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POSITIVE SOLUTIONS IN THE SPACE OF LIPSCHITZ FUNCTIONS TO A CLASS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE-POINT BOUNDARY VALUE CONDITIONS

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Abstract The purpose of this paper is to study the existence of positive solutions to a class of fractional differential equations with infinite-point boundary value conditions. Our solutions are placed in the space of Lipschitz functions and the main tools used in the proof of the results are a sufficient condition about the relative compactness in Holder spaces and the classical Schauder fixed point theorem.

Keywords Fractional differential equation, Holder spaces, Schauder fixed point theorem, positive solution.

MSC(2010) 47H10, 49L20.

1. Introduction

Fractional calculus has proved to be a valuable tool in the modeling of a great number of processes appearing in physics, control-theory, electron-analytical chemistry, etc ([6,7,9]).

In this paper, we study the existence of positive solutions to the following fractional differential equation with infinite-point boundary value conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t), (Hu)(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \displaystyle\sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases} \tag{1.1}$$

where $\alpha > 2$, $n-1 < \alpha \leq n$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$, $(j = 1, 2, \cdots)$, $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$, where $\Delta = (\alpha-1)(\alpha-2)\cdots(\alpha-i)$, H is an operator (not neccesarily linear) applying $\mathcal{C}[0,1]$ into itself satisfying certain assumptions and D_{0+}^{α} denotes de Riemann-Liouville fractional derivative. Our solutions are localized in the space of Holder functions.

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Our analysis relies in a sufficient condition about the relative compactness in Holder spaces recently proved in [1] and in the classical Schauder fixed point theorem.

To the best of our knowledgement, in the papers appearing in the literature and studying the same question that the one treated in this paper, the solutions are in the space $\mathcal{C}[0,1]$ and, perhaps, one of the main contributions of this work yields in the fact that the solutions of our paper are placed in the space of Lipschitz functions on $[0,1]$. A few papers studying solutions in Holder spaces have appeared in the literature (see [2–5, 8]).

2. Background

We start this section presenting some basic facts about the functions of Holder type. This material can be found in [1].

Let $[a,b]$ be a closed interval in $\mathbb{R}$, by $\mathcal{C}[a,b]$ we denote the space of continuous functions on $[a,b]$ with real values equipped with the classical norm of the supremum, that is, for $x \in \mathcal{C}[a,b]$,

$$\|x\|_\infty = \sup\{|x(t)| : t \in [a,b]\}.$$

For $0 < \alpha < 1$, by $H_\alpha[a,b]$ we denote the functions $x : [a,b] \to \mathbb{R}$ satisfying the Holder condition, that is, there exists a constants L_x^α such that

$$|x(t) - x(s)| \leq L_x^\alpha |t-s|^\alpha, \tag{2.1}$$

for any $t, s \in [a,b]$.

It is easily checked that $H_\alpha[a,b]$ is a linear subspace of $\mathcal{C}[a,b]$. For $x \in H_\alpha[a,b]$, we put L_x^α to the least constant satisfying Eq.(2.1), i.e.,

$$L_x^\alpha = \sup\{\frac{|x(t)-x(s)|}{|t-s|^\alpha} : \ \ t,s \in [a,b], \ \ t \neq s\}.$$

$H_\alpha[a,b]$ can be normed by

$$\|x\|_\alpha = |x(a)| + L_x^\alpha = |x(a)| + \sup\{\frac{|x(t)-x(s)|}{|t-s|^\alpha} : \ \ t,s \in [a,b], \ \ t \neq s\}.$$

In [1], it is proved that $(H_\alpha[a,b], \|\cdot\|_\alpha)$ is a Banach space.

Now, we need the following results which will be used later and they appear in [1].

Lemma 2.1. *For $x \in H_\alpha[a,b]$, we have*

$$\|x\|_\infty \leq \max(1, (b-a)^\alpha)\|x\|_\alpha.$$

Lemma 2.2. *For $0 < \alpha < \gamma \leq 1$ the following chain of inclusions*

$$H_\gamma[a,b] \subset H_\alpha[a,b] \subset \mathcal{C}[a,b]$$

holds. Moreover, for $x \in H_\gamma[a,b]$ we have

$$\|x\|_\alpha \leq \max(1, (b-a)^{\gamma-\alpha})\|x\|_\gamma.$$

For our study, the following result plays a very important role.

Theorem 2.1. *Suppose that $0 < \alpha < \beta \leq 1$ and X is a bounded subset in $H_\beta[a,b]$ (this means that $\|x\|_\beta \leq M$ for any $x \in X$, where M is a positive constant). Then X is a relatively compact subset of $H_\alpha[a,b]$.*

Next, we present some basic results about the fractional calculus which appear in [6].

Definition 2.1. Suppose $\alpha > 0$ and $f : (0,\infty) \to \mathbb{R}$ a function. The Riemann-Liouville fractional integral of order α of f is defined as

$$I^\alpha_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\alpha)$ is the classical gamma function, whenever the right-hand side is defined.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$D^\alpha_{a^+} f(t) = \frac{1}{\Gamma(\alpha - n)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α, provided that the right-hand side is defined.

Lemma 2.3. *Suppose that $\alpha > 0$ and $x \in \mathcal{C}(0,b) \cap L^1(0,b)$.*
The general solution to the homogeneous equation

$$D^\alpha_{0^+} x(t) = 0$$

is given by

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$ and $n = [\alpha] + 1$.

Lemma 2.4. *Suppose that $x \in \mathcal{C}(0,b) \cap L^1(0,b)$ and $D^\alpha_{0^+} x \in \mathcal{C}(0,b) \cap L^1(0,b)$.*
Then

$$I^\alpha_{0^+} D^\alpha_{0^+} x(t) = x(t) + c_1 t^{\alpha-1} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, and $n = [\alpha] + 1$.

By using Lemma 2.3 and Lemma 2.4, the authors proved in [10] the following result.

Lemma 2.5. *Suppose that $g \in \mathcal{C}[0,1]$. Then the unique solution to the following fractional boundary value problem*

$$\begin{cases} D^\alpha_{0^+} x(t) + g(t) = 0, & 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, & \\ x^{(i)}(1) = \displaystyle\sum_{j=1}^{\infty} \alpha_j x(\xi_j), & \end{cases} \tag{2.2}$$

where $\alpha > 2$, $n-1 < \alpha \leq n$, $i \in [1, n-2]$ *is a fixed integer,* $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < 1$, $\Delta = (\alpha-1)(\alpha-2)\cdots(\alpha-i)$, $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_i^{\alpha-1} > 0$ *and* $p(s) = \Delta - \sum_{\{s \leq \xi_j\}} \alpha_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s)^i$, *is given by*

$$x(t) = \int_0^1 G(t,s) g(s) ds,$$

where

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1} p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1} p(s)(1-s)^{\alpha-1-i}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Remark 2.1. Since $p(0) = \Delta - \sum_{j=1}^{\infty} \alpha_j \xi j^{\alpha-1} > 0$ and $p'(s) = \sum_{\{s \leq \xi_j\}} \alpha_j (\xi_j - s)^{\alpha-2} (1-s)^{-\alpha+1} ((\alpha-1)(1-\xi_j) + i(\xi_j - s)) > 0$, we infer that $p(s) > 0$ for $s \in [0,1]$.

Remark 2.2. It is clear that $G(t,s)$ is a continuous function on $[0,1] \times [0,1]$. Moreover, in [10] it is proved that $G(t,s) > 0$ for $t, s \in (0,1)$. Notice that $G(0,s) = 0$ for $s \in [0,1]$.

3. Main result

Our starting point in this section is the following lemma which will be used later and its proof is elemental and we omitt it.

Lemma 3.1. *Let* $\gamma : [0,1] \to \mathbb{R}$ *be the function given by* $\gamma(t) = t^\alpha$. *Then*

(*i*) $|t^\alpha - s^\alpha| \leq |t-s|^\alpha$, *for* $t, s \in [0,1]$ *if* $0 < \alpha < 1$.

(*ii*) $|t^\alpha - s^\alpha| \leq \alpha |t-s|$ *for* $t, s \in [0,1]$ *if* $\alpha \geq 1$.

In the following result, we will prove that the function $G(t,s)$ appearing in Lemma 3.1 is lipschitzian with respect to the first variable. This fact plays an important role in our study.

Lemma 3.2. *The function* $G(t,s)$ *appearing in Lemma 3.1 satisfies the following inequality*

$$|G(t,s) - g(\tau,s)| \leq \frac{(p(1) + p(0))(\alpha-1)}{p(0)\Gamma(\alpha)} |t-\tau|,$$

for any $t, \tau, s \in [0,1]$.

Proof. Fix $s \in [0,1]$ and we take $t, \tau \in [0,1]$. We distinguish three cases.

<u>Case 1</u>: $t, \tau \leq s$.

In this case, we get the following estimate,

$$\begin{aligned}&|G(t,s)-G(\tau,s)|\\=&\frac{1}{p(0)\Gamma(\alpha)}\left|t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}-\tau^{\alpha-1}p(s)(1-s)^{\alpha-1-i}\right|\\=&\frac{1}{p(0)\Gamma(\alpha)}p(s)(1-s)^{\alpha-1-i}|t^{\alpha-1}-\tau^{\alpha-1}|\\\leq&\frac{1}{p(0)\Gamma(\alpha)}p(1)|t^{\alpha-1}-\tau^{\alpha-1}|\leq\frac{p(1)}{p(0)\Gamma(\alpha)}(\alpha-1)|t-\tau|,\end{aligned}\tag{3.1}$$

where we have used the increasing character and nonnegativity of the function $p(s)$ (see Remark 2.1 and Lemma 3.1 (ii), since $\alpha-1>1$.

Case 2: $t,\tau\geq s$.

In this case, we have

$$\begin{aligned}|G(t,s)-G(\tau,s)|=&\frac{1}{p(0)\Gamma(\alpha)}\Big|t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}-p(0)(t-s)^{\alpha-1}\\&-\tau^{\alpha-1}p(s)(1-s)^{\alpha-1-i}+p(0)(\tau-s)^{\alpha-1}\Big|\\\leq&\frac{1}{p(0)\Gamma(\alpha)}[p(s)(1-s)^{\alpha-1-i}|t^{\alpha-1}-\tau^{\alpha-1}|\\&+p(0)|(t-s)^{\alpha-1}-(\tau-s)^{\alpha-1}|]\\\leq&\frac{1}{p(0)\Gamma(\alpha)}\left[p(1)(\alpha-1)|t-\tau|+p(0)(\alpha-1)|t-\tau|\right]\\=&\frac{1}{p(0)\Gamma(\alpha)}(p(1)+p(0))(\alpha-1)|t-\tau|,\end{aligned}\tag{3.2}$$

where we have used the same facts that in Case 1.

Case 3: $\tau\leq s\leq t$.

In this case, we infer

$$\begin{aligned}|G(t,s)-G(\tau,s)|=&\frac{1}{p(0)\Gamma(\alpha)}\Big|t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}-p(0)(t-s)^{\alpha-1}\\&-\tau^{\alpha-1}p(s)(1-s)^{\alpha-1-i}Big|\\\leq&\frac{1}{p(0)\Gamma(\alpha)}\left[p(s)(1-s)^{\alpha-1-i}|t^{\alpha-1}-\tau^{\alpha-1}|+p(0)(t-s)^{\alpha-1}\right]\\\leq&\frac{1}{p(0)\Gamma(\alpha)}[p(1)(\alpha-1)|t-\tau|+p(0)|t-\tau|^{\alpha-1}]\\\leq&\frac{1}{p(0)\Gamma(\alpha)}[p(1)(\alpha-1)|t-\tau|+p(0)|t-\tau|]\\=&\frac{1}{p(0)\Gamma(\alpha)}[p(1)(\alpha-1)+p(0)]|t-\tau|,\end{aligned}\tag{3.3}$$

where we have used that $t-s\leq t-\tau$, the decreasing character of the function $y=a^x$ with $0<a\leq1$ and that $\alpha-1>1$.

Summarizing, from Eq.(3.1), Eq.(3.2) and Eq.(3.3) and, since $\alpha - 1 > 1$, we get

$$\begin{aligned} |G(t,s) - G(\tau,s)| &\le \max\left(\frac{(p(1)(\alpha-1)}{p(0)\Gamma(\alpha)}, \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}, \right. \\ &\qquad \left. \frac{p(1)(\alpha-1)+p(0)}{p(0)\Gamma(\alpha)} \right) |t-\tau| \\ &= \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)} |t-\tau|. \end{aligned}$$

This completes the proof. □

Proposition 3.1. *Suppose that* $x \in H_1[0,1]$.

Let Tx *be the function defined on* $[0,1]$ *by*

$$(Tx)(t)) \int_0^1 G(t,s) f(s, x(s), (Hx)(s)) ds, \quad \textit{for} \quad t \in [0,1],$$

where $G(t,s)$ *is the function appearing in Lemma 2.5.*

Suppose that

- (i) $f : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous function and it satisfies*

  $$|f(t,x,y) - f(t,x_1,y_1)| \le \varphi(\max(|x-x_1|, |y-y_1|)),$$

  for any $t \in [0,1]$ *and* $x, x_1, y, y_1 \in \mathbb{R}_+$, *where* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing function.*
- (ii) $H : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ *(not neccesarily linear) satisfying*
  - (1) $Hx \ge 0$, *for* $x \ge 0$.
  - (2) $\|Hx\|_\infty \le \|x\|_\infty$.
  - (3) $\|Hx - Hy\|_\infty \le \|x - y\|_\infty$:

Under these assumptions, we have

- (1) $Tx \ge 0$, *for* $x \ge 0$.
- (2) $Tx \in H_1[0,1]$ *and* $\|Tx\|_1 \le \dfrac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)} \cdot [\varphi(\|x\|_1) + M]$, *where* $M = \sup\{f(t,0,0) : t \in [0,1]\}$.

Proof. By Remark 2.2 and assumption (i), it is clear that $Tx \ge 0$ for $x \ge 0$. This gives us (1).

In order to prove (2), we take $t, \tau \in [0,1]$ with $t \ne \tau$ and we get

$$\begin{aligned} \frac{|(Tx)(t) - (Tx)(\tau)|}{|t-\tau|} &= \frac{1}{|t-\tau|} \left| \int_0^1 G(t,s) f(s,x(s),(Hx)(s)) ds \right. \\ &\qquad \left. - \int_0^1 G(\tau,s) f(s,x(s),(Hx)(s)) ds \right| \\ &= \frac{1}{|t-\tau|} \left| \int_0^1 (G(t,s) - G(\tau,s)) f(s,x(s),(Hx)(s)) ds \right| \\ &\le \frac{1}{|t-\tau|} \int_0^1 |G(t,s) - G(\tau,s)| |f(s,x(s),(Hx)(s))| ds, \end{aligned}$$

by using Lemma 3.2,

$$
\begin{aligned}
&\le \frac{1}{|t-\tau|}\frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}|t-\tau|\int_0^1 |f(s,x(s),(Hx)(s))|ds\\
&= \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}\int_0^1 |f(s,x(s),(Hx)(s))-f(s,0,0)+f(s,0,0)|ds\\
&\le \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}\left[\int_0^1 [|f(s,x(s),(Hx)(s))-f(s,0,0)|+|f(s,0,0)|]ds\right],
\end{aligned}
$$

by using assumptions (i) and (ii),

$$
\begin{aligned}
&\le \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}\left[\int_0^1 (\varphi(\max(|x(s)|,|(Hx)(s)|))+|f(s,0,0)|)ds\right]\\
&\le \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}\int_0^1 (\varphi(\max(\|x\|_\infty,\|Hx\|_\infty))+M)ds\\
&= \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}(\varphi(\|x\|_\infty)+M).
\end{aligned}
$$

Now, since φ is an increasing function and $\|x\|_\infty \le \|x\|_1$ (see Lemma 2.1), from the last inequality it follows

$$\frac{|(Tx)(t)-(Tx)(\tau)|}{|t-\tau|} \le \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}[\varphi(\|x\|_1)+M].$$

This proves that $Tx \in H_1[0,1]$.

On the other hand, since $G(0,s)=0$ for $s\in[0,1]$ (see Remark 2.2), $(Tx)(0)=0$, and, this gives us

$$
\begin{aligned}
\|Tx\|_1 &= |(Tx)(0)| + \sup\left\{\frac{|(Tx)(t)-(Tx)(\tau)|}{|t-\tau|} : t,\tau\in[0,1], t\ne\tau\right\}\\
&\le \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}[\varphi(\|x\|_1)+M].
\end{aligned}
$$

This finishes the proof. □

Remark 3.1. Notice that, under assumptions of Proposition 3.1, if there exists $r_0>0$ such that

$$\frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}[\varphi(r_0)+M] \le r_0,$$

then the operator T appearing in Proposition 3.1 applies the ball $B_1^{r_0}$ of $H_1[0,1]$ into itself (by $B_1^{r_0}$ it is denoted the ball centered at zero with radius r_0 in the space $H_1[0,1]$).

In the sequel, we present the main result of the paper.

Theorem 3.1. *Suppose the following hypotheses:*

$(H1)$ $f:[0,1]\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ *is a continuous function and it satisfies*

$$|f(t,x,y)-f(t,x_1,y_1))| \le \varphi(\max(|x-x_1|,|y-y_1|)),$$

for any $t\in[0,1]$ *and* $x,x_1,y,y_1\in\mathbb{R}_+$, *where* $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ *is an increasing function and continuous at* $t_0=0$ *with* $\varphi(0)=0$.

$(H2)$ *$H : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ satisfying:*

(i) $Hx \geq 0 \quad if \quad x \geq 0.$

(ii) $\|Hx\|_\infty \leq \|x\|_\infty.$

(iii) $\|Hx - Hy\|_\infty \leq \|x - y\|_\infty.$

$(H3)$ *There exits $r_0 > 0$ such that*

$$\frac{(p(1)+p(0))}{p(0)\Gamma(\alpha)}(\alpha-1)[\varphi(r_0)+M] \leq r_0.$$

Then Problem (Eq.(1.1)) has at least one nonnegative solution x^* in $H_1[0,1]$.

Proof. Let P be the cone defined by $P = \{u \in H_1[0,1] : u \geq 0\}$. It is easily checked that P is closed in $H_1[0,1]$.

Now, we consider the operator T defined on P as

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s),(Hx)(s))ds, \quad \text{for} \quad t \in [0,1].$$

In virtue of Lemma 2.5, a solution to Problem (Eq.(1.1)) is a fixed point of the operator T.

By Proposition 3.1, T applies P into itself.

Moreover, by assumption $(H3)$ and Remark 3.1, T applies $P \cap B_1^{r_0}$ into itself.

Taking into account that $P \cap B_1^{r_0}$ is a bounded subset of $H_1[0,1]$, Theorem 2.1 says us that $P \cap B_1^{r_0}$ is a relatively compact subset of $H_\beta[0,1]$ for $0 < \beta < 1$.

Following a similar argument to the one used in Theorem 7 of [1], $P \cap B_1^{r_0}$ is closed in $H_\beta[0,1]$ for $0 < \beta < 1$ and, consequently, $P \cap B_1^{r_0}$ is a compact subset of $H_\beta[0,1]$ for $0 < \beta < 1$.

Therefore, we have that $T : P \cap B_1^{r_0} \to P \cap B_1^{r_0}$ and $P \cap B_1^{r_0}$ is a convex and compact subset of $H_\beta[0,1]$ for $0 < \beta < 1$.

In order to apply Schauder's fixed point theorem, we only need to prove that T is continuous for the norm $\|\cdot\|_\beta$.

To do this, we take $(x_n) \subset P \cap B_1^{r_0}$ such that $x_n \overset{\|\cdot\|_\beta}{\longrightarrow} x$ with $x \in P \cap B_1^{r_0}$, and we have that to prove that $Tx_n \overset{\|\cdot\|_\beta}{\longrightarrow} Tx$.

In fact, for $t, \tau \in [0,1]$ with $t \neq \tau$, we have the following estimate,

$$\begin{aligned}
&\frac{|(Tx_n)(t) - (Tx)(t)) - ((Tx_n)(\tau) - (Tx)(\tau))|}{|t-\tau|^\beta} \\
=&\frac{1}{|t-\tau|^\beta}\left|\left[\int_0^1 G(t,s)f(s,x_n(s),Hx_n(x))ds\right.\right. \\
&\left. - \int_0^1 G(t,s)f(s,x(s),(Hx)(s))ds\right] \\
&- \left[\int_0^1 G(\tau,s)f(s,x_n(s),Hx_n(s))ds\right. \\
&\left.\left. - \int_0^1 G(\tau,s)f(s,x(s),(Hx)(s))ds\right]\right| \\
=&\frac{1}{|t-\tau|^\beta}\left|\int_0^1 (G(t,s)-G(\tau,s))(f(s,x_n(s),Hx_n(s))\right.
\end{aligned}$$

$$-f(s,x(s),(Hx)(s)))ds|$$
$$\leq\frac{1}{|t-\tau|^\beta}\int_0^1 |G(t,s)-G(\tau,s)|\,|f(s,x_n(s),Hx_n(s))$$
$$-f(s,x(s),(Hx)(s))|\,ds,$$

by using Lemma 3.2,

$$\leq \frac{1}{|t-\tau|^\beta}\frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}|t-\tau|\int_0^1 |f(s,x_n(s),H_n(s))$$
$$- f(s,x(s),(Hx)(s))|ds,$$

by using assumptions $(H1)$ and $(H2)$ and Lemma 2.1 we have

$$\leq \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}|t-\tau|^{1-\beta}\int_0^1 \varphi(\max(|x_n(s)-x(s)|,$$
$$|(Hx_n)(s)-(Hx)(s)|))ds$$
$$\leq \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}\int_0^1 \varphi(\max(\|x_n-x\|_\infty,\|Hx_n-Hx\|_\infty))ds$$
$$= \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}\varphi(\|x_n-x\|_\infty)$$
$$\leq \frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}\varphi(\|x_n-x\|_\beta).$$

From the last inequality, it follows

$$\|Tx_n-Tx\|_\beta$$
$$=|(Tx_n)(0)-(Tx)(0)|$$
$$+\sup\left\{\frac{|(Tx_n(t)-(Tx)(t))-((Tx_n)(\tau)-(Tx)(\tau))|}{|t-\tau|^\beta},t,\tau\in[0,1],t\neq\tau\right\}$$
$$\leq\frac{(p(1)+p(0))(\alpha-1)}{p(0)\Gamma(\alpha)}\varphi(\|x_n-x\|_\beta),$$

where we have used the fact that $Tx_n(0)=Tx(0)=0$.

Since φ is continuous at $t_0=0$ (assumption $(H1)$), from the above obtained estimate, we infer that $\|Tx_n-Tx\|_\beta\to 0$ as $n\to\infty$.

This proves that T is continuous for the norm $\|\cdot\|_\beta$, with $0<\beta<1$.

Finally, by using Schauder's fixed point theorem, T has at least one fixed point $x^*\in P\cap B_1^{r_0}$, that is, x^* is a solution to Problem (Eq.(1.1)). Moreover, since $x^*\in P\cap B_1^{r_0}$, this solution x^* belongs to $H_1[0,1]$.

This completes the proof. □

From a practical point of view, it is interesting that the solutions to Problem (Eq.(1.1)) are positive, i.e., $x(t)>0$ for $t\in(0,1)$.

The following result presents a sufficient condition for that the solution to Problem (Eq.(1.1)) is positive.

Theorem 3.2. *If to assumptions of Theorem 3.1 we add the following ones:*

$(H4)$ *There exists $t_0\in[0,1]$ such that $f(t_0,0,0)>0$,*

($H5$) $f : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing with respect to the second and third variables,*

then Problem (Eq.(1.1)) has at least one positive solution x^* *belonging to* $H_1[0,1]$.

Proof. Let x^* be a nonnegative solution to Problem (Eq.(1.1)) whose existence is guaranteed by Theorem 3.1.

Since x^* is a fixed point of operator T appearing in Theorem 3.1, we have

$$x^*(t) = \int_0^1 G(t,s) f(s, x^*(s), (Hx^*)(s)) ds$$

for $t \in [0,1]$.

Suppose that $x^*(t)$ is not a positive solution.

In this case, we can find $t^* \in (0,1)$ such that $x^*(t^*) = 0$, and, therefore,

$$0 = x^*(t^*) = \int_0^1 G(t^*,s) f(s, x^*), (Hx^*)(s)) ds.$$

Since the integrand is nonnegative and by assumption ($H5$), it follows

$$0 = \int_0^1 G(t^*,s) f(s, x^*(s), (Hx^*)(s)) ds \geq \int_0^1 G(t^*,s) f(s,0,0) ds \geq 0,$$

and, this gives us

$$\int_0^1 G(t^*,s) f(s,0,0) ds = 0.$$

As $G(t^*,s)f(s,0,0)$ is a nonnegative function, we infer

$$G(t^*,s) f(s,0,0) = 0 \text{ almost everywhere in } s.$$

Since $G(t^*,s)$ is of polynomial type, $G(t^*,s) \neq 0$ almost everywhere in s, and, consequently,

$$f(s,0,0) = 0 \text{ almost everywhere in } s.$$

On the other hand, the continuity of f and assumption ($H4$), that is, $f(t_0,0,0) > 0$ for certain $t_0 \in [0,1]$, give us the existence of a subset $A \subset [0,1]$ with $t_0 \in A$ and $\mu(A) > 0$, where μ is the Lebesgue measure on $[0,1]$, such that

$$f(t,0,0) > 0 \text{ for any } t \in A.$$

This fact contraducts to $f(s,0,0) = 0$ almost everywhere in s, above obtained.

This contradiction says us that $x^*(t) > 0$ for $t \in (0,1)$.

This finishes the proof. □

Next, we present some examples of operators $H : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ satisfying assumption ($H2$) of Theorem 3.1.

Let $\varphi : [0,1] \to [0,1]$ be a continuous function.

Consider the composition operator C_φ defined on $\mathcal{C}[0,1]$ by

$$(C_\varphi x)(t) = x(\varphi(t)),$$

and the multliplication operator M_φ defined on $\mathcal{C}[0,1]$ by

$$(M_\varphi x)(t) = \varphi(t) x(t).$$

Let I be the integral operator defined by on $\mathcal{C}[0,1]$ by

$$(Ix)(t) = \int_0^t x(s)ds,$$

and Q the operator defined on $\mathcal{C}[0,1]$ by

$$(Qx)(t) = \max_{0\le\tau\le t} |x(\tau)|.$$

It is easily checked that C_φ, M_φ, I and Q are operators satisfying assumption $(H2)$ of Theorem 3.1.

Remark 3.2. It is easily checked that if $F, G : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ are two operators satisfying assumption $(H2)$ of Theorem 3.1, then $F \circ G$ also satisfies it.

Next, we present a numerical example where our results can be applied.

Example 3.1. Consider the following nonlinear fractional boundary value problem

$$\begin{cases} D_{0^+}^{7/2} x(t) + \alpha(\arctan(x(t) + (Hx)(t)) + t) = 0, \quad t \in [0,1], \\ x(0) = x'(0) = x''(0) = 0, \\ x'(1) = \displaystyle\sum_{j=1}^{\infty} \frac{2}{j^2} x\left(\frac{1}{j}\right), \end{cases} \tag{3.4}$$

with $\alpha > 0$ and H any operator C_φ, M_φ, I or Q above mentioned.

Problem (Eq.(3.4)) is a particular case of Problem (Eq.(1.1)) with $\alpha = 7/2$, $i = 1$, $\alpha_j = \dfrac{2}{j^2}$, $\xi_j = \dfrac{1}{j}$, $\Delta = 2,5$ and $f(t,x,y) = \alpha(\arctan(x+y) + t)$.

A simple calculation gives us that $\displaystyle\sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} \cong 2,109$ and, consequently, $\Delta - \displaystyle\sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} \cong 2,5 - 2,109 > 0$ and $p(1) = \Delta = 2,5$.

It is clear that $f : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and it is continuous. Moreover, for $t \in [0,1]$, $x, x_1, y, y_1 \in \mathbb{R}_+$, we have

$$\begin{aligned} |f(t,x,y) - f(t,x_1,y_1)| &= \alpha |\arctan(x+y) - \arctan(x_1+y_1)| \\ &\le \alpha |x + y - (x_1 + y_1)| \le \alpha(|x - x_1| + |y - y_1|) \\ &\le 2\alpha \max(|x - x_1| + |y - y_1|). \end{aligned}$$

This proves that assumption $(H1)$ of Theorem 3.1 is satisfied with $\varphi(t) = 2\alpha t$.

Moreover, $M = \sup\{f(t,0,0) : t \in [0,1]\} = \sup\{\alpha t : t \in [0,1]\} = \alpha$.

In our case, the inequality appearing in $(H3)$ of Theorem 3.1 has the following expression,

$$\frac{2,5 + 0,3910}{0,3910 \cdot 3,3234} \cdot 2,5 \cdot [2\alpha r_0 + \alpha] \le r_0,$$

or, equivalently,

$$\alpha \cdot 5,556198 \cdot (2r_0 + 1) \le r_0.$$

If $\alpha = 0,01$ then $r_0 = 1$ satisfies this equation.

Moreover, for this α, $f(1/2,0,0) = \alpha \cdot 1/2 = 0,005 > 0$ and, notice that f is increasing in x and y.

By using Theorem 3.2, it follows that Problem (Eq.(3.4)) for $\alpha = 0,01$ has at least one positive solution x^* with $x^* \in H_1[0,1]$, and $\|x^*\|_1 \leq 1$.

Finally, notice that the same argument works for any operator H satisfying assumption $(H2)$ of Theorem 3.1.

Remark 3.3. As our solution x^* to Problem (Eq.(1.1)) is placed in $H_1[0,1]$ by Rademacher Theorem, this solution x^* is almost everywhere differentiable and this question is very interesting and useful from a practical point of view.

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