Journal of Applied Analysis and Computation
Volume 11, Number 1, February 2021, 445–463
Website:http://www.jaac-online.com
DOI:10.11948/20200055

ASYMPTOTIC DYNAMIC OF THE NONCLASSICAL DIFFUSION EQUATION WITH TIME-DEPENDENT COEFFICIENT*

Jing Wang[1] and Qiaozhen Ma[1,†]

Abstract We study the asymptotic behavior of solutions for a nonclassical diffusion equation with polynomial growth condition of arbitrary order $p \geq 2$ on bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$. Firstly, the existence and uniqueness of weak solution are obtained in the time-dependent space $\mathcal{H}_t$ with the norm depending on time t explicitly. Then we establish the existence, regularity and asymptotic structure of the time-dependent global attractor.

Keywords Nonclassical diffusion equation, time-dependent global attractor, polynomial growth of arbitrary order, asymptotic structure, regularity.

MSC(2010) 35B25, 35B40, 35B41, 37L30, 45K05.

1. Introduction

In this paper, we are concerned with the following nonclassical diffusion equation with time-dependent coefficient :

$$\begin{cases} u_t - \varepsilon(t)\triangle u_t - \triangle u + \lambda u + f(u) = g, \quad x \in \Omega, t > \tau, \\ u|_{\partial\Omega} = 0, \quad t > \tau, \\ u(x,\tau) = u_\tau, \quad x \in \Omega, \end{cases} \tag{1.1}$$

where the unknown variable $u = u(x,t) : \Omega \times [\tau, \infty) \to \mathbb{R}$, $\lambda > 0$, $\tau \in \mathbb{R}$ and $g \in H^{-1}(\Omega)$. Let $\varepsilon(t)$ be a decreasing bounded function satisfying

$$\lim_{t \to +\infty} \varepsilon(t) = 0. \tag{1.2}$$

In particular, there is constant $L > 0$ such that

$$\sup_{t \in \mathbb{R}} (|\varepsilon(t)| + |\varepsilon'(t)|) \leq L. \tag{1.3}$$

The nonlinear function $f \in C^1(\mathbb{R})$ with $f(0) = 0$ satisfies the polynomial growth condition of arbitrary order

$$\gamma_1 |s|^p - \beta_1 \leq f(s)s \leq \gamma_2 |s|^p + \beta_2, \quad p \geq 2, \tag{1.4}$$

†The corresponding author. Email address:maqzh@nwnu.edu.cn(Q. Ma)
[1]College of Mathematics and Statistics, Northwest Normal University, Anning-Dong Road, Lanzhou 730070, China
*The authors were supported by National Natural Science Foundation of China (11961059, 11561064, 11761062).

and the dissipation condition

$$f'(s) \geq -l, \tag{1.5}$$

where γ_i, β_i $(i = 1, 2)$ and l are positive constants. Let $F(s) = \int_0^s f(y)dy$, then from (1.4), there is constants $\tilde{\gamma_i}$, $\tilde{\beta_i} > 0$ $(i = 1, 2)$ such that

$$\tilde{\gamma_1}|s|^p - \tilde{\beta_1} \leq F(s) \leq \tilde{\gamma_2}|s|^p + \tilde{\beta_2}. \tag{1.6}$$

As an important mathematical model, the nonclassical diffusion equation has been used to describe several physical phenomena, such as heat conduction, solid mechanics, non-Newtonian flows, see [1, 14, 23] and the reference and therein.

Therefore, when ε is a positive constant independent of time t in Eq. (1.1), this kind of equation has been studied by many researchers and several excellent results have been obtained in the recent twenty years, see [2–5, 7, 12, 16, 27–30, 33, 34, 36, 38] and the references therein. In particular, by using the decomposition technique, Xiao [33] obtained the existence of global attractor for the nonclassical diffusion equation with subcritical nonlinearity in $H_0^1(\Omega)$. Sun and Yang [28] considered the dynamical behavior of the nonclassical diffusion equations with critical nonlinearity for both autonomous and non-autonomous cases, they obtained not only the existence of global attractor when the time-independent forcing term belongs to $H^{-1}(\Omega)$, but also the existence of a uniform attractor and exponential attractor when the time-dependent forcing term is translation bounded instead of translation compact. Later, Xie et al. [34] studied the existence of global attractor of the nonclassical diffusion equation in $H^1(\mathbb{R}^N)$. The method they used is the method of Asymptotic Contractive Semigroup, which was introduced by themselves.

Provided that ε is a positive decreasing function vanishing at infinity, very few people study the problem (1.1) in the time-dependent space. Note that the time-dependent space mentioned, which the norm of the space depends on the time explicitly, is very important. Since the norm of space depends on time t explicitly, the considered problem is still non-autonomous even when the forcing term g is independent of t. If not, the time-dependent coefficient leads to the lose of the dissipation of the natural energy as $t \to \pm\infty$, which affects the existence of absorbing set in the general sense. So, in order to avoid this obstacle, Plinio et al. [24] first introduced the concept of the time-dependent global attractor in the time-dependent space. Until 2013, Conti et al. [10] applied other condition, that is, the invariance was replaced by the minimality in this concept of the attractor. By using the decomposition technique, they showed the existence of the time-dependent global attractor for wave equation. Recently, some authors have extensively studied the corresponding results for wave equations Berger equation and plate equations, see [9, 15, 17, 19–22] and the references therein. Especially, Meng et al. [21] introduced a technical method (contractive function) for verifying compactness of the process. They got the existence of the time-dependent global attractor for wave equation. As for the nonclassical diffusion equation, Ding and Liu [11] proved the existence of time-dependent global attractor for (1.1) by using the decomposition technique, where the forcing term $g \in L^2(\Omega)$ $(\Omega \subset \mathbb{R}^3)$. Using the same method, Ma et al. [18] investigated the existence, regularity and asymptotic structure of time-dependent global attractor, when the forcing term $g \in H^{-1}(\Omega)$ $(\Omega \subset \mathbb{R}^N, N \geq 3)$ and the nonlinearity f satisfies the more weaker condition than [31]. Compared to the above mentioned equations, the related results of the nonclassical diffusion equation in the time-dependent space are not abundant.

Inspired by [11, 18, 21, 24, 31] we continue discussing the asymptotic behavior for the Eq. (1.1). It is worth noting that the nonlinearity f satisfies critical or subcritical growth conditions about the above articles. The problem (1.1) with the nonlinearity of polynomial growth condition in the space $\mathcal{H}_t$ has not been studied, except by us. Even more interesting, when ε is positive constant, the result of (1.1) can be reduced to the previous result (from [27]). Besides, the existence and uniqueness of weak solution for the partial differential equation in such time-dependent space have not been explicitly proved until now. Thus, a natural problem is: can we get the asymptotic behavior of the problem (1.1) with any space dimension N in $\mathcal{H}_t$ when nonlinearity f satisfies the polynomial growth of arbitrary order and $g \in H^{-1}(\Omega)$?

In order to answer the above problem, we need to overcome two obstacles. In the Eq. (1.1), the presence of the term $-\varepsilon(t)\triangle u_t$ makes it different from the usual reaction diffusion equation ($i.e., \varepsilon(t) \equiv 0$). For instance, the reaction diffusion equation has some smoothing property: although the initial data only belongs to a weaker topological space, the solution will belong to a stronger topological space with the higher regularity. Consequently, for the problem (1.1), we can not use the compact Sobolev embedding to verify the asymptotic compactness. On the other hand, we can not understand its dynamics in the standard semigroup framework because the coefficient $\varepsilon(t)$ of $-\triangle u_t$ depends on the time t, which makes the problem more complex. For this purpose, we will go on our this problem.

The rest of the paper is organized as follows. Sect.2 is devoted to notations of function spaces involved, standard conclusions and some abstract results for the time-dependent global attractor. In Sect.3, we will prove the existence and uniqueness of solution. It is mentioned that the problem is a non-autonomous case because of the space norm depending on time. Hence, based on the existence result of the solution, we get a process generated by a weak solution. In Sect.4, we show the existence of the time-dependent attractor by contractive function method. In Sect.5, the uniform boundedness of the time-dependent attractor is obtained. Finally, combining with the estimates of Sect.5, we study the limit relation between the time-dependent attractor for this nonclassical diffusion equation and the (classical) global attractor for the reaction diffusion equation with the same conditions.

For the sake of convenience, we choose C as the positive constant depending on the subscript which may be different from line to line or in the same line throughout the paper.

2. Preliminaries

Firstly, we give some spaces and corresponding norms used in the following paper. Without loss of generality, the norm in $L^p(\Omega)$ $(p \geq 1)$ is denoted as $\|\cdot\|_{L^p(\Omega)}$. Especially, set $\mathrm{H} = L^2(\Omega)$, the scalar product and norm on H are denoted as $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. Recall that $A = -\triangle$, the Laplacian with Dirichlet boundary conditions, is a positive operator on H with domain $H^2(\Omega) \cap H_0^1(\Omega)$. Then, we introduce the family of Hilbert spaces $\mathrm{H}_s = \mathrm{D}(A^{s/2})$, $\forall s \in \mathbb{R}$, with the standard inner products and norms, respectively,

$$(\cdot,\cdot)_{\mathrm{D}(A^{s/2})} = (\cdot,\cdot)_s = (A^{s/2}\cdot, A^{s/2}\cdot), \qquad \|\cdot\|_s = \|A^{s/2}\cdot\|.$$

In particular, $\mathrm{H}_{-1} = H^{-1}(\Omega)$, $\mathrm{H}_0 = \mathrm{H}$, $\mathrm{H}_1 = H_0^1(\Omega)$, $\mathrm{H}_2 = H^2(\Omega) \cap H_0^1(\Omega)$.

Now, for any $t \in \mathbb{R}$, $-1 \le s \le 1$, we have the spaces $\mathcal{H}_t^s$ with the time-dependent norm

$$\|u\|_{\mathcal{H}_t^s}^2 = \|u\|_s^2 + \varepsilon(t)\|u\|_{s+1}^2,$$

where the symbol s is always omitted whenever zero. Especially,

$$\|u\|_{\mathcal{H}_t}^2 = \|u\|^2 + \varepsilon(t)\|u\|_1^2.$$

Here, the dual space of X is denoted as X^*.

Secondly, we give some notations and recall some standard conclusions (see, [13, 26]). For every $t \in \mathbb{R}$, let X_t be a family of normed spaces, we introduce the $R-$ball of X_t

$$\mathbb{B}_{X_t}(R) = \{u \in X_t : \|u\|_{X_t}^2 \le R\}.$$

We denote the Hausdorff semi-distance of two (nonempty) sets $B,\ C \subset X_t$ by

$$\operatorname{dist}_{X_t}(B, C) = \sup_{x\in B} \inf_{y \in C} \|x - y\|_{X_t}.$$

We also focus on the particular, case of a process $\{U(t,\tau)\}_{t\ge\tau}$ acting on a family of spaces $\{\mathcal{Z}_t\}_{t\in\mathbb{R}}$, endowed with the product norm

$$\|x\|_{\mathcal{Z}_t}^2 = \|x\|_{\mathcal{X}}^2 + \xi(t)\|x\|_{\mathcal{Y}}^2,$$

where $\xi(t)$ is a function. Let $\Pi_t : \mathcal{Z}_t \to \mathcal{X}$ be the projection. Accordingly, if $J_t \subset \mathcal{Z}_t$, then $\Pi_t J_t = \{x \in \mathcal{X} : x \in J_t\}$. And if $\mathfrak{J} = \{J_t\}_{t\in\mathbb{R}}$, then $\Pi\mathfrak{J} = \{\Pi_t J_t\}_{t\in\mathbb{R}}$.

Lemma 2.1 (Aubin-Lions Lemma)**.** *Assume that X, B and Y are three Banach spaces with $X \hookrightarrow\hookrightarrow B$ and $B \hookrightarrow Y$. Let f_n be bounded in $L^p([0,T], B)$ $(1 \le p < \infty)$. If f_n satisfies*

(i) f_n is bounded in $L^p([0,T], X)$;

(ii) $\frac{\partial f_n}{\partial t}$ is bounded in $L^p([0,T], Y)$.

Then, f_n relatively compact in $L^p([0,T], B)$.

Lemma 2.2. *Assume that X, B and Y are three Banach spaces with $X \hookrightarrow\hookrightarrow B$ and $B \hookrightarrow Y$. Let f_n be bounded in $L^\infty(0,T;X)$ and $\frac{\partial f_n}{\partial t}$ is bounded in $L^p(0,T;Y)$ $(p > 1)$. Then, f_n is relatively compact in $C(0,T;B)$.*

Finally, we recall some abstrsct results about the theory of the time-dependent global attractor, see [10, 21, 22, 24] for more details.

Definition 2.1. Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. A process is a two-parameter family of mappings $\{U(t,\tau) : X_\tau \to X_t,\ t \ge \tau \in \mathbb{R}\}$ with properties

(i) $U(\tau,\tau) = Id$ is the identity on $X_\tau,\ \tau \in \mathbb{R}$;

(ii) $U(t,s)U(s,\tau) = U(t,\tau),\ \forall\ t \ge s \ge \tau$.

Definition 2.2. A family $\mathfrak{D} = \{D_t\}_{t\in\mathbb{R}}$ of bounded sets $D_t \subset X_t$ is called uniformly bounded if there exist a constant $R > 0$ such that $D_t \subset \mathbb{B}_{X_t}(R),\ \forall\ t \in \mathbb{R}$.

Definition 2.3. A time-dependent absorbing set for the process $\{U(t,\tau)\}_{t\geq\tau}$ is a uniformly bounded family $\mathfrak{B}=\{B_t\}_{t\in\mathbb{R}}$ with the following property: for every $R>0$ there exists a t_0 such that

$$\tau\leq t-t_0\Rightarrow U(t,\tau)\mathbb{B}_{X_\tau}(R)\subset B_t.$$

Definition 2.4. The time-dependent global attractor for $\{U(t,\tau)\}_{t\geq\tau}$ is the smallest family $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$ such that

(i) each A_t is compact in X_t;

(ii) $\mathfrak{A}$ is pullback attracting, i.e. it is uniformly bounded and the limit

$$\lim_{\tau\to-\infty} dist_{X_t}(U(t,\tau)D_\tau, A_t)=0$$

holds for every uniformly bounded family $\mathfrak{D}=\{D_t\}_{t\in\mathbb{R}}$ and every fixed $t\in\mathbb{R}$.

Definition 2.5. We say $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$ is invariant if

$$U(t,\tau)A_\tau=A_t,\ \forall\ t\ \geq\ \tau.$$

Definition 2.6. We say that a process $\{U(t,\tau)\}_{t\geq\tau}$ in a family of normed spaces $\{X_t\}_{t\in\mathbb{R}}$ is pullback asymptotically compact if and only if for any fixed $t\in\mathbb{R}$, bounded sequence $\{x_n\}_{n=1}^\infty\subset X_{\tau_n}$ and any $\{\tau_n\}_{n=1}^\infty\subset\mathbb{R}^{-t}$ with $\tau_n\to-\infty$ as $n\to\infty$, sequence $\{U(t,\tau_n)x_n\}_{n=1}^\infty$ has a convergent subsequence, where $\mathbb{R}^{-t}=\{\tau:\tau\in\mathbb{R},\tau\leq t\}$.

Definition 2.7. Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process in a family of Banach spaces $\{X_t\}_{t\in\mathbb{R}}$. Then $U(\cdot,\cdot)$ has a time-dependent global attractor $\mathcal{A}=\{A_t\}_{t\in\mathbb{R}}$ satisfying $A_t=\bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s}U(t,\tau)B_\tau}$ if and only if

(i) $\{U(t,\tau)\}_{t\geq\tau}$ has a pullback absorbing family $\mathfrak{B}=\{B_t\}_{t\in\mathbb{R}}$;

(ii) $\{U(t,\tau)\}_{t\geq\tau}$ is pullback asymptotically compact.

Definition 2.8. Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces and $\mathfrak{C}=\{C_t\}_{t\in\mathbb{R}}$ be a family of uniformly bounded subsets of $\{X_t\}_{t\in\mathbb{R}}$. We call a function $\psi_\tau^t(\cdot,\cdot)$, defined on $X_t\times X_t$, a contractive function on $C_\tau\times C_\tau$ if for any fixed $t\in\mathbb{R}$ and any sequence $\{x_n\}_{n=1}^\infty\subset C_\tau$, there is a subsequence $\{x_{n_k}\}_{n=1}^\infty\subset\{x_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\psi_\tau^t(x_{n_k},x_{n_l})=0.$$

Theorem 2.1. *Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process $\{X_t\}_{t\in\mathbb{R}}$ and has a pullback absorbing family $\mathfrak{B}=\{B_t\}_{t\in\mathbb{R}}$. Moreover, assume that for any $\epsilon>0$ there exists $T(\epsilon)\leq t$, $\psi_T^t\in\mathfrak{C}(B_T)$ such that*

$$\|U(t,T)x-U(t,T)y\|_{X_t}\leq\epsilon+\psi_T^t(x,y),\ \forall x,y\in B_T,$$

for any fixed $t\in\mathbb{R}$. Then $\{U(t,\tau)\}_{t\geq\tau}$ is pullback asymptotically compact.

In order to prove the asymptotic structure of the time-dependent global attractors for the process $\{U(t,\tau)\}_{t\geq\tau}$, we also need the following results.

Theorem 2.2. *A function $z:t\mapsto z(t)\in X_t$ is a complete bounded trajectories of $\{U(t,\tau)\}_{t\geq\tau}$ if and only if* *(i)* $\sup_{t\in\mathbb{R}} \|z(t)\|_{X_t} < \infty$;

(ii) $z(t) = U(t,\tau)z(\tau), \quad \forall t \geq \tau, \quad \tau \in \mathbb{R}$.

Theorem 2.3. *Let $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ be the time-dependent global attractor of $\{U(t,\tau)\}_{t\geq\tau}$. If $\mathfrak{A}$ is invariant, then $A_t = \{z(t) \in X_t : \ z \ CBT \ of \ U(t,\tau)\}$. Accordingly, we can write*

$$\mathfrak{A} = \{z : t \to z(t) \in X_t \ with \ z \ CBT \ of \ U(t,\tau)\}.$$

Theorem 2.4. *For any sequence $z_n = (x_n, y_n)$ of the complete bounded trajectory of the process $\{U(t,\tau)\}_{t\geq\tau}$ and any $t_n \to +\infty$, there exists a complete bounded trajectory w of the semigroup $\{S(t)\}_{t\geq 0}$ and any $s \in \mathbb{R}$ for which*

$$\|x_n(s+t_n) - w(s)\|_{\mathcal{X}} \to 0$$

as $n \to +\infty$ up to a subsequence. Then

$$\lim_{t\to+\infty} dist_{\mathcal{X}}(\Pi_t A_t, A_\infty) = 0.$$

3. Existence and uniqueness of solution

First we shall give the definition of a weak solution.

Definition 3.1. The function $u = u(x,t)$ defined in $\Omega \times [\tau, T]$ is said to be a weak solution for the problem (1.1) if for any $T > \tau$, $u \in C([\tau,T], \mathcal{H}_t)$ and u also satisfies the initial data $u(\tau) = u_\tau \in \mathbb{B}_{\mathcal{H}_\tau}(R_0) \subset \mathcal{H}_\tau$. Furthermore, the following identity hold

$$(u_t, v) + \varepsilon(t)(\nabla u_t, \nabla v) + (\nabla u, \nabla v) + \lambda(u,v) + (f(u), v) = (g, v),$$

for *a.e.* $[\tau, T]$.

We are now ready to state the existence and uniqueness of the weak solution for the problem (1.1).

Theorem 3.1. *Assume that* (1.2)-(1.6) *hold and $g \in H^{-1}(\Omega)$, then for any initial data $u_\tau \in \mathbb{B}_{\mathcal{H}_\tau}(R_0) \subset \mathcal{H}_\tau$ and any $\tau \in \mathbb{R}$, there exists a unique solution u for the problem* (1.1) *such that $u \in C(\tau, T; \mathcal{H}_t)$ for all fixed $T > \tau$. Furthermore, the solution depends on the initial data continuously in $\mathcal{H}_t$.*

Proof. The existence of solution for the problem (1.1) can be obtained by Faedo-Galerkin method [6, 13, 25]. We know that $\{\omega_k\}_{k=1}^\infty$ which consists of the eigenfunctions of $A = -\triangle$ with Dirichlet boundary value in H_1 is a standard orthogonal basis of H and is also an orthogonal basis in H_1. The corresponding eigenvalues are denoted by $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$, $\lambda_j \to \infty$ with $A\omega_k = \lambda_k \omega_k$, $\forall k \in \mathbb{N}$. We will finish our proof through the following steps.

• Faedo-Galerkin scheme.

Given an integer m, we denote by P_m the projection on the subspace $span\{\omega_1, \cdots, \omega_m\}$ in $H_0^1(\Omega)$. For every fixed m, we look for a function $u^m(t) = P_m u = \Sigma_{k=1}^m a_m^k(t)\omega_k$, where a_m^k satisfies

$$\begin{cases} (u_t^m, \omega_k) + (\varepsilon(t)Au_t^m, \omega_k) + (Au^m, \omega_k) + \lambda(u^m, \omega_k) + (f(u^m), \omega_k) = (g, \omega_k), \\ a_m^k(\tau) = (u_\tau, \omega_k). \end{cases} \tag{3.1}$$

Exploiting the standard existence theory for ordinary differential equations, there exists a continuous solution $u^m(t)$ of the problem (3.1) on an interval $[\tau, T]$.

• Energy estimates.

Multiplying the first equation of (3.1) by a_m^k and summing from 1 to k, we have

$$\frac{d}{dt}(\|u^m\|^2+\varepsilon(t)\|u^m\|_1^2)+(2-\varepsilon'(t))\|u^m\|_1^2+2\lambda\|u^m\|^2+2(f(u^m),u^m)=2(g,u^m). \quad (3.2)$$

It follows from (1.4), Hölder's inequality and Young's inequality that

$$(f(u^m),u^m) \geq \gamma_1 \int_\Omega |u^m|^p dx - \beta_1|\Omega|, \quad (3.3)$$

and

$$(g,u^m) \leq \|g\|_{-1}^2 + \frac{1}{4}\|u^m(t)\|_1^2. \quad (3.4)$$

Together with (3.2)-(3.4), we get

$$\begin{aligned}&\frac{d}{dt}(\|u^m\|^2 + \varepsilon(t)\|u^m\|_1^2) + (\frac{3}{2} - \varepsilon'(t))\|u^m\|_1^2 + 2\lambda\|u^m\|^2 + 2\gamma_1 \int_\Omega |u^m|^p dx \\ \leq& 2\|g\|_{-1}^2 + 2\beta_1|\Omega|. \quad (3.5)\end{aligned}$$

Due to the properties of $\varepsilon(t)$, (1.2) and (1.3), then for $t \in \mathbb{R}$,

$$(1-\varepsilon'(t))\|u^m\|_1^2 \geq \|u^m\|_1^2 \geq \frac{\varepsilon(t)}{L}\|u^m\|_1^2.$$

Choosing $\sigma = \min\{\frac{1}{L}, 2\lambda\}$ $(L, \lambda > 0)$, we arrive at

$$\begin{aligned}&\frac{d}{dt}(\|u^m\|^2 + \varepsilon(t)\|u^m\|_1^2) + \sigma(\|u^m\|^2 + \varepsilon(t)\|u^m\|_1^2) + \frac{1}{2}\|u^m\|_1^2 + 2\gamma_1 \int_\Omega |u^m|^p dx \\ \leq& 2\|g\|_{-1}^2 + 2\beta_1|\Omega|. \quad (3.6)\end{aligned}$$

By Gronwall's lemma, we have

$$\|u^m\|^2+\varepsilon(t)\|u^m\|_1^2 \leq e^{-\sigma(t-\tau)}(\|u^m(\tau)\|^2+\varepsilon(\tau)\|u^m(\tau)\|_1^2)+\frac{2}{\sigma}(\|g\|_{-1}^2+\beta_1|\Omega|). \quad (3.7)$$

In addition, integrating from τ to t on both sides of (3.6), we yields

$$\begin{aligned}&\int_\tau^t \|u^m(s)\|_1^2 ds + 2\gamma_1 \int_\tau^t \int_\Omega |u^m(s)|^p dx ds \\ \leq& \|u^m(\tau)\|^2 + \varepsilon(\tau)\|u^m(\tau)\|_1^2 + 2(t-\tau)(\|g\|_{-1}^2 + \beta_1|\Omega|). \quad (3.8)\end{aligned}$$

Hence, by (3.7) and (3.8), we infer that

$$\{u^m\}_m^\infty \text{ is bounded in } L^\infty([\tau,T],\mathcal{H}_t)) \cap L^2([\tau,T],\mathrm{H}_1) \cap L^p([\tau,T],L^p(\Omega)), \quad (3.9)$$

for all $T > \tau$. In virtue of (1.4),

$$\int_\tau^t \int_\Omega |f(u^m(s))|^q dx dt \leq C_{q,\gamma_2} \int_\tau^t \|u^m(s)\|_{L^p(\Omega)}^p ds + C_{q,\beta_2,|\Omega|,t-\tau}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Hence, we find that

$$\{f(u^m)\}_{m=1}^{\infty} \text{ is bounded in } L^q([\tau,T],L^q(\Omega)) \text{ for all } T>\tau. \tag{3.10}$$

Next, we need to achieve uniform estimate for the time derivatives.

Multiplying the first equation of (3.1) by $\partial_t a_m^k$ and summing from 1 to k, we get

$$\frac{d}{dt}(\frac{1}{2}\|u^m\|_1^2+\frac{\lambda}{2}\|u^m\|^2+\int_\Omega F(u^m)dx-\int_\Omega gu^m dx)+\|u_t^m\|^2+\varepsilon(t)\|u_t^m\|_1^2=0. \tag{3.11}$$

Let

$$E(t)=\frac{1}{2}\|u^m\|_1^2+\frac{\lambda}{2}\|u^m\|^2+\int_\Omega F(u^m)dx-\int_\Omega gu^m dx.$$

Applying (1.6), (3.4) and embedding inequality ($c>0$ is embedding constant), we have

$$\begin{aligned} E(t) &\geq \frac{1}{4}\|u^m\|_1^2+\frac{\lambda}{2}\|u^m\|^2+\tilde{\gamma_1}\|u^m\|_{L^p(\Omega)}^p-\tilde{\beta_1}|\Omega|-\|g\|_{-1}^2 \\ &\geq \frac{1}{4}\|u^m\|_1^2+\tilde{\gamma_1}\|u^m\|_{L^p(\Omega)}^p-\tilde{\beta_1}|\Omega|-\|g\|_{-1}^2, \end{aligned} \tag{3.12}$$

$$\begin{aligned} E(t) &\leq \frac{3}{4}\|u^m\|_1^2+\frac{\lambda}{2}\|u^m\|^2+\tilde{\gamma_2}\|u^m\|_{L^p(\Omega)}^p+\tilde{\beta_2}|\Omega|+\|g\|_{-1}^2 \\ &\leq C_{\lambda,\tilde{\gamma_2},c}(\|u^m\|_1^2+\|u^m\|_{L^p(\Omega)}^p)+\tilde{\beta_2}|\Omega|+\|g\|_{-1}^2. \end{aligned} \tag{3.13}$$

Integrating from s to t at the sides of (3.11), we obtain that

$$E(t)\leq E(s).$$

Then, integrating over $[t,t+1]$ about variable s, we know

$$E(t)\leq \int_t^{t+1} E(s)ds. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$E(t)\leq C_{\lambda,\tilde{\gamma}_2,c}\int_t^{t+1}(\|u^m(s)\|_1^2+\|u^m(s)\|_{L^p(\Omega)}^p)ds+\tilde{\beta_2}|\Omega|+\|g\|_{-1}^2. \tag{3.15}$$

By (3.7), (3.8), (3.12) and (3.15), there exists $R>0$ such that

$$\|u^m\|_1^2+\|u^m\|_{L^p(\Omega)}^p\leq R. \tag{3.16}$$

Integrating from τ to T about (3.11), we can get

$$\int_\tau^T \|u_t^m(s)\|^2+\varepsilon(s)\|u_t^m(s)\|_1^2 ds=E(\tau)-E(T). \tag{3.17}$$

From (3.7) and (3.13), we have the following conclusion

$$E(\tau)\leq C_{\lambda,\tilde{\gamma_2},c,R}+\tilde{\beta_2}|\Omega|+\|g\|_{-1}^2, \tag{3.18}$$

for any $u_\tau\in\mathbb{B}_{\mathcal{H}_\tau}(R_0)$ and all $T>\tau$.

Hence, it follows from (3.12), (3.17) and (3.18) that

$$\int_{\tau}^{T}(\|u_t^m\|^2+\varepsilon(s)\|u_t^m\|_1^2)ds \le C_{\lambda,\tilde{\gamma_2},c,R}+(\tilde{\beta_1}+\tilde{\beta_2})|\Omega|+2\|g\|_{-1}^2. \tag{3.19}$$

In view of (3.19),

$$\{u_t^m\}_{m=1}^{\infty} \text{ is bounded in } L^2([\tau,T],\mathcal{H}_t). \tag{3.20}$$

Then, we consider the limiting process of (3.1) as $m\to\infty$.

• Existence of solution.

Step 1. By (3.9), (3.10) and (3.20), we get that there exists $u\in L^\infty([\tau,T],\mathcal{H}_t)\cap L^2([\tau,T],\mathrm{H}_1)\cap L^p([\tau,T],L^p(\Omega))$, $\chi\in L^q([\tau,T],L^q(\Omega))$, $u_t\in L^2([\tau,T],\mathcal{H}_t)$ for all $T>\tau$ and a subsequence of $\{u^m\}_{m=1}^{\infty}$ (still denote as $\{u^m\}_{m=1}^{\infty}$) such that

$$u^m\to u \quad \text{weak-star in } L^\infty([\tau,T],\mathcal{H}_t)), \tag{3.21}$$

$$u^m\to u \quad \text{weakly in } L^2([\tau,T],\mathrm{H}_1), \tag{3.22}$$

$$u^m\to u \quad \text{weakly in } L^p([\tau,T],L^p(\Omega)), \tag{3.23}$$

$$f(u^m)\to \chi \quad \text{weakly in } L^q([\tau,T],L^q(\Omega)), \tag{3.24}$$

$$u_t^m\to u_t \quad \text{weakly in } L^2([\tau,T],\mathcal{H}_t). \tag{3.25}$$

Combining (3.8) with (3.19) and applying Lemma 2.1 (Aubin-Lions Lemma), we find that there exists a subsequence of $\{u^m\}_{m=1}^{\infty}$ (still denote as $\{u^m\}_{m=1}^{\infty}$) such that

$$u^m\to u \ \text{ in } L^2([\tau,T],L^2(\Omega)).$$

As a result,

$$u^m\to u, \quad \text{a.e. in } \Omega\times[\tau,T]. \tag{3.26}$$

Next, we claim $\chi=f(u)$. Indeed, it follows from (3.26) and the continuity of f that

$$f(u^m)\to f(u), \quad \text{a.e. in } \Omega\times[\tau,T].$$

In addition, we have

$$u_t^m-u_t^n-\varepsilon(t)\triangle(u_t^m-u_t^n)-\triangle(u^m-u^n)+\lambda(u^m-u^n)+f(u^m)-f(u^n)=0. \tag{3.27}$$

Multiplying the equation (3.27) by (u^m-u^n) and integrating on Ω, we know that

$$\begin{aligned}&\frac{d}{dt}\left(\|u^m-u^n\|^2+\varepsilon(t)\|u^m-u^n\|_1^2\right)+(2-\varepsilon'(t))\|u^m-u^n\|_1^2\\ =&-2\lambda\|u^m-u^n\|^2-(f(u^m)-f(u^n),2(u^m-u^n)).\end{aligned} \tag{3.28}$$

It follows from (1.5) and the monotony of $\varepsilon(t)$ that

$$\frac{d}{dt}\left(\|u^m-u^n\|^2+\varepsilon(t)\|u^m-u^n\|_1^2\right)\le 2l(\|u^m-u^n\|^2+\varepsilon(t)\|u^m-u^n\|_1^2).$$

By Gronwall's lemma, we have

$$\|u^m-u^n\|_{\mathcal{H}_t}^2\le e^{2l(t-\tau)}\|u^m(\tau)-u^n(\tau)\|_{\mathcal{H}_\tau}^2. \tag{3.29}$$

That is

$$\{u^m\}_{m=1}^{\infty} \text{ is a Cauchy sequence in } C([\tau,T],\mathcal{H}_t).$$

Therefore, by the uniqueness of the limit, we conclude that

$$u^m \to u \text{ uniformly in } C([\tau,T],\mathcal{H}_t), \quad \text{for all } T>\tau.$$

Thus, we have

$$u \in C([\tau,T],\mathcal{H}_t). \tag{3.30}$$

At the same time, when $m\to\infty$, $u^m(\tau)\to u_\tau$ in $\mathcal{H}_t$.

Step 2. Set a text function $v(t)=\Sigma_{k=1}^{N} a_m^k(t)\omega_k \in C^1([\tau,T],\mathcal{H}_t)$ for fixed $\bar{N}$. Choosing $m\geq \bar{N}$, multiplying the first equation of (3.1) by a_m^k, summing from 1 to $\bar{N}$ and integrating from τ to T, we can see

$$\begin{aligned}
&\int_\tau^T [(u_t^m,v)+\varepsilon(t)(\nabla u_t^m,\nabla v)]dt+\int_\tau^T(\nabla u^m,\nabla v)dt\\
&+\int_\tau^T \lambda(u^m,v)dt+\int_\tau^T(f(u^m),v)dt\\
=&\int_\tau^T (g,v)dt.
\end{aligned} \tag{3.31}$$

Then, applying (3.21)-(3.25) and (3.31), by passing to the limit, we conclude that u satisfies

$$\begin{aligned}
&\int_\tau^T [(u_t,v)+\varepsilon(t)(\nabla u_t,\nabla v)]dt+\int_\tau^T(\nabla u,\nabla v)dt\\
&+\int_\tau^T \lambda(u,v)dt+\int_\tau^T(f(u),v)]dt\\
=&\int_\tau^T (g,v)dt.
\end{aligned} \tag{3.32}$$

Due to the arbitrariness of T, for any $v\in\mathcal{H}_t$, *a.e.* $[\tau,T]$,

$$(u_t,v)+\varepsilon(t)(\nabla u_t,\nabla v)+(\nabla u,\nabla v)+\lambda(u,v)+(f(u),v)=(g,v). \tag{3.33}$$

Step 3. We next prove $u(\tau)=u_\tau$. By (3.30), we know that $u\in C([\tau,T],\mathcal{H}_t)$, which makes $u(\tau)$ meaningful. Choosing function $v(t)\in C^1([\tau,T],\mathcal{H}_t)$ with $v(T)=0$ in (3.32), we have

$$\begin{aligned}
&-\int_\tau^T [(u,v_t)-\varepsilon(t)(\nabla u_t,\nabla v)]dt+\int_\tau^T(\nabla u,\nabla v)dt\\
&+\int_\tau^T \lambda(u,v)dt-\int_\tau^T(f(u),v)dt\\
=&\int_\tau^T (g,v)dt+(u(\tau),v(\tau)).
\end{aligned} \tag{3.34}$$

Similarly, it follows from (3.31) that

$$-\int_\tau^T [(u^m,v_t)-\varepsilon(t)(\nabla u_t^m,\nabla v)]dt+\int_\tau^T(\nabla u_m,\nabla v)dt$$

$$+\int_\tau^T \lambda(u^m,v)dt+\int_\tau^T (f(u^m),v)dt$$
$$=\int_\tau^T (g,v)dt+(u^m(\tau),v(\tau)). \tag{3.35}$$

Owing to $u_m(\tau)\to u_\tau$ as $m\to\infty$, we deduce from (3.35) that

$$-\int_\tau^T [(u,v_t)-\varepsilon(t)(\nabla u_t,\nabla v)]dt+\int_\tau^T (\nabla u,\nabla v)dt$$
$$+\int_\tau^T \lambda(u,v)dt+\int_\tau^T (f(u),v)dt$$
$$=\int_\tau^T (g,v)dt+(u_\tau,v(\tau)). \tag{3.36}$$

According to (3.34), (3.36) and the arbitrariness of $v(\tau)$, we get

$$u(\tau)=u_\tau. \tag{3.37}$$

So, the existence of solution follows from (3.30), (3.33) and (3.37).

• Uniqueness of solution.

Let $u_1,\ u_2$ are two solutions of the problem (1.1) with the initial data $u_\tau^1,\ u_\tau^2$, respectively. We define $\bar u(t)=u^1(t)-u^2(t)$, then $\bar u(t)$ satisfies the following equation

$$\bar u_t-\varepsilon(t)\triangle\bar u_t-\Delta\bar u+\lambda\bar u+f(u^1)-f(u^2)=0$$

with initial data

$$\bar u(x,\tau)=\bar u_\tau=u_\tau^1-u_\tau^2.$$

Repeating the arguments used in the proof of (3.29), we can obtain that

$$\|u_1-u_2\|^2_{\mathcal{H}_t}\le e^{2l(t-\tau)}\|u_\tau^1-u_\tau^2\|^2_{\mathcal{H}_\tau},$$

which implies the uniqueness and continuous dependence of solution on initial value. □

According to Theorem 3.1, we can define a continuous process $\{U(t,\tau)\}_{t\ge\tau}$ by

$$U(t,\tau):\mathcal{H}_\tau\to\mathcal{H}_t,\quad t\ge\tau\in\mathbb{R}$$

acting as $U(t,\tau)u_\tau=u(t)$.

4. The time-dependent global attractor

In this subsection, we first consider a time-dependent absorbing family for the solution process to prove the existence of the time-dependent global attractor.

Theorem 4.1. *Assume that* (1.2)-(1.4) *hold,* $g\in H^{-1}(\Omega)$. *For any* $u_\tau\in\mathbb{B}_{H_\tau}(R_0)\subset\mathcal{H}_\tau$, *there exists* $R_1>0$ *such that* $\mathfrak{B}=\{B_t\}_{t\in\mathbb{R}}=\{\mathbb{B}_{\mathcal{H}_t}(R_1)\}_{t\in\mathbb{R}}$ *is a time-dependent absorbing family in* $\mathcal{H}_t$ *for the process* $\{U(t,\tau)\}_{t\ge\tau}$ *corresponding to the problem* (1.1).

Proof. Multiplying the equation (1.1) by $2u$ and repeating the arguments used in the proof of (3.6), we get

$$\begin{aligned}&\frac{d}{dt}(\|u\|^2+\varepsilon(t)\|u\|_1^2)+\sigma(\|u\|^2+\varepsilon(t)\|u\|_1^2)+\frac{1}{2}\|u\|_1^2+2\gamma_1\int_\Omega |u|^p dx\\ \le& 2\|g(x)\|_{-1}^2+2\beta_1|\Omega|. \end{aligned}\tag{4.1}$$

Let $E_1(t)=\|u\|^2+\varepsilon(t)\|u\|_1^2$, then

$$\frac{d}{dt}E_1(t)+\sigma E_1(t)\le 2\|g(x)\|_{-1}^2+2\beta_1|\Omega|. \tag{4.2}$$

By Gronwall's lemma, we see that

$$E_1(t)\le e^{-\sigma(t-\tau)}E(\tau)+\frac{2}{\sigma}(\|g(x)\|_{-1}^2+\beta_1|\Omega|),$$

that is,

$$\|u\|^2+\varepsilon(t)\|u\|_1^2\le R_1$$

for any $t\ge t_1=\tau+\frac{2}{\sigma}\ln\frac{E_1(\tau)}{R_1}$, where $R_1=\frac{4}{\sigma}(\|g(x)\|_{-1}^2+\beta_1|\Omega|)$ and σ is given in Theorem 3.1.

So, $B_t=\{u\in\mathcal{H}_t:\|u(t)\|^2+\varepsilon(t)\|u(t)\|_1^2\le R_1\}$ is a time-dependent absorbing set in $\mathcal{H}_t$ for the solution process $\{U(t,\tau)\}_{t\ge\tau}$. The proof is finished. □

Next, we will show that the process $\{U(t,\tau)\}_{t\ge\tau}$ corresponding to the problem (1.1) is pullback asympotically compact by using the method of the contractive function.

Theorem 4.2. *Assume that* (1.2), (1.3) *and* (1.5) *hold. Then the process* $\{U(t,\tau)\}_{t\ge\tau}$ *of the problem* (1.1) *is pullback asymptotic compact in* $\mathcal{H}_t$.

Proof. Let u_n,u_m be the corresponding two solutions of the problem (1.1) with initial data $u^{n\tau},u^{m\tau}\in\mathbb{B}_{\mathcal{H}_\tau}(R_0)$, respectivly. For the sake of convenience, we denote $w(t)=u^n(t)-u^m(t)$, then $w(t)$ satisfies the following equation

$$w_t-\varepsilon(t)\triangle w_t-\triangle w+\lambda w+f(u^n)-f(u^m)=0,\quad t>\tau, \tag{4.3}$$

with

$$w(x,T_1)=w_{T_1}=u_{T_1}^n-u_{T_1}^m.$$

Multiplying the equation (4.3) by w and repeating the arguments used in the proof of (3.29), we can get

$$\frac{d}{dt}\left(\|w\|^2+\varepsilon(t)\|w\|_1^2\right)+\frac{\varepsilon(t)}{L}\|w\|_1^2+2\lambda\|w\|^2\le 2l\|w\|^2. \tag{4.4}$$

Let $E_2(t)=\|w\|^2+\varepsilon(t)\|w\|_1^2$, then

$$\frac{d}{dt}E_2(t)+\sigma E_2(t)\le 2l\|w\|^2. \tag{4.5}$$

Integrating from s to t at both sides of (4.4), we obtain that

$$E_2(t)\le E_2(s)+2l\int_s^t\|w(r)\|^2dr. \tag{4.6}$$

Similarly, integrating from T_1 to t at both sides of (4.5), we have

$$\int_{T_1}^{t} E_2(r)dr \le \frac{1}{\sigma}E_2(T_1) + \frac{2l}{\sigma}\int_{T_1}^{t} \|w(r)\|^2 dr. \tag{4.7}$$

Then, integrating over $[T_1, t]$ about variable s at both sides of (4.6), we find that

$$(t - T_1)E_2(t) \le \int_{T_1}^{t} E_2(r)dr + 2l\int_{T_1}^{t}\int_{s}^{t} \|w(r)\|^2 drds. \tag{4.8}$$

Combing with (4.7) and (4.8), we have

$$E_2(t) \le \frac{E_2(T_1)}{\sigma(t-T_1)} + \frac{2l}{\sigma(t-T_1)}\int_{T_1}^{t} \|w(r)\|^2 dr + \frac{2l}{t-T_1}\int_{T_1}^{t}\int_{s}^{t} \|w(r)\|^2 drds,$$

that is,

$$\|w\|^2 + \varepsilon(t)\|w\|_1^2 \le \frac{E_2(T_1)}{\sigma(t-T_1)} + \psi_{T_1}^{t}(u_{T_1}^{n}, u_{T_1}^{m}). \tag{4.9}$$

For any fixed $\epsilon > 0$ and some fixed t, set $t > T_1$ such that $t - T_1$ enough large, $\frac{E_2(T_1)}{\sigma(t-T_1)} \le \epsilon$. Next, we only need to verify $\psi_{T_1}^{t} \in \mathfrak{C}(B_{T_1})$ for each fixed T_1. In fact, if u^k is solution of the problem (1.1) with initial data $u_\tau^k \in \mathbb{B}_{\mathcal{H}_\tau}(R_0)$. Using the same arguments of Theorem 3.1, we know that $u_t^k \in L^2([T_1, t], \mathcal{H}_t)$. In addition, we have $u^k \in L^2([T_1, t], H_0^1(\Omega))$. Hence, according to Aubin-Lions lemma, there exists a convergent subsequence of u^k (denoted as u^{k_i}) such that

$$\lim_{i\to\infty}\lim_{j\to\infty}\int_{T_1}^{t} \|u^{k_i}(r) - u^{k_j}(r)\|^2 dr = 0. \tag{4.10}$$

At the same time, for some fixed t, $\int_s^t \|u^{k_i}(r) - u^{k_j}(r)\|^2 dr$, $(s \in [\tau, t])$ is bounded. According to the Lebesgue dominated convergence theorem, we know

$$\lim_{i\to\infty}\lim_{j\to\infty}\int_{T_1}^{t}\int_{s}^{t} \|u^{k_i}(r) - u^{k_j}(r)\|^2 drds = 0. \tag{4.11}$$

Together with (4.10) and (4.11), we infer that $\psi_{T_1}^{t} \in \mathfrak{C}(B_{T_1})$. Thus,

$$\|U(t, T_1)|u_{T_1}^{n} - U(t, T_1)u_{T_1}^{m}\| \le \epsilon + \psi_{T_1}^{t}(u_{T_1}^{m}, u_{T_1}^{m}).$$

Consequently, by Theorem 2.1, the process $\{U(t, \tau)\}_{t\ge\tau}$ is pullback asymptotic compact in $\mathcal{H}_t$. The proof is finished. □

Thus, we gain the following result.

Theorem 4.3. *Assume that the conditions* (1.2)-(1.6) *hold, then the process* $U(t, \tau)$: $\mathcal{H}_\tau \to \mathcal{H}_t$ *generated by the problem* (1.1) *has an invariant time-dependent global attractor* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$.

Proof. It follows from Theorem 4.1 and Theorem 4.2 that the problem (1.1) exists a unique time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$. So the proof is completed. □

Remark 4.1. Since $\mathfrak{A}$ is invariant, it follows from Theorem 2.3 that

$$\mathfrak{A} = \{u : t \mapsto u(t) \in \mathcal{H}_t \ with\ u\ CBT\ of\ U(t, \tau)\}.$$

5. Regularity of the attractor

In what follows, we use the idea from [8, 32, 35] to study the regularity of the time-dependent global attractor.

Theorem 5.1. *Assume that* (1.2)-(1.2) *hold,* $g \in H^{-1}(\Omega)$. *Then* $\{A_t\}_{t\in\mathbb{R}}$ *is bounded in* $\mathcal{H}_t^1$.

Proof. Since $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is dense, for every $g \in H^{-1}(\Omega)$ and any $\eta > 0$, there exists a $g^\eta \in L^2(\Omega)$ such that

$$\|g - g^\eta\|_{H^{-1}(\Omega)} < \eta. \tag{5.1}$$

Now, fix $\tau \in \mathbb{R}$, we split the solution $U(t,\tau)u_\tau = u(t)$ with $u_\tau \in A_\tau$ into the sum

$$U(t,\tau)u_\tau = U_0(t,\tau)u_\tau + U_1(t,\tau)u_\tau,$$

where $U_0(t,\tau)u_\tau = v(t)$, $U_1(t,\tau)u_\tau = y(t)$ solve the following equations, respectively,

$$\begin{cases} v_t + \varepsilon(t)Av_t + Av + \lambda v = g - g^\eta, \quad x \in \Omega, \\ v|_{\partial\Omega} = 0, \quad t \geq \tau, \\ v(x,\tau) = u_\tau(x), \quad \tau \in \mathbb{R}, \end{cases} \tag{5.2}$$

and

$$\begin{cases} y_t + \varepsilon(t)Ay_t + Ay + \lambda y + f(u) = g^\eta, \quad x \in \Omega, \\ y|_{\partial\Omega} = 0, \quad t \geq \tau, \\ y(x,\tau) = 0, \quad \tau \in \mathbb{R}. \end{cases} \tag{5.3}$$

According to Theorem 4.1 and (5.1), we know

$$\|U_0(t,\tau)u_\tau\|_{\mathcal{H}_t}^2 \leq e^{-\sigma(t-\tau)}\|u_\tau\|_{\mathcal{H}_t}^2 + \frac{\eta^2}{\sigma}. \tag{5.4}$$

Multiplying the equation (5.3) by Ay and integrating on Ω, we get

$$\frac{d}{dt}(\|y\|_1^2 + \varepsilon(t)\|y\|_2^2) - \varepsilon'(t)\|y\|_2^2 + 2\|y\|_2^2 + 2\lambda\|y\|_1^2 = -2(f(u), Ay) + 2(g^\eta, Ay).$$

It follows from (1.5) and Young's inequality that

$$-(f(u), Ay) = -(f(u) - f(0), Ay) \leq l^2\|u\|^2 + \frac{1}{4}\|y\|_2^2, \tag{5.5}$$

and

$$|(g^\eta, Ay)| \leq \|g^\eta\|^2 + \frac{1}{4}\|y\|_2^2. \tag{5.6}$$

Combining with (5.5) and (5.6), we arrive at

$$\frac{d}{dt}(\|y\|_1^2 + \varepsilon(t)\|y\|_2^2) + (1 - \varepsilon'(t))\|y\|_2^2 + 2\lambda\|y\|_1^2 \leq 2l^2\|u\|^2 + 2\|g^\eta\|^2.$$

Further, it follows from Theorem 4.1 that

$$\frac{d}{dt}(\|y\|_1^2+\varepsilon(t)\|y\|_2^2)+\sigma(\|y\|_1^2+\varepsilon(t)\|y\|_2^2)\leq 2l^2R_1+2\|g^\eta\|^2. \tag{5.7}$$

By Gronwall's lemma, we conclude that

$$\|y\|_1^2+\varepsilon(t)\|y\|_2^2\leq\frac{2}{\sigma}(l^2R_1+\|g^\eta\|^2),$$

that is,

$$\sup_{t\geq\tau}\|U_1(t,\tau)u_\tau\|^2_{\mathcal{H}_t^1}\leq R_2, \tag{5.8}$$

where $R_2=\frac{2}{\sigma}(l^2R_1+\|g^\eta\|^2)$.

For any $t\in\mathbb{R}$, thanks to (5.4) and (5.8), and then

$$dist_{\mathcal{H}_t}(A_t,\mathbb{B}_{\mathcal{H}_t^1}(R_2))=dist_{\mathcal{H}_t}(U(t,\tau)A_\tau,\mathbb{B}_{\mathcal{H}_t^1}(R_2))\leq Ce^{-\sigma_1(t-\tau)}\to 0,\quad \tau\to-\infty,$$

where $\sigma_1>0$,

$$\mathbb{B}_{\mathcal{H}_t^1}(R_2)=\{u(t)\in\mathcal{H}_t^1:\|u(t)\|^2_{\mathcal{H}_t^1}\leq R_2\}.$$

Hence, $A_t\subseteq\mathbb{B}_{\mathcal{H}_t^1}(R_2)$, that is, the time-dependent global attractor $\{A_t\}_{t\in\mathbb{R}}$ is bounded in $\mathcal{H}_t^1$. The proof is completed. □

6. Asymptotic regularity of the attractors

In this subsection, We investigate the relationship between the time-dependent global attractor of $\{U(t,\tau)\}_{t\geq\tau}$ and the global attractor of the limit equation formally corresponding to (1.1) when $t\to+\infty$.

If $\varepsilon(t)\equiv 0$ in (1.1), we obtain the following classical reaction-diffusion equation

$$\begin{cases} u_t-\triangle u+\lambda u+f(u)=g(x), & x\in\Omega,\\ u|_{\partial\Omega}=0, & t>\tau,\\ u(x,\tau)=u_\tau(x), & x\in\Omega.\end{cases} \tag{6.1}$$

Under the conditions (1.2), (1.4), (1.5) and (1.6), it is well known that the problem (6.1) has an unique solution $u(t)$. At the same time, it generates a continuous semigroup $\{S(t)\}_{t\geq 0}$ (see [37]) acting on the space H associated with the problem (6.1), such that $u(t)=S(t)u_\tau$. Further, $\{S(t)\}_{t\geq 0}$ admits the (classical) global attractor A_∞ in H. In addition, we also know that, for any fixed $t\in\mathbb{R}$,

$$A_\infty=\{z(t):\mathbb{R}\to \mathrm{H}\ \textit{with}\ z\ \mathrm{CBT\ of}\ S(t)\}.$$

Now, we establish the asymptotic regularity of the time-dependent global attractors $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$ of the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by (1.1) and the global attractor A_∞ of the semigroup $\{S(t)\}_{t\geq 0}$ generated by (6.1). To prove this result, we first need the following lemmas.

Lemma 6.1. *Assume that* (1.2)-(1.6) *hold,* $g \in H^{-1}(\Omega)$. *Then there exists* $R_3,\ R_4 > 0$ *such that the solution of the problem* (1.1) *with intial data* $u_\tau \in \mathbb{B}_{\mathcal{H}_\tau}(R_0)$ *satisfies*

$$\sup_{u_\tau \in A_\tau} \sup_{t>\tau} (\|u\|_1^2 + \varepsilon(t)\|u\|_2^2) \le R_3, \tag{6.2}$$

and

$$\int_\tau^\infty \|u_t(s)\|^2 dt \le R_4. \tag{6.3}$$

Proof. Note that we can easily verify (6.2) by using Theorem 5.1. We use the same argument of (3.19), then (6.3) holds. □

Lemma 6.2. *Assume that* (1.2)-(1.6) *hold,* $g \in H^{-1}(\Omega)$. *For any sequence* $\{u^n\}_{n=1}^\infty$ *of CBT for the process* $\{U(t,\tau)\}_{t\ge\tau}$ *associated with* (1.1) *and any* $t_n \to +\infty$ $(n \to +\infty)$, *there exists a CBT* z *of the semigroup* $\{S(t)\}_{t\ge\tau}$ *associated with* (6.1) *such that for every* $T_2 > 0$,

$$\sup_{t\in[-T_2,T_2]} \|u^n(t+t_n) - z(t)\| \to 0. \tag{6.4}$$

Proof. Firstly, according to (6.2) and (6.3), for every $T_2 > 0$, $u_\tau \in A_\tau$, we have the boundedness of the sequence $u_n(\cdot+t_n)$ in $L^\infty([-T_2,T_2],\mathrm{H}_1)$ and the sequence $u_t^n(\cdot+t_n)$ in $L^2([-T_2,T_2],\mathrm{H})$, respectively. Then, it follows from the embedding inequality that $u_t^n(\cdot+t_n)$ is bounded in $L^2([-T_2,T_2],H^{-1}(\Omega))$. Therefore, by Lemma 2.2, we find that $u^n(\cdot+t_n)$ is relatively compact in $C([-T_2,T_2],\mathrm{H})$. Then, there exists the function z in H such that $u^n(t+t_n)\to z(t)$ holds in the sense of (6.4). Especially, $z \in C(\mathbb{R},\mathrm{H})$. In addition, there exists $M>0$, such that

$$\sup_{t\in\mathbb{R}} \|z\| \le M. \tag{6.5}$$

Next, we show that z solves (6.1). Set

$$v^n(t) = u^n(t+t_n), \quad \varepsilon_n(t) = \varepsilon(t+t_n),$$

and we rewrite (1.1) for v^n in the form

$$v_t^n = \varepsilon_n(t)\triangle v_t^n + \triangle v^n - \lambda v^n + f(v^n) + g(x).$$

Indeed, for every fixed $T_2>0$, there exists H-valued function φ supported on $(-T_2,T_2)$ such that

$$\int_{-T_2}^{T_2} \varepsilon_n(t)(\triangle v^n,\varphi)dt = -\int_{-T_2}^{T_2} \varepsilon_n'(t)(\triangle v^n,\varphi)dt + \int_{-T_2}^{T_2} \varepsilon_n(t)(\triangle v_t^n,\varphi_t)dt.$$

Using Lemma 6.1, we have

$$\begin{aligned} \left| \int_{-T_2}^{T_2} \varepsilon_n(t)(\triangle v_t^n,\varphi)dt \right| =& \int_{-T_2}^{T_2} \sqrt{\varepsilon_n(t)}\sqrt{\varepsilon_n(t)}\|\triangle v^n\|\|\varphi_t\|dt \\ &+ \int_{-T_2}^{T_2} \frac{\varepsilon_n'(t)}{\sqrt{\varepsilon_n(t)}}\varepsilon_n(t)\|\triangle v^n\|\|\varphi\|dt \\ \le& C_{R_3,c}\int_{-T_2}^{T_2}\sqrt{\varepsilon_n(t)}dt + C_{R_3,c}\int_{-T_2}^{T_2}\frac{\varepsilon_n'(t)}{\sqrt{\varepsilon_n(t)}}dt \end{aligned}$$

$$\leq C_{R_3,c}(\sqrt{\varepsilon_n(T_2)}-\sqrt{\varepsilon_n(-T_2)})+C_{R_3,c,T_2}\sup_{t\in[-T_2,T_2]}\sqrt{\varepsilon_n(t)}.$$

Since

$$\lim_{n\to+\infty}\sup_{t\in[-T_2,T_2]}\varepsilon_n(t)=0,$$

we get

$$\lim_{n\to+\infty}\int_{-T_2}^{T_2}\varepsilon_n(t)(\triangle v_t^n,\varphi)dt=0.$$

Moreover, it follows from (1.4) and (6.4) that

$$\triangle v^n-\lambda v^n+f(v^n)\to\triangle z-\lambda z+f(z),$$

in the topology of $L^\infty([-T_2,T_2],H^{-1}(\Omega))$. So, the convergence

$$v_t^n(t)\to z_t(t),$$

holds in the distributional sense. Thus, we obtain the equality

$$z_t-\triangle z+\lambda z=f(z)+g.$$

Consequently, z is the solution of the problem (6.1). Combining with (6.5), we find that z is a CBT of the semigroup $\{S(t)\}_{t\geq\tau}$. The proof is finished. □

Then, we get immediately the following result by Lemma 2.4.

Theorem 6.1. *If the process $\{U(t,\tau)\}_{t\geq\tau}$ has an invariant time-dependent global attractor, then*

$$\lim_{t\to\infty}dist_{\mathrm{H}}(\Pi_t A_t,A_\infty)=0.$$

References

[1] E. A. Aifantis, *On the problem of diffusion in solids*, Acta Mechanica, 1980, 37, 265–296.

[2] C. T. Anh, D. T. P. Thanh and N. D. Toan, *Global attractors for nonclassical diffusion euqations with hereditary memory and a new class of nonlinearities*, Ann. Polon. Math., 2017, 119, 1–21.

[3] C. T. Anh, D. T. P. Thanh and N. D. Toan, *Averaging of nonclasssical diffusion equations with memory and singularly oscillating forces*, Z. Anal. Anwend., 2018, 37, 299–314.

[4] C. T. Anh and N. D. Toan, *Nonclassical diffusion equations on $\mathbb{R}^N$ with singularly oscillating external forces*, Appl. Math. Lett., 2014, 38, 20–26.

[5] T. Caraballo, A. M. Marquez-Durán and F. Rivero, *Asymptotic behaviour of a non-classical and non-autonomous diffusion equation containing some hereditary characteristic*, Discrete Contin. Dyn. Syst. Ser. B, 2017, 22, 1817–1833.

[6] V. V. Chpyzhov and M. I. Vishik, *Attractor for Equations of Mathematical Physics*, Amer Mathematical Society, Providence, RI, 2002.

[7] M. Conti, V. Danese, C. Giorgi and V. Pata, *A model of viscoelasticity with time-dependent memory kernels*, Amer. J. Math., 2018, 140, 349–389.

[8] M. Conti and V. Pata, *On the regularity of global attractors*, Discrete Contin. Dyn. Syst. Ser., 2009, 25, 1209–1217.

[9] M. Conti and V. Pata, *On the time-dependent cattaneo law in space dimension one*, Appl. Math. Comput., 2015, 259, 32–44.

[10] M. Conti, V. Pata and R. Temam, *Attractors for the process on time-dependent spaces, applications to wave equation*, J. Differential Equations, 2013, 255, 1254–1277.

[11] T. Ding and Y. Liu, *Time-dependent global attractor for the nonclassical diffusion equations*, Appl. Anal., 2015, 94, 1439–1449.

[12] K. Kuttler and E. C. Aifantis, *Existence and uniqueness in nonclassical diffusion*, Quart. Appl. Math., 1987, 45, 549–560.

[13] J. L. Lions, *Quelques méthodes de Résolutions Des Problèms Aus Limites Nonlinéaries*, Dunod Gauthier-Villars, Paris, 1969.

[14] J. L. Lions and E. Magenes, *Non-homogeneous Boundary value Problem and Applications*, Springer-Verlag, Berlin, 1972.

[15] T. Liu and Q. Ma, *Time-dependent asymptotic behavior of the solution for plate equations with linear memory*, Discrete Contin. Dyn. Syst. Ser. B, 2018, 23, 4595–4616.

[16] Q. Ma, Y. Liu and F. Zhang, *Global attractors in* $H_1(\mathbb{R}^N)$ *for nonclassical diffusion equation*, Discrete Dyn. Nat. Soc., 2012, 2012. Article ID 672762.

[17] Q. Ma, J. Wang and T. Liu, *Time-dependent asymptotic behavior of the solution for wave equations with linear memory*, Comput. Math. Appl., 2018, 76, 1372–1387.

[18] Q. Ma, X. Wang and L. Xu, *Existence and regularity of time-dependent global attractors for the nonclassical reaction-diffusion equations with lower forcing term*, Bound. Value Probl., 2016, 2016, 1–11.

[19] F. Meng and C. Liu, *Necessary and sufficient condition for the existence of time-dependent global attractor and application*, J. Math. Phys., 2017, 58, 1–9.

[20] F. Meng, J. Wu and C. Zhao, *Time-dependent global attractor for extensible berger equation*, J. Math. Anal. Appl., 2019, 469, 1045–1069.

[21] F. Meng, M. Yang and C. Zhong, *Attractors for wave equations with nonlinear damping on time-dependent on time-dependent space*, Discrete Contin. Dyn. Syst. Ser. B, 2015, 21, 205–225.

[22] V. Pata and M. Conti, *Asymptotic structure of the attractor for process on time-dependent spaces*, Nonlinear Anal. Real World Appl., 2014, 19, 1–10.

[23] J. G. Peter and M. E. Gurtin, *On the theory of heat condition involving two temperatures*, Z. Angew. Math. Phys., 1968, 19(4), 614–627.

[24] F. D. Plinio, G. Duan and R. Temam, *Time dependent attractor for the oscillon equation*, Discrete Dyn. Nat. Soc., 2011, 29, 141–167.

[25] J. C. Robinson, *Infinite-Dimensional Dynamical Systens*, Cambridge University press, 2011.

[26] J. Simon, *Compact sets in the space* $L^p(0, t; b)$, Ann. Mat. Pura Appl., 1987, 146, 65–96.

[27] C. Sun, S. Wang and C. Zhong, *Global attractors for a nonclassical diffusion equation*, Acta Math. Appl. Sin. Engl. Ser., 2007, 23(7), 1271–1280.

[28] C. Sun and M. Yang, *Dynamics of the nonclassical diffusion equations*, Asymptot. Anal., 2008, 59, 51–81.

[29] D. T. P. Thanh and N. D. Toan, *Existence and long-time behavior of solutions to a class of nonclassical diffusion equations with infinite delays*, Vietnam J. Math., 2019, 47, 309–325.

[30] S. Wang, D. Li and C. Zhong, *On the dynamics of a class of nonclassical parabolic equations*, J. Math. Anal. Appl., 2006, 317, 565–582.

[31] X. Wang and Q. Ma, *Asymptotic structure of time-dependent global attractors for the nonclassical diffusion equations*, J. Sichuan Univ. Eng. Sci. Ed., 2016, 53, 508–511.

[32] Y. Wang, Z. Zhu and P. Li, *Regularity of pullback attractors for nonautonomous nonclassical diffusion equations*, J. Math. Anal. Appl., 2018, 459, 16–31.

[33] Y. Xiao, *Attractors for a nonclassical diffusion equation*, Acta Math. Appl. Sin. Engl. Ser., 2002, 18(2), 273–276.

[34] Y. Xie, Q. Li and K. Zhu, *Attractors for nonclassical diffusion equations with arbitrsry polynomial growth nonlinearity*, Nonlinear Anal. Real World Appl., 2016, 31, 23–37.

[35] S. Zelik, *Asymptotic regularity of solutions of a nonautonomous damped wave equation with a critical growth exponent*, Commun. Pure Appl. Anal., 2004, 3, 921–934.

[36] F. Zhang and L. Bai, *Attractors for the nonclassical diffusion equations of kirchhoff type with critical nonlinearity on unbounded domain*, Dyn. Syst., 2019, 56, 1–24.

[37] C. Zhong, M. Yang and C. Sun, *The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction diffusion equations*, J. Differential Equations, 2006, 223, 367–399.

[38] K. Zhu and C. Sun, *Pullback attractors for nonclassical diffusion equations with delays*, J. Math. Phys., 2015, 56, 1–20.