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BIFURCATIONS OF TRAVELLING WAVE SOLUTIONS IN THREE MODIFIED CAMASSA-HOLM EQUATIONS*

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Abstract This paper studies traveling wave solutions of three modified Camassa-Holm equations posed by Anco and Recio in 2019. The corresponding traveling system is a singular system of second class. The bifurcations of traveling wave solutions in the parameter space are investigated from a dynamical systems theoretical point of view. The existence of solitary wave solution, periodic wave solution and so-called M-shape-solution are proved.

Keywords Solitary wave solution, M-shape-wave solution, periodic wave solution, bifurcation, modified Camassa-Holm equation.

MSC(2010) 34C37, 34C23, 74J30, 58Z05.

1. Introduction

In 2006, Qiao Zhijun [10] proposed the following new completely integrable wave equation

$$u_t - u_{xxt} + 3u^2 u_x - u_x^3 = (4u - 2u_{xx})u_x u_{xx} + (u^2 - u_x^2)u_{xxx}, \tag{1.1}$$

namely,

$$m_t + m_x(u^2 - u_x^2) + 2m^2 u_x = 0, \quad m = u - u_x. \tag{1.2}$$

This equation is derived from the two dimensional Euler equation. The author proved that (1.1) has Lax pair and bi-Hamiltonian structures. In addition, the author found so called "W/M"-shape-peaks solitons and claimed that there exist no smooth solitons for this integrable water wave equation. In 2009, by using the method of dynamical systems, Li and Zhang [9] showed that there exists a smooth solitary solution of equation (1.1) when some parameter conditions are satisfied. In addition, they explained why so called "W/M"-shape-peaks solitons can be created and gave the determined parameter conditions and exact explicit parametric representations for all solitary wave solutions of equation (1.1).

More recently, in Anco and Recio [4], a general family of peakon equations is introduced, involving two arbitrary functions of the wave amplitude and the wave gradient, in which one-parameter subfamilies of the CH-mCH Hamiltonian family

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are explored. The authors suggested to study the following "family of Hamiltonian multi-peakon equations"

$$m_t + u_x f_1(u^2 - u_x^2)m + [uf_1(u^2 - u_x^2) + g_1(u^2 - u_x^2))m]_x = 0, \tag{1.3}$$

which involves two arbitrary functions f_1 and g_1 of $u^2 - u_x^2$.

Two subfamilies represent nonlinear generalizations of the mCH equation given by

$$m_t + ((u^2 - u_x^2)^p m)_x = 0, \ m = u - u_{xx}, \quad p \geq 1 \tag{1.4}$$

and

$$m_t + (u^2 - u_x^2)^{p-1})u_x m + [u(u^2 - u_x^2)^{p-1})m]_x = 0, \quad p \geq 1. \tag{1.5}$$

A unified generalization of these two subfamilies is

$$m_t + au_x(u^2 - u_x^2)^{\frac{1}{2}k} m + [au(u^2 - u_x^2)^{\frac{1}{2}k} m + b((u^2 - u_x^2)^{\frac{1}{2}(k+1)} m]_x = 0, \ k \geq 0. \tag{1.6}$$

By putting $k = 2p - 2, b = 0, a = 1$ in the three-parameter family (1.6), we obtain the one-parameter family of generalized CH equation (1.5). Taking $a = 0, b = 1, k = 2p - 1$, equation (1.6) becomes equation (1.4).

The authors in [4] stated that "in the case of the $p = 2$ generalized mCH equation (1.4), the peakon and anti-peakon can form a bound pair which has a maximum finite separation in the asymptotic past and future." Unfortunately, they did not study the corresponding traveling wave systems of equations (1.4), (1.5) and (1.6). In this paper, we consider the bifurcations problems of the solutions of the corresponding traveling wave systems of equations (1.4), (1.5) and (1.6) depending on the parameters of systems.

To study the traveling wave solutions of equations (1.4), (1.5) and (1.6), setting $u(x,t) = u(x - ct) \equiv \phi(\xi)$, where $\xi = x - ct$ and c is the wave speed. Substituting it into (1.4), (1.5) and (1.6) and integrating the obtained equations once, we obtain

$$[(\phi^2 - \phi_\xi^2)^p - c]\phi_{\xi\xi} = \phi(\phi^2 - \phi_\xi^2)^p - c\phi + g, \tag{1.7}$$

$$[\phi(\phi^2 - \phi_\xi^2)^{p-1} - c]\phi_{\xi\xi} = \frac{1}{2p}(\phi^2 - \phi_\xi^2)^p + \phi^2(\phi^2 - \phi_\xi^2)^{p-1} - c\phi + g, \tag{1.8}$$

and

$$\begin{aligned} &[a\phi(\phi^2 - \phi_\xi^2)^{\frac{1}{2}k} + b(\phi^2 - \phi_\xi^2)^{\frac{1}{2}(k+1)} - c]\phi_{\xi\xi} \\ =&g - c\phi + \frac{a}{k+2}(\phi^2 - \phi_\xi^2)^{\frac{1}{2}k+1} + a\phi^2(\phi^2 - \phi_\xi^2)^{\frac{1}{2}k} + b\phi(\phi^2 - \phi_\xi^2)^{\frac{1}{2}(k+1)}, \end{aligned} \tag{1.9}$$

where g is an integral constant. For equation (1.7), we suppose that $g \neq 0$. Otherwise, (1.7) becomes a linear equation. Equations (1.7), (1.8) and (1.9) are respectively equivalent to the planar dynamical systems

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{[c - (\phi^2 - y^2)^p]\phi - g}{c - (\phi^2 - y^2)^p}, \tag{1.10}$$

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\frac{1}{2p}(\phi^2 - y^2)^p - \phi^2(\phi^2 - y^2)^{p-1} + c\phi - g}{c - \phi(\phi^2 - y^2)^{p-1}} \tag{1.11}$$

and

$$\frac{d\phi}{d\xi} = y, \ \frac{dy}{d\xi} = \frac{g - c\phi + \frac{a}{k+2}(\phi^2 - y^2)^{\frac{1}{2}k+1} + a\phi^2(\phi^2 - y^2)^{\frac{1}{2}k} + b\phi(\phi^2 - y^2)^{\frac{1}{2}(k+1)}}{a\phi(\phi^2 - y^2)^{\frac{1}{2}k} + b(\phi^2 - y^2)^{\frac{1}{2}(k+1)} - c}. \tag{1.12}$$

System (1.10), (1.11) and (1.12) have the first integrals, respectively,

$$H_1(\phi,y)=\frac{1}{p+1}(\phi^2-y^2)^{p+1}+cy^2-c\phi^2+2g\phi=h, \tag{1.13}$$

$$H_2(\phi,y)=\frac{1}{p}\phi(\phi^2-y^2)^{p}+cy^2-c\phi^2+2g\phi=h, \tag{1.14}$$

and

$$\begin{aligned}H_3(\phi,y)=&\frac{2a}{k+2}\phi(\phi^2-y^2)^{\frac{1}{2}k+1}+\frac{2b}{k+3}(\phi^2-y^2)^{\frac{1}{2}(k+3)}+cy^2\\&+\frac{a}{k+2}\int(\phi^2-y^2)^{\frac{1}{2}k+1}d\phi-\frac{1}{2}c\phi^2+g\phi=h,\end{aligned} \tag{1.15}$$

where the calculation of integral $\int(\phi^2-y^2)^{\frac{1}{2}k+1}d\phi$ can use the recursive formula as follows:

$$\int(\phi^2-y^2)^{\frac{1}{2}k+1}d\phi=\frac{1}{k+3}\left[\phi(\phi^2-y^2)^{\frac{1}{2}k+1}-(k+2)y^2\int(\phi^2-y^2)^{\frac{1}{2}k}d\phi\right]. \tag{1.16}$$

Clearly, for $c>0$, on the curves $c-(\phi^2-y^2)^p=0, c-\phi(\phi^2-y^2)^{p-1}=0$ and $a\phi(\phi^2-y^2)^{\frac{1}{2}k}+b(\phi^2-y^2)^{\frac{1}{2}(k+1)}-c=0$, respectively, systems (1.10), (1.11) and (1.12) are discontinuous. Such systems are called a singular traveling wave systems of the second class defined by Li [6] and Li and Chen [7].

We notice that equation (1.1) has the same singular traveling wave system as (1.10) with $p=1$ (see Li and Zhang [9]), namely,

$$\frac{d\phi}{d\xi}=y,\quad \frac{dy}{d\xi}=\frac{[c-\phi^2+y^2]\phi-g}{c-\phi^2+y^2}. \tag{1.17}$$

with the first integral

$$H_1(\phi,y)=\frac{1}{2}(c-\phi^2+y^2)^2+2g\phi=h. \tag{1.18}$$

In addition, when $p=1$, system (1.11) becomes the traveling wave system of Camassa-Holm equation as follows:

$$\frac{d\phi}{d\xi}=y,\quad \frac{dy}{d\xi}=\frac{-y^2+3\phi^2-2c\phi+2g}{2(\phi-c)}, \tag{1.19}$$

with the first integral

$$H_2(\phi,y)=(\phi-c)y^2-\phi^3+c\phi^2-2g\phi=h. \tag{1.20}$$

The bifurcations and exact solutions of system (1.19) can be seen in Li [6]. So that, in this paper, for equation (1.5), we only consider the case $p\geq 2$.

This paper is organized as follows. In section 2, we discuss the bifurcations of phase portraits of systems (1.10), (1.11) and (1.12) depending on the changes of parameter g when $c>0$ is fixed. In section 3 and section 4, we investigate the existence of solitary wave solution, periodic wave solutions and M-shape wave solutions of equations (1.4), (1.5) and (1.6).

2. Bifurcation of phase portraits of systems (1.10), (1.11) and (1.12)

In this paper, we always assume that $c > 0$, $g > 0$, and p is a positive integer.

2.1 We first consider all possible phase portraits of system (1.10). It is known that system (1.11) has the same invariant curve solutions as the associated regular system:

$$\frac{d\phi}{d\zeta} = y[c - (\phi^2 - y^2)^p], \quad \frac{dy}{d\zeta} = \phi[c - (\phi^2 - y^2)^p] - g, \tag{2.1}$$

where $d\xi = [c - (\phi^2 - y^2)^p]d\zeta$, for $c - (\phi^2 - y^2)^p \neq 0$. Denote that $f_1(\phi) = \phi^{2p+1} - c\phi + g$, $f_1'(\phi) = (2p+1)\phi^{2p} - c$. Thus, when $\phi = \mp\tilde{\phi}_0 = \mp\left(\frac{c}{2p+1}\right)^{\frac{1}{2p}}$, $f_1'(\mp\tilde{\phi}_0) = 0$.

The equilibrium points $E_j(z_j, 0)$ of system (2.1) satisfy $f_1(z_j) = 0$. Write that $f_a = f_1(-\tilde{\phi}_0) = g + \frac{2pc}{2p+1}\tilde{\phi}_0$, $f_b = f_1(\tilde{\phi}_0) = g - \frac{2pc}{2p+1}\tilde{\phi}_0$. It is easy to show for a fixed $c > 0$, the following fact holds.

(1) When $g > \frac{2pc}{2p+1}\tilde{\phi}_0 = 2p\left(\frac{c}{2p+1}\right)^{1+\frac{1}{2p}}$, $f_1(\phi)$ only has a negative zero $z_1 < -c^{\frac{1}{2p}} < -\tilde{\phi}_0 < 0$;

(2) When $g = \frac{2pc}{2p+1}\tilde{\phi}_0$, $f_1(\phi)$ has one simple zero $z_1 < -c^{\frac{1}{2p}} < -\tilde{\phi}_0 < 0$ and a double zero $z_{23} = \tilde{\phi}_0$;

(3) When $0 < g < \frac{2pc}{2p+1}\tilde{\phi}_0$, $f_1(\phi)$ has three simple zeros: $z_1 < -c^{\frac{1}{2p}} < -\tilde{\phi}_0 < 0 < z_2 < \tilde{\phi}_0 < z_3 < c^{\frac{1}{2p}}$.

Let $M(z_j, 0)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point $E_j(z_j, 0)$. We have

$$J(z_j, 0) = det M(z_j, 0) = -f_1'(z_j)(z_j^{2p} - c). \tag{2.2}$$

By the theory of planar dynamical systems (see [5,7,8]), for an equilibrium point of a planar integral system, if $J < 0$, then the equilibrium point is a saddle point; If $J > 0$, then it is a center point; if $J = 0$ and the Poincaré index of the equilibrium point is 0, then this equilibrium point is a cusp.

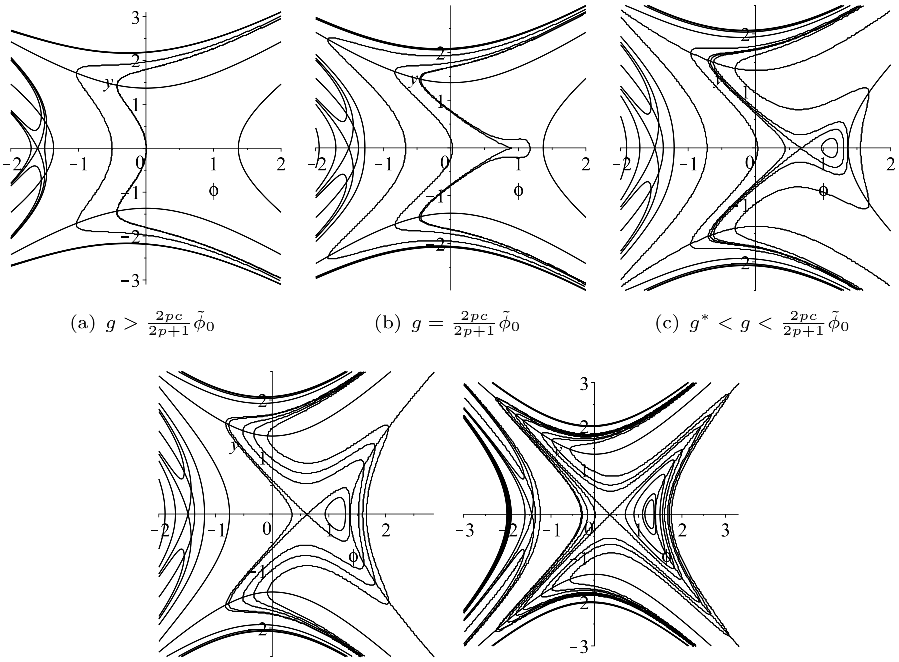
We write that $h_j = H_1(z_j, 0), j = 1, 2, 3$ where H_1 is given by (1.13).

Obviously, we see from (2.2) that when $g > \frac{2pc}{2p+1}\tilde{\phi}_0$, the unique equilibrium point $E_1(z_1, 0)$ of system (2.1) is a saddle point. When $0 < g < \frac{2pc}{2p+1}\tilde{\phi}_0$, the point $E_3(z_3, 0)$ is a center; $E_1(z_1, 0)$ and $E_2(z_2, 0)$ are saddle points. When $g = g^* = \frac{1}{2}\left(h_2 + \frac{p}{p+1}c^{1+\frac{1}{p}}\right)c^{-\frac{1}{2p}}$, the homoclinic orbit defined by $H_1(\phi, y) = h_2$ to the saddle point $E_2(z_2, 0)$ passes through the point $E_s(c^{\frac{1}{2p}}, 0)$. By using the above result to do qualitative analysis, with the change of the parameter g, we have the bifurcations of phase portraits of (2.1) shown in Fig.1. We also draw the graph of the hyperbola $(\phi^2 - y^2)^p - c = 0$ in every phase portrait in order to show the position of the singular curve.

When $g < 0$, the bifurcations of phase portraits of system (1.10) are just the reflections of Fig.1 with respect to the $y-$axis. Therefore, we only need to discuss the case $g > 0$.

2.2 We next consider all possible phase portraits of system (1.11). System (1.11) has the same invariant curve solutions as the associated regular system:

$$\frac{d\phi}{d\zeta} = y[c - \phi(\phi^2 - y^2)^{p-1}], \quad \frac{dy}{d\zeta} = -\frac{1}{2p}(\phi^2 - y^2)^p - \phi^2(\phi^2 - y^2)^{p-1} + c\phi - g, \tag{2.3}$$



(a) $g > \frac{2pc}{2p+1}\tilde{\phi}_0$ (b) $g = \frac{2pc}{2p+1}\tilde{\phi}_0$ (c) $g^* < g < \frac{2pc}{2p+1}\tilde{\phi}_0$

(d) $g = g^*$ (e) $0 < g < g^*$

Figure 1. The bifurcations of phase portraits of system (1.10) for a fixed $c > 0, p \geq 1$.

where $d\xi = [c - \phi(\phi^2 - y^2)^{p-1}]d\zeta$, for $c - \phi(\phi^2 - y^2)^{p-1} \neq 0$.

Write that $f_2(\phi) = \left(1 + \frac{1}{2p}\right)\phi^{2p} - c\phi + g,\ f_2'(\phi) = (2p+1)\phi^{2p-1} - c$. Thus, when $\phi = \hat{\phi}_0 = \left(\frac{c}{2p+1}\right)^{\frac{1}{2p-1}}, f_2'(\hat{\phi}_0) = 0, f_2(\hat{\phi}_0) = g - \left(1 - \frac{1}{2p}\right)c\hat{\phi}_0$.

The equilibrium points $E_j(z_j, 0)$ of system (2.3) satisfy $f_2(z_j) = 0$. It is easy to show for a fixed $c > 0$, the following fact holds.

(1) When $g > \left(1 - \frac{1}{2p}\right)c\hat{\phi}_0 = \left(2p - \frac{1}{2p}\right)\left(\frac{c}{2p+1}\right)^{1+\frac{1}{2p-1}}, f_2(\phi)$ has no real zero;

(2) When $g = \left(1 - \frac{1}{2p}\right)c\hat{\phi}_0, f_2(\phi)$ has a double zero $z_{12} = \hat{\phi}_0$;

(3) When $0 \leq g < \left(1 - \frac{1}{2p}\right)c\hat{\phi}_0, f_2(\phi)$ has two simple zeros: $0 \leq z_1 < \hat{\phi}_0 < z_2 < c^{\frac{1}{2p-1}}$.

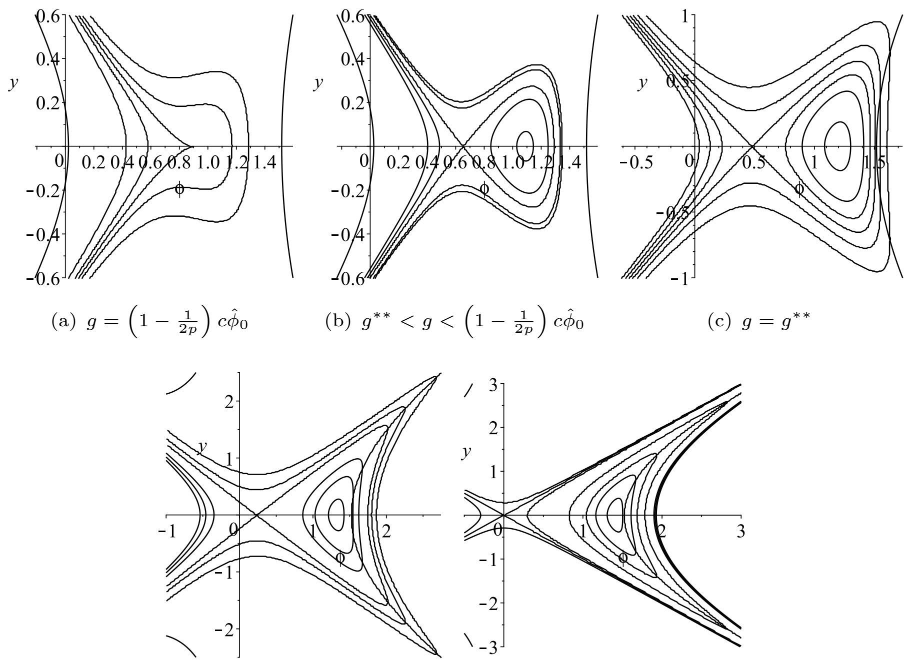
Let $M(z_j, 0)$ be the coefficient matrix of the linearized system of (2.3) at an equilibrium point $E_j(z_j, 0)$. We have

$$J(z_j, 0) = detM(z_j, 0) = -f_2'(z_j)(z_j^{2p-1} - c). \tag{2.4}$$

We write that $h_j = H_2(z_j, 0), j = 1, 2$ where H_2 is given by (1.14).

We see from (2.4) that when $0 < g < \left(1 - \frac{1}{2p}\right)c\hat{\phi}_0$, the point $E_1(z_1, 0)$ is a saddle point; $E_2(z_2, 0)$ is a center point. When $g = g^{**} = \frac{1}{2}\left[h_1 + \left(1 - \frac{1}{p}\right)c^{1+\frac{2}{2p-1}}\right]c^{-\frac{1}{2p-1}}$, the homoclinic orbit defined by $H_2(\phi, y) = h_1$ to the saddle point $E_1(z_1, 0)$ passes through the point $E_s(c^{\frac{1}{2p-1}}, 0)$.

By using the above result to do qualitative analysis, with the change of the parameter g, we have the bifurcations of phase portraits of system (1.11) shown in Fig.2. We also draw the graph of the hyperbola $\phi(\phi^2-y^2)^{p-1}-c=0$ in every phase portrait in order to show the position of the singular curve.



(a) $g=\left(1-\frac{1}{2p}\right)c\hat{\phi}_0$ (b) $g^{**}<g<\left(1-\frac{1}{2p}\right)c\hat{\phi}_0$ (c) $g=g^{**}$

(d) $0<g<g^{**}$ (e) $g=0$

Figure 2. The bifurcations of phase portraits of system (1.11) for a fixed $c>0, p\geq 2$.

2.3 We now consider all possible phase portraits of system (1.12). System (1.1) has the same invariant curve solutions as the associated regular system:

$$\begin{aligned} \frac{d\phi}{d\zeta} &= y[a\phi(\phi^2-y^2)^{\frac{1}{2}k}+b(\phi^2-y^2)^{\frac{1}{2}(k+1)}-c], \\ \frac{dy}{d\zeta} &= g_1-c\phi+\frac{a}{k+2}(\phi^2-y^2)^{\frac{1}{2}k+1}+a\phi^2(\phi^2-y^2)^{\frac{1}{2}k}+b\phi(\phi^2-y^2)^{\frac{1}{2}(k+1)}, \end{aligned} \tag{2.5}$$

where $d\xi=[a\phi(\phi^2-y^2)^{\frac{1}{2}k}+b(\phi^2-y^2)^{\frac{1}{2}(k+1)}-c]d\zeta$, for $a\phi(\phi^2-y^2)^{\frac{1}{2}k}+b(\phi^2-y^2)^{\frac{1}{2}(k+1)}-c\neq 0$.

Write that $f_3(\phi)=\left(\frac{a}{k+2}+a+b\right)\phi^{k+2}-c\phi+g,\ f_3'(\phi)=(a+(k+2)(a+b))\phi^{k+1}-c$. Thus, when $a+(k+2)(a+b)>0$, if $\phi=\bar{\phi}_0=\left(\frac{c}{a+(k+2)(a+b)}\right)^{\frac{1}{k+1}}, f_3'(\bar{\phi}_0)=0, f_3(\bar{\phi}_0)=g-\left(1-\frac{1}{k+2}\right)c\bar{\phi}_0$.

The equilibrium points $E_j(z_j,0)$ of system (2.5) satisfy $f_3(z_j)=0$. It is easy to show for a fixed $c>0$, the following fact holds.

(1) When $g>\left(1-\frac{1}{k+2}\right)c\bar{\phi}_0, f_3(\phi)$ has no positive real zero;

(2) When $g = \left(1 - \frac{1}{k+2}\right) c\bar{\phi}_0, f_3(\phi)$ has a positive double real zero $z_{23} = \bar{\phi}_0$;

(3) When $0 \le g < \left(1 - \frac{1}{k+2}\right) c\bar{\phi}_0, f_3(\phi)$ has two positive simple zeros: $0 < z_2 < \bar{\phi}_0 < z_3 < \left(\frac{c}{a+b}\right)^{\frac{1}{k+1}}$.

Let $M(z_j, 0)$ be the coefficient matrix of the linearized system of (2.3) at an equilibrium point $E_j(z_j, 0)$. We have

$$J(z_j, 0) = detM(z_j, 0) = -f_3'(z_j)[(a+b)z_j^{k+1} - c]. \tag{2.6}$$

We write that $h_j = H_3(z_j, 0), j = 2, 3$ where H_3 is given by (1.15).

By using the above result to do qualitative analysis, with the change of the parameter g, we have the bifurcations of phase portraits of system (2.5) like Fig.2 when k is even number. When k is a odd number, the bifurcations of phase portraits of system (2.5) like Fig.1.

Notice that when $g = g^{***}$, the homoclinic orbit defined by $H_3(\phi, y) = h_2$ contacts to a branch of singular curve $a\phi(\phi^2 - y^2)^{\frac{1}{2}k} + b(\phi^2 - y^2)^{\frac{1}{2}(k+1)} - c = 0$, where g^{***} is a solution of $H_3(\bar{\phi}_0, 0) = h_2$.

3. The solitary wave solutions determined by the homocinic orbits of systems (1.10), (1.11) and (1.12)

It well know that a smooth homoclinic orbit of a traveling system gives rise to a solitary wave solution of the corresponding nonlinear wave equation. We always assume that $c > 0$ and is fixed.

We know from section 2 that for systems (1.10), (1.11) and (1.12), when conditions $g^* < g < 2p\left(\frac{c}{2p+1}\right)^{1+\frac{1}{2p}}$, $g^{**} < g < \left(1 - \frac{1}{2p}\right) c\hat{\phi}_0$ and $g^{***} < g < \left(1 - \frac{1}{k+2}\right) c\bar{\phi}_0$ hold, respectively. Then, the homoclinic orbits defined by $H_1(\phi, y) = h_2$, $H_2(\phi, y) = h_1$ and $H_3(\phi, y) = h_2$ have no intersection points with the one of three singular curves, respectively (see Fig.1 (c) and Fig.2 (b)). Thus, we immediately obtain the following conclusion.

Theorem 3.1. *Assume that $c > 0$ is fixed.*

(i) When $g^* < g < 2p\left(\frac{c}{2p+1}\right)^{1+\frac{1}{2p}}$, equation (1.4) has a smooth solitary wave solution given by the homoclinic orbit of system (1.10) defined by $H_1(\phi, y) = h_2$. In addition, equation (1.4) has a family of smooth periodic wave solutions defined by the closed branch of $H_1(\phi, y) = h$, $h \in (h_3, h_2)$ (see Fig.1 (c)).

(ii) When $g^{**} < g < \left(1 - \frac{1}{2p}\right) c\hat{\phi}_0$, equation (1.5) has a smooth solitary wave solution given by the homoclinic orbit of system (1.11) defined by of $H_2(\phi, y) = h_1$. In addition, equation (1.5) has a family of smooth periodic wave solutions defined by the closed branch of $H_2(\phi, y) = h$, $h \in (h_2, h_1)$ (see Fig.2 (b)).

(iii) $g^{***} < g < \left(1 - \frac{1}{k+2}\right) c\bar{\phi}_0$, equation (1.6) has a smooth solitary wave solution given by the homoclinic orbit of system (1.12) defined by of $H_3(\phi, y) = h_2$. In addition, equation (1.6) has a family of smooth periodic wave solutions defined by the closed branch of $H_3(\phi, y) = h$, $h \in (h_3, h_2)$ (see Fig.2 (b)).

For $p = 1$, to find the exact explicit parametric representation of solitary wave solution, we have from (1.18) that

$$y^2 = \phi^2 \pm \sqrt{h - 4g\phi} - c. \tag{3.1}$$

The signs $\pm$ before the term $\sqrt{h - 4g\phi}$ are dependent with the interval of ϕ. Let $\psi^2 = h - 4g\phi$, i.e., $\phi = \frac{1}{4g}(h - \psi^2)$. Then, we obtain

$$y^2 = \frac{1}{16g^2}[\psi^4 - 2h\psi^2 \pm 16g^2\psi + (h^2 - 16g^2c)]. \tag{3.2}$$

Under the condition $g^* < g < \frac{2c}{3}\sqrt{\frac{c}{3}}$, for $\phi \in (\phi_2, \sqrt{c})$, we need to take $+$ before the term $16g^2\psi$. By the first equation of (1.17), corresponding to the homoclinic orbit defined by $H_1(\phi, y) = h_2$, we have

$$\frac{\psi d\psi}{\sqrt{\psi^4 - 2h_2\psi^2 + 16g^2\psi + (h_2^2 - 16g^2c)}} = -\frac{1}{2}d\xi. \tag{3.3}$$

Denote that $(c - \phi^2)^2 + 4g\phi - h_2 = (\phi - \phi_2)^2(\phi - \phi_M)(\phi - \phi_3)$, where $\phi_3 < \phi_2 < \phi_M$. The point $(\phi_M, 0)$ is the intersection point of the homoclinic orbit to $(\phi_2, 0)$ of system (1.10) with the positive $\phi-$axis. Thus, we have $\psi^4 - 2h_2\psi^2 + 16g^2\psi + (h_2^2 - 16g^2c) = (\psi - \psi_1)^2(\psi - \psi_m)(\psi - \psi_3)$, where $\psi_1 = \sqrt{h_2 - 4g\phi_2}$, $\psi_m = \sqrt{h_2 - 4g\phi_M}$, $\psi_3 = -\sqrt{h_2 - 4g\phi_3}$. By introducing a parametric variable χ and integrating (3.3), we obtain

$$\begin{aligned} \psi(\chi) &= \psi_1 - \frac{2(\psi_1 - \psi_m)(\psi_1 - \psi_3)}{(\psi_m - \psi_3)\cosh(\sqrt{(\psi_1 - \psi_m)(\psi_1 - \psi_3)}\chi) - (2\psi_1 - \psi_m - \psi_3)}, \\ \xi(\chi) &= -2\left[\psi_1\chi + \ln\left(\frac{2\sqrt{(\psi - \psi_m)(\psi - \psi_3)} + 2\psi - (\psi_m - \psi_3)}{\psi_m - \psi_3}\right)\right]. \end{aligned} \tag{3.4}$$

Thus, we have the following exact explicit parametric representations of smooth solitary solution of equation (1.3)(see [6], [9]):

$$\begin{aligned} \phi(\chi) &= \frac{1}{4g}(1 - \psi^2(\chi)), \\ \xi(\chi) &= -2\left[\psi_1\chi + \ln\left(\frac{2\sqrt{(\psi - \psi_m)(\psi - \psi_3)} + 2\psi - (\psi_m - \psi_3)}{\psi_m - \psi_3}\right)\right]. \end{aligned} \tag{3.5}$$

For $p \geq 2$, because we can not explicitly solve the exact y from $H(\phi, y) = h_2$ given by (1.13), therefore, we can not give the exact solitary wave solutions of equation (1.4).

4. The M-shape traveling wave solutions determined by the homocinic orbit of sysems (1.10), (1.11) and (1.12)

In this section, we assume that $0 < g < g^*, 0 < g < g^{**}$ and $0 < g < g^{***}$, respectively. We notice that unlike the case of there exists a singular straight line in the book [5], for system sysems (1.10), (1.11) with $p \geq 2$ and (1.12), the curves $(\phi^2-y^2)^p-c=0, \phi(\phi^2-y^2)^{p-1}-c=0$ and $a\phi(\phi^2-y^2)^{\frac{1}{2}k}+b(\phi^2-y^2)^{\frac{1}{2}(k+1)}-c=0$ are not the solutions of system (1.10), (1.11) and (1.12), respectively. For every fixed $c>0$, when $0<g<g^*$, $0<g<g^{**}$ and $0<g<g^{***}$ hold, respectively, there exist uncountable infinite many periodic orbits and a homoclinic orbit of system (1.10), (1.11) and (1.12), which are transversely intersecting with an open branch of the above three singular curves (see Fig.1 (e) and Fig.2 (d)). These phase portraits Fig.1 (e) of system (2.1) and Fig.2 (d) of systems (2.3) are redrown in the left of Fig.3, while the vector fields defined by systems (1.10), (1.11) have been represented in the right of Fig.3. Obviously, the vector fields defined by systems (1.10) and (2.1) (or syetams (1.11) and (2.3)) are different (see Fig.3).

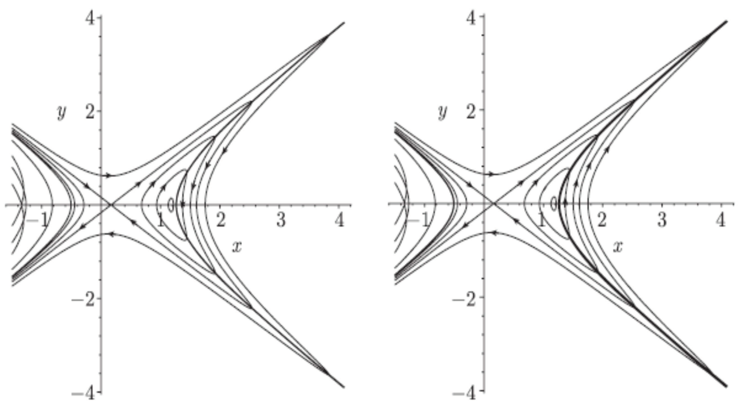
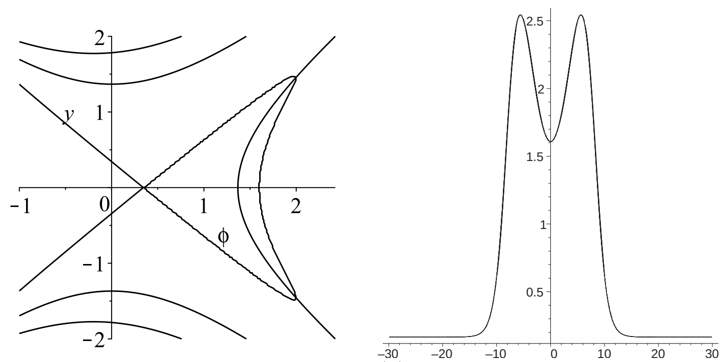


Figure 3. The graph of $(\phi^2-y^2)^p-c=0$ and the vector fields defined by (2.1) and (1.10)

The curves $c-(\phi^2-y^2)^p=0$ is the infinite isocline of the vector field of system (2.1). When ζ is varied along the homoclinic orbit defined by $H(\phi,y)=h_2$, the vector field of system (2.1) has the same direction, as shown in left of Fig.2. On the left-hand side of a branch of the curve $c-(\phi^2+y^2)^p=0$ in the first quadrant, one has $\frac{d\phi}{d\zeta}=y[c-(\phi^2-y^2)^p]>0, \frac{dy}{d\zeta}=-g+\phi[c-(\phi^2+y^2)^p]>0$. On the right-hand side of the hyperbola $(\phi^2-y^2)^p-c=0$ in the first quadrant, one has $\frac{d\phi}{d\zeta}=y[c-(\phi^2-y^2)^p]<0, \frac{dy}{d\zeta}=-g+\phi[c-(\phi^2+y^2)^p]<0$. Corresponding to the homoclinic orbit of system (1.10) (see Fig.4 (a)), we have the $M-$shape wave profile as Fig.4 (b).

Similarly, we can discuss the vector fields for systems (1.11) and (2.3), (1.12) and (2.5).

Differing from system (2.1), for system (1.10), the hyperbola $(\phi^2-y^2)^p-c=0$ is a singular curve of the vector field of the system. Here, consider the case of $\phi>0$. Clearly, on both the left-hand and the right-hand sides of the hyperbola $(\phi^2-y^2)^p-c=0$, when ξ is varied along the loop orbit defined by $H_1(\phi,y)=h_2$, the vector field of system (1.10) has a different direction, as shown in the right of Fig.3. In fact, on the right-hand side of the hyperbola $(\phi^2-y^2)^p-c=0$ in the first quadrant, one has $\frac{d\phi}{d\xi}=y>0, \frac{dy}{d\xi}=\frac{-g+\phi[c-(\phi^2-y^2)]}{c-(\phi^2-y^2)^p}>0$. This implies that the loop orbit of system (1.10) defined by $H_1(\phi,y)=h_2$ consists of three breaking solutions of system (1.10) (see Fig.5).



(a) Homoclinic orbit of system (2.1)

(b) The M-shape solitary wave of system (2.1)

Figure 4. $M-$shape wave solution of system (2.1) in the $(\xi, \phi)-$plane when $0 < g < g^*$.

The above statement is also holds for the systems (2.3) and (1.11).

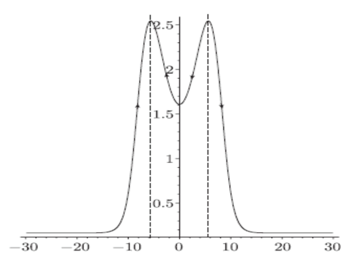


Figure 5. The M-shape wave consisting of three breaking waves of equation (1.4)

When $p = 1$ in system (1.10), the exact parametric representation of M-shape wave solution had been given in Li [6] and Li and Zhang [9].

To sum up, we has the following conclusion.

Theorem 4.1. *Assume that $c > 0$ is fixed.*

(i) *When $0 < g < g^*$, equation* (1.4) *has an M-shape wave solution defined by a branch of the level curves $H_1(\phi, y) = h_2$, which consists of three breaking wave solutions.*

(ii) *When $0 < g < g^{**}$, equation* (1.5) *has a M-shape wave solution defined by a branch of the level curves $H_2(\phi, y) = h_1$, which consists of three breaking wave solutions.*

(iii) *When $0 < g < g^{***}$, equation* (1.6) *has a M-shape wave solution defined by a branch of the level curves $H_3(\phi, y) = h_2$, which consists of three breaking wave solutions.*

We see from the discussion of this section that for the singular travelling wave systems of the second class, the dynamics of the orbits are different from the singular travelling wave systems of the first class. Because every singular curve intersects an orbit of the singular travelling wave systems of the second class only in a few isolated points, so that in systems (1.10) and (2.1), the variables ξ and ζ have the same ¡°time scale¡±. The singular curves make the direction of vector field defined by the singular travelling wave systems of the second class swiftly change. It derives new breaking wave solutions. Equation (1.4), (1.5) with $p \geq 2$ and (1.6) have no peakon solution in the sense of peakon in Camassa-Holm equation. Therefore, a lot new traveling wave models with peakons for the equations posed by [1]-[4] need to be studied.

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