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DYNAMICS OF A HIGH-ORDER NONLINEAR FUZZY DIFFERENCE EQUATION*

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Abstract This paper is concerned with the following high-order nonlinear fuzzy difference system

$$x_{n+1} = \frac{Ax_{n-m}}{B + C\prod_{i=0}^{m} x_{n-i}}, \quad n = 0, 1, 2, \cdots,$$

where x_n is a sequence of positive fuzzy numbers, the parameters and the initial conditions $x_{-m},\ x_{-m+1},\ \cdots, x_0$ are positive fuzzy numbers, m is non-negative integer. More accurately, our main purpose is to study the existence and uniqueness of the positive solutions, the boundedness of the positive solutions, the instability, local asymptotic stability and global asymptotic stability of the equilibrium points for the above equation by using the iteration method, the inequality skills, the mathematical induction, and the monotone boundedness theorem. Moreover, some numerical examples to the difference system are given to verify our theoretical results.

Keywords Fuzzy difference equation, boundedness, equilibrium point, asymptotic behavior.

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1. Introduction

Nonlinear difference equation which can be used in mathematical models describing the real world phenomenon has been studied in the fields of population biology, economics, probability theory, genetics, control engineering etc (see, e.g. [16, 17, 21]and the references therein). At percent, the research of nonlinear difference equation has been rapidly pushed forward, for a detail study of the theory of difference equations see [1, 13, 32].

Next, making a historical flash back for the equation we study in this paper, firstly, we should mention that in paper [6], Cinar investigated the global behavior

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of all positive solutions of the following rational second-order difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, 2, \cdots, \tag{1.1}$$

where the x_n is a sequence of real numbers and the initial values $x_0, x_{-1}$are positive real numbers. Similarly, in paper [2], Bajo and Liz investigated the asymptotic behavior and the stability properties of all solutions to the nonlinear second-order difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + b x_n x_{n-1}}, \quad n = 0, 1, 2, \cdots, \tag{1.2}$$

for all values of the real parameters a, b and any initial conditions $x_0, x_{-1} \in R$. In addition, Shojaei, Saadati, and Adibi [25] investigated the stability and periodic character of the following rational third-order difference equation

$$x_{n+1} = \frac{\alpha x_{n-2}}{\beta + \gamma x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, 2, \cdots, \tag{1.3}$$

where the parameters α, β, γ and the initial conditions x_0, x_{-1}, x_{-2} are real numbers. Moreover, in 2017, Wang et al. [30] considered the following nonlinear high order difference equation

$$x_{n+1} = \frac{A x_{n-k}}{B + C \prod\limits_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, 2, \cdots, \tag{1.4}$$

where the initial conditions $x_{-k}, \cdots, x_0$ are real numbers, the parameters A, B, C are positive real numbers, and k is nonnegative integer. Firstly, a sufficient and necessary condition for the existence and uniqueness of solutions for the initial value problem (1.4) is given. And then the local stability, asymptotic behavior, periodicity and oscillation of solutions for the system (1.4) are investigated. More related difference equations readers also can refer to the references [7, 15, 18, 29, 31, 33].

In recent years, the study of difference equations has attracted more and more attention from scholars. In paper [8], Clark and Kulenovic investigated the dynamics of a system of rational difference equations

$$x_{n+1} = \frac{x_n}{a + c\, y_n}, \quad y_{n+1} = \frac{y_n}{b + d\, x_n}, \quad n = 0, 1, 2, \cdots, \tag{1.5}$$

where the parameters a, b, c, d are arbitrary positive real numbers, and the initial conditions x_0, y_0 are arbitrary nonnegative real numbers. In addition, Zhang et al. [39] studied the dynamics of a system of rational third-order difference equations

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \quad y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, 2, \cdots, \tag{1.6}$$

where the parameters A, B and the initial conditions are arbitrary positive numbers. Din et al. [12] investigated the dynamics of a system of fourth-order rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}, n = 0, 1, 2, \cdots, \tag{1.7}$$

where the parameters$\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and the initial conditions are positive real numbers.

On the other hand, as we know, a amount of difference equation models usually be used to describe many practical problems [14,19,23], but the information of the difference equation model to describe many practical problems is incomplete. In view of the fact that the fuzzy set theory is a powerful tool for simulating uncertainty and processing fuzzy or subjective information in mathematical models [34], it is more meaningful to study the behavior of solutions of a class of fuzzy difference systems where the parameters and initial values are fuzzy numbers, and the solution is a sequence of fuzzy numbers (see, e.g., [22,26,27] and the references therein). As the origin of the study of fuzzy difference equations, we must mention that in 1996, Deeba et al. [10] studied the following first-order linear fuzzy difference equation

$$x_{n+1} = wx_n + q, \; n = 0, \; 1, \; \cdots, \tag{1.8}$$

where x_n is a sequence of fuzzy numbers and $x_0, \; q, \; w$ are fuzzy numbers, which arise in population genetics. Moreover, Deeba and Korvin [9] studied the following second-order linear fuzzy difference equation

$$C_{n+1} = C_n - abC_{n-1} + m, \; n = 0, \; 1, \; \cdots, \tag{1.9}$$

where $a, \; b, \; m, \; C_0, \; C_1$ are fuzzy numbers and C_n is a sequence of fuzzy numbers. The equation is a linearization of a nonlinear model that determines the level of carbon dioxide in the blood.

Recently, Zhang et al. [37] study the existence, asymptotic behavior of the positive solutions of the following fuzzy nonlinear difference equation

$$x_{n+1} = \frac{Ax_n + x_{n-1}}{B + x_{n-1}}, \;\; n = 0, \; 1, \; 2, \; \cdots, \tag{1.10}$$

where $\{x_n\}$ is a sequence of positive fuzzy number, A, B and the initial conditions x_{-1}, x_0 are positive fuzzy numbers. Moreover, in 2014, Zhang et al. [38] continuously deal with the existence, the boundedness and the asymptotic behavior of the positive solutions for a first order fuzzy Ricatti difference equation

$$x_{n+1} = \frac{A + x_n}{B + x_n}, \;\; n = 0, \; 1, \; 2, \; \cdots, \tag{1.11}$$

where $\{x_n\}$ is a sequence of positive fuzzy numbers, A, B and the initial value x_0 are positive fuzzy numbers.

More recently, in 2017, Wang et al. [28] consider the existence and uniqueness of the positive solutions and the asymptotic behavior of the equilibrium points of the following five-order fuzzy nonlinear difference equation

$$x_{n+1} = \frac{Ax_{n-1}x_{n-2}}{D + Bx_{n-3} + Cx_{n-4}}, n = 0, 1, 2, \cdots, \tag{1.12}$$

where $\{x_n\}$ is a sequence of positive fuzzy numbers, the initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 and the parameters A, B, C, D are positive fuzzy numbers. In 2018, Zhang et al. [35] investigate the dynamical behavior of the following nonlinear fuzzy logistic discrete time system

$$x_{n+1} = Ax_n(\tilde{1} - x_n), \; n = 1, 2, \cdots, \tag{1.13}$$

where parameter A, $\tilde{1}$ and the initial condition x_0 are positive fuzzy numbers. In 2019, Zhang et al. [36] consider the following discrete time Beverton-Holt model with fuzzy uncertainty parameters

$$x_{n+1} = \frac{Ax_n}{\tilde{1} + Bx_n}, \ n = 1, 2, \cdots, \tag{1.14}$$

where x_n is population at the nth generation, A denotes a productivity parameter, and B controls the level of density dependence. Furthermore A, $\tilde{1}$, B and the initial value x_0 are positive fuzzy numbers.

Inspired with the previous works, in this paper, we consider the dynamics of the following nonlinear high-order fuzzy difference equation

$$x_{n+1} = \frac{Ax_{n-m}}{B + C \prod\limits_{i=0}^{m} x_{n-i}}, \quad n = 0, 1, \cdots, \tag{1.15}$$

where the parameters A, B, C and the initial conditions $x_{-m}, \cdots, x_0$ are positive fuzzy numbers, m is positive integer.

This paper is arranged as follows. In Section 2, we give some definitions and preliminary results. The main results and their proofs are given in Section 3. Finally, some numerical simulations are given in Section 4 to illustrate our theoretical analysis.

2. Preliminaries and notations

For the convenience of the readers, we give the following definitions and preliminary results, see ([3–5, 11, 20, 24])

Definition 2.1. For a set B we denote by $\bar{B}$ the closure of B. We say that a function $A : R \to [0, 1]$ is a fuzzy number if the following conditions hold

(i) A is normal, i.e., there exists $x \in R$ such that $A(x) = 1$;

(ii) A is a fuzzy convex set, i.e., $A(tx_1 + (1-t)x_2) \geqslant \min\{A(x_1), A(x_2)\}$, $\forall t \in [0, 1]$, $x_1, x_2 \in R$;

(iii) A is upper semicontinuous on R;

(iv) A is compactly supported, i.e. supp $A = \overline{\cup_{\alpha \in (0,1]} [A]_\alpha} = \overline{\{x \in R : A\,(x) > 0\}}$ is compact.

For $\alpha \in (0,\ 1]$ the $\alpha - cuts$ of A on R is defined as $[A]_\alpha = \{x \in R : A\,(x) \geqslant \alpha\}$. It is clear that $[A]_\alpha$ is a bounded closed interval in R, we say that a fuzzy number A is positive if supp $A \subset\ (0,\ \infty)$. It is obvious that if A is a positive real number then A is a positive fuzzy number and $[A]_\alpha = [A,\ A]\,,\ \alpha \in (0,\ 1]$. We say that A is a trivial fuzzy number when A is a positive fuzzy number.

For $u,\ \nu \in R_f$, $[u]_\alpha = [u_{l,\,\alpha},\ u_{r,\,\alpha}]$, $[\nu]_\alpha = [\nu_{l,\,\alpha},\ \nu_{r,\,\alpha}]$, and $\lambda \in R$, the sum $\mu + \nu$, the scalar product $\lambda\mu$, multiplication $u\nu$ and division $\frac{u}{\nu}$ in the standard interval arithmetic (SIA) setting are defined by

$$\begin{aligned}
&[\mu + \nu]_\alpha = [\mu]_\alpha + [\nu]_\alpha, \quad [\lambda\,\mu]_\alpha = \lambda\,[\mu]_\alpha, \quad \forall \alpha \in [0, 1].\\
&[u\nu]_\alpha = [\min\{u_{l,\alpha}, \nu_{l,\alpha}, u_{l,\alpha}\nu_{r,\alpha}, u_{r,\alpha}\nu_{l,\,\alpha}, u_{r,\alpha}\nu_{r,\alpha}\},\\
&\qquad\qquad \max\{u_{l,\alpha}\nu_{l,\alpha}, u_{l,\alpha}\nu_{r,\alpha}, u_{r,\,\alpha}\nu_{l,\,\alpha}, u_{r,\alpha}\nu_{r,\alpha}\}],
\end{aligned}$$

$$[\frac{u}{\nu}]_\alpha = [\min\{\frac{u_{l,\alpha}}{\nu_{l,\alpha}}, \frac{u_{l,\alpha}}{\nu_{r,\alpha}}, \frac{u_{r,\alpha}}{\nu_{l,\alpha}}, \frac{u_{r,\alpha}}{\nu_{r,\alpha}}\},$$
$$\max\{\frac{u_{l,\alpha}}{\nu_{l,\alpha}}, \frac{u_{l,\alpha}}{\nu_{r,\ \alpha}}, \frac{u_{r,\alpha}}{\nu_{l,\alpha}}, \frac{u_{r,\alpha}}{\nu_{r,\alpha}}\}], \quad 0 \notin [\nu]_\alpha.$$

Definition 2.2. Let $u,\ v$ be fuzzy numbers with $[u]_\alpha = [u_{l,\ \alpha},\ u_{r,\ \alpha}]$, $[\nu]_\alpha = [\nu_{l,\ \alpha},\ \nu_{r,\ \alpha}]$, $\alpha \in [0,\ 1]$. Then we define the metric on the fuzzy numbers set as follows

$$D(u,v) = \sup\ \max\{|u_{l,\ \alpha} - v_{l,\ \alpha}|\,,\ |u_{r,\ \alpha} - v_{r,\ \alpha}|\}\,,$$

where sup is taken for all $\alpha \in [0,\ 1]$. Then (R_f, D) is a complete metric space. For future use we define $\hat{0} \in R_f$ as

$$\hat{0}(x) = \begin{cases} 1,\ x = 0, \\ 0,\ x \neq 0. \end{cases}$$

Thus

$$[\hat{0}]_\alpha = [0,0],\ 0 < \alpha \leqslant 1.$$

Lemma 2.1. *Let $I_x,\ I_y$ be some intervals of real numbers and let $f: I_x^{k+1} \times I_y^{l+1} \to I_x$ be continuously differentiable functions. Then for every set of initial conditions $(x_i,\ y_j) \in I_x \times I_y, (i = -k,\ -k+1,\ \cdots,\ 0,\ j = -l,\ -l+1,\ \cdots,\ 0)$, the following system of difference equations*

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1},\ \cdots, x_{n-k}, y_n, y_{n-1},\ \cdots, y_{n-l}), \\ y_{n+1} = g(x_n, x_{n-1},\ \cdots, x_{n-k}, y_n, y_{n-1},\ \cdots, y_{n-l}), \end{cases} \quad n = 0,\ 1,\ 2, \cdots, \quad (2.1)$$

has a unique solution $\{\ (x_i,y_j)\}_{i=-k,j=-l}^{+\infty,\ +\infty}$.

Definition 2.3. A point $(\bar{x}, \bar{y}) \in I_x \times I_y$ is called an equilibrium point of system (2.1) if

$$\bar{x} = f(\bar{x}, \bar{x}, \cdots, \bar{x}, \bar{y}, \bar{y}, \cdots, \bar{y}), \bar{y} = g\,(\bar{x}.\bar{x}, \cdots, \bar{x}, \bar{y}, \bar{y}, \cdots, \bar{y})\,.$$

That is, $(x_n, y_n) = (\bar{x}, \bar{y})$ for $n \geq 0$ is the solution of difference system (2.1), or equivalently, $(\bar{x}, \bar{y})$ is a fixed point of the vector map $(f,\ g)$.

Definition 2.4. Assume that $(\bar{x}, \bar{y})$ be an equilibrium point of the system (2.1). Then, we have

(i) An equilibrium point $(\bar{x}, \bar{y})$ is called locally stable if for every $\delta > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y (i = -k, \cdots, 0,\ j = -l, \cdots, 0)$, with $\sum_{i=-k}^{0} |x_i - \bar{x}| < \delta$, $\sum_{j=-l}^{0} |y_j - \bar{y}| < \delta$, we have $|x_n - \bar{x}| < \varepsilon,\ |y_n - \bar{y}| < \varepsilon$ for any $n > 0$;

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is called attractor if $lim_{n\to\infty} x_n = \bar{x}$, $lim_{n\to\infty} y_n = \bar{y}$ for any initial conditions $(x_i, y_i) \in I_x \times I_y (i = -k, \cdots, 0,\ j = -l, \cdots, 0)$;

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotically stable if it is stable, and $(\bar{x}, \bar{y})$ is also attractor;

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called unstable if it is not locally stable.

Definition 2.5. Let $(\bar{x}, \bar{y})$ be an equilibrium point of the vector map $F = (f,\ x_n, \cdots, x_{n-k}, g,\ y_n, \cdots, y_{n-l})$ where f and g are continuously differential functions at $(\bar{x}, \bar{y})$. The linearized system of (2.1) about the equilibrium point $(\bar{x}, \bar{y})$ is $X_{n+1} = F(X_n) = F_j \cdot X_n$, where F_j is the Jacobian matrix of the system (2.1) about $(\bar{x}, \bar{y})$ and $X_n = (x_n, \cdots, x_{n-k}, y_n, \cdots, y_{n-l})^T$.

Definition 2.6. Let p, q, s, t be four nonnegative integers such that $p+q=n,\ s+t=m$. Splitting $x=(x_1, x_2, \cdots, x_n)$ into $x=([x]_p,\ [x]_q)$ and $y=(y_1,\ y_2, \cdots, y_m)$ into $y=([y]_s,\ [y]_t)$, where $[x]_\sigma$ denotes a vector with σ-components of x. We say that the function $f(x_1,\ x_2, \cdots, x_n,\ y_1,\ y_2, \cdots, y_m)$ possesses a mixed monotone property in subsets $I_x^n \times I_y^m$ of $R^n \times R^m$ if $f(\ [x]_p,\ [x]_q,\ [y]_s,\ [y]_t)$ is monotone non-decreasing in each component of $([x]_p, [y]_s)$ and is monotone non-increasing in each component of $([x]_q, [y]_t)$ for $(x, y) \in I_x^n \times I_y^m$. In particular, if $q=0,\ t=0,$, then it is said to be monotone non-decreasing in $I_x^n \times I_y^m$.

Lemma 2.2. *Assume that $X(n+1)=F(X(n)), n=0,1,\cdots,$ is a system of difference equations and $\bar{X}$ is the equilibrium point of this system i.e., $F(\bar{X})=\bar{X}$. Then we have*

(i) If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk $|\lambda|<1$, then $\bar{X}$ is locally asymptoticddally stable.

(ii) If one of eigenvalues of the Jacobian matrix J_F about $\bar{X}$ has norm greater than one, then $\bar{X}$ is unstable.

Lemma 2.3. *Assume that $X(n+1)=F(X(n)), n=0,1,\cdots,$ is a system of difference equations and $\bar{X}$ is the equilibrium point of this system, the characteristic polynomial of this system about the equilibrium point $\bar{X}$ is $P(\lambda)=a_0\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n$, with the real coefficients and $a_0>0$. Then all roots of the polynomial $P(\lambda)$ lie inside the open unit disk $|\lambda|<1$ if and only if*

$$\Delta_k > 0 \quad for \quad k=1,2,\cdots,n,$$

where Δ_k is the principal minor of order k of the $n \times n$ matrix

$$\Delta_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

3. Main results

Firstly, we study the existence and uniqueness of the positive solutions of Eq. (1.15).

Theorem 3.1. *Consider equation (1.15), suppose that A, B, C is positive fuzzy number, then for every positive fuzzy numbers $x_{-m},\ x_{-m+1}, \cdots, x_0$, there exists a unique positive solution x_n of Eq. (1.15) with initial values $x_{-m},\ x_{-m+1}, \cdots, x_0$.*

Proof. The proof is similar to Theorem 3.3 of [28], so we omit the proof of Theorem 3.1.

In the following theorem we investigate the asymptotic behavior of the equilibrium point of Eq. (1.15).

If x_n is the unique positive solution of Eq. (1.15) with the initial values x_{-m}, $x_{-m+1}, \cdots,\ x_0$ such that $[x_n]_\alpha=[L_{n,\alpha},\ R_{n,\alpha}],\ \alpha \in (0,1],\ n=0,1,\cdots$, then we obtain that $(L_{n,\alpha},\ R_{n,\alpha})$ satisfies the following ordinary difference equations

$$L_{n+1,\alpha}=\frac{A_{l,\alpha}L_{n-m,\alpha}}{B_{r,\alpha}+C_{r,\alpha}\prod_{i=0}^{m}R_{n-i,\alpha}},\quad R_{n+1,\alpha}=\frac{A_{r,\alpha}R_{n-m,\alpha}}{B_{l,\alpha}+C_{l,\alpha}\prod_{i=0}^{m}L_{n-i,\alpha}},\tag{3.1}$$

where $\alpha\in(0,1],\quad n=0,1,\cdots$.

In order to study the asymptotic behavior of Eq. (1.15), from (3.1), we will consider the following systems of ordinary parametric difference equations

$$y_{n+1}=\frac{A_1y_{n-m}}{B_2+C_2\prod_{i=0}^{m}z_{n-i}},\quad z_{n+1}=\frac{A_2z_{n-m}}{B_1+C_1\prod_{i=0}^{m}y_{n-i}},\quad n=0,1,\cdots,\tag{3.2}$$

where the parameters A_1,A_2,B_1,B_2,C_1,C_2 are positive real constants and $A_1\leqslant A_2,B_1\leqslant B_2,C_1\leqslant C_2$, the initial values $y_{-m},\ y_{-m+1},\cdots,\ y_0,\ z_{-m},\ z_{-m+1},\ \cdots,\ z_0$ are also positive real constants and $y_i\leq z_i,i=-m,\cdots,0$. From Theorem 3.1, we know that the systems of ordinary parametric difference equations (3.2) has a unique solution (y_n,z_n) for any initial values. Moreover, we can easily obtain that the systems (3.2) has an equilibrium point $\overline{X}_1=(\bar{y}_1,\bar{z}_1)=(0,0)$. If $A_1=A_2=A>B=B_1=B_2,C=C_1=C_2$, then Eq. (3.2) has another positive equilibrium point $\overline{X}_2$

$$\overline{X}_2=(\bar{y}_2,\bar{z}_2)=(\sqrt[m+1]{\frac{A-B}{C}},\ \sqrt[m+1]{\frac{A-B}{C}}).$$

□

Theorem 3.2. *For the equilibrium point $\overline{X}_1$ of Eq. (3.2), we have the following results:*

(i) if $A_1<B_2,A_2<B_1$, then the equilibrium point $\overline{X}_1$ is locally asymptotically stable.

(ii) if $A_1>B_2$ or $A_2>B_1$, then the equilibrium point $\overline{X}_1$ is unstable.

Proof. Let $F:(R^+)^{m+2}\to R^+,\ H:(R^+)^{m+2}\to R^+$ be multivariate function defined by

$$F(y_{n-m},z_{n-m},z_{n-m+1},\cdots,z_n)=\frac{A_1y_{n-m}}{B_2+C_2\prod_{i=0}^{m}z_{n-i}},$$

$$H(z_{n-m},y_{n-m},y_{n-m+1},\cdots,y_n)=\frac{A_2z_{n-m}}{B_1+C_1\prod_{i=0}^{m}y_{n-i}}.$$

Thus, we have

$$\begin{aligned}F_{y_{n-m}}&=\frac{A_1}{B_2+C_2\prod_{i=0}^{m}z_{n-i}},\quad F_{z_{n-i}}=-\frac{A_1C_2y_{n-m}}{(B_2+C_2\prod_{i=0}^{m}z_{n-i})^2}\prod_{j=0,j\neq i}^{m}z_{n-j},\\ H_{z_{n-m}}&=\frac{A_2}{B_1+C_1\prod_{i=0}^{m}y_{n-i}},\quad H_{y_{n-i}}=-\frac{A_2C_1z_{n-m}}{(B_1+C_1\prod_{i=0}^{m}y_{n-i})^2}\prod_{j=0,j\neq i}^{m}y_{n-j}.\end{aligned}\tag{3.3}$$

Moreover, we can easily obtain that the linearized equations of the system (3.2) about the equilibrium point $\overline{X}_1$ is

$$\varphi_{n+1} = D_1 \varphi_n, \tag{3.4}$$

where

$$\phi_n = \begin{bmatrix} y_n \\ y_{n-1} \\ \vdots \\ y_{n-m} \\ z_n \\ z_{n-1} \\ \vdots \\ z_{n-m} \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 & 0 \cdots 0 & \frac{A_1}{B_2} & 0 \cdots 0 & 0 & 0 \\ 1 & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & 0 & 0 \\ 0 & 1 & 0 \cdots 0 & 0 & 0 \cdots 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 \cdots 1 & 0 & 0 \cdots 0 & 0 & 0 \\ 0 & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & 0 & \frac{A_2}{B_1} \\ 0 & 0 & 0 \cdots 0 & 0 & 1 \cdots 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & 1 & 0 \end{bmatrix}.$$

The characteristic equation with Eq. (3.4) is

$$P(\lambda) = (\lambda^m + \frac{A_1}{B_2})(\lambda^m + \frac{A_2}{B_1}).$$

In view of $A_1 < B_2, A_2 < B_1$, this shows that we have all $|\lambda| < 1$, from Lemma 2.2, we have that the equilibrium point $\overline{X}_1$ of (3.2) is locally asymptotically stable. If $A_1 > B_2$ or $A_2 > B_1$, then at least one characteristic root $|\lambda^*| > 1$. Thus, the equilibrium point $\overline{X}_1$ of Eq. (3.2) is unstable. The proof is completed. □

Lemma 3.1. *Let I_x, I_y be some intervals of real numbers and assume that $f : I_x^{k+1} \times I_y^{l+1} \to I_x$ and $g : I_x^{k+1} \times I_y^{l+1} \to I_y$ be continuously differentiable functions satisfying mixed monotone property. If there exits*

$$\begin{cases} m_0 \leqslant \min\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \leqslant \max\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \leqslant M_0, \\ n_0 \leqslant \min\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \leqslant \max\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \leqslant N_0, \end{cases}$$

such that

$$\begin{cases} m_0 \leqslant f([m_0]_p, [M_0]_q, [n_0]_s, [N_0]_t) \leqslant f([M_0]_p, [m_0]_q, [N_0]_s, [n_0]_t) \leqslant M_0, \\ n_0 \leqslant g([m_0]_{p_1}, [M_0]_{q_1}, [n_0]_{s_1}, [N_0]_{t_1}) \leqslant g([M_0]_{p_1}, [m_0]_{q_1}, [N_0]_{s_1}, [n_0]_{t_1}) \leqslant N_0, \end{cases}$$

then there exit $(m, M) \in [m_0, M_0]^2$ and $(n, N) \in [n_0, N_0]^2$ satisfying

$$\begin{cases} M = f([M]_p, [m]_q, [N]_s, [n]_t), \quad m = f([m]_p, [M]_q, [n]_s, [N]_t), \\ N = g([M]_{p_1}, [m]_{q_1}, [N]_{s_1}, [n]_{t_1}), \quad n = g([m]_{p_1}, [M]_{q_1}, [n]_{s_1}, [N]_{t_1}). \end{cases}$$

Moreover, if $m = M$, $n = N$, then the equations (2.1) has a unique equilibrium point $(\bar{x}, \bar{y}) \in [m_0, M_0] \times [n_0, N_0]$ and every solution of (2.1) converges to $(\bar{x}, \bar{y})$.

Proof. The proof is similar to Theorem 3.8 of [28], so we omit the proof of Lemma 3.1. □

Theorem 3.3. *When m of system (3.2) is positive even numbers. If $A_1 = A_2 < B_1 = B_2, C_1 = C_2$, then the equilibrium point $\overline{X_1} = (0,\ 0)$ of the system (3.2) is global attractor for any initial conditions*

$$(\ y_{-m}, y_{-m+1}, \cdots, y_0, z_{-m}, z_{-m+1}, \cdots, z_0) \in (0, \infty)^{2m+2}.$$

Proof. Since $A_1 = A_2 = A < B_1 = B_2 = B,\ C_1 = C_2$, then the system (3.2) is changed to

$$y_{n+1} = \frac{A y_{n-m}}{B + C \prod_{i=0}^{m} z_{n-i}}, \qquad z_{n+1} = \frac{A z_{n-m}}{B + C \prod_{i=0}^{m} y_{n-i}},\ \ n = 0, 1, \cdots.$$

Let $(f, g) : (0, \infty)^{m+1} \times (0, \infty)^{m+1} \to (0, \infty) \times (0, \infty)$ be a function defined by

$$f(y_{n-m}, y_{n-m+1}, \cdots, y_n, z_{n-m}, z_{n-m+1}, \cdots, z_n) = \frac{A y_{n-m}}{B + C \prod_{i=0}^{m} z_{n-i}},$$

$$g(y_{n-m}, y_{n-m+1}, \cdots, y_n, z_{n-m}, z_{n-m+1}, \cdots, z_n) = \frac{A z_{n-m}}{B + C \prod_{i=0}^{m} y_{n-i}},$$

we can easily see that the functions f and g possess a mixed monotone property in subsets $(0, \infty)^{2m+2}$ of R^{2m+2}. Let

$$P_0 = Q_0 = \max\{y_{-m},\ y_{-m+1},\ \cdots,\ y_0,\ z_{-m},\ z_{-m+1},\ \cdots,\ z_0\},$$

and $\sqrt[m+1]{\frac{A-B}{C}} < p_0 = q_0 < 0$, we have

$$p_0 \leqslant \frac{Ap_0}{B + CQ_0^{m+1}} \leqslant \frac{AP_0}{B + C\,q_0^{m+1}} \leqslant P_0,\ q_0 \leqslant \frac{Aq_0}{B + CP_0^{m+1}} \leqslant \frac{AQ_0}{B + Cp_0^{m+1}} \leqslant Q_0.$$

Thus, from the system (3.2) and Lemma 3.1, there exist $p, P \in [p_0, P_0], q, Q \in [q_0, Q_0]$ satisfying

$$p = \frac{Ap}{B + CQ^{m+1}}, P = \frac{AP}{B + C\,q^{m+1}}, q = \frac{Aq}{B + CP^{m+1}}, Q = \frac{AQ}{B + Cp^{m+1}}.$$

Thus, we have

$$P = p = Q = q = 0.$$

It follows by Lemma 3.1 that the unique equilibrium point $(0, 0)$ of the system (3.2) is global attractor. The proof is therefore completed. □

Theorem 3.4. *If $A_1 = A_2 = A > B_1 = B_2 = B,\ C_1 = C_2$, then the positive equilibrium point $\overline{X}_2$ of the system (3.2) is unstable.*

Proof. From Eq. (3.3), we have that the linearized equation of Eq. (3.2) about the equilibrium point $\overline{X}_2$ is

$$\varphi_{n+1} = D_2 \varphi_n, \tag{3.5}$$

where

$$\phi_n = \begin{bmatrix} y_n \\ y_{n-1} \\ \vdots \\ y_{n-m} \\ z_n \\ z_{n-1} \\ \vdots \\ z_{n-m} \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & M & \cdots & M & M & M \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ M & M & M & \cdots & M & M & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

where $M = 1 - \frac{B}{A}$. The characteristic equation with (3.5) is

$$\begin{aligned} P(\lambda) = \lambda^{2m} - (1 - \frac{B}{A})^2 [(\lambda^{2m-2} + 1) + 2(\lambda^{2m-3} + \lambda) + \cdots \\ +(m+1)(\lambda^{2m-m} + \lambda^{2m-m-2}) + m\lambda^{2m-m-1}] - 2\lambda^m + 1, \end{aligned}$$

from Lemma 2.3, we easily know

$$\Delta_1 = (0) = 0, \quad \Delta_2 = \begin{pmatrix} 0 & -2M^2 \\ 1 & -M^2 \end{pmatrix} = 2M^2 > 0,$$

$$\Delta_3 = \begin{pmatrix} 0 & -2M^2 & -4M^2 \\ 1 & -M^2 & -3M^2 \\ 0 & 0 & -2M^2 \end{pmatrix} = -4M^2 < 0,$$

thus, there is not all $\Delta_k > 0,\ k = 1, 2, \cdots, 2m$, from **Lemma** 2.2 and **Lemma** 2.3, we obtain that the equilibrium point $\bar{X}_2$ is unstable, and then the proof is completed. □

Lemma 3.2. *Let (y_n, z_n) be a positive solution of the system (3.2), if $A_1 \leqslant A_2 \leqslant B_1 \leqslant B_2$, the positive solution of (3.2) is bounded.*

Proof. If the following system of inequalities (3.6) are true, then the positive solution of the system (3.2) is bounded.

$$\begin{aligned}
&0 \leqslant y_n \leqslant (A_1/B_2)^{q+1} y_{-m} \leqslant y_{-m}, && \text{if } n = (m+1)q+1,\\
&0 \leqslant y_n \leqslant (A_1/B_2)^{q+1} y_{-m+1} \leqslant y_{-m+1}, && \text{if } n = (m+1)q+2,\\
&0 \leqslant y_n \leqslant (A_1/B_2)^{q+1} y_{-m+2} \leqslant y_{-m+2}, && \text{if } n = (m+1)q+3,\\
&\qquad\vdots\\
&0 \leqslant y_n \leqslant (A_1/B_2)^{q+1} y_0 \leqslant y_0, && \text{if } n = (m+1)q+(m+1),\\
&0 \leqslant z_n \leqslant (A_2/B_1)^{q+1} z_{-m} \leqslant z_{-m}, && \text{if } n = (m+1)q+1,\\
&0 \leqslant z_n \leqslant (A_2/B_1)^{q+1} z_{-m+1} \leqslant z_{-m+1}, && \text{if } n = (m+1)q+2,\\
&0 \leqslant z_n \leqslant (A_2/B_1)^{q+1} z_{-m+2} \leqslant z_{-m+2}, && \text{if } n = (m+1)q+3,\\
&\qquad\vdots\\
&0 \leqslant z_n \leqslant (A_2/B_1)^{q+1} z_0 \leqslant z_0, && \text{if } n = (m+1)q+(m+1).
\end{aligned} \tag{3.6}$$

Now, we need prove the (3.6), the inequalities are obviously true for $q = 0$. Suppose that inequalities are true for $q = k \geqslant 1$, i.e.,

$$\begin{aligned}
&0 \leqslant y_n \leqslant (A_1/B_2)^{k+1} y_{-m} \leqslant y_{-m}, && \text{if } n = (m+1)k+1,\\
&0 \leqslant y_n \leqslant (A_1/B_2)^{k+1} y_{-m+1} \leqslant y_{-m+1}, && \text{if } n = (m+1)k+2,\\
&0 \leqslant y_n \leqslant (A_1/B_2)^{k+1} y_{-m+2} \leqslant y_{-m+2}, && \text{if } n = (m+1)k+3,\\
&\qquad\vdots\\
&0 \leqslant y_n \leqslant (A_1/B_2)^{k+1} y_0 \leqslant y_0, && \text{if } n = (m+1)k+(m+1),\\
&0 \leqslant z_n \leqslant (A_2/B_1)^{k+1} z_{-m} \leqslant z_{-m}, && \text{if } n = (m+1)k+1,\\
&0 \leqslant z_n \leqslant (A_2/B_1)^{k+1} z_{-m+1} \leqslant z_{-m+1}, && \text{if } n = (m+1)k+2,\\
&0 \leqslant z_n \leqslant (A_2/B_1)^{k+1} z_{-m+2} \leqslant z_{-m+2}, && \text{if } n = (m+1)k+3,\\
&\qquad\vdots\\
&0 \leqslant z_n \leqslant (A_2/B_1)^{k+1} z_0 \leqslant z_0, && \text{if } n = (m+1)k+(m+1).
\end{aligned}$$

Then, for $q = k+1$, one has

$$\begin{aligned}
&0 \leqslant y_{(m+1)(k+1)+1} \leqslant \frac{A_1 y_{(m+1)(k+1)-m}}{B_2} = \frac{A_1 y_{(m+1)k+1}}{B_2}\\
&\quad \leqslant (A_1/B_2)^{k+2} y_{-m} \leqslant y_{-m},\\
&0 \leqslant y_{(m+1)(k+1)+2} \leqslant \frac{A_1 y_{(m+1)(k+1)+1-m}}{B_2} = \frac{A_1 y_{(m+1)k+2}}{B_2}\\
&\quad \leqslant (A_1/B_2)^{k+2} y_{-m+1} \leqslant y_{-m+1},\\
&0 \leqslant y_{(m+1)(k+1)+3} \leqslant \frac{A_1 y_{(m+1)(k+1)+2-m}}{B_2} = \frac{A_1 y_{(m+1)k+3}}{B_2}\\
&\quad \leqslant (A_1/B_2)^{k+2} y_{-m+2} \leqslant y_{-m+2},\\
&\vdots\\
&0 \leqslant y_{(m+1)(k+1)+(m+1)} \leqslant \frac{A_1 y_{(m+1)(k+1)}}{B_2} = \frac{A_1 y_{(m+1)k+(m+1)}}{B_2}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (A_1/B_2)^{k+2}y_0 \leqslant y_0,\\
0 \leqslant z_{(m+1)(k+1)+1} &\leqslant \frac{A_2 z_{(m+1)(k+1)-m}}{B_1} = \frac{A_2 z_{(m+1)k+1}}{B_1}\\
&\leqslant (A_2/B_1)^{k+2}z_{-m} \leqslant z_{-m},\\
0 \leqslant z_{(m+1)(k+1)+2} &\leqslant \frac{A_2 z_{(m+1)(k+1)+1-m}}{B_1} = \frac{A_2 z_{(m+1)k+2}}{B_1}\\
&\leqslant (A_2/B_1)^{k+2}z_{-m+1} \leqslant z_{-m+1},\\
0 \leqslant z_{(m+1)(k+1)+3} &\leqslant \frac{A_2 z_{(m+1)(k+1)+2-m}}{B_1} = \frac{A_2 z_{(m+1)k+3}}{B_1}\\
&\leqslant (A_2/B_1)^{k+2}z_{-m+2} \leqslant z_{-m+2},\\
&\vdots\\
0 \leqslant z_{(m+1)(k+1)+(m+1)} &\leqslant \frac{A_2 z_{(m+1)(k+1)}}{B_1} = \frac{A_2 z_{(m+1)k+(m+1)}}{B_1}\\
&\leqslant (A_2/B_1)^{k+2}z_0 \leqslant z_0.
\end{aligned}
$$

By mathematical induction, the proof is completed. □

Theorem 3.5. *Consider the fuzzy difference system (1.15), where the parameters A, B, C are positive fuzzy numbers, and the initial values $x_i, i = -m, \cdots, 0$ are positive fuzzy numbers. If $A_{l,\alpha} \leqslant A_{r,\alpha} \leqslant B_{l,\alpha} \leqslant B_{r,\alpha}$, then every positive solution of (1.15) is bounded.*

Proof. Let x_n be a positive solution of (1.15) with initial values $x_i, i = -m, \cdots, 0$, such that $[x_n]_\alpha = [L_{n,\alpha},\ R_{n,\alpha}],\ \alpha \in (0,1],\ n = 0, 1, \cdots$ hold, and A, B, C is positive fuzzy numbers, thus

$$[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}], \quad [B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]\ , \quad [C]_\alpha = [C_{l,\alpha}, C_{r,\alpha}]\ ,\ \alpha \in (0,1].$$

From **Theorem** 3.1, $(L_{n,\alpha}, R_{n,\alpha}), i = 1, 2, 3, \cdots, \alpha \in (0,1]$ satisfies system (3.1). From **Lemma** 3.5, we have that

$$L_{(m+1)q+i,\alpha} \leqslant L_{-m+(i-1),\alpha}, R_{(m+1)q+i,\alpha} \leqslant R_{-m+(i-1),\alpha}, i = 1, \cdots, (m+1),$$

where q is a positive integer, i.e., there exist two positive real numbers μ, ν such that $0 \leqslant L_{n,\alpha} \leqslant \mu, 0 \leqslant R_{n,\alpha} \leqslant \nu$ for all $n = 0, 1, 2, \cdots$, where $\mu = \max\{L_{-m,\alpha}, L_{-m+1,\alpha}, \cdots, L_{0,\alpha}\}$, and $\nu = \max\{R_{-m,\alpha}, R_{-m+1,\alpha}\}, \cdots, R_{0,\alpha}\}$. The proof is completed. □

Theorem 3.6. *Consider the fuzzy difference system (1.15), where the parameters A, B, C are positive fuzzy numbers, and the initial values $x_i,\ i = -m,\ \cdots,\ 0$ are positive fuzzy numbers. if $A_{l,\alpha} < A_{r,\alpha} < B_{l,\alpha} < B_{r,\alpha}$, then the equilibrium point $[\hat{0}]_\alpha = [0,0]$ of the system (1.15) is global asymptotically stable.*

Proof. From **Theorem** 3.2, if $A_{l,\alpha} < A_{r,\alpha} < B_{l,\alpha} < B_{r,\alpha}$, it is easily to see that the equilibrium point $[\hat{0}]_\alpha = [0,0]$ of the difference equation (1.15) is locally asymptotically stable. From **Theorem** 3.5, we know that every positive solution x_n is bounded. Now, it is sufficient to prove that x_n is decreasing. For the system

(3.1), one has

$$L_{n+1,\alpha} = \frac{A_{l,\alpha} L_{n-m,\alpha}}{B_{r,\alpha} + C_{r,\alpha} \prod_{i=0}^{m} R_{n-i,\alpha}} \leqslant \frac{A_{l,\alpha} L_{n-m,\alpha}}{B_{r,\alpha}} < L_{n-m,\alpha}.$$

This implies that $L_{(m+1)n+1,\alpha} < L_{(m+1)n-m,\alpha}$ and $L_{(m+1)n+(m+1),\alpha} < L_{(m+1)n,\alpha}$. Hence, the subsequences $\left\{L_{(m+1)n+1,\alpha}\right\}, \left\{L_{(m+1)n+2,\alpha}\right\}, \cdots, \left\{L_{(m+1)n+(m+1),\alpha}\right\}$ are decreasing, i.e., the sequence $\{L_{n,\alpha}\}$ is decreasing. Similarly, we have

$$R_{n+1,\alpha} = \frac{A_{r,\alpha} R_{n-m,\alpha}}{B_{l,\alpha} + C_{l,\alpha} \prod_{i=0}^{m} L_{n-i,\alpha}} \leqslant \frac{A_{r,\alpha} R_{n-m,\alpha}}{B_{l,\alpha}} < R_{n-m,\alpha}.$$

This implies that $R_{(m+1)n+1,\alpha} < R_{(m+1)n-m,\alpha}$ and $R_{(m+1)n+(m+1),\alpha} < R_{(m+1)n,\alpha}$. Hence, the subsequences $\left\{R_{(m+1)n+1,\alpha}\right\}, \left\{R_{(m+1)n+2,\alpha}\right\}, \cdots, \left\{R_{(m+1)n+(m+1),\alpha}\right\}$ are decreasing, i.e., the sequence $\{R_{n,\alpha}\}$ is decreasing. Hence, from the monotone boundedness theorem and Eq. (1.15), we have $\lim_{n\to\infty} L_{n,\alpha} = \lim_{n\to\infty} R_{n,\alpha} = 0$, i.e., $\lim_{n\to\infty} x_n = \hat{0}$. The proof is completed. □

4. Numerical Simulation

In this section some numerical examples are given in order to confirm the results of the previous sections and support our theoretical discussions. These examples represents different types of the asymptotically behavior of solutions for the fuzzy difference system (1.15).

Example 4.1. when $m = 1$, we consider the following fuzzy difference equation

$$x_{n+1} = \frac{A x_{n-1}}{B + C x_n x_{n-1}}, \quad n = 0, 1, 2, \cdots, \tag{4.1}$$

where A, B, C are positive fuzzy numbers. From **Theorem** 3.6, we take

$$A = [A_1, A_2] = [3 + 2\alpha, 8 - 3\alpha], \; B = [B_1, B_2] = [12 + 2\alpha, 15 - \alpha],$$
$$C = [C_1, C_2] = [4 + 4\alpha, 9 - \alpha], \; \alpha \in (0, 1].$$

We also denote the initial conditions $x_{-1} = [1 + 4\alpha, 7 - 2\alpha]$, $x_0 = [2 + 5\alpha, 8 - \alpha]$. From Eq. (4.1), it results in a coupled system of difference equation with parameter α

$$L_{n+1} = \frac{A_{l,\alpha} L_{n-1,\alpha}}{B_{r,\alpha} + C_{r,\alpha} R_{n-1,\alpha}},$$
$$R_{n+1,\alpha} = \frac{A_{r,\alpha} R_{n-1,\alpha}}{B_{l,\alpha} + C_{l,\alpha} L_{n-1,\alpha}}, \quad \alpha \in (0, 1], \; n = 0, 1, 2, \cdots. \tag{4.2}$$

It is easy to see that $A_{l,\alpha} < B_{r,\alpha}, A_{r,\alpha} < B_{l,\alpha}$ for $\alpha \in (0, 1]$, namely, the conditions of **Theorem** 3.6 are satisfied. So from Theorem 3.6, we have that the trivial solution $x = \widehat{0}$ of Eq. (4.1) is global asymptotically stable with respect to D as $n \to \infty$ (see Figure 1-4).

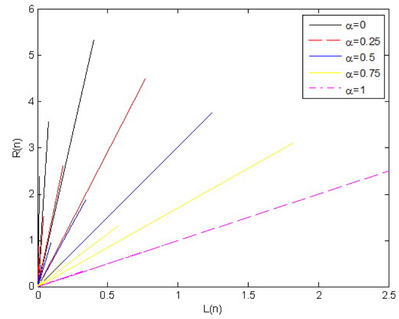

Figure 1. The dynamics of system (4.2).

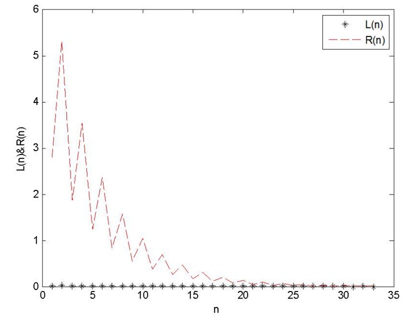

Figure 2. The solution of system (4.2) when $\alpha = 0$.

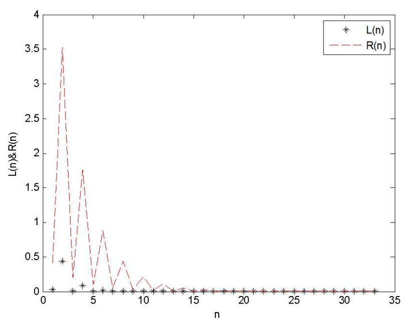

Figure 3. The solution of system (4.2) when $\alpha = 0.5$.

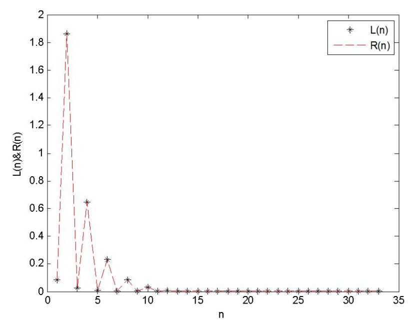

Figure 4. The solution of system (4.2) when $\alpha = 1$.

Example 4.2. When $m = 5$ we consider the following fuzzy difference equation

$$x_{n+1} = \frac{Ax_{n-5}}{B + Cx_n x_{n-1} x_{n-2} x_{n-3} x_{n-4} x_{n-5}}, \quad n = 0, 1, 2, \cdots, \tag{4.3}$$

where the parameters A, B, C and the initial conditions are positive fuzzy numbers. From **Theorem** 3.6, we take

$$\begin{aligned}
&A = [5 + 3\alpha, 11 - 3\alpha], B = [13 + 3\alpha, 18 - 2\alpha], C = [3 + 2\alpha, 9 - 4\alpha],\\
&x_0 = [1 + 7\alpha, 13 - 5\alpha], x_{-1} = [2 + 5\alpha, 12 - 5\alpha], x_{-2} = [3 + 4\alpha, 11 - 4\alpha],\\
&x_{-3} = [4 + 3\alpha, 10 - 3\alpha], x_{-4} = [5 + 2\alpha, 9 - 2\alpha], x_{-5} = [6 + \alpha, 8 - \alpha], \alpha \in (0, 1].
\end{aligned}$$

From Eq. (4.3), it results in a coupled system of difference equation with parameter

$$\begin{aligned}
L_{n+1} &= \frac{A_{l,\alpha} L_{n-5,\alpha}}{B_{r,\alpha} + C_{r,\alpha} \prod\limits_{i=0}^{5} R_{n-i,\alpha}},\\
R_{n+1,\alpha} &= \frac{A_{r,\alpha} R_{n-5,\alpha}}{B_{l,\alpha} + C_{l,\alpha} \prod\limits_{i=0}^{5} L_{n-i,\alpha}}, \alpha \in (0, 1], n = 0, 1, 2, \cdots.
\end{aligned} \tag{4.4}$$

Similarly, we have $A_{l,\alpha} < B_{r,\alpha}, A_{r,\alpha} < B_{l,\alpha}$ for $\alpha \in (0,1]$, so from Theorem 3.6, we have that the trivial solution $x = \widehat{0}$ of (4.3) is global asymptotically stable with respect to D as $n \to \infty$(see Figure 5-8).

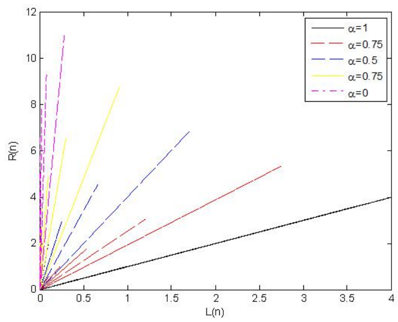

Figure 5. The dynamics of system (4.4).

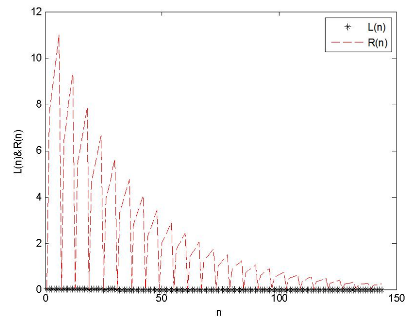

Figure 6. The solution of system (4.4) when $\alpha = 0$.

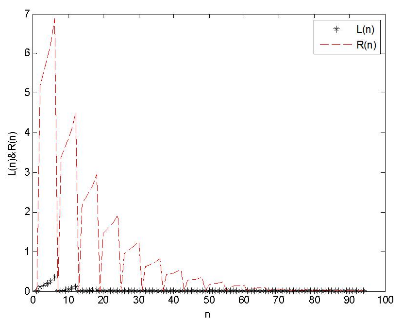

Figure 7. The solution of system (4.4) when $\alpha = 0.5$.

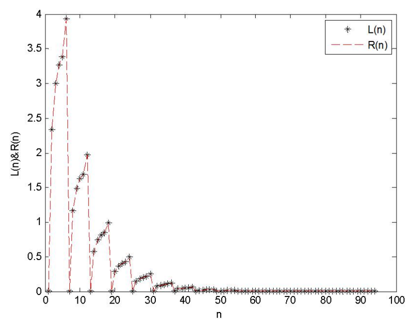

Figure 8. The solution of system (4.4) when $\alpha = 1$.

5. Conclusion

In this paper, we have dealt with the dynamics behavior for a class of high-order nonlinear fuzzy difference equation. Firstly, the existence and uniqueness of the positive fuzzy solutions for the Eq. (1.15) is proved. Secondly, we also obtain some conditions to ensure the nonzero equilibrium points of the corresponding ordinary difference equations (3.2) is unstable or locally asymptotically stable by using linearization method. Moreover, it is proved that If $A_{l,\alpha} \leqslant A_{r,\alpha} \leqslant B_{l,\alpha} \leqslant B_{r,\alpha}$, then every positive solution of system (1.15) is bounded when the parameters A, B, C and the initial values $x_i, i = -m, \cdots, 0$ are positive fuzzy numbers. Finally, we find that if$A_{l,\alpha} < A_{r,\alpha} < B_{l,\alpha} < B_{r,\alpha}$, then the equilibrium point $[\hat{0}]_\alpha = [0,0]$ of the system (1.15) is global asymptotically stable when the parameters A, B, C, D and the initial values $x_i, i = -m, \cdots, 0$ are positive fuzzy numbers. In particular, some examples are given to illustrate the effectiveness of the obtained results. In addition, the obtained sufficient conditions are very simple and provide some flexibility for the application and analysis of nonlinear fuzzy difference equations.

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