$

$$M=\begin{pmatrix} 2-i & 1+i & 3-i \\ 1-i & 1+2i & 2i \\ 3-2i & -3i & 2-5i \\ 3+i & 2-i & 1-2i \\ 1-i & 3-i & 1-i \end{pmatrix}, \quad N=\begin{pmatrix} 1-2i & 1-i & 2-2i & 2+3i & 1-2i \\ 3i & 2-2i & 2i & -4i & -3i \\ 2i & -3i & -2i & 1+i & 1-i \\ 1+i & 1-i & i & 3i & 2i \end{pmatrix},$$

$$G=\begin{pmatrix} 2+i & 1-2i & 1+2i & 3-i \\ 2+2i & 3+i & 2-i & 2+i \\ 1+i & 2-i & 3+i & 6-i \\ 1-i & -1+i & 1+i & 3+i \\ 4-2i & 2+i & 1-3i & 1-i \end{pmatrix}, H=\begin{pmatrix} 1-2i & 1+2i & 3+i & 1-i & 1+i \\ 2-i & 5+i & 1-4i & 2i & 1+2i \\ 2-2i & 1-3i & 2-i & 1-i & 1+i \end{pmatrix},$$

where

$$R=\begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}, \quad S=\begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix},$$

and $E,\ F$ are chosen by the exact solution $(X^*,\ Y^*)$.

Analogously, choosing the initial matrix pair $X_1=0,\ Y_1=0$ for the sake of getting the least Frobenius norms solution, running Algorithm 1, it gets the solution also only with 31 iteration steps:

$$X_{31} = \begin{pmatrix} 1-i & 2+i & 3-i \\ 1+i & 3+i & 2-i \\ 4-i & 5+i & 1-i \\ 4+i & 1+i & 5-i \end{pmatrix} \in \mathcal{RSC}^{4\times 3},$$

$$Y_{31} = \begin{pmatrix} 3-2i & 5+i & 1-i \\ 3+2i & 1+i & 5-i \\ 2-i & 6+i & 1-3i \\ 2+i & 1+3i & 6-i \end{pmatrix} \in \mathcal{RSC}^{4\times 3}.$$

The elapsed CPU time is 0.0096 (t), the corresponding residual error, relative error and least norms are

$$\begin{aligned} &\mathrm{RES} = 1.0084e-011, \quad \delta_k = 5.3890e-015, \\ &\|X_{31}\| = 11.1355, \quad \|Y_{31}\| = 13.6382, \end{aligned}$$

respectively. The results are presented in Fig. 2.

Moreover, we will find the optimal approximation solution $\widetilde{X}^*$, $\widetilde{Y}^*$ of the problem (3.1). Firstly, it sets

$$\widehat{X} = \begin{pmatrix} 3-i & 1+i & 2-i \\ 4+i & 2+i & 3-i \\ 5-i & 6+i & 3+i \\ 4-i & 1+2i & 2+i \end{pmatrix}, \ \widehat{Y} = \begin{pmatrix} 1-2i & 2-i & 4-i \\ 3+i & 2+i & 3-i \\ 1-i & 3+i & 1-2i \\ 3+i & 2+i & 4-i \end{pmatrix}.$$

From the analysis in Section 3 and (3.4), it products

$$\widetilde{X}^* = \begin{pmatrix} 1.0000-1.0000i & 2.0000+1.0000i & 3.0000-1.0000i \\ 1.0000+1.0000i & 3.0000+1.0000i & 2.0000-1.0000i \\ 4.0000-1.0000i & 5.0000+1.0000i & 1.0000-1.0000i \\ 4.0000+1.0000i & 1.0000+1.0000i & 5.0000-1.0000i \end{pmatrix},$$

$$\widetilde{Y}^* = \begin{pmatrix} -0.5000-1.5000i & 2.5000+0.0000i & 4.0000-1.0000i \\ -0.5000+1.5000i & 4.0000+1.0000i & 2.5000-0.0000i \\ 1.5000-2.0000i & 4.5000+2.0000i & 0.5000-2.0000i \\ 1.5000+2.0000i & 0.5000+2.0000i & 4.5000-2.0000i \end{pmatrix}.$$

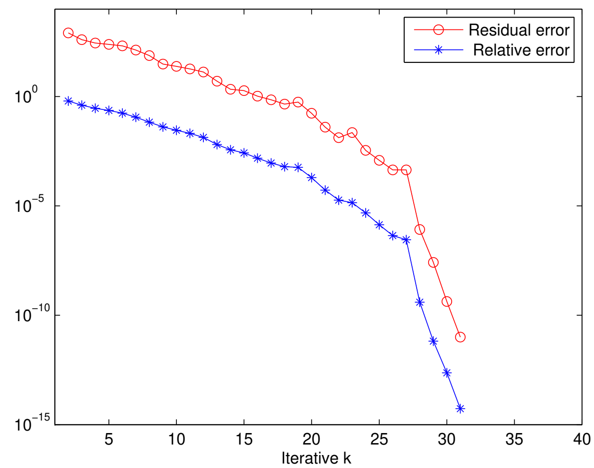

Figure 2. The Relative error and Residual error for Example 4.2.

Example 4.3. Consider Eqs. (1.1) with the following matrices:

$$\begin{aligned}
A &= \mathrm{tril}(\mathrm{rand}(m,p),1) * i - 0.4 * \mathrm{ones}(m,p) \in \mathbb{C}^{m\times p},\\
B &= \mathrm{tril}(\mathrm{rand}(q,n),1) * i - 1.5 * \mathrm{ones}(q,n) \in \mathbb{C}^{q\times n},\\
C &= \mathrm{tril}(\mathrm{rand}(m,q),1) * i - 1.12 * \mathrm{ones}(m,q) \in \mathbb{C}^{m\times q},\\
D &= \mathrm{triu}(\mathrm{rand}(p,n),1) * i - 1.09 * \mathrm{ones}(p,n) \in \mathbb{C}^{p\times n},\\
M &= \mathrm{tril}(\mathrm{rand}(m,q),1) * i \in \mathbb{C}^{m\times q},\\
N &= \mathrm{tril}(\mathrm{rand}(p,n),1) \in \mathbb{C}^{p\times n},\\
G &= \mathrm{tril}(\mathrm{rand}(m,p),1) - 0.4 * \mathrm{ones}(m,p) \in \mathbb{C}^{m\times p},\\
H &= \mathrm{triu}(\mathrm{rand}(q,n),1) * i \in \mathbb{C}^{q\times n},
\end{aligned}$$

where

$$R = fliplr(eye(p)) \in \mathbb{R}^{p\times p},\ S = fliplr(eye(q)) \in \mathbb{R}^{q\times q},$$

and the right matrices E, F are generated by the (R,S)-conjugate rectangular matrices $X^* = \mathrm{fourdiag}(-i,2,2,i) \in \mathbb{C}^{p\times q}$, $Y^* = \mathrm{fourdiag}(1-i,1,1,i+i) \in \mathbb{C}^{p\times q}$.

In this example, the coefficient matrices are randomly generated, so we test the problem by taking the average value with 100 experiments. The concrete numerical results are listed in Tables 1-3. Also, we can see the error curve with respect to the iterative number k in Figs. 3-9. All the numerical examples demonstrate the proposed Algorithm is very efficient.

Table 1. Numerical results for Example 4.3 with $n = 50,\ p = 10,\ q = 11$.

	$m = 1000$	$m = 200$	$m = 100$	$m = 50$
IT	477	460	450	430
CPU	3.2804	0.7025	0.5261	0.3124
RES	$8.2625e-011$	$9.7891e-011$	$8.9053e-011$	$5.8936e-011$
δ_k	$6.2234e-013$	$5.5674e-013$	$4.5219e-013$	$3.2235e-013$

Table 2. Numerical results for Example 4.3 with $m = 1000$, $p = 10$, $q = 11$.

	$n = 500$	$n = 200$	$n = 100$	$n = 50$
IT	675	670	520	485
CPU	46.2749	20.3371	8.1740	3.6709
RES	$9.8690e-011$	$5.6114e-011$	$8.9810e-011$	$7.5620e-011$
δ_k	$7.13503e-013$	$6.1003e-013$	$4.1063e-013$	$3.0383e-013$

Table 3. Numerical results for Example 4.3 with $m = 500$, $n = 300$.

	$p = 20$, $q = 21$	$p = 10$, $q = 11$	$p = 8$, $q = 9$	$p = 6$, $q = 7$
IT	1006	512	430	227
CPU	32.5398	10.8271	7.7464	4.9403
RES	$8.0427e-011$	$9.4303e-011$	$9.4683e-011$	$6.7878e-011$
δ_k	$6.3533e-013$	$4.1203e-013$	$3.0503e-013$	$4.0381e-013$

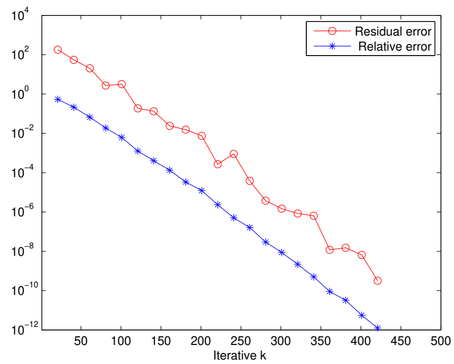

Figure 3. The Relative error and Residual error for Example 4.3 with $m = 200, n = 50, p = 10, q = 11$.

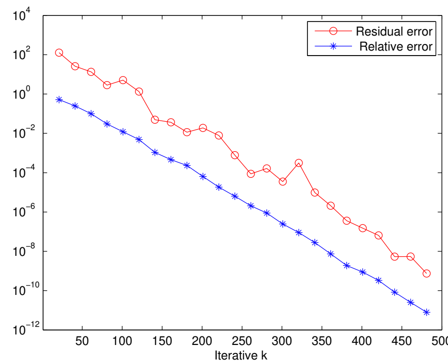

Figure 4. The Relative error and Residual error for Example 4.3 with $m = 100, n = 50, p = 10, q = 11$.

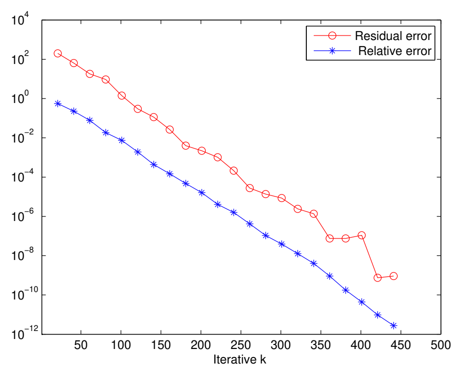

Figure 5. The Relative error and Residual error for Example 4.3 with $m = 50, n = 50, p = 10, q = 11$.

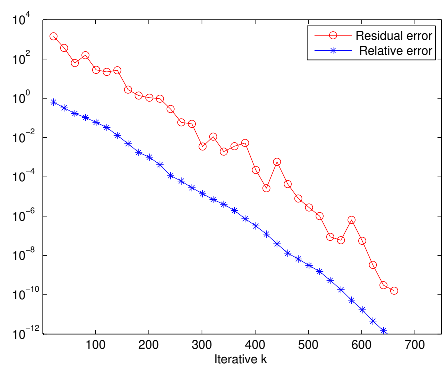

Figure 6. The Relative error and Residual error for Example 4.3 with $m = 1000, n = 500, p = 10, q = 11$.

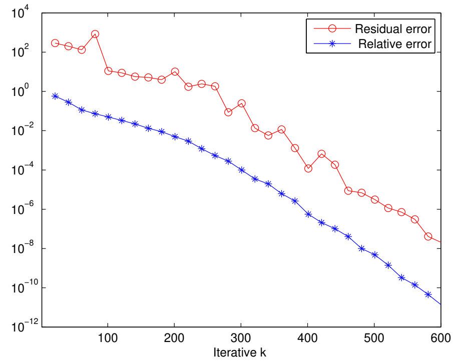

Figure 7. The Relative error and Residual error for Example 4.3 with $m = 1000, n = 200, p = 10, q = 11$.

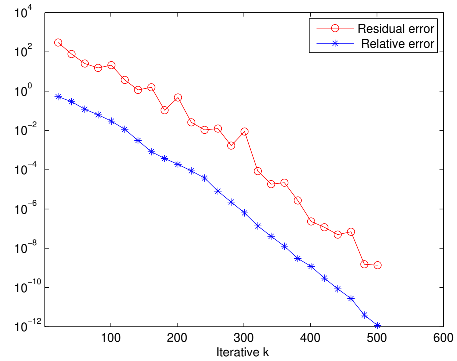

Figure 8. The Relative error and Residual error for Example 4.3 with $m = 1000, n = 100, p = 10, q = 11$.

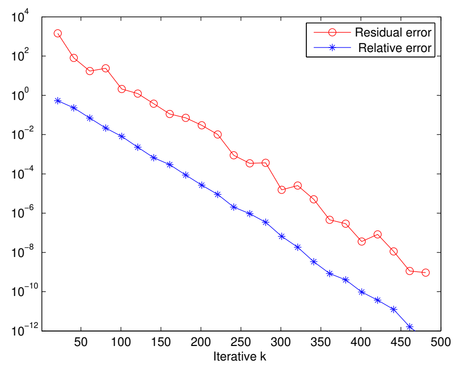

Figure 9. The Relative error and Residual error for Example 4.3 with $m = 1000, n = 50, p = 10, q = 11$.

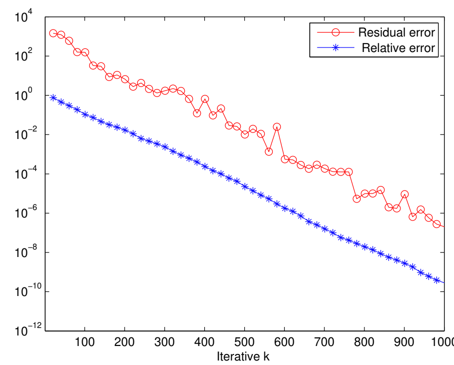

Figure 10. The Relative error and Residual error for Example 4.3 with $m = 500, n = 300, p = 20, q = 21$.

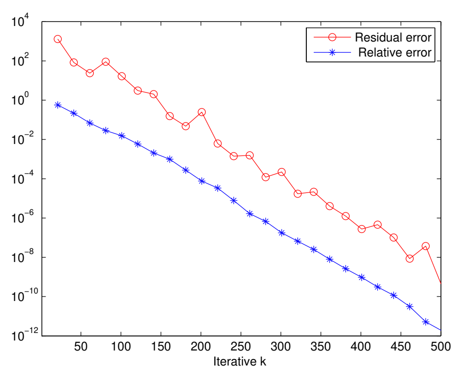

Figure 11. The Relative error and Residual error for Example 4.3 with $m = 500, n = 300, p = 10, q = 11$.

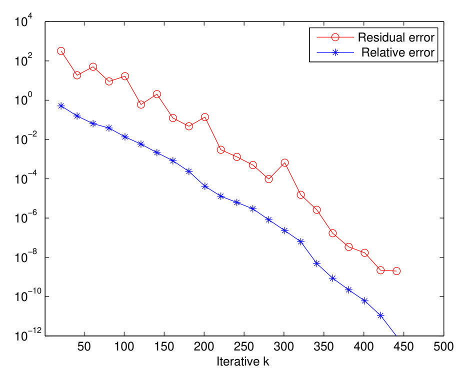

Figure 12. The Relative error and Residual error for Example 4.3 with $m = 500, n = 300, p = 8, q = 9$.

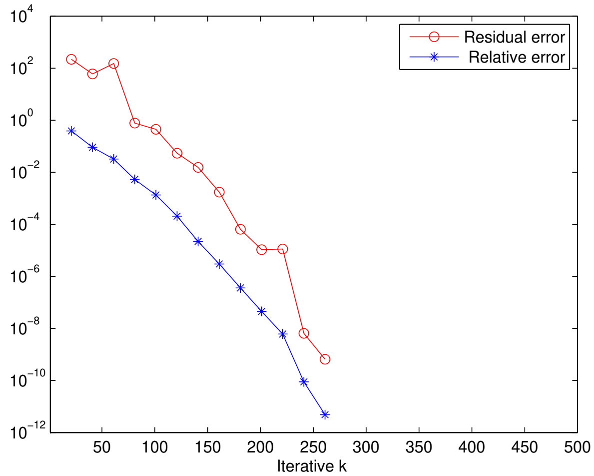

Figure 13. The Relative error and Residual error for Example 4.3 with $m = 500, n = 300, p = 6, q = 7$.

5. Conclusion

An efficient iterative method is investigated for solving the generalized complex coupled Sylvester-transpose linear matrix equations $AXB + CY^TD = E,\ EX^TF + GYH = F$ over (R,S)-conjugate matrix solutions. Using the properties of Kronecker product, vec operator as well as permutation matrix, the exact solution of the above matrix equations can be determined by the MCG method within finite iterative steps in the absence of roundoff-error for any given initial value of (R,S)-conjugate matrix. Moreover, we also consider the least Frobenius norm solution by choosing a special initial iteration guess. Furthermore, we analyze fully the optimal approximate solution problem by exploiting the least Frobenius norm solution of a new generalized complex coupled Sylvester matrix equations. Finally, a number of numerical examples are provided to demonstrate the presented iterative algorithm is effective.

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