Journal of Applied Analysis and Computation
Volume 11, Number 2, April 2021, 1029–1038

Website:http://www.jaac-online.com
DOI:10.11948/20200227

EXISTENCE OF PERIODIC AND KINK WAVES IN A PERTURBED DEFOCUSING MKDV EQUATION*

Hong Li[1,†], Lilin Ma[2] and Mei Liu[1]

Abstract In this paper, we consider the existence of periodic and kink wave solutions of a perturbed defocusing mKdV equation. Based on geometric singular perturbation theory, Chebyshev criteria and bifurcation theory of dynamic system, the wave speed conditions for the periodic and kink solutions are given. The monotonicity of the wave speed is proved, and moreover the upper and lower bounds of the limiting wave speeds are obtained. The uniqueness of the periodic waves is established by showing that the Abelian integrals form a Chebyshev set. In addition, there is no coexistence of one periodic and one solitary waves. The proof process does not need any explicit expression of the original defocusing mKdV periodic wave or kink wave solutions.

Keywords Defocusing mKdV equation, traveling wave, Abelian integral, Chebyshev system.

MSC(2010) 34C60, 37C27, 37C29.

1. Introduction

Traveling wave solution is an important type of solutions for nonlinear wave equations. Many nonlinear wave equations have a number of different traveling wave solutions. However, the traveling wave solution is very sensitive to external influences [1]. In practice, there are often small external perturbations. In order to simulate these small perturbations, the perturbed terms are often added to the wave equation to generate the perturbed nonlinear wave equation.

In this paper, we consider the perturbed defocusing mKdV equation

$$\psi_t - \psi^2\psi_x + \psi_{xxx} + \varepsilon(q\psi_{xx} + r(\psi\psi_x)_x + s\psi_{xxxx}) = 0, \tag{1.1}$$

where q, r, s are constants and $\varepsilon > 0$ is a perturbation parameter. When $\varepsilon = 0$, the equation is a defocusing mKdV equation. System (1.1) is more complex than the defocusing mKdV equation, and some methods [6, 9, 10] for finding exact solutions usually do not work. In recent years, the existence of many kinds of traveling wave solutions for some perturbed wave equations have been obtained by using the geometric singular perturbation theory [7, 8, 12, 13].

†The corresponding author. Email: Lchzxz3@126.com(H. Li)
1Department of Mathematics, Jiujiang University, Jiujiang 332005, China
2Information Technology Center, Jiujiang University, Jiujiang 332005, China
*The authors were supported by National Natural Science Foundation of China (11861040, 11961037) and Program for Innovation training of Jiangxi Province (S202011843012).

Recently, Chen et al. [3] considered the perturbed defocusing mKdV equation

$$\psi_t - \psi^2\psi_x + \psi_{xxx} + \varepsilon(\psi_{xx} + \psi_{xxxx}) = 0. \tag{1.2}$$

It is proved that kink waves and periodic waves exist when ε is small enough. Compared with Eq.(1.2), Eq.(1.1) contains the nonlinear term $(\psi\psi_x)_x$ due to the Marangoni effect, describing the opposite to the Bénard convection. Sun et al. [15] showed that the existence of traveling wave solutions for the perturbed mKdV equation is related to the Marangoni effect, but is it the same for the perturbed defocusing mKdV equation? What about the existence of traveling wave solutions? Is there coexistence? In this paper, we study these problems and give the necessary and sufficient conditions for the existence of traveling wave solutions.

In order to study traveling wave solutions of Eq.(1.1), we set

$$\psi(x,t) = \psi(\xi), \qquad \xi = x + ct, \tag{1.3}$$

where $c(c > 0)$ is the wave speed.

Substituting (1.3) into system (1.1), it is reduced to

$$c\psi'(\xi) - \psi^2\psi'(\xi) + \psi'''(\xi) + \varepsilon(q\psi''(\xi) + r(\psi\psi'(\xi))' + s\psi''''(\xi)) = 0. \tag{1.4}$$

Integrating (1.4) with respect to ξ and omitting the integral constant, we get

$$c\psi - \frac{\psi^3}{3} + \psi''(\xi) + \varepsilon(q\psi'(\xi) + r\psi\psi'(\xi) + s\psi'''(\xi)) = 0. \tag{1.5}$$

After transformation $\xi = \sqrt{\frac{1}{c}}\tau$ and $\psi = \sqrt{3c}\phi$, Eq.(1.5) is transformed into

$$\phi(\tau) - \phi^3(\tau) + \phi''(\tau) + \frac{\varepsilon}{\sqrt{c}}(q\phi'(\tau) + r\sqrt{3c}\phi\phi'(\tau) + sc\phi'''(\tau)) = 0, \tag{1.6}$$

which has an equivalent form

$$\frac{d\phi}{d\tau} = y, \quad \frac{dy}{d\tau} = -\phi + \phi^3 - \frac{\varepsilon}{\sqrt{c}}(q\phi'(\tau) + r\sqrt{3c}\phi\phi'(\tau) + sc\phi'''(\tau)). \tag{1.7}$$

$(1.6)_{\varepsilon=0}$ has an equivalent form

$$\frac{d\phi}{d\tau} = y, \quad \frac{dy}{d\tau} = -\phi + \phi^3, \tag{1.8}$$

which is a Hamiltonian system and the Hamiltonian function is

$$H(\phi, y) = -\frac{y^2}{2} - \frac{\phi^2}{2} + \frac{\phi^4}{4}. \tag{1.9}$$

It is easy to see that system (1.8) has three equilibrium points at $E_0(0,0)$, $E_1(1,0)$ and $E_2(-1,0)$. E_0 is a center , E_1 and E_2 are saddle points. For the phase portrait $H(\phi, y) = h$, there are a family of closed orbits surrounding the center E_0 when $h \in (-\frac{1}{4}, 0)$ and a heteroclinic orbit to E_1 and E_2 enclosing the closed orbits when $h = -\frac{1}{4}$. (see Figure 1).

The remaining part is organized as follows. In Section 2, we present some perturbation theories and derive a special form of Abelian integral for periodic and kink waves. In Section 3, the monotonicity of Abelian integral ratio is analyzed and the range of its value is obtained. In section 4, we state and prove our main result. This paper ends with a brief conclusion.

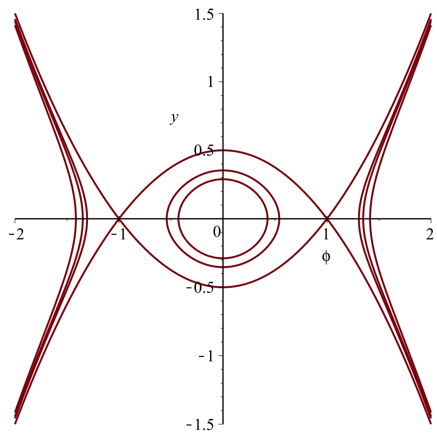

Figure 1. Phase portraits of Eq. (1.8) on the (ϕ, y) plane.

2. Perturbation Analysis

Lemma 2.1 (Fenichel Criteria)**.** *Consider the system*

$$\dot{x} = g_1(x, y, \varepsilon), \quad \dot{y} = \varepsilon g_2(x, y, \varepsilon), \tag{2.1}$$

where $x \in \mathbb{R}^l$, $y \in \mathbb{R}^m$, $0 < \varepsilon \ll 1$ is a real parameter, g_1 and g_2 are c^∞ on the set $V \times I$, where $V \in \mathbb{R}^{l+m}$ and I is an open interval containing zero. Assume that for $\varepsilon = 0$, system (2.1) has a compact normally hyperbolic manifold M_0 which is contained in the set $g_1(x, y, 0) = 0$. The manifold M_0 is said to be normally hyperbolic if the linearization of (2.1) at each point in M_0 has exactly $dim(M_0)$ eigenvalues on the imaginary axis. Then for any $0 < r < +\infty$, there exists a manifold M_ε such that the following conclusions hold.

(i) M_ε is locally invariant under the flow of (2.1);

(ii) M_ε is C^r in x, y and ε;

(iii) $M_\varepsilon = \{(x, y)|x = h_\varepsilon(y)\}$ for any C^r function h^ε and y in some compact set K;

(iv) There exist locally invariant stable and unstable manifolds $W^s(M_\varepsilon)$, $W^u(M_\varepsilon)$, that lie within $\mathcal{O}(\varepsilon)$ of, and are diffeomorphic to $W^s(M_0)$ and $W^u(M_0)$.

In order to study the dynamical behavior of the perturbed system (1.7) , we express it as

$$\frac{d\phi}{d\tau} = y, \quad \frac{dy}{d\tau} = v, \quad \varepsilon s\sqrt{c}\frac{dv}{d\tau} = -\phi + \phi^3 - v - \frac{\varepsilon}{\sqrt{c}}(qy + r\sqrt{3c}\phi y), \tag{2.2}$$

which is the slow system. Letting $\sigma = \frac{\tau}{\varepsilon}$, the system(2.2) is converted to the fast system

$$\frac{d\phi}{d\sigma} = \varepsilon y, \quad \frac{dy}{d\sigma} = \varepsilon v, \quad s\sqrt{c}\frac{dv}{d\sigma} = -\phi + \phi^3 - v - \frac{\varepsilon}{\sqrt{c}}(qy + r\sqrt{3c}\phi y). \tag{2.3}$$

For $\varepsilon > 0$, two systems (2.2) and (2.3) are equivalent. Let

$$M_0 = \{(\phi, y, v) \in R^3 | v = -\phi + \phi^3\}$$

which is a 2-dimensional critical manifold of the slow system (2.2). The Jacobian matrix of the fast system (2.3) restricted to M_0 is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{s\sqrt{c}}(3\phi^2-1) & 0 & -\frac{1}{s\sqrt{c}} \end{pmatrix},$$

which has three eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -\frac{1}{s\sqrt{c}}$. The number of eigenvalues on the imaginary axis is exactly equal to the dimension of M_0, This implies that M_0 is a normal hyperbolic. According to Lemma (2.1), if $\varepsilon > 0$ is sufficiently small, there is an invariant manifold M_ε for the slow system (2.2)

$$M_\varepsilon = \{(\phi, y, v) \in R^3 | v = -\phi + \phi^3 + \zeta(\phi, y, \varepsilon)\},$$

where $\zeta(\phi, y, \varepsilon)$ depends smoothly on ε and satisfies $\zeta(\phi, y, 0) = 0$. Therefore, we can let $\zeta(\phi, y, \varepsilon) = \varepsilon\zeta_1(\phi, y) + \mathcal{O}(\varepsilon^2)$. Substituting it into the third equation of the slow system (2.2), we can get $\zeta_1(\phi, y) = -s\sqrt{c}(3\phi^2 - 1)y - \frac{qy}{\sqrt{c}} - \sqrt{3}r\phi y$. The dynamics of the slow system (2.2) will be

$$\frac{d\phi}{d\tau} = y, \quad \frac{dy}{d\tau} = -\phi + \phi^3 + \varepsilon[-s\sqrt{c}(3\phi^2 - 1)y - \frac{qy}{\sqrt{c}} - \sqrt{3}r\phi y] + \mathcal{O}(\varepsilon^2). \tag{2.4}$$

Now we can check the existence of periodic orbits by the following methods. For $h \in (-\frac{1}{4}, 0)$, $H(\phi, y) = h$ defines a period orbit Γ_h of system (1.9). Let $(a, 0)$ denote the intersection of Γ_h and the negative ϕ-axis. Let $(\phi(\tau), y(\tau))$ be a solution to system (2.4) with $(\phi(0), y(0)) = (a, 0)$, then τ_1 and $\tau_2(\tau_2 < 0 < \tau_1)$ exist so that

$$y(\tau) > 0 \quad \text{for} \quad 0 < \tau < \tau_1, \quad y(\tau_1) = 0$$

and

$$y(\tau) < 0 \quad \text{for} \quad \tau_2 < \tau < 0, \quad y(\tau_2) = 0.$$

Then we can define a function

$$\Phi(a, c, \varepsilon) = \int_{\tau_2}^{\tau_1} \dot{H}(\phi, y) d\tau$$

and

$$\dot{H}(\phi, y) = \frac{\partial H}{\partial \phi}\frac{d\phi}{d\tau} + \frac{\partial H}{\partial y}\frac{dy}{d\tau} = \frac{\varepsilon}{\sqrt{c}} y^2(sc(3\phi^2 - 1) + q + \sqrt{3c}r\phi) + \mathcal{O}(\varepsilon^2).$$

The system (2.4) has a periodic solution passing through $(a, 0)$ if and only if $\Phi(a, c, \varepsilon) = 0$. Since $\Phi(a, c, 0) = 0$, we have

$$\Phi(a, c, \varepsilon) = \varepsilon\widetilde{\Phi}(a, c, \varepsilon) = \frac{\varepsilon}{\sqrt{c}}(\int_{\tau_2}^{\tau_1} (sc(3\phi^2 - 1) + q + \sqrt{3c}r\phi)y^2 d\tau + \mathcal{O}(\varepsilon)).$$

Let

$$\widetilde{\Phi}_0(a, c) = \lim_{\varepsilon \to 0} \widetilde{\Phi}(a, c, \varepsilon) = \frac{1}{\sqrt{c}} \int_{\tau_2}^{\tau_1} (sc(3\phi^2 - 1) + q + \sqrt{3c}r\phi) y d\phi. \tag{2.5}$$

The existence of periodic wave of Eq.(1.6) depends on whether the Abelian integral $\widetilde{\Phi}_0(a,c)$ has zero points.The condition for the limit speed c_0 of a periodic orbit is $\widetilde{\Phi}_0(a,c_0)=0$.

For $h=-\frac{1}{4}$, $H(\phi,y)=h$ defines a heteroclinic orbit $\Gamma_{-\frac{1}{4}}$ of system (1.9). We can define a similar function for the heteroclinic orbit as

$$\Psi(c,\varepsilon)=\int_{-\infty}^{0}\dot{H}(\phi,y)d\tau+\int_{0}^{+\infty}\dot{H}(\phi,y)d\tau.$$

Here the first part is the integral along with a solution $(\phi(\tau),y(\tau))$ on the one dimensional unstable manifold of the saddle point $E_2(-1,0)$ with $y(\tau)>0$ for $-\infty<\tau<0$ and $y(0)=\frac{\sqrt{2}}{2}$. The second part is the integral along the stable manifold. $\widetilde{\Psi}(c,\varepsilon)$ and $\widetilde{\Psi}_0(c)$ are similar to the case of periodic solutions. Consequently, we get

$$\widetilde{\Psi}_0(c)=\frac{1}{\sqrt{c}}\oint_{\Gamma_{-\frac{1}{4}}}(sc(3\phi^2-1)+q+\sqrt{3c}r\phi)yd\phi. \tag{2.6}$$

Therefore the condition for the limit speed c_0 of a heteroclinic orbit is $\widetilde{\Psi}_0(c_0)=0$.

3. Analysis by the Abelian integral theory

Let

$$J_i(h)=\oint_{\Gamma_h}\phi^i yd\phi,\quad i=0,1,2. \tag{3.1}$$

According to the symmetry, we know $J_1(h)=0$ for $h\in(-\frac{1}{4},0)$.

Then $\widetilde{\Phi}_0(a,c)=\frac{1}{\sqrt{c}}((q-sc)J_0(h)+3scJ_2(h))$.

Lemma 3.1. *For $h\in(-\frac{1}{4},0)$, $J_0'(h)<0$ and $J_0(h)>0$.*

Proof. By $y^2=-\phi^2+\frac{\phi^4}{2}-2h$, we have

$$J_0'(h)=\oint_{\Gamma_h}-\frac{d\phi}{y}=-\int_0^{T(h)}d\tau=-T(h)<0,$$

where $T(h)$ is the period of the periodic orbit Γ_h for $h\in(-\frac{1}{4},0)$.

$$J_0(0)=\lim_{h\to 0}\oint_{\Gamma_h}yd\phi=\lim_{h\to 0}\int_0^{T(h)}y^2d\tau=0.$$

Because $J_0(h)$ is strictly monotonically decreasing, so $J_0(h)>0$ for $h\in(-\frac{1}{4},0)$. □

By Lemma 3.1 and $\widetilde{\Phi}_0(a,c)=0$, we know the existence of periodic wave of Eq.(1.6) depends on whether $(3\frac{J_2}{J_0}(h)-1)sc+q=0$ has zero points.

Through some simple integral operations, we have

$$J_0(-\frac{1}{4})=\frac{4\sqrt{2}}{3},\qquad J_2(-\frac{1}{4})=\frac{4\sqrt{2}}{15}. \tag{3.2}$$

Let $\alpha(h)$ be the a non-negative real root of $-\frac{\phi^2}{2}+\frac{\phi^4}{4}=h$, where $0\le\alpha(h)<1$.

Lemma 3.2. $\lim_{h\to 0} \frac{J_2(h)}{J_0(h)} = 0$.

Proof. $\lim_{h\to 0} \frac{J_2(h)}{J_0(h)} = \lim_{h\to 0} \frac{\int_{-\alpha(h)}^{\alpha(h)} \phi^2 y d\phi}{\int_{-\alpha(h)}^{\alpha(h)} y d\phi} = \lim_{\phi\to 0} \phi^2 = 0.$ □

In general, it is very complicated to prove the monotonicity of Abelian integral ratio (Carr, Chow and hale [2], and chow and Sanders [4]), and sometimes it is difficult to find an appropriate analysis method. In fact, this kind of proof can be transformed into proving that a group of Abelian integral has Chebyshev property [11]. In this way, the problem can be solved by pure algebra [14]. For completeness, the following definitions are introduced:

Definition 3.1. Let $j_0, j_1, ..., j_{n-1}$ be analytic functions on a real open interval K.

(i) $\{j_0, j_1, ..., j_{n-1}\}$ is a Chebyshev system (abbreviated T-system) on K if any nontrivial linear combination

$$k_0 j_0 + k_1 j_1 + ... + k_{n-1} j_{n-1} = 0$$

has at most $n-1$ isolated zeros on K .

(ii) An ordered set of n functions $\{j_0, j_1, ..., j_{n-1}\}$ is called a complete Chebyshev system (abbreviated CT-system) on K if $\{j_0, j_1, ..., j_{r-1}\}$ is a T-system for each $r = 1, ..., n$. Moreover it is an extended complete Chebyshev system (abbreviated ECT-system) if any nontrivial linear combination

$$k_0 j_0 + k_1 j_1 + ... + k_{r-1} j_{r-1} = 0$$

has at most $r-1$ zeros on K counted with multiplicities.

(iii) The Wronskian of $\{j_0, j_1, ..., j_{n-1}\}$ at $x \in K$ is

$$W[j_0, j_1, ..., j_{n-1}](x) = \begin{vmatrix} j_0(x) & j_1(x) & ... & j_{n-1}(x) \\ j_0'(x) & j_1'(x) & ... & j_{n-1}'(x) \\ ... & ... & ... & ... \\ j_0^{n-1}(x) & j_1^{n-1}(x) & ... & j_{n-1}^{n-1}(x) \end{vmatrix}.$$

Lemma 3.3 (see [5]). *$\{j_0, j_1, ..., j_{n-1}\}$ is an ECT-system on K if and only if, for each $r = 1, ..., n$,*

$$W[j_0, j_1, ..., j_{r-1}](x) \neq 0$$

for all $x \in K$.

For Eq.(1.9) $H(\phi, y) = -\frac{y^2}{2} + A(\phi)$, where $A(\phi) = -\frac{\phi^2}{2} + \frac{\phi^4}{4}$, there exists a punctured neighborhood w of the origin foliated by ovals $\Gamma_h \subset \{H(\phi, y) = h| -\frac{1}{4} < h < 0\}$. We call them *period annulus* and the projection of w on the $\phi-$axis is $(-1, 1)$. Obviously $\phi A'(\phi) < 0$ for any $\phi \in (-1, 1)\backslash\{0\}$ and $A(\phi)$ have even multiplicity at $\phi = 0$. Then, there exists an analytical function $z(\phi)$ such that $A(\phi) = A(z(\phi))$ for $\phi \in (-1, 1)$, where $z(\phi)$ is the involution associated to A. Factorizing $A(\phi) - A(z)$ gives $\frac{1}{4}(\phi - z)(\phi + z)(\phi^2 + z^2 - 2)$. Therefore, the involution $z(\phi) = -\phi$ and $z'(\phi) = -1$. If we restrict $\phi \in (-1, 0)$, then $0 < z(\phi) < 1$.

For the Abelian integral

$$J_i(h) = \oint_{\Gamma_h} f_i(\phi) y d\phi, \quad h \in (-\frac{1}{4}, 0),$$

where $i = 0, 1, 2$ and $f_0(\phi) = 1, f_1(\phi) = \phi, f_2(\phi) = \phi^2$. We define the criterion function

$$g_i(\phi) := \frac{f_i(\phi)}{A'(\phi)} - \frac{f_i(z(\phi))}{A'(z(\phi))}, \quad i = 0, 1, 2.$$

Then, we have Lemma 3.4.

Lemma 3.4 (see [5]). *Under the above assumption,* $\{J_0, J_2\}$ *is an ECT system on* $(-\frac{1}{4}, 0)$ *if* $\{g_0, g_2\}$ *is an ECT system on* $(-1, 0)$ *or* $(0, 1)$.

Lemma 3.5. $\{J_0, J_2\}$ *is an ECT-system.*

Proof. Based on the previous discussion, we know that we only need to prove $\{g_0, g_2\}$is ECT-system. By Maple, we have:

$$g_0 = -\frac{(\phi - z)(\phi^2 + \phi z + z^2 - 1)}{\phi(\phi - 1)(\phi + 1)z(z - 1)(z + 1)},$$

$$W[g_0, g_2] = -\frac{(\phi - z)^3 q_1(\phi, z)}{\phi^2(\phi - 1)^2(\phi + 1)^2 z^2 (z - 1)^2 (z + 1)^2},$$

where $q_1(\phi, z) = 2\phi^3 z + 3\phi^2 z^2 + 2\phi z^3 + z^2 + \phi^2 - 1$ and $z = -\phi$.

Because $z = -\phi$, study whether g_0 has zero solutions as long as studies whether the following equations (3.3)

$$z = -\phi, \quad \phi^2 + \phi z + z^2 - 1 = 0, \tag{3.3}$$

have roots for $-1 < \phi < 0 < z < 1$. After simple calculation, it is not difficult to get $g_0 \neq 0$ for $\phi \in (-1, 0)$. Similarly, we can get $W[g_0, g_2] \neq 0$ for $\phi \in (-1, 0)$.

By Lemma 3.3, we know that $\{g_0, g_2\}$ is ECT system for $\phi \in (-1, 0)$. Then we can conclude that $\{J_0, J_2\}$ is ECT-system by Lemma 3.4. □

Lemma 3.6. *For* $h \in [-\frac{1}{4}, 0)$, $\frac{J_2(h)}{J_0(h)}$ *monotonously decreases from* $\frac{1}{5}$ *to* 0.

Proof. By Lemma 3.5, we have shown that $\{J_0, J_2\}$ is an extended Chebyshev system, and therefore, $\frac{J_2(h)}{J_0(h)}$ is monotonic on $(-\frac{1}{4}, 0)$, By Lemma 3.2 and Eq.(3.2), we have $\lim\limits_{h \to -\frac{1}{4}} \frac{J_2(h)}{J_0(h)} = \frac{1}{5}$ and $\lim\limits_{h \to 0} \frac{J_2(h)}{J_0(h)} = 0$. So, $\frac{J_2(h)}{J_0(h)}$ monotonously decreases from $\frac{1}{5}$ to 0 on $[-\frac{1}{4}, 0)$. □

4. Main Result

For the perturbed defocusing mKdV equation (1.1), we have the following results about the traveling wave propagating with speed $c(c > 0)$ from left to right.

Theorem 4.1. *If* $\frac{q}{s} > 0$, *for any sufficiently small* $\varepsilon > 0$,

(i) The perturbed defocusing mKdV equation (1.1) *has only one isolated periodic traveling wave*

$$\psi = \sqrt{3c}\phi(\tau, \varepsilon, h, c(\varepsilon, h)) \quad (c > 0)$$

in a sufficiently small neighborhood of Γ_h, *where* $\phi(\tau, \varepsilon, h, c)$ *is a solution of Eq.*(1.6) *and*

$$\lim_{\varepsilon \to 0} c(\varepsilon, h) = c_0(h), \quad \frac{q}{s} < c_0(h) < \frac{5}{2}\frac{q}{s}.$$

(ii) *The perturbed defocusing mKdV equation* (1.1) *have kink, anti-kink waves*

$$\psi = \pm\sqrt{3c}\phi(\tau, \varepsilon, 0, c(\varepsilon, 0)) \quad (c > 0)$$

in a sufficiently small neighborhood of $\Gamma_{-\frac{1}{4}}$, *where* $\phi(\tau, \varepsilon, 0, c)$ *is a solution of Eq.*(1.6) *and*

$$\lim_{\varepsilon \to 0} c(\varepsilon, 0) = \frac{5q}{2s}.$$

Proof. By Eq.(2.5) and Eq.(2.6), we can get the limit speed

$$c_0(h) = \frac{q}{s}\frac{1}{1 - 3\frac{J_2}{J_0}(h)}. \tag{4.1}$$

For $h \in (-\frac{1}{4}, 0)$,

$$\frac{\partial}{\partial c}\widetilde{\Phi}(a(h), c_0(h), 0) = \frac{sJ_0(h)}{2c_0(h)\sqrt{c_0(h)}}(c_0(h)(3\frac{J_2}{J_0}(h) - 1) - \frac{q}{s}).$$

For $h = -\frac{1}{4}$,

$$\frac{\partial}{\partial c}\widetilde{\Psi}(c_0(-\frac{1}{4}), 0) = \frac{sJ_0(-\frac{1}{4})}{2c_0(-\frac{1}{4})\sqrt{c_0(-\frac{1}{4})}}(c_0(-\frac{1}{4})(3\frac{J_2}{J_0}(-\frac{1}{4}) - 1) - \frac{q}{s}).$$

By Lemma 3.6, we have $\frac{\partial}{\partial c}\widetilde{\Phi}(a(h), c_0, 0) < 0$ and $\frac{\partial}{\partial c}\widetilde{\Psi}(c_0(-\frac{1}{4}), 0) < 0$. We can solve the equation $\widetilde{\Phi} = 0$ and $\widetilde{\Psi} = 0$ by the implicit function theorem. That is, there exists a unique smooth function $c(\varepsilon, h) = c_0(h) + \mathcal{O}(\varepsilon)$ makes $\widetilde{\Phi}_0(a(h), c) + \mathcal{O}(\varepsilon) = 0$ and a unique smooth function $c(\varepsilon, -\frac{1}{4}) = c_0(-\frac{1}{4}) + \mathcal{O}(\varepsilon)$ makes $\widetilde{\Psi}_0(c) + \mathcal{O}(\varepsilon) = 0$.

By Lemma 3.6, and Eq.(4.1), we can get the range of the limit speed $c_0(h)$ for one isolated periodic traveling wave is $\frac{q}{s} \leq c_0(h) < \frac{5}{2}\frac{q}{s}$ and the limit speed for kink, anti-kink waves is $c_0(-\frac{1}{4}) = \frac{5q}{2s}$. □

Remark 4.1. The existence of periodic waves and kink waves of the system (1.1) is independent of the Marangoni effect parameter r.

Remark 4.2. System (1.1) can not possess the coexistence phenomenon of periodic and kink waves.

5. Conclusions

In this paper, we study the existence of periodic traveling wave solutions and kink wave solutions for the perturbed defocusing mKdV equation (1.1). The uniqueness of periodic wave is established, that is, the system does not have multiple periodic solutions. We also find that there is no coexistence of periodic solution and kink solution. Chebyshev criterion is used to transform the analysis of the ratio of Abelian integral into an algebraic problem, and then Sturm's theory is used to solve the problem with the help of the function RealRootCounting in maple. When the degree of Hamiltonian function of the unperturbed evolution equation is higher, this method is superior to Picard-Fuchs equation method. This method is also useful in the study of other types of wave equations. The wave speed conditions for the existence of periodic solutions and kink solutions are given, which are necessary and sufficient conditions.

Acknowledgements

Any comments and suggestions from referees and editors are highly appreciated.

References

[1] F. H. Busse, *Non-linear properties of thermal convection,* Rep. Prog. Phys., 1978, 41, 1929–1967.

[2] J. Carr, S. N. Chow and J. K. Hale, *Abelian integrals and bifurcation theory,* J. Differential Equations, 1985, 59, 413–436.

[3] A. Chen, L. Guo and W. Huang, *Existence of kink waves and periodic waves for a perturbed defocusing mkdv equation,* Qual Theor Dyn Syst, 2018, 17(3), 495–517.

[4] S. N. Chow and J. A. Sanders, *On the number of critical points of the period,* J. Differential Equations, 1986, 64, 51–66.

[5] M. Grau, F. Maenosas and J. Villadelprat, *A chebyshev criterion for abelian integrals,* Trans. Amer. Math. Soc., 2011, 363, 109–129.

[6] S. Geomez, A. Cesar, M. Molati and M. P. Ramollo, *Exact solutions of a generalized kdv-mkdv equation,* Int. J. Nonlinear Sci., 2012, 13, 94–98.

[7] L. Hattam, *Travelling wave solutions of the perturbed mkdv equation that represent traffic congestion,* Wave Motion, 2018, 79, 57–72.

[8] F. d. Heinrich and F. Jan, *Nonlinear waves and polarization in diffusive directed particle flow,* Siam J Appl Math, 2018, 78(2), 759–773.

[9] J. Li and M. Han, *Exact peakon solutions given by the generalized hyperbolic functions for some nonlinear wave equations,* J Appl Anal Comput, 2020, 10(4), 1708–1719.

[10] J. Liang and J. Li, *Bifurcations and exact solutions of nonlinear Schrödinger equation with an anti-cubic nonlinearity,* J Appl Anal Comput, 2018, 8(4), 1194–1210.

[11] D. Marin and J. Villadelprat, *On the chebyshev property of certain abelian integrals near a polycycle,* Qual Theor Dyn Syst, 2018, 17(1), 261–270.

[12] J. R. Miller, *Invasion waves and pinning in the kirkpatrick-barton model of evolutionary range dynamics,* J Math Biol, 2019, 78, 257–292.

[13] J. C. da Mota and A. J. de Souza, *Multiple traveling waves for dry forward combustion through a porous medium,* Siam J Appl Math, 2018, 78(2), 1056–1077.

[14] X. Sun, *Exact bound on the number of limit cycles arising from a periodic annulus bounded by a symmetric heteroclinic loop,* J Appl Anal Comput, 2020, 10, 378–390.

[15] X. Sun, W. Huang and J. Cai, *Coexistence of the solitary and periodic waves in convecting shallow water fluid,* Nonlinear Anal-Real, 2020, 53, 103067.