Journal of Applied Analysis and Computation
Volume 11, Number 2, April 2021, 1062–1073
Website:http://www.jaac-online.com
DOI:10.11948/20210004

ON PIECEWISE MONOTONE FUNCTIONS WITH HEIGHT BEING INFINITY

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Abstract It is known that every piecewise monotone function with height finity has a characteristic interval after finite times iteration, and then the study of dynamics for such functions is able to be restricted to their characteristic intervals, which becomes monotone case. To the opposite, the description for piecewise monotone functions with height being infinity is much more complicated since the theory of characteristic interval does not work anymore. In this paper, we consider the problem of topological conjugacy for piecewise monotone functions with height being infinity. Some necessary and sufficient conditions are given for the existence of conjugacies between these functions. Moreover, the height of infinity under composition is also discussed. The fact shows a kind of symmetry for the height.

Keywords Piecewise monotone function, height, topological conjugacy, fort.

MSC(2010) 39B12, 37E05, 54H20, 68W30.

1. Introduction

The complexity of iteration does not come from nonlinearity but arise from non-monotonicity. A monotone function, no matter whether it is nonlinear, is as good as a linear one by defining a homeomorphism, which are the same in topological sense. In contrast, a non-monotonic continuous function has more than one monotone piece and the number of monotone pieces may increase under iteration, which generates complicated dynamical behaviors. Let I be a compact interval which can also be the whole real line $\mathbb{R}$. A continuous function $F : I \to I$ is said to be a *piecewise monotone function* (abbreviated as *PM function* in [20, 21]) which is also called modal map [11, 12] if F has finitely many non-monotone points or *forts* (or turning points in [11, 12]). Let $\mathcal{PM}(I, I)$ be the set of all PM functions mapping I into itself. Furthermore, let $S(F)$ be the set of all forts of F and $N(F)$ be the cardinality of $S(F)$. For each function $F \in \mathcal{PM}(I, I)$, it is known that the sequence $(N(F^n))_{n\in\mathbb{N}}$ is increasing, i.e., we have the following ascending relation

$$0 = N(F^0) \le N(F) \le N(F^2) \le \cdots \le N(F^n) \le N(F^{n+1}) \cdots, \ n \in \mathbb{N}. \quad (1.1)$$

Thus, the number $H(F)$ of F, which denotes the least integer $k \in \mathbb{N}$ (exists or ∞ otherwise) such that $N(F^k) = N(F^{k+1})$, is called the *non-monotonicity height* (or

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simply *height*) of F (see [9]). The simplest case $H(F) = 0$ implies that F is monotone on the whole interval I. When $H(F) = 1$, there exists a closed maximal monotone piece of I covering the range of F and bounded by either forts or endpoint [21, Lemma 2.4]. Such a closed interval is unique and called characteristic interval of F, denoted by $K(F)$. For the height greater than 1, namely $H(F) = n > 1$, it is known that $H(F^n) = 1$ and F^n has a characteristic interval.

Generally speaking, the height is a measure of the dynamical complexity for PM functions, which leads an important problem in the field of dynamical systems, i.e., the question about topological relations among these functions. Let $F, G \in \mathcal{PM}(I, I)$. Recall that a homeomorphism (continuous function) $\varphi : I \to I$ satisfying the conjugacy equation

$$\varphi \circ F(x) = G \circ \varphi(x), \qquad x \in I, \tag{1.2}$$

is said to be a *topological conjugacy* (semi-conjugacy) between F and G (F and G are then called *topologically conjugate* (topologically semi-conjugate)). We use the notation $F \sim G$ for the topological conjugate relation between F and G.

In 1966, Parry [15] proved the existence of a conjugacy between a topologically transitive PM function and a piecewise linear map. It is also known that any unimodal map (a PM function with precisely one fort) is semi-conjugate to a quadratic map [6]. Furthermore, by adding the smoothness on the given maps, Melo and Strien [11] proved that any continuously differentiable PM function is semi-conjugate to a polynomial with the same number of forts. More results about semi-conjugacy, see references [1, 3, 13, 14]. Another approach comes from Baldwin [2], by applying the invariance of itineraries who presented a general classification for PM functions via order-preserving conjugacy. Recently, regarding the index of height and using the theory of characteristic interval, the third author gave a complete classification for all PM functions with height being finite [8]. As noted at the end of the paper [8], this method is unavailable for the functions with height being infinity since they have no characteristic interval anymore even under iteration. Therefore, an open problem raised naturally: *Give a description of the topological relations between PM functions with height being infinity.*

In this paper, based on the above presented question, we first give necessary and sufficient conditions for the existence of topological conjugacy for PM functions with height being infinity under some regularity conditions in section 2. In section 3, a topological relation between two cubic and two quintic polynomials (whose heights are infinity) is discussed separately. In the last section, we consider the height of infinity under composition, which shows a symmetry of such a height.

2. Topological relations on a finite interval

Let $J := [a, b]$ for $a, b \in \mathbb{R}$ be a compact interval. Then we have the following useful lemmas.

Lemma 2.1 ([9]). *Let $F : J \to \mathbb{R}$ and $G : [\alpha, \beta] \to \mathbb{R}$ be continuous functions such that $F(J) \subseteq [\alpha, \beta]$. Then*

$$S(G \circ F) = S(F) \cup \{c \in (a, b) : F(c) \in S(G)\}.$$

Lemma 2.2 ([7]). *Let $F, G \in \mathcal{PM}(J, J)$ and $F \sim G$ via a topological conjugacy $\varphi : J \to J$. Then φ maps $S(F) \cup \{a, b\}$ onto $S(G) \cup \{a, b\}$.*

Lemma 2.3 ([7]). *Let $F, G \in \mathcal{PM}(J, J)$ and $F \sim G$ via a topological conjugacy $\varphi : J \to J$, then $N(F^i) = N(G^i)$ for all $i \in \mathbb{N}$ and $H(F) = H(G)$.*

Lemma 2.1 tells us the regularity of increasing number of forts under iteration and Lemmas 2.2-2.3 show an equivalent relation between those forts and endpoints. Now, we present an effective classification for the forts of all functions $F \in \mathcal{PM}(J, J)$ with $H(F) = \infty$ as follows. For every $n \in \mathbb{N}$, define

$$S_n(F) := \{x \in S(F) : \min_{n\in\mathbb{N}} F^n(y) \neq x, \forall y \in (a, b)\backslash S(F^n)\}.$$

Consequently, $S_\infty(F) := S(F)\backslash \bigcup_{i\in\mathbb{N}} S_i(F)$. It is clear that the set $S_n(F)$ is countable for $n \in \mathbb{N}$ since $\#S(F)$ is finite, $S_n(F) \subset S(F^n)$ and $S_i(F) \bigcap S_j(F) = \emptyset$ if $i \neq j$. Moreover, $S_\infty(F) \neq \emptyset$ if and only if $H(F) = \infty$.

Proposition 2.1 ([8]). *Let $F, G \in \mathcal{PM}(J, J)$ with $H(F) = H(G) = \infty$. If $F \sim G$ via a topological conjugacy $\varphi : J \to J$, then φ maps $S_n(F)$ onto $S_n(G)$ for all $n \in \mathbb{N} \cup \{\infty\}$.*

Proposition 2.1 presents a necessary condition for the existence of a topological conjugacy between PM functions with height being infinity, which is a further description of Lemma 2.2 and enables us to find the sufficient conditions.

For a given $F \in \mathcal{PM}(J, J)$, note that each sequence $\{F^{-n}(x)\}_{n\in\mathbb{N}}$ for every $x \in S_\infty(F)$ is bounded. Then there exists a subsequence $\{F^{-n_k}(x)\}_{k\in\mathbb{N}}$ that is monotone and convergent. Let $S(F) := \{c_1, c_2, ..., c_n\}$ and $S(G) := \{d_1, d_2, ..., d_n\}$ with $c_1 < c_2 < ... < c_n$ and $d_1 < d_2 < ... < d_n$. In order to illustrate the construction of conjugacies easily, we first give the following hypotheses:

($\mathcal{H}_1$) $F(b) = b$, $\{F^{-i}(c_n)\}_{i\in\mathbb{N}}$ is strictly increasing and converges to b (Figure 1);

($\mathcal{H}_2$) $F(a) = a$, $\{F^{-i}(c_1)\}_{i\in\mathbb{N}}$ is strictly decreasing and converges to a (Figure 2).

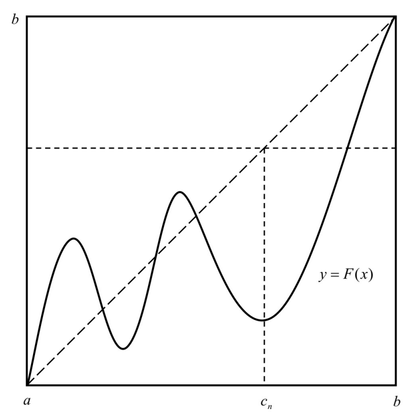

Figure 1. the case of ($\mathcal{H}_1$)

Without loss of generality, in what follows we only discuss under the hypothesis ($\mathcal{H}_1$) since every function satisfying ($\mathcal{H}_2$) is topologically conjugate to a function fulfilling ($\mathcal{H}_1$) by $\varphi(x) = a+b-x$. In ($\mathcal{H}_1$) the strictly monotonicity of $\{F^{-i}(c_n)\}_{i=0}^{\infty}$ implies that $F([a, c_n]) \subset [a, c_n]$. Then, the whole interval J is partitioned into

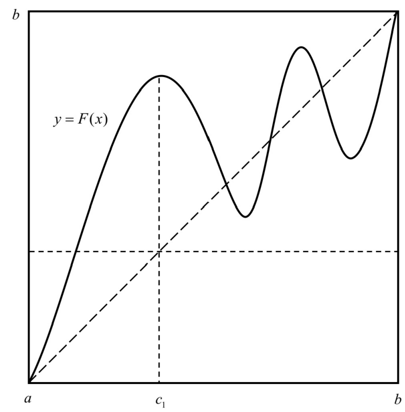

Figure 2. the case of $(\mathcal{H}_2)$

infinitely many subintervals, namely,

$$J = [a, b] = [a, c_n) \bigcup \cup_{i=0}^{+\infty} [F^{-i}(c_n), F^{-(i+1)}(c_n)] \bigcup \{b\}.$$

Then, we have the following results.

Theorem 2.1. *Let $F, G \in \mathcal{PM}(J, J)$ with $H(F) = H(G) = \infty$. Assume that the functions F, G satisfy $(\mathcal{H}_1)$ (resp. $(\mathcal{H}_2)$). Then F is topologically conjugate to G if and only if there exists an increasing homeomorphism $\varphi_0 : [a, c_n] \to [a, d_n]$ (resp. $\varphi_0 : [c_n, b] \to [d_n, b]$) satisfying equation (1.2). Furthermore, any homeomorphism φ_0 can be extended uniquely to a topological conjugacy between F and G.*

Proof. *Sufficiency.* Assume that there exists an increasing homeomorphism $\varphi_0 : [a, c_n] \to [a, d_n]$ satisfying $\varphi_0 \circ F = G \circ \varphi_0$. By Proposition 2.1, we have

$$\varphi_0(c_n) = d_n. \tag{2.1}$$

Then for each subinterval $[F^{-i}(c_n), F^{-(i+1)}(c_n)], i = 0, 1, 2, ...$, define an increasing homeomorphism $\varphi_i : [F^{-i}(c_n), F^{-(i+1)}(c_n)] \to [G^{-i}(d_n), G^{-(i+1)}(d_n)]$ by

$$\varphi_i(x) := G^{-(i+1)} \circ \varphi_0 \circ F^{i+1}(x). \tag{2.2}$$

Clearly, the functions φ_is in (2.2) are well defined. Moreover, it follows from (2.1)-(2.2) that

$$\varphi_i(F^{-(i+1)}(c_n)) = G^{-(i+1)} \circ \varphi_0(c_n) = G^{-(i+1)}(d_n),$$
$$\varphi_{i+1}(F^{-(i+1)}(c_n)) = G^{-(i+2)} \circ \varphi_0(F(c_n)) = G^{-(i+1)}(d_n).$$

This implies that all those φ_is are continuous at each junction. We further define the function $\varphi : J \to J$ as

$$\varphi(x) := \begin{cases} \varphi_0(x), & x \in [a, c_n) \\ \varphi_i(x), & x \in [F^{-i}(c_n), F^{-(i+1)}(c_n)], \quad i = 0, 1, 2, ..., \\ b, & x = b. \end{cases} \tag{2.3}$$

Obviously, φ is strictly increasing on $[a,b]$. Moreover, we infer from $(\mathcal{H}_1)$ and (2.3) that

$$\lim_{x\to b}\varphi(x) = \lim_{i\to+\infty}\varphi_i(F^{-(i+1)}(c_n)) = \lim_{i\to+\infty}G^{-(i+1)}(d_n) = b = \varphi(b),$$

which shows the continuity of φ on the whole interval J. Finally, it suffices to prove that φ satisfies equation (1.2). Actually, for every $x \in [F^{-i}(c_n), F^{-(i+1)}(c_n)]$ we have

$$\varphi\circ F(x) = G^{-(i+1)}\circ\varphi_0\circ F^{i+2}(x) = G^{-(i+1)}\circ G^{i+2}\circ\varphi_0(x) = G\circ\varphi(x).$$

Therefore, the function φ defined in (2.3) is a topological conjugacy between F and G, which is extended uniquely by the homeomorphism φ_0. The part of sufficiency is proved.

Necessity. Assume that F is topologically conjugate to G via an increasing homeomorphism $\varphi : J \to J$. According to Proposition 2.1 we get equality (2.1). Let $\varphi_0 := \varphi_{[a,c_n]}$. It suffices to prove that φ_0 is a solution of equation (1.2) on $[a, c_n]$. In fact, for every $x \in [a, c_n]$ we get $F(x) \in [a, c_n]$ and

$$G\circ\varphi_0(x) = G\circ\varphi(x) = \varphi\circ F(x) = \varphi_0\circ F(x).$$

Therefore, φ_0 is an increasing topological conjugacy between $F_{[a,c_n]}$ and $G_{[a,d_n]}$. The whole proof is completed. □

Remark 2.1. Note that the authors in [7] investigated a topological conjugate relation for a class of PM functions with height being infinity, only those functions whose graphs under the diagonal line, are considered there, which is a special case of our Theorem 2.1.

Example 2.1. Consider two maps F and G defined by

$$F(x) := \begin{cases} \frac{6}{5}x + \frac{1}{4}, & x\in[0,\frac{1}{8}),\\ -\frac{2}{5}x + \frac{9}{20}, & x\in[\frac{1}{8},\frac{1}{2}),\\ \frac{3}{2}x - \frac{1}{2}, & x\in[\frac{1}{2},1], \end{cases} \quad\text{and}\quad G(x) := \begin{cases} \frac{2}{5}x + \frac{2}{5}, & x\in[0,\frac{1}{4}),\\ -\frac{1}{5}x + \frac{11}{20}, & x\in[\frac{1}{4},\frac{3}{4}),\\ \frac{12}{5}x - \frac{7}{5}, & x\in[\frac{3}{4},1], \end{cases}$$

respectively. One checks that $S(F) = \{\frac{1}{8},\frac{1}{2}\}$, $S_1(F) = \{\frac{1}{8}\}$, $S_\infty(F) = \{\frac{1}{2}\}$ and $S(G) = \{\frac{1}{4},\frac{3}{4}\}$, $S_1(G) = \{\frac{1}{4}\}$, $S_\infty(G) = \{\frac{3}{4}\}$. Clearly, F and G satisfy condition $(\mathcal{H}_1)$, where $c_n = \frac{1}{2}, d_n = \frac{3}{4}$. Note that $F_{[0,\frac{1}{2}]}$ and $G_{[0,\frac{3}{4}]}$ are self-maps. It is easy to verify that $H(F_{[0,\frac{1}{2}]}) = H(G_{[0,\frac{3}{4}]}) = 1$, $K(F_{[0,\frac{1}{2}]}) = [\frac{1}{8},\frac{1}{2}]$ and $K(G_{[0,\frac{3}{4}]}) = [\frac{1}{4},\frac{3}{4}]$. Moreover, there exists an increasing homeomorphism $H : [\frac{1}{8},\frac{1}{2}] \to [\frac{1}{4},\frac{3}{4}]$ defined by $H(x) := \frac{2}{3}x + \frac{7}{30}$, which satisfies $H\circ F(0) = G(0)$, $H\circ F(\frac{1}{8}) = G(\frac{1}{4})$ and $H\circ F(\frac{1}{2}) = G(\frac{3}{4})$. Hence, all conditions in [8, Lemma 2.1] are fulfilled, and then $F_{[0,\frac{1}{2}]}$ is topologically conjugate to $G_{[0,\frac{3}{4}]}$. Therefore, by Theorem 2.1 we conclude that F is topologically conjugate to G on $[0,1]$.

According to the proof of Theorem 2.1, we have the following result of a decreasing conjugacy.

Corollary 2.1. *Let $F, G \in \mathcal{PM}(J,J)$ with $H(F) = H(G) = \infty$. Assume that the function F satisfies $(\mathcal{H}_1)$ and G satisfies $(\mathcal{H}_2)$. Then F is topologically conjugate* *to G if and only if there exists a decreasing homeomorphism $\varphi_0 : [a, c_n] \to [d_1, b]$ satisfying equation (1.2). Furthermore, any homeomorphism φ_0 can be extended uniquely to a topological conjugacy between F and G.*

By Theorem 2.1 and Corollary 2.1, for those PM functions with two unique fixed points a, b, we get the following results directly.

Corollary 2.2. *Let $F, G \in \mathcal{PM}(J, J)$ with $H(F) = H(G) = \infty$. Assume that $F(a) = G(a) = a$, $F(b) = G(b) = b$ and $F(x), G(x) > x$ for all $x \in (a, b)$ (resp. $F(x), G(x) < x$ for all $x \in (a, b)$) holds. Then F is topologically conjugate to G via an increasing conjugacy if and only if there exists an increasing homeomorphism $\varphi_0 : [c_1, b] \to [d_1, b]$ (resp. $\varphi_0 : [a, c_n] \to [a, d_n]$) satisfying equation (1.2). Furthermore, any homeomorphism φ_0 can be extended uniquely to a topological conjugacy between F and G.*

Corollary 2.3. *Let $F, G \in \mathcal{PM}(J, J)$ with $H(F) = H(G) = \infty$. Assume that $F(a) = G(a) = a$, $F(b) = G(b) = b$ and $F(x), G(x) > x$ for all $x \in (a, b)$ (resp. $F(x), G(x) < x$ for all $x \in (a, b)$) holds. Then F is topologically conjugate to G via a decreasing conjugacy if and only if there exists a decreasing homeomorphism $\varphi_0 : [c_1, b] \to [a, d_n]$ (resp. $\varphi_0 : [a, c_n] \to [d_1, b]$) satisfying equation (1.2). Furthermore, any homeomorphism φ_0 can be extended uniquely to a topological conjugacy between F and G.*

Recall that each sequence $\{F^{-n}(x)\}_{n\in\mathbb{N}}$ for every $x \in S_\infty(F)$ has a subsequence $\{F^{-n_i}(x)\}_{i\in\mathbb{N}}$ that is monotone and convergent. Choose a point $x_0 \in S_\infty(F)$ and let $x_i := F^{-n_i}(x_0)$ for each $i \in \mathbb{N}$. At the last part of this section, compared with conditions $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ we put some regularities on such subsequences.

$(\mathcal{H}_3)$ $\{x_i\}_{i\in\mathbb{N}}$ is strictly increasing fulfilling $\lim_{i\to+\infty} x_i = b$ and $F^j(x_0) \in [a, F^{-n_1}(x_0)]$ for $j = 0, 1, 2, ...$ (Figure 3);

$(\mathcal{H}_4)$ $\{x_i\}_{i\in\mathbb{N}}$ is strictly decreasing fulfilling $\lim_{i\to+\infty} x_i = a$ and $F^j(x_0) \in [F^{-n_1}(x_0), b]$ for $j = 0, 1, 2, ...$ (Figure 4).

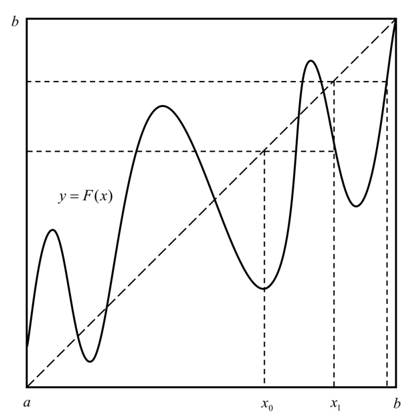

Figure 3. the case of $(\mathcal{H}_3)$

Theorem 2.2. *Let $F, G \in \mathcal{PM}(J, J)$ with $H(F) = H(G) = \infty$. Assume that the functions F, G satisfy $(\mathcal{H}_3)$ (resp. $(\mathcal{H}_4)$). Then F is topologically conjugate to G if and only if there exists an increasing homeomorphism $\varphi_0 : [a, F^{-n_1}(x_0)] \to$*

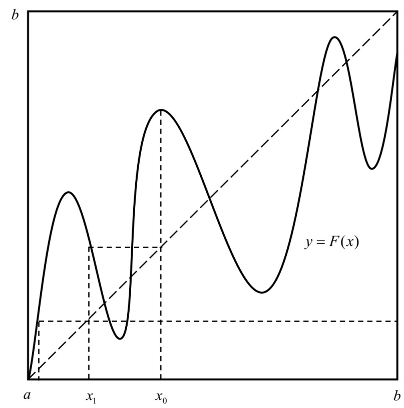

Figure 4. the case of $(\mathcal{H}_4)$

$[a, G^{-n_1}(y_0)]$ (resp. $\varphi_0 : [F^{-n_1}(x_0), b] \to [G^{-n_1}(y_0), b]$) for some $x_0 \in S_\infty(F), y_0 \in S_\infty(G)$ satisfying equation (1.2). Furthermore, any homeomorphism φ_0 can be extended uniquely to a topological conjugacy between F and G.

Proof. *Sufficiency.* Assume that there exists an increasing homeomorphism $\varphi_0 : [a, F^{-n_1}(x_0)] \to [a, G^{-n_1}(y_0)]$ satisfying $\varphi_0 \circ F = G \circ \varphi_0$. The other case that $\varphi_0 : [F^{-n_1}(x_0), b] \to [G^{-n_1}(y_0), b]$ can be proved similarly.

Since $F^j(x_0) \in [a, F^{-n_1}(x_0)]$ for $j = 0, 1, 2, ...$, it follows from Proposition 2.1 that

$$\varphi_0(F^j(x_0)) = G^j(x_0), j = 0, 1, 2, ... \tag{2.4}$$

Then, define an increasing homeomorphism $\varphi_i : [F^{-n_i}(x_0), F^{-n_{i+1}}(x_0)] \to [G^{-n_i}(y_0), G^{-n_{i+1}}(y_0)]$ by

$$\varphi_i(x) := G^{-n_{i+1}} \circ \varphi_0 \circ F^{n_{i+1}}(x). \tag{2.5}$$

According to (2.4), it is easy to check that

$$\varphi_i(F^{-n_i}(x_0)) = G^{-n_{i+1}} \circ \varphi_0 \circ F^{n_{i+1}}(F^{-n_i}(x_0)) = G^{-n_{i+1}} \circ G^{n_{i+1}-n_i}(x_0) = G^{-n_i}(x_0)$$

since $n_{i+1} - n_i > 0$. Moreover,

$$\varphi_i(F^{-n_{i+1}}(x_0)) = G^{-n_{i+1}} \circ \varphi_0 \circ F^{n_{i+1}}(F^{-n_{i+1}}(x_0)) = G^{-n_{i+1}}(x_0)$$

by (2.4) again. Hence, the functions φ_i in (2.5) is well defined for each $i \in \mathbb{N}$. By the proof of Theorem 2.1, the function $\varphi : I \to I$ defined by

$$\varphi(x) := \begin{cases} \varphi_0(x), & x \in [a, F^{-n_1}(x_0)) \\ \varphi_i(x), & x \in [F^{-n_i}(x_0), F^{-n_{i+1}}(x_0)], \quad i = 1, 2, ..., \\ b, & x = b, \end{cases}$$

is a topological conjugacy between F and G.

The part of necessity is obtained obviously. $\square$

According to the proof of Theorem 2.2, we have the following result of a decreasing conjugacy.

Corollary 2.4. *Let $F, G \in \mathcal{PM}(J, J)$ with $H(F) = H(G) = \infty$. Assume that the functions F, G satisfy $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$), respectively. Then F is topologically conjugate to G if and only if there exists an decreasing homeomorphism $\varphi_0 : [a, F^{-n_1}(x_0)] \to [G^{-n_1}(y_0), b]$ for some $x_0 \in S_\infty(F), y_0 \in S_\infty(G)$ satisfying equation (1.2). Furthermore, any homeomorphism φ_0 can be extended uniquely to a topological conjugacy between F and G.*

3. Topological relations on the whole real line

It is clear that every real polynomial is a PM function, which is defined on the whole real line $\mathbb{R}$ into itself. Furthermore, we infer from [17, Theorem 1] that $H(F) = \infty$ for all polynomials $F : \mathbb{R} \to \mathbb{R}$ with odd degree greater than 2. In this section, we mainly consider a topological classification for two kinds of polynomials, i.e., cubic polynomials and quintic polynomials, separately.

Theorem 3.1. *$f(x) := a_3x^3 + a_2x^2 + a_1x + a_0$ is topologically conjugate to $g(x) := b_3x^3 + b_2x^2 + b_1x + b_0$ via a conjugacy*

$$h(x) := \frac{u_1x + u_0}{v_1x + v_0}, \quad (u_1x + u_0, v_1x + v_0) = 1, \quad v_1 \neq 0$$

if and only if

$$a_0 = \frac{a_2(a_2^2 - 9a_3)}{27a_3^2},\ a_1 = \frac{a_2^2}{3a_3},$$

where

$$u_0 = \frac{(-b_2a_2 + 9\sqrt{a_3b_3})\kappa_1}{9a_3b_3},\ u_1 = -\frac{\kappa_1 b_2}{3b_3},\ v_0 = \frac{\kappa_1 a_2}{3a_3},\ v_1 = \kappa_1,\ a_3b_3 > 0,\ \kappa_1 \in \mathbb{R}$$

or

$$u_0 = \frac{u_1a_2}{3a_3},\ v_0 = \frac{a_2\kappa_2}{3a_3},\ v_1 = \kappa_2,\ \text{and } u_1 \text{ is one of the real roots of equation}$$

$$b_3x^3 + b_2\kappa_2x^2 + (b_1\kappa_2^2 - \kappa_2^2)x + b_0\kappa_2^3 = 0,$$

$\kappa_2 \in \mathbb{R}$.

Proof. Let $\nu(x) := u_1x + u_0$ and $\delta(x) := v_1x + v_0$. Since $(\nu(x), \delta(x)) = 1$, by [4, Chapter 12, pp.397-399], neither the resultant of $A_2(x)$ and $B_2(x)$ nor the resultant of $\nu(x)$ and $\delta(x)$ vanishes, i.e.,

$$\mathfrak{R}_1 := \text{resultant}(\nu(x), \delta(x), x) = u_0v_1 - v_0u_1 \neq 0.$$

Consider the conjugation between f and g. One can compute

$$\frac{\hat{L}_3x^3 + \hat{L}_2x^2 + \hat{L}_1x + \hat{L}_0}{\check{L}_3x^3 + \check{L}_2x^2 + \check{L}_1x + \check{L}_0} = g(h(x)) = h(f(x)) = \frac{\hat{R}_3x^3 + \hat{R}_2x^2 + \hat{R}_1x + \hat{R}_0}{\check{R}_3x^3 + \check{R}_2x^2 + \check{R}_1x + \check{R}_0}. \quad (3.1)$$

By [10, Lemma 3], the fractions on both sides are irreducible. On account of $\check{L}_3 = v_1^3 \neq 0$ and $\check{R}_3 = a_3v_1 \neq 0$, in order to simplify the calculation, we divide $\check{L}_3$ and $\check{R}$ on the left and right sides of equation (3.1) respectively, and equate the corresponding coefficients of both sides of (3.1), which gives the following semi-algebraic system

$$\begin{aligned}
f_1 &:= b_0v_1^3 + b_1u_1v_1^2 + b_2u_1^2v_1 + b_3u_1^3 - u_1v_1^2 = 0;\\
f_2 &:= -3a_3b_0v_1^2v_2 - 2a_3b_1u_1v_1v_2 - a_3b_1u_2v_1^2 - a_3b_2u_1^2v_2 - 2a_3b_2u_1u_2v_1\\
&\quad - 3a_3b_3u_1^2u_2 + a_2u_1v_1^2 = 0;\\
f_3 &:= -3a_3b_0v_1v_2^2 - a_3b_1u_1v_2^2 - 2a_3b_1u_2v_1v_2 - 2a_3b_2u_1u_2v_2 - a_3b_2u_2^2v_1\\
&\quad - 3a_3b_3u_1u_2^2 + a_1u_1v_1^2 = 0;\\
f_4 &:= -a_3b_0v_2^3 - a_3b_1u_2v_2^2 - a_3b_2u_2^2v_2 - a_3b_3u_2^3 + a_0u_1v_1^2 + u_2v_1^2 = 0;\\
f_5 &:= a_2v_1 - 3a_3v_2 = 0;\\
f_6 &:= a_1v_1^2 - 3a_3v_2^2 = 0;\\
f_7 &:= a_0v_1^3 - a_3v_2^3 + v_1^2v_2 = 0;\\
\mathfrak{R}_1 &:= u_0v_1 - v_0u_1 \neq 0;\\
\mathfrak{R}_2 &:= a_3b_3 \neq 0;\\
\mathfrak{R}_3 &:= v_1 \neq 0;
\end{aligned}$$

denoted by $\widetilde{\mathbf{PS}}$. From which we expect to find a simplest algebraic relation among coefficients a_i s and b_i s for f to be conjugate to g by eliminating u_0, v_0, u_1 and v_1, that are coefficients of h. Since it is usually difficult to eliminate variables from a semi-algebraic system, we consider the following algebraic system

$$\begin{aligned}
\widehat{\mathbf{PS}} :=&\{f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0, f_5 = 0, f_6 = 0, f_7 = 0,\\
&1 - r\mathfrak{R}_1 = 0, 1 - s\mathfrak{R}_2 = 0, 1 - t\mathfrak{R}_3 = 0\}
\end{aligned}$$

in the ring $\mathbb{C}[a_i\,\mathrm{s}, b_i\,\mathrm{s}, u_0, v_0, u_1, v_1, r, s, t]$ instead because $\mathfrak{R}_1 \neq 0$, $\mathfrak{R}_2 \neq 0$ and $\mathfrak{R}_3 \neq 0$ if and only if there exist $r, s, t \in \mathbb{C}$ such that $1 - r\mathfrak{R}_1 = 0$, $1 - s\mathfrak{R}_2 = 0$ and $1 - t\mathfrak{R}_3 = 0$. Thus, the problem is converted to compute the variety $V(\mathbf{I})$ as done in [16, 18], where $\mathbf{I} := \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7, 1 - r\mathfrak{R}_1, 1 - s\mathfrak{R}_2, 1 - t\mathfrak{R}_3\rangle$. Aiming at the above polynomials in $\mathbf{I}$, we use computer algebra system *Singular* to find minimal associate primes of the obtained corresponding ideal with the routine `minAssGTZ` (see [5]). Computations show that in our case there are no embedded components. Furthermore, we get the conditions to guarantee the conjugation between f and g which is shown in the presentation of our theorem and u_0, u_1, v_0, v_1 can be obtained accordingly. This completes the whole proof. □

Using the same idea, we can also obtain

Theorem 3.2. *A quintic polynomial $f(x) := a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ is topologically conjugate to another one $g(x) := b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$ via a homeomorphism*

$$h(x) := \frac{u_1x + u_0}{v_1x + v_0}, \quad (u_1x + u_0, v_1x + v_0) = 1, \quad v_1 \neq 0$$

if and only if

$$a_0 = \frac{a_4(a_4^4 - 625a_5^3)}{3125a_5^4},\ a_1 = \frac{a_4^4}{125a_5^3},\ a_2 = \frac{2a_4^3}{25a_5^2}\ a_3 = \frac{2a_4^2}{5a_5},$$

where

$$u_0 = \frac{\tau_1(-a_4b_4 \pm 25(a_5^3b_5^3)^{\frac{1}{4}})}{25a_5b_5},\ u_1 = -\frac{\tau_1b_4}{5b_5},\ v_0 = \frac{\tau_1a_4}{5a_5},\ v_1 = \tau_1,\ a_5b_5 > 0,\ \tau \in \mathbb{R}$$

or

$u_0 = \dfrac{\tau_2\tau_3 a_4}{5a_5}, u_1 = \tau_2\tau_3, v_0 = \dfrac{\tau_2 a_4}{5a_5}, v_1 = \tau_2,$ *and* τ_3 *is one of the real roots of equation*

$$b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + (b_1 - 1)x + b_0 = 0,$$

$\tau_2 \in \mathbb{R}$.

Proof. The proof of this theorem is similar to that of Theorem 3.1. □

4. Height under composition

In the last section of this paper, we consider the non-monotonicity height of PM functions under composition. Since the case of height 1 was investigated in [19], we mainly discuss the height under composition equals to infinity as follows.

Before presenting the main results of this section, we need a useful lemma and notation, as given in [19].

Lemma 4.1 ([19]). *Let $F \in \mathcal{PM}(J, J)$. If there exists a subinterval $J' \subsetneq J$ such that $J' \subseteq F(J')$ and $S(F) \cap \mathrm{int}J' \neq \emptyset$, then $H(F) = \infty$. Particularly, if $F(J) = J$, then $H(F) = \infty$.*

For a given function $F \in \mathcal{PM}(J, J)$, the closed subinterval $J' \subsetneq J$ is referred to as a *spanning interval* of F if $F(J') \supseteq J'$ and $\mathrm{int}J' \cap S(F) \neq \emptyset$ (see [19]). In particular, J' is called a *unimodal interval* if J' contains precisely one fort of F, $F(J') = J'$ and no subinterval of J' has these properties [11].

According to Lemma 4.1, we get the following results.

Theorem 4.1. *Let $F, G \in \mathcal{PM}(J, J)$. If the function $F \circ G$ has a spanning interval $J' \subset J$, then $H(G \circ F) = \infty$.*

Proof. By reduction to absurdity, we assume that $H(G \circ F) = n < \infty$ for some positive integer n. Define

$$J_0 := (G \circ F)^n(J). \tag{4.1}$$

Clearly, the function $G \circ F$ is monotone on J_0, which implies that F and G is strictly monotone on J_0 and $F(J_0)$, respectively. By the definition of spanning interval, there exists a subinterval $J' \subseteq J$ such that

$$F \circ G(J') = J' \quad \text{and} \quad \mathrm{int}J' \cap S(F \circ G) \neq \emptyset.$$

Consequently,

$$S(G) \cap \mathrm{int}J' \neq \emptyset \quad \text{or} \quad S(F) \cap G(\mathrm{int}J') \neq \emptyset. \tag{4.2}$$

Note that

$$J' \subseteq (F \circ G)^{n+1}(J) = F \circ (G \circ F)^n \circ G(J) \subseteq F \circ (G \circ F)^n(J) = F(J_0). \tag{4.3}$$

Then, we further obtain that

$$F(\mathrm{int}J_0) = \mathrm{int}F(J_0) \ \text{ and } \ G(\mathrm{int}J') = \mathrm{int}G(J') \subseteq \mathrm{int}(G \circ F(J_0)) \subseteq \mathrm{int}J_0. \tag{4.4}$$

In view of (4.2), if $S(G) \cap \mathrm{int}J' \neq \emptyset$, then

$$S(G) \cap F(\mathrm{int}J_0) = S(G) \cap \mathrm{int}F(J_0) \supseteq S(G) \cap \mathrm{int}J' \neq \emptyset$$

by (4.3), and thus we infer from Lemma 2.1 that

$$\begin{aligned} S(G \circ F) \cap \mathrm{int}J_0 &= \{S(F) \cup \{c \in (a,b) : F(c) \in S(G)\}\} \cap \mathrm{int}J_0 \\ &\supseteq \{c \in (a,b) : F(c) \in S(G)\} \cap \mathrm{int}J_0 \neq \emptyset. \end{aligned}$$

This is a contradiction to the assumption that $H(G \circ F) = n$. For the other case, i.e., $S(F) \cap G(\mathrm{int}J') \neq \emptyset$ in (4.2), using Lemma 2.1 again, we have

$$S(G \circ F) \cap \mathrm{int}J_0 \supseteq S(F) \cap \mathrm{int}J_0 \supseteq S(F) \cap G(\mathrm{int}J') \neq \emptyset$$

according to (4.4), a contradiction to the assumption of $H(G \circ F) = n$ again. Therefore, the proof is completed. □

Remark 4.1. Note that $H(F \circ G) = \infty$ if the function $F \circ G$ has a spanning interval. Hence, Theorem 4.1 shows the symmetry of height being infinity.

Corollary 4.1. *Let $F, G \in \mathcal{PM}(J, J)$ and $H(F) = H(G) = 1$. If $G(K(F)) = K(G)$, $F(K(G)) = K(F)$ and either $S(G) \cap \mathrm{int}K(F) \neq \emptyset$ or $S(F) \cap \mathrm{int}K(G) \neq \emptyset$, then $H(F \circ G) = \infty$.*

Proof. By the assumption, it is clear that $F \circ G|_{K(F)}$ is a surjective self-map on $K(F)$. If $S(G) \cap \mathrm{int}K(F) \neq \emptyset$, we have $S(F \circ G) \cap \mathrm{int}K(F) \neq \emptyset$ by Lemma 4.1. Then it follows from Lemma 4.1 again that $H(F \circ G|_{K(F)}) = \infty$, implying $H(F \circ G) = \infty$. In the other case that $S(F) \cap intK(G) \neq \emptyset$, by a similar discussion we get $H(F \circ G) = \infty$ by Theorem 4.1. □

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