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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR A SYSTEM OF NONLINEAR FRACTIONAL MULTI-POINT BOUNDARY VALUE PROBLEMS WITH *P*-LAPLACIAN OPERATOR*

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Abstract In this paper, we deal with a coupled system of nonlinear fractional multi-point boundary value problems with p-Laplacian operator. The existence and multiplicity of positive solutions are obtained by employing Leray-Schauder alternative theory, Leggett-Williams fixed point theorem and Avery-Henderson fixed point theorem. As an application, two examples are given to illustrate the effectiveness of our main results.

Keywords fractional differential system, p-Laplacian operator, coupled boundary conditions, fixed point theorem.

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1. Introduction

This paper deals with the existence of multiple positive solutions for the following system nonlinear fractional differential equations multi-point boundary problems with p-Laplacian operator:

$$D_{0^+}^{\beta_1}(\varphi_{p_1}(D_{0^+}^{\alpha_1}u(t))) = f(t,u(t),v(t)), \quad t\in(0,1), \tag{1.1}$$

$$D_{0^+}^{\beta_2}(\varphi_{p_2}(D_{0^+}^{\alpha_2}v(t))) = g(t,u(t),v(t)), \quad t\in(0,1), \tag{1.2}$$

subject to the boundary conditions

$$\begin{cases} u(0)=0, \quad D_{0^+}^{\gamma_1}u(1)=\sum_{i=1}^{m-2}\xi_{1i}D_{0^+}^{\gamma_1}u(\eta_{1i}), \\ D_{0^+}^{\alpha_1}u(0)=0, \quad \varphi_{p_1}(D_{0^+}^{\alpha_1}u(1))=\sum_{i=1}^{m-2}\zeta_{1i}\varphi_{p_1}(D_{0^+}^{\alpha_1}u(\eta_{1i})), \end{cases} \tag{1.3}$$

$$\begin{cases} v(0)=0, \quad D_{0^+}^{\gamma_2}v(1)=\sum_{i=1}^{m-2}\xi_{2i}D_{0^+}^{\gamma_2}u(\eta_{2i}), \\ D_{0^+}^{\alpha_2}v(0)=0, \quad \varphi_{p_2}(D_{0^+}^{\alpha_2}v(1))=\sum_{i=1}^{m-2}\zeta_{2i}\varphi_{p_2}(D_{0^+}^{\alpha_2}v(\eta_{2i})), \end{cases} \tag{1.4}$$

where $1<\alpha_i,\beta_i\leqslant 2$, $0<\gamma_i\leqslant 1$, $D_{0^+}^{\alpha_i}$, $D_{0^+}^{\beta_i}$, $D_{0^+}^{\gamma_i}$ are the standard Riemann-Liouville fractional derinatives, $\varphi_{p_i}(s)=|s|^{p_i-2}s$, $p_i>1$, $\varphi_{p_i}^{-1}=\varphi_{q_i}$, $\frac{1}{p_i}+\frac{1}{q_i}=1(i=1,2)$. We make the following assumptions:

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(s_0) $3 < \alpha_i + \beta_i \leqslant 4$, $\alpha_i - \gamma_i - 1 > 0$, i=1,2 ;
(s_1) $0 < \xi_{1i}, \eta_{1i}, \zeta_{1i} < 1$ $(i = 1, 2, \cdots, m-2)$ satisfy that

$$A_1 = 1 - \sum_{i=1}^{m-2} \xi_{1i}\eta_{1i}^{\alpha_1-\gamma_1-1} > 0, \quad B_1 = 1 - \sum_{i=1}^{m-2} \zeta_{1i}\eta_{1i}^{\beta_1-1} > 0;$$

(s_2) $0 < \xi_{2i}, \eta_{2i}, \zeta_{2i} < 1$ $(i = 1, 2, \cdots, m-2)$ satisfy that

$$A_2 = 1 - \sum_{i=1}^{m-2} \xi_{2i}\eta_{2i}^{\alpha_2-\gamma_2-1} > 0, \quad B_2 = 1 - \sum_{i=1}^{m-2} \zeta_{2i}\eta_{2i}^{\beta_2-1} > 0.$$

Fractional calculus provides an excellent tool in various fields of scientists and mathematicians due to high profile accuracy and usability. Fractional calculus has made great advances in the past years. Compared with integer order differential, fractional differential can better describe some physical phenomenons, that is why academics of different areas have paid great attention to study it. For more details of some results on fractional differential equations, we refer the readers to see [2–7, 11–17, 28].

In the last years, many scholars study the fractional order differential equation boundary value problems with p-Laplacian operator, see [4,9,10,15,18–20,22–27,29–32]. In [26], the authors consider the following boundary value problem of nonlinear fractional differential equation with p-Laplacian operator:

$$\begin{cases} D^{\alpha}(\phi_p(D^{\alpha}u(t))) = f(t, u(t)), \quad t \in [0,1]_T, \\ u(0) = u(\sigma(1)) = D^{\alpha}u(0), \quad D^{\alpha}u(\sigma(1)) = 0, \end{cases}$$

where $1 < \alpha \leqslant 2$ is a real number, the time scale T is a nonempty closed subset of R. D^{α} is the comfortable fractional derivative on time scale, and $f, g \in C([0, \sigma(1)] \times [0,\infty), [0,\infty))$. By the use of the approach method and fixed-point theorems on cone, some existence and multiplicity results of positive solutions are acquired. Li et al. [23] considered the positive solutions for p-Laplacian fractional differential equation with a parameter:

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}u(t))) = \lambda f(t, u(t)), \quad t \in (0,1), \\ [\phi_p(D_{0^+}^{\alpha}u(0))]^{(i)} = 0, \quad i = 0, 1, 2, \ldots, l-2, \\ [\phi_p(D_{0^+}^{\alpha}u(t))]'|_{t=1} = b[\phi_p(D_{0^+}^{\alpha}u(t))]'|_{t=\xi}, \\ u^{(j)}(0) = 0, \quad u'(1) = au'(\xi), \quad j = 0, 1, 2, \ldots, n-2, \end{cases}$$

where $\lambda > 0$, $3 < n-1 < \alpha \leqslant n$, $3 < l-1 < \beta \leqslant l$, and $l + n - 1 < \alpha + \beta \leqslant l + n$. The existence and nonexistence of positive solutions are obtained for the boundary value problems based on the properties of Green's function and Guo-Krasnosel'skill fixed point theorem.

On the other hand, the system of fractional differential equations boundary value problems with p-Laplacian have developed very rapidly. More and more researchers pay attention to consider the existence results for coupled systems involving fractional differential equations, see [9, 10, 20, 22, 25, 30]. In [10], the authors deal with a coupled system of singular p-Laplacian differential equations involving fractional differential-integral conditions

$$\begin{cases} -D^{\beta_1}(\varphi_{p_1}(-D^{\alpha_1}u_1))(t) = \lambda f_1(u_1(t), D^{\gamma_1}u_1(t), D^{\gamma_2}u_2(t)), \quad t \in [0,1], \\ -D^{\beta_2}(\varphi_{p_2}(-D^{\alpha_2}u_2))(t) = f_2(t, u_2(t)), \quad t \in [0,1], \\ D^{\alpha_i}u_i(0) = D^{\alpha_i}u_i(1) = 0, \\ D^{\gamma_i}u_i(0) = 0, \quad D^{\alpha_i-1}u(1) = \xi_i I^{w_i}(D^{\gamma_i}u_i(\eta_i)), \quad i = 1, 2, \end{cases}$$

where the nonlinearity $f_1(x, y, z)$ may be singular at $x = 0, y = 0, z = 0$. An eigenvalue interval for the existence of positive solutions were obtained via the Schauder's fixed point theorem and the upper and lower solution method. Hao et.al [9] considered the following system of nonlinear fractional differential equations nonlocal boundary value problems with parameters

$$\begin{cases} -D^{\alpha_1}_{0+}(\varphi_{p_1}(D^{\beta_1}_{0+}u(t))) = \lambda f(t, u(t), v(t)), \quad t \in [0,1], \\ -D^{\alpha_2}_{0+}(\varphi_{p_2}(D^{\beta_2}_{0+}v(t))) = \mu g(t, u(t), v(t)), \quad t \in [0,1], \\ u(0) = u(1) = u'(0) = u'(1), \quad D^{\beta_1}_{0+}u(0) = 0, \quad D^{\beta_1}_{0+}u(1) = b_1 D^{\beta_1}_{0+}u(\eta_1), \\ v(0) = v(1) = v'(0) = v'(1), \quad D^{\beta_2}_{0+}v(0) = 0, \quad D^{\beta_2}_{0+}v(1) = b_2 D^{\beta_1}_{0+}v(\eta_2), \end{cases}$$

where $\alpha_i \in (1, 2]$, $\beta_i \in (3, 4]$, $\eta_i \in (0, \eta_i^{\frac{1-\alpha_i}{p_i-1}})$, $i = 1, 2$. $f, g \in C([0, 1] \times [0, \infty)^2, [0, \infty))$, λ and μ are positive parameters. The authors derived various existence results in terms of different combinations of superlinearity and sublinearity of the nonlinearities.

Motivated by the aforementioned papers, we investigate the existence and multiplicity of positive solutions for a system of nonlinear fractional differential equations multi-point boundary value problems with p-Laplacian operator. By employing Leray-Schauder alternative theory, Avery-Henderson fixed point theorem and Legget-Williams fixed point theorem, we will discuss the existence and multiplicity of positive solutions for the system (1.1)-(1.4). The result obtained in this paper it is possible to replace multi-point boundary conditions by integral boundary conditions with minor modifications. As application, two examples are presented to illustrate the main results.

2. Preliminaries

In this section, we will present some preliminaries and lemmas that will be used in the proof of our main results.

Definition 2.1 ([21]). The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

where $n - 1 < \alpha < n$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([21]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to R$ is given by

$$D^\alpha_{0+} y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0, +\infty)$.

Lemma 2.1 ([19]). *Let $y \in C[0,1]$. Then the fractional order BVP*

$$\begin{cases} D_{0^+}^{\beta_1}(\varphi_{p_1}(D_{0^+}^{\alpha_1} u(t))) = y(t), \quad t \in (0,1), \\ u(0) = 0, \quad D_{0^+}^{\gamma_1} u(1) = \sum_{i=1}^{m-2} \xi_{1i} D_{0^+}^{\gamma_1} u(\eta_{1i}), \\ D_{0^+}^{\alpha_1} u(0) = 0, \quad \varphi_{p_1}(D_{0^+}^{\alpha_1} u(1)) = \sum_{i=1}^{m-2} \zeta_{1i} \varphi_{p_1}(D_{0^+}^{\alpha_1} u(\eta_{1i})) \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)y(\tau)d\tau\right) ds,$$

where

$$G_1(t,s) = G_{11}(t,s) + \bar{G}_{12}(t,s),$$
$$H_1(t,s) = H_{11}(t,s) + H_{12}(t,s).$$

In which

$$G_{11}(t,s) = \begin{cases} \dfrac{t^{\alpha_1-1}(1-s)^{\alpha_1-\gamma_1-1} - (t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{t^{\alpha_1-1}(1-s)^{\alpha_1-\gamma_1-1}}{\Gamma(\alpha_1)}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$\bar{G}_{12}(t,s) = \frac{t^{\alpha_1-1}}{A_1\Gamma(\alpha_1)}\Bigg[\sum_{\eta_{1i}>s} \xi_{1i}[\eta_{1i}^{\alpha_1-\gamma_1-1}(1-s)^{\alpha_1-\gamma_1-1} - (\eta_{1i}-s)^{\alpha_1-\gamma_1-1}] + \sum_{\eta_{1i}\leq s} \xi_{1i}\eta_{1i}^{\alpha_1-\gamma_1-1}(1-s)^{\alpha_1-\gamma_1-1}\Bigg], \quad t,s \in [0,1],$$

$$H_{11}(t,s) = \begin{cases} \dfrac{t^{\beta_1-1}(1-s)^{\beta_1-1} - (t-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{t^{\beta_1-1}(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$H_{12}(t,s) = \frac{t^{\beta_1-1}}{B_1\Gamma(\beta_1)}\Bigg[\sum_{\eta_{1i}>s} \zeta_{1i}[\eta_{1i}^{\beta_1-1}(1-s)^{\beta_1-1} - (\eta_{1i}-s)^{\beta_1-1}] + \sum_{\eta_{1i}\leq s} \zeta_{1i}\eta_{1i}^{\beta_1-1}(1-s)^{\beta_1-1}\Bigg], \quad t,s \in [0,1].$$

It is easy to see that

$$\bar{G}_{12}(t,s) = \frac{1}{A_1}\sum_{i=1}^{m-2} \xi_{1i} G_{12}(\eta_{1i}, s)t^{\alpha_1-1}, \quad H_{12}(t,s) = \frac{1}{B_1}\sum_{i=1}^{m-2} \zeta_{1i} H_{11}(\eta_{1i}, s)t^{\beta_1-1},$$

where

$$G_{12}(t,s) = \begin{cases} \dfrac{t^{\alpha_1-\gamma_1-1}(1-s)^{\alpha_1-\gamma_1-1} - (t-s)^{\alpha_1-\gamma_1-1}}{\Gamma(\alpha_1)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{t^{\alpha_1-\gamma_1-1}(1-s)^{\alpha_1-\gamma_1-1}}{\Gamma(\alpha_1)}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

In a similar manner, the conclusion of functions $G_2(t,s)$ and $H_2(t,s)$ for the homogeneous BVPs consistenting with the fractional differential equation (1.2) and (1.4) are gain.

Lemma 2.2 ([19]). *The function $G_k(t,s)(k=1,2)$ is continuous on $[0,1]\times[0,1]$ and has the following properties:*

(i) $G_k(t,s) \geqslant 0$, $\forall\ t,s \in (0,1)\times(0,1)$;

(ii) $G_k(t,s) \leqslant \rho_k(s)$, *and where* $\rho_k(s) = \frac{1}{A_k\Gamma(\alpha_k)}(1-s)^{\alpha_k-\gamma_k-1}$, $\forall\ t,s\in[0,1]$;

(iii) $t^{\alpha_k-1}\Omega_k(s) \leqslant G_k(t,s) \leqslant \Omega_k(s)$, $\forall t,s \in [0,1]$ *and where* $\Omega_k(s) = h_k(s) + \frac{1}{A_k}\sum_{i=1}^{m-2}\xi_{ki}G_{k2}(\eta_{ki},s)$, $h_k(s) = (1-s)^{\alpha_k-\gamma_k-1}(1-(1-s)^{\gamma_k})/\Gamma(\alpha_k)$, $s\in[0,1]$.

Lemma 2.3 ([19]). *The function $H_k(t,s)(k=1,2)$ satisfies the following inequalities:*

(i) $H_k(t,s) \geqslant 0$, $\forall (t,s)\in(0,1)\times(0,1)$;

(ii) $H_k(t,s) \leqslant \omega_k(s)$, *where* $\omega_k(s) = \frac{1}{B_k\Gamma(\beta_k)}(1-s)^{\beta_k-1}$ *for all* $(t,s)\in(0,1)\times(0,1)$;

(iii) $H_k(t,s) \geqslant \sigma_k\nu_k(s)$, *where* $\nu_k(s) = \frac{1}{B_k}\sum_{i=1}^{m-2}\zeta_{ki}H_{k1}(\eta_{ki},s)$, $\forall\ t\in[\theta_1,\theta_2], s\in(0,1)$, $\sigma_k = \min_{t\in[\theta_1,\theta_2]} t^{\beta_k-1}$.

Lemma 2.4 ([8]). *Let E be a Banach space, $K\subset E$. Suppose that $T: K\to K$ is a completely continuous operator. Let $\epsilon(T) = \{x\in K : x = \varepsilon T(x), 0<\varepsilon<1\}$. Then either*

(i) T has at least a fixed point, or

(ii) the set $\varepsilon(T)$ is unbounded.

Lemma 2.5 ([1]). *Let K be a cone in a real Banach space E. If ζ and ϕ are increasing, non-negative continuous functional on K. Let χ be a non-negative continuous functional on K with $\chi(0)=0$ such that, for some positive constants c and λ,*

$$\zeta(u) \leqslant \chi(u) \leqslant \phi(u),\ \text{and}\ \|u\| \leqslant \lambda\zeta(u),$$

for all $u\in\overline{K(\zeta,c)}$. Suppose that there exist positive numbers $a<b<r$ such that

$$\chi(\tau u) \leqslant \tau\chi(u),\ \text{for all}\ 0\leqslant\tau\leqslant 1\ \text{and}\ u\in\partial K(\chi,b).$$

If $T:\overline{K(\zeta,c)}\to K$ is a completely continuous operator satisfying

(i) $\zeta(Tu) > r$ for all $u\in\partial K(\zeta,r)$;

(ii) $\chi(Tu) < b$ for all $u\in\partial K(\chi,b)$;

(iii) $K(\phi,a)\neq\emptyset$ and $\phi(Tu) > a$ for all $u\in\partial K(\phi,a)$.

Then T has at least two fixed points u_1 and u_2 such that $a<\phi(u_1)$ with $\chi(u_1)<b$ and $b<\chi(u_2)$ with $\zeta(u_2)<c$.

Lemma 2.6 ([18]). *(Leggett-Williams) Let $E = (E, \|\cdot\|_1)$ be a cone of E, and $K_r = \{x\in K : \|x\| < r\}$. Suppose there exists a concave nonnegative continuous functional ψ on K with $\psi(x)\leqslant\|x\|$ for $x\in\overline{K}_r$. Let $T:\overline{K}_r\to\overline{K}_r$ be a completely continuous operator. Assume there are numbers r_1, r_2 and r_3 with $0<r_1<r_2<r_3\leqslant r$ such that*

(i) $\{x \in K(\psi, r_2, r_3) | \psi(x) > r_2\} \neq \emptyset$, *and* $\psi(Tx) > r_2$ *for* $x \in K(\psi, r_2, r_3)$;

(ii) $\|Tx\| < r_1$ *for* $\|x\| \leqslant r_1$;

(iii) $\psi(Tx) > r_2$ *for* $x \in K(\psi, r_2, r)$ *with* $\|x\| > r_3$.

Then T *has at least three fixed points* x_1, x_2, x_3 *with* $\|x_1\| < r_1$, $r_2 < \psi(x_2)$, $r_1 < \|x_3\|$ *with* $\psi(x_3) < r_2$.

Let $X = C[0,1]$, X is a Banach space with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Let the Banach space $E = X \times X$ be endowed with the norm $\|(u,v)\|_1 = \|u\| + \|v\|$. For $\theta_1, \theta_2 \in (0,1)$ and $\theta_1 < \theta_2$, denote

$$K = \big\{(u,v) \in E : u(t) \geqslant 0, v(t) \geqslant 0, \forall\ t \in [0,1], \min_{t\in I}\{u(t)+v(t)\} \geqslant \delta \|(u,v)\|_1\big\},$$

where $I = [\theta_1, \theta_2]$, $\delta = \min\{\delta_1, \delta_2\}$ and $\delta_k = \min_{t\in I} t^{\alpha_k - 1}$, $k = 1,2$, then K is a cone of E. Define the operators $T_1, T_2 : E \to X$ and $T : E \to E$ as follows:

$$\begin{aligned}
&T_1(u,v)(t) = \int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau) f(\tau, u(\tau), v(\tau)) d\tau\right) ds, \quad t \in [0,1],\\
&T_2(u,v)(t) = \int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^1 H_2(s,\tau) g(\tau, u(\tau), v(\tau)) d\tau\right) ds, \quad t \in [0,1],\\
&T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \quad (u,v) \in E.
\end{aligned}$$

It is clear that if (u,v) is a fixed point of the operator T in K, then (u,v) is a positive solution of system (1.1)-(1.4).

3. Main results

Denote

$$\begin{aligned}
&M = \frac{1}{A_1\Gamma(\alpha_1)\Gamma(\alpha_1-\gamma_1)(B_1\Gamma(\beta_1+1))^{q_1-2}},\\
&N = \frac{1}{A_2\Gamma(\alpha_2)\Gamma(\alpha_2-\gamma_2)(B_2\Gamma(\beta_2+1))^{q_2-2}},\\
&W = \min\left\{\frac{1}{2}\left(\int_0^1 \Omega_1(s)\varphi_{q_1}(\int_0^1 \omega_1(\tau)d\tau)ds\right)^{-1}, \frac{1}{2}\left(\int_0^1 \Omega_2(s)\varphi_{q_2}(\int_0^1 \omega_2(\tau)d\tau)ds\right)^{-1}\right\},\\
&D = \max\left\{\frac{1}{2}\left(\int_{\theta_1}^{\theta_2} \Omega_1(s)\varphi_{q_1}(\int_{\theta_1}^{\theta_2} \sigma_1\nu_1(\tau)d\tau)ds\right)^{-1}, \frac{1}{2}\left(\int_{\theta_1}^{\theta_2} \Omega_2(s)\varphi_{q_2}(\int_{\theta_1}^{\theta_2} \sigma_2\nu_2(\tau)d\tau)ds\right)^{-1}\right\}.
\end{aligned} \tag{3.1}$$

Theorem 3.1. *Suppose the conditions* (s_0)–(s_2) *holds,* $f, g \in C([0,1]\times[0,\infty)^2, [0,\infty))$ *and there exist real constants* $m_k, n_k \geqslant 0$, $k = 1,2$ *and* $m_0 > 0$, $n_0 > 0$ *such that for all* $u, v \in K$, *we have*
(H_1) $f(t,u,v) \leqslant \varphi_{p_1}(m_0 + m_1 u + m_2 v)$, $\quad g(t,u,v) \leqslant \varphi_{p_2}(n_0 + n_1 u + n_2 v)$ *and it is assumed that* $Mm_1 + Nn_1 < 1$, $Mm_2 + Nn_2 < 1$.
Then the the systems (1.1)-(1.4) has at least one solution.

Proof. Firstly, we show that the operator $T : K \to K$ is completely continuous. For $u, v \in K$ and $t \in [0,1]$, we have

$$\begin{aligned}
\|T(u,v)\|_1 &= \max_{t\in[0,1]} |T_1(u,v)(t)| + \max_{t\in[0,1]} |T_2(u,v)(t)| \\
&= \max_{t\in[0,1]} \left\{ \int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right) ds \right\} \\
&\quad + \max_{t\in[0,1]} \left\{ \int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^1 H_2(s,\tau)g(\tau,u(\tau),v(\tau))d\tau\right) ds \right\} \\
&\leqslant \int_0^1 \Omega_1(s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right) ds \\
&\quad + \int_0^1 \Omega_2(s)\varphi_{q_2}\left(\int_0^1 H_2(s,\tau)g(\tau,u(\tau),v(\tau))d\tau\right) ds.
\end{aligned}$$

On the other hand, for $t \in I$, we have

$$T_1(u,v)(t) \geqslant \int_0^1 t^{\alpha_1-1}\Omega_1(s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right) ds,$$

then $T_1(u,v)(t) \geqslant \delta_1\|T_1(u,v)\|$. Similarly, $T_2(u,v)(t) \geqslant \delta_2\|T_2(u,v)\|$. Therefore $\min\left\{T_1(u,v)(t) + T_2(u,v)(t)\right\} \geqslant \delta\|T(u,v)\|_1$. It is well know that $T(K) \subset K$. By the continuous of functions f and g, the operator T is continuous.

Let $\Omega \subset K$ be bounded. Then there exists L_1 and L_2 such that

$$f(t,u(t),v(t)) \leqslant \varphi_{p_1}(L_1), \quad g(t,u(t),v(t)) \leqslant \varphi_{p_2}(L_2).$$

Then for any $(u,v) \in \Omega$, it follows from Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
T_1(u,v)(t) &= \int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right) ds \\
&\leqslant \int_0^1 \rho_1(s)\varphi_{q_1}\left(\int_0^1 \omega_1(\tau)f(\tau,u(\tau),v(\tau))d\tau\right) ds \\
&\leqslant \frac{L_1}{A_1\Gamma(\alpha_1)\Gamma(\alpha_1-\gamma_1)(B_1\Gamma(\beta_1+1))^{q_1-1}} = L_1 M.
\end{aligned}$$

And also

$$T_2(u,v)(t) \leqslant \frac{L_2}{A_2\Gamma(\alpha_2)\Gamma(\alpha_2-\gamma_2)(B_2\Gamma(\beta_2+1))^{q_2-1}} = L_2 N.$$

Hence, from the above inequalities, the operator T is uniformly bounded.

Next, we shall show that T is equicontinuous. Let $0 \leqslant t_1 < t_2 \leqslant 1$, we get

$$\begin{aligned}
|T_2(u,v)(t) - T_1(u,v)(t)| =& \left| \int_0^1 G_1(t_2,s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right) ds \right. \\
&\left. - \int_0^1 G_1(t_1,s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right) ds \right| \\
\leqslant& \frac{L_1}{(B_1\Gamma(\beta_1+1))^{q_1-1}} \left| \int_0^1 [G_1(t_2,s) - G_2(t_1,s)]ds \right|
\end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
&\left|\int_0^1[G_1(t_2,s)-G_2(t_1,s)]ds\right|\\
=&\left|\int_{t_1}^1(t_2^{\alpha_1-\gamma_1-1}-t_1^{\alpha_1-\gamma_1-1})(1-s)^{\alpha_1-\gamma_1-1}ds\right.\\
&+\int_{t_1}^{t_2}[t_2^{\alpha_1-\gamma_1-1}(1-s)^{\alpha_1-\gamma_1-1}-(t_2-s)^{\alpha_1-\gamma_1-1}]ds\\
&+\int_{t_1}^1[(t_1-s)^{\alpha_1-\gamma_1-1}-(t_2-s)^{\alpha_1-\gamma_1-1}]ds\\
&+\frac{1}{A_1}(t_2^{\alpha_1-1}-t_1^{\alpha_1-1})\int_0^1\sum_{i=1}^{m-2}\xi_iG_{12}(\eta_i,s)ds\\
&+\int_{t_2}^1(t_2^{\alpha_1-1}-t_1^{\alpha_1-1})(1-s)^{\alpha_1-\gamma_1-1}ds\\
&\left.-\int_{t_1}^{t_2}t_1^{\alpha_1-1}(1-s)^{\alpha_1-\gamma_1-1}ds\right|\\
\leqslant&\frac{1}{\Gamma(\alpha_1-\gamma_1)}\left|t_1^{\alpha_1-\gamma_1-1}(1-t_1)^{\alpha_1-\gamma_1}-t_2^{\alpha_1-\gamma_1-1}(1-t_2)^{\alpha_1-\gamma_1}\right|\\
&+\frac{1}{\Gamma(\alpha_1-\gamma_1)}\left|t_2^{\alpha_1-1}(1-t_2)^{\alpha_1-\gamma_1}-t_1^{\alpha_1-1}(1-t_1)^{\alpha_1-\gamma_1}\right|\\
&+\frac{\sum_{i=1}^{m-2}\xi_i}{A_1\Gamma(\alpha_1)}\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right|.
\end{aligned}
\tag{3.3}
$$

From (3.2) and (3.3), it is easy to see that $|T_2(u,v)(t)-T_1(u,v)(t)|\to 0$ as $t_2-t_1\to 0$. The operator T is equicontinuous. Therefore, $T:K\to K$ is a completely continuous operator.

Finally, it will be verified that the set $\epsilon=\{(u,v)\in K:(u,v)=\varepsilon T(u,v),0<\varepsilon<1\}$ is bounded. Let $(u,v)\in\epsilon$, we have $(u,v)=\varepsilon T(u,v)$. For $\forall\ t\in[0,1]$, we get

$$u(t)=\varepsilon T_1(u,v)(t),\quad v(t)=\varepsilon T_2(u,v)(t).$$

From (H_1), we can obtain

$$
\begin{aligned}
u(t)=\varepsilon T_1(u,v)(t)\leqslant T_1(u,v)(t)=&\int_0^1G_1(t,s)\varphi_{q_1}\left(\int_0^1H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds\\
\leqslant&\int_0^1\rho_1(s)\varphi_{q_1}\left(\int_0^1\omega_1(\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds\\
\leqslant&\frac{m_0+m_1u+m_2v}{A_1\Gamma(\alpha_1)\Gamma(\alpha_1-\gamma_1)(B_1\Gamma(\beta_1+1))^{q_1-1}}
\end{aligned}
$$

and

$$v(t)\leqslant\frac{n_0+n_1u+n_2v}{A_2\Gamma(\alpha_2)\Gamma(\alpha_2-\gamma_2)(B_2\Gamma(\beta_2+1))^{q_2-1}}.$$

We get $\|u\|\leqslant M(m_0+m_1u+m_2v)$, $\|v\|\leqslant N(n_0+n_1u+n_2v)$. Thus,

$$\|u\|+\|v\|\leqslant(Mm_0+Nn_0)+(Mm_1+Nn_1)\|u\|+(Mm_2+Nn_2)\|v\|.$$

Therefore,

$$\|(u,v)\|_1 \leqslant \frac{Mm_0+Nn_0}{S},$$

where $S=\min\{1-(Mm_1+Nn_1),1-(Mm_2+Nn_2)\}$. The set ϵ is bounded. By Lemma 2.4, T has at least one fixed point. Hence, the system (1.1)-(1.4) has at least one positive solution. □

Theorem 3.2. *Suppose the conditions* (s_0)–(s_2) *holds,* $f,g\in C([0,1]\times[0,\infty)^2,[0,\infty))$ *and there exist positive real numbers* $0<a<b<c$ *such that the functions* f,g *satisfying the following conditions:*
(H_2) $f(t,u,v)>\varphi_{p_1}(\frac{cD}{\delta})$, $g(t,u,v)>\varphi_{p_2}(\frac{cD}{\delta})$ *for* $t\in I$ *and* $(u,v)\in[c,\frac{c}{\delta}]$*;*
(H_3) $f(t,u,v)<\varphi_{p_1}(bW)$, $g(t,u,v)<\varphi_{p_1}(bW)$ *for* $t\in[0,1]$ *and* $(u,v)\in[0,\frac{b}{\delta}]$*;*
(H_4) $f(t,u,v)>\varphi_{p_1}(\frac{aD}{\delta})$, $g(t,u,v)>\varphi_{p_2}(\frac{aD}{\delta})$ *for* $t\in I$ *and* $(u,v)\in[\delta a,a]$*.*
Then the system (1.1)-(1.4) has at least two positive solutions (u_1,v_1) *and* (u_2,v_2) *such that*

$$a<\max_{t\in[0,1]}\{u_1(t)+v_1(t)\},\ \textit{with}\ \max_{t\in I}\{u_1(t)+v_1(t)\}<b,$$
$$b<\max_{t\in I}\{u_2(t)+v_2(t)\},\ \textit{with}\ \min_{t\in I}\{u_2(t)+v_2(t)\}<c.$$

Proof. Due to Theorem 3.1, we know $T:K\to K$ is a completely continuous operator. Let

$$\zeta(u,v)=\min_{t\in I}\{u(t)+v(t)\},\quad \chi(u,v)=\max_{t\in I}\{u(t)+v(t)\},$$
$$\phi(u,v)=\max_{t\in[0,1]}\{u(t)+v(t)\},\quad K(\zeta,c)=\{(u,v)\in K:\zeta(u,v)<c\}.$$

Obviously, $\zeta(u,v)\leqslant\chi(u,v)\leqslant\phi(u,v)$ and

$$\|(u,v)\|_1\leqslant\frac{1}{\delta}\min_{t\in I}\{u(t)+v(t)\}=\frac{1}{\delta}\zeta(u,v).$$

For all $(u,v)\in K$, $\mu\in[0,1]$, we have

$$\chi(\mu u,\mu v)=\max_{t\in I}\{\mu u(t)+\mu v(t)\}=\mu\chi(u,v).$$

It is clear that $\chi(0,0)=0$. Next, we shall verify that condition (i) of Lemma 2.5 is satisfied. Since $(u,v)\in\partial K(\zeta,c)$, we get

$$\min_{t\in I}\{u(t)+v(t)\}=c,\ \text{and}\ c\leqslant\|u\|+\|v\|\leqslant\frac{c}{\delta}.$$

From (H_2), one has

$$\begin{aligned}\zeta(T(u,v))=&\min_{t\in I}T(u,v)(t)\\=&\min_{t\in I}\left\{\int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds\right.\\&\left.+\int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^1 H_2(s,\tau)g(\tau,u(\tau),v(\tau))d\tau\right)ds\right\}\\\geqslant&\delta\int_{\theta_1}^{\theta_2}\Omega_1(s)\varphi_{q_1}\left(\int_{\theta_1}^{\theta_2}\sigma_1\nu_1(\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds\end{aligned}$$

$$
\begin{aligned}
&+\delta\int_{\theta_1}^{\theta_2}\Omega_2(s)\varphi_{q_2}\left(\int_{\theta_1}^{\theta_2}\sigma_2\nu_2(\tau)g(\tau,u(\tau),v(\tau))d\tau\right)ds\\
\geqslant& c.
\end{aligned}\tag{3.4}
$$

Now, we will show that condition (ii) of Lemma 2.5 is contented. Since $(u,v)\in\partial K(\chi,b)$, we have

$$0\leqslant u(t)+v(t)\leqslant\|u\|+\|v\|\leqslant\frac{b}{\delta}\ \text{for}\ t\in[0,1].$$

By (H_3), we have

$$
\begin{aligned}
\chi(T(u,v))=&\max_{t\in I}T(u,v)(t)\\
=&\max_{t\in I}\left\{\int_0^1G_1(t,s)\varphi_{q_1}\left(\int_0^1H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds\right.\\
&\left.+\int_0^1G_2(t,s)\varphi_{q_2}\left(\int_0^1H_2(s,\tau)g(\tau,u(\tau),v(\tau))d\tau\right)ds\right\}\\
\leqslant&\int_0^1\Omega_1(s)\varphi_{q_1}\left(\int_0^1\omega_1(\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds\\
&+\int_0^1\Omega_2(s)\varphi_{q_2}\left(\int_0^1\omega_2(\tau)g(\tau,u(\tau),v(\tau))d\tau\right)ds\\
\leqslant& b.
\end{aligned}
$$

Finally, we shall show that condition (iii) of Lemma 2.5 is contented. Since $(0,0)\in K$ and $a>0$. $K(\phi,a)\neq\emptyset$. Let $(u,v)\in\partial K(\phi,a)$,

$$\delta a\leqslant u(t)+v(t)\leqslant\|u\|+\|v\|=a\quad\text{for}\ t\in I.$$

It follows from (H_4) that

$$\phi(T(u,v))=\max_{t\in[0,1]}T(u,v)(t)\geqslant a.$$

The process of proof is same as (3.4), so we omit it.

Therefore, the hypotheses of Lemma 2.5 have been satisfied. Thus, the operator $T(u,v)$ has at least two fixed points (u_1,v_1) and (u_2,v_2) such that

$$
\begin{aligned}
&a<\max_{t\in[0,1]}\{u_1(t)+v_1(t)\},\ \text{with}\ \max_{t\in I}\{u_1(t)+v_1(t)\}<b,\\
&b<\max_{t\in I}\{u_2(t)+v_2(t)\},\ \text{with}\ \min_{t\in I}\{u_2(t)+v_2(t)\}<c.
\end{aligned}
$$

Hence, the system (1.1)-(1.4) has at least two positive solutions (u_1,v_1) and (u_2,v_2). □

Theorem 3.3. *Suppose the conditions $(s_0)-(s_2)$ holds and there exist constants $0<r_1<r_2<r_3\leqslant r$ such that*

(H_5) $\limsup\limits_{u+v\to0}\max\limits_{t\in[0,1]}\dfrac{f(t,u,v)}{\varphi_{p_1}(u+v)}<\varphi_{p_1}(W),\quad\limsup\limits_{u+v\to0}\max\limits_{t\in[0,1]}\dfrac{g(t,u,v)}{\varphi_{p_2}(u+v)}<\varphi_{p_2}(W)$;

(H_6) $f(t,u,v)\geqslant\varphi_{p_1}(\frac{r_2D}{\delta})$, $g(t,u,v)\geqslant\varphi_{p_2}(\frac{r_2D}{\delta})$ *for* $t\in I$, $(u,v)\in[r_2,\frac{r_2}{\delta}]\times[r_2,\frac{r_2}{\delta}]$.

Then the systems (1.1)-(1.4) has at least three positive solutions (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with* $\|(u_1, v_1)\|_1 < r_1$, $r_2 < \zeta(u_2, v_2) < \|(u_2, v_2)\|_1 < r$, $r_3 < \|(u_3, v_3)\|_1$ *with* $\zeta(u_3, v_3) < r_2$.

Proof. Due to Theorem 3.1, there exists enough $r > r_1 > 0$, $T : \overline{K}_r \to \overline{K}_r$ is completely continuous. Since (H_5), we get

$$\begin{aligned} f(t,u,v) &\leqslant \varphi_{p_1}(W(u+v)), \quad t \in [0,1], \quad 0 \leqslant u+v \leqslant r_1, \\ g(t,u,v) &\leqslant \varphi_{p_2}(W(u+v)), \quad t \in [0,1], \quad 0 \leqslant u+v \leqslant r_1. \end{aligned}$$

Suppose $(u, v) \in \overline{K}_{r_1}$, then $\|(u,v)\|_1 \leqslant r_1$, we have

$$\begin{aligned} \|T(u,v)\|_1 = &\max_{0\leqslant t\leqslant 1} \big\{T_1(u,v)(t) + T_2(u,v)(t)\big\} \\ = &\max_{0\leqslant t\leqslant 1} \bigg\{ \int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds \\ &+ \int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^1 H_2(s,\tau)g(\tau,u(\tau),v(\tau))d\tau\right)ds\bigg\} \\ \leqslant &r_1 W \int_0^1 \Omega_1(s)\varphi_{q_1}\left(\int_0^1 \omega_1(\tau)d\tau\right)ds \\ &+ r_1 W \int_0^1 \Omega_2(s)\varphi_{q_2}\left(\int_0^1 \omega_2(\tau)d\tau\right)ds \\ \leqslant &r_1. \end{aligned}$$

This shows that condition (ii) of Lemma 2.6 is fulfilled.

Denote $r_2 > 0, r_3 = \frac{r_2}{\delta} < r$, $K(\zeta, r_2, r_3) = \{(u,v) \in K : r_2 \leqslant \zeta(u,v), \|(u,v)\|_1 \leqslant r_3\}$. The definition of ζ is defined as in above Theorem 3.2. We choose $u(t) + v(t) = \frac{r_2}{\delta}$ for $t \in [\theta_1, \theta_2]$. It is clear that $u(t) + v(t) = \frac{r_2}{\delta} \in K(\zeta, r_2, \frac{r_2}{\delta})$, and $\zeta(u,v) = \frac{r_2}{\delta} > r_2$, and so $\{(u,v) \in K(\zeta, r_2, \frac{r_2}{\delta}))|\zeta(u,v) > r_2\} \neq \emptyset$. Thus, for all $(u,v) \in K(\zeta, r_2, \frac{r_2}{\delta}))$, one has

$$\begin{aligned} \zeta(T(u,v)(t)) = &\min_{t\in I} \big|T_1(u,v)(t) + T_2(u,v)(t)\big| \\ = &\min_{t\in I} \bigg\{ \int_0^1 G_1(t,s)\varphi_{q_1}\left(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds \\ &+ \int_0^1 G_2(t,s)\varphi_{q_2}\left(\int_0^1 H_2(s,\tau)g(\tau,u(\tau),v(\tau))d\tau\right)ds\bigg\} \\ \geqslant &\frac{r_2 D}{\delta}\int_{\theta_1}^{\theta_2} \delta_1\Omega_1(s)\varphi_{q_1}\left(\int_{\theta_1}^{\theta_2} \sigma_1\nu_1(\tau)d\tau\right)ds \\ &+ \frac{r_2 D}{\delta}\int_{\theta_1}^{\theta_2} \delta_2\Omega_2(s)\varphi_{q_2}\left(\int_{\theta_1}^{\theta_2} \sigma_2\nu_2(\tau)d\tau\right)ds \\ \geqslant &r_2. \end{aligned}$$

Hence the condition (i) of Lemma 2.6 is verified. Next, we prove that (iii) of Lemma 2.6.

$\min_{t\in I}\big|T_1(u,v)(t) + T_2(u,v)(t)\big| > \delta\|T(u,v)\|_1 > r_2$ for $(u,v) \in K(\zeta, r_2, r)$ with $\|T(u,v)\|_1 > \frac{r_2}{\delta}$.

To sum up, all the conditions of Lemma 2.6 are fulfilled, then there exist three positive solutions (u_1,v_1), (u_2,v_2) and (u_3,v_3) satisfying $\|(u_1,v_1)\|_1 < r_1$, $r_2 < \zeta(u_2,v_2) < \|(u_2,v_2)\|_1 < r$, $r_3 < \|(u_3,v_3)\|_1$ with $\zeta(u_3,v_3) < r_2$. □

Corollary 3.1. *Suppose the conditions $(s_0)-(s_2)$ and (H_6) hold. The function $f(t,u,v), g(t,u,v)$ satisfies*

$$\limsup_{u+v\to 0}\max_{t\in[0,1]}\frac{f(t,u,v)}{\varphi_{p_1}(u+v)}=0,\quad \limsup_{u+v\to 0}\max_{t\in[0,1]}\frac{g(t,u,v)}{\varphi_{p_2}(u+v)}<\varphi_{p_2}(W).$$

Then the systems (1.1)-(1.4) has at least three positive solutions.

Corollary 3.2. *Suppose the conditions $(s_0)-(s_2)$ and (H_6) hold. The function $f(t,u,v), g(t,u,v)$ satisfies*

$$\limsup_{u+v\to 0}\max_{t\in[0,1]}\frac{f(t,u,v)}{\varphi_{p_1}(u+v)}<\varphi_{p_1}(W),\quad \limsup_{u+v\to 0}\max_{t\in[0,1]}\frac{g(t,u,v)}{\varphi_{p_2}(u+v)}=0.$$

Then the systems (1.1)-(1.4) has at least three positive solutions.

4. Examples

Example 4.1. Consider the following fractional differential systems

$$\begin{cases} D_{0^+}^{\frac{5}{2}}(\varphi_{p_1}(D_{0^+}^{\frac{3}{2}}u(t)))=f(t,u(t),v(t)), \quad t\in(0,1),\\ D_{0^+}^{\frac{3}{2}}(\varphi_{p_2}(D_{0^+}^{\frac{5}{2}}v(t)))=g(t,u(t),v(t)), \quad t\in(0,1),\\ u(0)=0, \quad D_{0^+}^{\frac{1}{4}}u(1)=\frac{1}{10}D_{0^+}^{\frac{1}{4}}u(\frac{1}{4})+\frac{2}{10}D_{0^+}^{\frac{1}{4}}u(\frac{1}{2}),\\ D_{0^+}^{\frac{3}{2}}u(0)=0, \quad \varphi_{p_1}(D_{0^+}^{\frac{3}{2}}u(1))=\frac{1}{6}\varphi_{p_1}(D_{0^+}^{\frac{3}{2}}u(\frac{1}{4}))+\frac{1}{3}\varphi_{p_1}(D_{0^+}^{\frac{3}{2}}u(\frac{1}{2})),\\ v(0)=0, \quad D_{0^+}^{\frac{1}{2}}v(1)=\frac{1}{3}D_{0^+}^{\frac{1}{2}}v(\frac{3}{4})+\frac{1}{5}D_{0^+}^{\frac{1}{2}}v(\frac{3}{2}),\\ D_{0^+}^{\frac{5}{2}}u(0)=0, \quad \varphi_{p_2}(D_{0^+}^{\frac{5}{2}}v(1))=\frac{1}{7}\varphi_{p_2}(D_{0^+}^{\frac{5}{2}}v(\frac{3}{4}))+\frac{1}{2}\varphi_{p_2}(D_{0^+}^{\frac{5}{2}}v(\frac{3}{2})), \end{cases} \tag{4.1}$$

where $\alpha_1=\frac{3}{2}$, $\alpha_2=\frac{5}{2}$, $\beta_1=\frac{5}{2}$, $\beta_2=\frac{3}{2}$, $\gamma_1=\frac{1}{4}$, $\gamma_2=\frac{1}{2}$, $\zeta_{11}=\frac{1}{6}$, $\zeta_{12}=\frac{1}{3}$, $\zeta_{21}=\frac{1}{7}$, $\zeta_{22}=\frac{1}{2}$, $\eta_{11}=\frac{1}{4}$, $\eta_{12}=\frac{1}{2}$, $\eta_{21}=\frac{3}{4}$, $\eta_{22}=\frac{3}{2}$, $\xi_{11}=\frac{1}{10}$, $\xi_{12}=\frac{1}{5}$, $\xi_{21}=\frac{1}{3}$, $\xi_{22}=\frac{1}{5}$, $p_1=2$, $p_2=3$, $q_1=2$, $p_1=\frac{3}{2}$, $m=4$.

Simple computation shows that

$$A_1=1-\sum_{i=1}^{2}\xi_{1i}\eta_{1i}^{\alpha_1-\gamma_1-1}=0.7611>0,\quad B_1=1-\sum_{i=1}^{2}\zeta_{1i}\eta_{1i}^{\beta_1-1}=0.8613>0;$$

$$A_2=1-\sum_{i=1}^{2}\xi_{2i}\eta_{2i}^{\alpha_2-\gamma_2-1}=0.45>0,\quad B_2=1-\sum_{i=1}^{2}\zeta_{2i}\eta_{2i}^{\beta_2-1}=0.2639>0.$$

It is clear that $(s_0)-(s_2)$ holds.

Let $f(t,u(t),v(t))=t+u(t)+v(t)$, $g(t,u(t),v(t))=(t+0.5u(t)+0.1v(t))^{\frac{1}{2}}$. If we choose $m_0=m_1=m_2=1$, $n_0=n_1=n_2=1$. Clearly, (H_1) holds. Thus, all the hypotheses of Theorem 3.1 are satisfied. Hence, the systems (4.1) has at least one solutions.

Example 4.2. Consider the following fractional differential systems

$$\begin{cases} D^{\frac{5}{3}}_{0+}(\varphi_2(D^{\frac{4}{3}}_{0+}u(t))) = f(t,u(t),v(t)), \quad t\in(0,1),\\ D^{2}_{0+}(\varphi_3(D^{\frac{5}{3}}_{0+}v(t))) = g(t,u(t),v(t)), \quad t\in(0,1),\\ u(0)=0, \quad D^{\frac{1}{3}}_{0+}u(1)=\frac{1}{2}D^{\frac{1}{3}}_{0+}u(\frac{1}{7}),\\ D^{\frac{5}{3}}_{0+}u(0)=0, \quad \varphi_2(D^{\frac{5}{3}}_{0+}u(1))=\frac{1}{4}\varphi_2(D^{\frac{5}{3}}_{0+}u(\frac{1}{7})),\\ v(0)=0, \quad D^{\frac{1}{3}}_{0+}v(1)=D^{\frac{1}{3}}_{0+}u(\frac{1}{4}),\\ D^{2}_{0+}v(0)=0, \quad \varphi_3(D^{2}_{0+}v(1))=\frac{1}{2}\varphi_3(D^{2}_{0+}v(\frac{1}{4})), \end{cases} \tag{4.2}$$

where $\alpha_1=\frac{5}{3}$, $\alpha_2=2$, $\beta_1=\frac{4}{3}$, $\beta_2=\frac{5}{3}$, $\gamma_1=\frac{1}{3}$, $\gamma_2=\frac{1}{3}$, $\zeta_{11}=\frac{1}{4}$, $\zeta_{21}=\frac{1}{2}$, $\eta_{11}=\frac{1}{7}$, $\eta_{21}=\frac{1}{4}$, $\xi_{11}=\frac{1}{2}$, $\xi_{21}=1$, $p_1=2$, $p_2=3$, $q_1=2$, $q_1=\frac{3}{2}$, $m=3$, $\theta_1=\frac{1}{3}$, $\theta_2=\frac{1}{2}$.

Simple computation shows that

$$A_1 = 1-\xi_{11}\eta_{11}^{\alpha_1-\gamma_1-1}=0.7386>0, \quad B_1=1-\zeta_{11}\eta_{11}^{\beta_1-1}=0.8693>0;$$

$$A_2 = 1-\xi_{21}\eta_{21}^{\alpha_2-\gamma_2-1}=0.6032>0, \quad B_2=1-\zeta_{21}\eta_{21}^{\beta_2-1}=0.9213>0.$$

It is clear that $(s_0)-(s_2)$ holds.

Let $f(t,u,v)=(10^6+t)(u+v)^2$, $g(t,u,v)=(10^6+t)(u+v)^3$. We obtain $\sigma_1=3^{-\frac{1}{3}}$, $\sigma_2=\frac{1}{3}$, $D=200$, $W=1.0865$. Choose $r_2=\frac{1}{10}$, then

$$f(t,u,v)=(10^6+t)(u+v)^2\geqslant 10^4>60=\varphi_{p_1}(\frac{r_2D}{\delta}),\ (u,v)\in\left[\frac{1}{10},\frac{3}{10}\right]\times\left[\frac{1}{10},\frac{3}{10}\right],$$
$$g(t,u,v)=(10^6+t)(u+v)^3\geqslant 8000>3600=\varphi_{p_2}(\frac{r_2D}{\delta}),\ (u,v)\in\left[\frac{1}{10},\frac{3}{10}\right]\times\left[\frac{1}{10},\frac{3}{10}\right].$$

So condition (H_6) was satisfied.

$$\begin{aligned}\limsup_{u+v\to 0}\max_{t\in[0,1]}\frac{f(t,u,v)}{\varphi_{p_1}(u+v)} &= \limsup_{u+v\to 0}\max_{t\in[0,1]}(10^6+t)(u+v)\\ &= \limsup_{u+v\to 0}(10^6+1)(u+v)<1.8065=\varphi_{p_1}(W),\\ \limsup_{u+v\to 0}\max_{t\in[0,1]}\frac{g(t,u,v)}{\varphi_{p_2}(u+v)} &= \limsup_{u+v\to 0}\max_{t\in[0,1]}(10^6+t)(u+v)\\ &= \limsup_{u+v\to 0}(10^6+1)(u+v)<3.2635=\varphi_{p_2}(W).\end{aligned}$$

So condition (H_5) holds. By the use of Theorem 3.3, the systems (4.2) has at least three positions solutions.

5. Conclusion

In this paper, we obtained several sufficient conditions for the existence and multiplicity of positive solutions for a coupled system of nonlinear fractional multi-point boundary value problems with p-Laplacian operator. Our results will be a useful contribution to the existing literature on the topic of fractional-order nonlocal differential equations. The results of the existence and multiplicity are demonstrated on two relevant examples.

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