Journal of Applied Analysis and Computation
Volume 11, Number 2, April 2021, 656–673
Website:http://www.jaac-online.com
DOI:10.11948/20190164

NEW PRECONDITIONED GAOR METHODS FOR BLOCK LINEAR SYSTEM ARISING FROM WEIGHTED LINEAR LEAST SQUARES PROBLEMS*

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Abstract In this paper, new preconditioned GAOR methods are proposed for solving a class of 2×2 block structure linear systems arising from the weighted linear least squares problems. Comparison theorems are derived. Comparison results show that the convergence rates of the new preconditioned GAOR methods are better than those of the preconditioned GAOR methods in the previous literatures whenever these methods are convergent. A numerical example is given to confirm our theoretical results.

Keywords Preconditioner, GAOR method, preconditioned GAOR method, weighted linear least squares problem, comparison theorem.

MSC(2010) 65F10, 65F50.

1. Introduction

In this paper, we consider the 2×2 block structure linear systems of the form

$$Hy = f, \quad y, f \in \mathbb{R}^n, \tag{1.1}$$

where

$$H = \begin{bmatrix} I_p - B & U \\ L & I_q - C \end{bmatrix}$$

is a nonsingular matrix with $B = (b_{ij}) \in \mathbb{R}^{p\times p}$, $C = (c_{ij}) \in \mathbb{R}^{q\times q}$, $L = (l_{ij}) \in \mathbb{R}^{q\times p}$, $U = (u_{ij}) \in \mathbb{R}^{p\times q}$, and $p + q = n$. The linear system (1.1) arises from the solution of the weighted linear least squares problem [14, 15]

$$\min_{w\in\mathbb{R}^n} (Aw - b)^T W^{-1}(Aw - b),$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. The weighted linear least squares problem has many scientific applications, a typical

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*The authors were supported by the Natural Science Foundation of China (No. 11861059), Natural Science Foundation of Northwest Normal University (No. NWNU-LKQN-17-5) and the Science and Technology Program of Shandong Colleges (J16LI04).

source is the parameter estimation in mathematical modelling, see [4, 13, 15] for details. Here and elsewhere in the paper, I_k denotes the identity matrix with dimension k.

Many classical iterative methods for solving the linear system (1.1) have been studied by many authors, see for example [3, 10, 14, 15]. To avoid the inverses of the matrices $I_p - B$ and $I_q - C$, Yuan and Jin [15] proposed the generalized AOR (GAOR) method for solving the linear system (1.1). Splitting the coefficient matrix H of the linear system (1.1) as

$$H = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -L & 0 \end{bmatrix} - \begin{bmatrix} B & -U \\ 0 & C \end{bmatrix},$$

the GAOR method is defined by [15]

$$y^{(k+1)} = L_{\omega\gamma} y^{(k)} + \omega g, \quad k = 0, 1, 2, \cdots, \tag{1.2}$$

where

$$\begin{aligned} L_{\omega\gamma} &= \begin{bmatrix} I_p & 0 \\ \gamma L & I_q \end{bmatrix}^{-1} \left\{ (1-\omega) I_n + (\omega - \gamma) \begin{bmatrix} 0 & 0 \\ -L & 0 \end{bmatrix} + \omega \begin{bmatrix} B & -U \\ 0 & C \end{bmatrix} \right\} \\ &= \begin{bmatrix} (1-\omega) I_p + \omega B & -\omega U \\ \omega(\gamma - 1) L - \omega\gamma L B & (1-\omega) I_q + \omega C + \omega\gamma L U \end{bmatrix} \end{aligned} \tag{1.3}$$

is the iteration matrix and

$$g = \begin{bmatrix} I_p & 0 \\ -\gamma L & I_q \end{bmatrix} f$$

with real parameters $\omega \neq 0$ and γ. Darvishi and Hessari [3] studied the convergence of the GAOR method when the coefficient matrix H is a diagonally dominant matrix.

It is known that the smaller the spectral radius of the iteration matrix $L_{\omega\gamma}$, the faster the GAOR method converges. For improving the convergent rate of the corresponding iterative method, preconditioning techniques are used [2]. Especially, we consider the following equivalent left preconditioned linear system of (1.1)

$$PHy = Pf, \tag{1.4}$$

where $P \in \mathbb{R}^{n \times n}$, called the left preconditioner, is nonsingular. If we express PH as

$$PH = \begin{bmatrix} I_p - \widehat{B} & \widehat{U} \\ \widehat{L} & I_q - \widehat{C} \end{bmatrix},$$

then the GAOR method for solving the preconditioned linear system (1.4), which is also called the preconditioned GAOR method [17] for solving the linear system (1.1), is defined as

$$y^{(k+1)} = \widehat{L}_{\omega\gamma} y^{(k)} + \omega \widehat{g}, \quad k = 0, 1, 2, \cdots, \tag{1.5}$$

where

$$\widehat{L}_{\omega\gamma} = \begin{bmatrix} (1-\omega)I_p + \omega\widehat{B} & -\omega\widehat{U} \\ \omega(\gamma-1)\widehat{L} - \omega\gamma\widehat{L}\widehat{B} & (1-\omega)I_q + \omega\widehat{C} + \omega\gamma\widehat{L}\widehat{U} \end{bmatrix}$$

and

$$\widehat{g} = \begin{bmatrix} I_p & 0 \\ -\gamma\widehat{L} & I_q \end{bmatrix} \widehat{P}f.$$

Recently, many preconditioners have been proposed for accelerating the convergence rate of the GAOR method, such as [6–9, 12, 16, 17]. In this paper, two new preconditioners are proposed to accelerate the convergence rate of the GAOR method for solving the linear system (1.1). Some comparison theorems are established to demonstrate the effectiveness of the proposed preconditioners theoretically, and a numerical example is given to show the correctness of theoretical analysis.

2. Preliminaries

For $A = (a_{ij}),\ B = (b_{ij}) \in \mathbb{R}^{n\times n}$, we write $A \geq B$ (or $A > B$) if $a_{ij} \geq b_{ij}$ (or $a_{ij} > b_{ij}$) holds for all $i, j = 1, 2 \cdots, n$. We say that A is nonnegative (positive) if $A \geq 0 (A > 0)$, and $A - B \geq 0$ if and only if $A \geq B$. These definitions carry immediately over to vectors by identifying them with $n \times 1$ matrices. $\rho(*)$ denotes the spectral radius of a square matrix. A is called irreducible if the directed graph of A is strongly connected [11].

Some useful results which we refer to later are provided below.

Lemma 2.1 ([11]). *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative and irreducible matrix. Then*

(a). *A has a positive eigenvalue equal to $\rho(A)$;*

(b). *A has an eigenvector $x > 0$ corresponding to $\rho(A)$.*

Lemma 2.2 ([1]). *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Then*

(a). *If $\alpha x \leq Ax$ for a vector $x \geq 0$ and $x \neq 0$, then $\alpha \leq \rho(A)$.*

(b). *If $Ax \leq \beta x$ for a vector $x > 0$, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$, equality excluded, for a vector $x \geq 0$ and $x \neq 0$, then $\alpha < \rho(A) < \beta$ and $x > 0$.*

3. Preconditioned GAOR methods

In this section, on the basis of reviewing the existing preconditioners, we will propose two kinds of new preconditioners and corresponding preconditioned GAOR methods.

In 2012, Shen *et al.* [9] proposed the preconditioner of the form $P_1 = \begin{bmatrix} I_p + S_1 & 0 \\ 0 & I_q \end{bmatrix}$

with

$$S_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \beta_2 b_{21} & 0 & \cdots & 0 & 0 \\ 0 & \beta_3 b_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_p b_{p,p-1} & 0 \end{bmatrix},$$

where $\beta_i > 0$, $i = 2, 3, \cdots, p$. In 2014, Zhao *et al.* [16] considered the preconditioner

$$\widetilde{P}_1 = \begin{bmatrix} I_p + S_1 & 0 \\ 0 & I_q + V_1 \end{bmatrix},$$

where S_1 is defined as above and for $\tau_j > 0$, $j = 2, 3, \cdots, q$,

$$V_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \tau_2 c_{21} & 0 & \cdots & 0 & 0 \\ 0 & \tau_3 c_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \tau_q c_{q,q-1} & 0 \end{bmatrix}.$$

Similar preconditioned techniques are considered in [5,7]. The preconditioned matrix $\widetilde{P}_1 H$ can be expressed as

$$\widetilde{P}_1 H = \begin{bmatrix} I_p - B_1^* & U_1^* \\ \widetilde{L}_1 & I_q - \widetilde{C}_1 \end{bmatrix},$$

where $B_1^* = B - S_1(I_p - B)$, $U_1^* = (I_p + S_1)U$, $\widetilde{L}_1 = (I_q + V_1)L$, $\widetilde{C}_1 = C - V_1(I_q - C)$, and the corresponding preconditioned GAOR method is defined as

$$y^{(k+1)} = \widetilde{L}_{\omega\gamma 1} y^{(k)} + \omega \widetilde{g}_1, \quad k = 0, 1, 2, \cdots \tag{3.1}$$

with the iteration matrix

$$\widetilde{L}_{\omega\gamma 1} = \begin{bmatrix} (1-\omega) I_p + \omega B_1^* & -\omega U_1^* \\ \omega(\gamma - 1)\widetilde{L}_1 - \omega\gamma \widetilde{L}_1 B_1^* & (1-\omega) I_q + \omega \widetilde{C}_1 + \omega\gamma \widetilde{L}_1 U_1^* \end{bmatrix} \tag{3.2}$$

and the corresponding known vector $\widetilde{g}_1 = \begin{bmatrix} I_p & 0 \\ -\gamma \widetilde{L}_1 & I \end{bmatrix} \widetilde{P}_1 H$. The preconditioner introduced by Zhou *et al.* [17] in 2009 has the form $\bar{P}_1 = \begin{bmatrix} I_p & 0 \\ K_1 & I_q \end{bmatrix}$, where $\theta > 0$

and

$$K_1 = \begin{bmatrix} 0 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ -\frac{l_{q1}}{\theta} & 0 \cdots 0 \end{bmatrix}.$$

Recently, in view of the preconditioners P_1 and $\bar{P}_1$, Huang *et al.* [6] considered the following preconditioner

$$\overline{P}_1 = \begin{bmatrix} I_p + S_1 & 0 \\ K_1 & I_q \end{bmatrix},$$

where S_1 and K_1 are as above, the matrix $\overline{P}_1 H$ can be written as

$$\overline{P}_1 H = \begin{bmatrix} I_p - B_1^* & U_1^* \\ \overline{L}_1 & I_q - \overline{C}_1 \end{bmatrix},$$

where B_1^* and U_1^* are as above, and $\overline{L}_1 = L + K_1(I_p - B)$, $\overline{C}_1 = C - K_1 U$. Then the corresponding preconditioned GAOR method is defined as

$$y^{(k+1)} = \overline{L}_{\omega\gamma 1} y^{(k)} + \omega \overline{g}_1, \quad k = 0, 1, 2, \cdots \tag{3.3}$$

with the iteration matrix

$$\overline{L}_{\omega\gamma 1} = \begin{bmatrix} (1-\omega) I_p + \omega B_1^* & -\omega U_1^* \\ \omega(\gamma - 1)\overline{L}_1 - \omega\gamma \overline{L}_1 B_1^* & (1-\omega) I_q + \omega \overline{C}_1 + \omega\gamma \overline{L}_1 U_1^* \end{bmatrix} \tag{3.4}$$

and the corresponding known vector $\overline{g}_1 = \begin{bmatrix} I_p & 0 \\ -\gamma \overline{L}_1 & I \end{bmatrix} \overline{P}_1 f$. Based on the idea of [7,8], we propose our first new preconditioner $\widehat{P}_1$ of the form

$$\widehat{P}_1 = \begin{bmatrix} I_p + S_1 & 0 \\ K_1 & I_q + V_1 \end{bmatrix}, \tag{3.5}$$

where S_1, V_1 and K_1 are defined as above. Let the preconditioned matrix $\widehat{P}_1 H$ be expressed as

$$\widehat{P}_1 H = \begin{bmatrix} I_p - \widehat{B}_1 & \widehat{U}_1 \\ \widehat{L}_1 & I_q - \widehat{C}_1 \end{bmatrix},$$

where $\widehat{B}_1 = B - S_1(I_p - B)$, $\widehat{U}_1 = (I_p + S_1)U$, $\widehat{L}_1 = (I_q + V_1)L + K_1(I_p - B)$ and $\widehat{C}_1 = C - V_1(I_q - C) - K_1 U$, then applying the GAOR method to the preconditioned linear system (1.4) with the preconditioners $\widehat{P}_1$, we have the following new preconditioned GAOR method

$$y^{(k+1)} = \widehat{L}_{\omega\gamma 1} y^{(k)} + \omega \widehat{g}_1, \quad k = 0, 1, 2, \cdots, \tag{3.6}$$

where

$$\widehat{L}_{\omega\gamma 1}=\begin{bmatrix}(1-\omega)I_p+\omega\widehat{B}_1 & -\omega\widehat{U}_1\\ \omega(\gamma-1)\widehat{L}_1-\omega\gamma\widehat{L}_1\widehat{B}_1 & (1-\omega)I_q+\omega\widehat{C}_1+\omega\gamma\widehat{L}_1\widehat{U}_1\end{bmatrix} \tag{3.7}$$

is the iteration matrix and $\widehat{g}_1=\begin{bmatrix}I_p & 0\\ -\gamma\widehat{L}_1 & I_q\end{bmatrix}\widehat{P}_1 f$ is the corresponding known vector.

Based on the works in [6, 17], we will propose our second preconditioner. In 2013, Wang *et al.* [12] proposed the preconditioner $P_2=\begin{bmatrix}I_p+S_2 & 0\\ 0 & I_q\end{bmatrix}$ with

$$S_2=\begin{bmatrix}0 & b_{12} & 0 & \cdots & 0\\ 0 & 0 & b_{23} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & b_{p-1,p}\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}.$$

In 2015, Huang *et al.* [6] considered the preconditioner

$$\widetilde{P}_2=\begin{bmatrix}I_p+S_2 & 0\\ 0 & I_q+V_2\end{bmatrix},$$

where S_2 is defined as above and

$$V_2=\begin{bmatrix}0 & c_{12} & 0 & \cdots & 0\\ 0 & 0 & c_{23} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & c_{q-1,q}\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}.$$

Applying the GAOR method to the peconditioned linear system (1.4) with the preconditioner $\widetilde{P}_2$, the corresponding preconditioned GAOR method is defined as

$$y^{(k+1)}=\widetilde{L}_{\omega\gamma 2}y^{(k)}+\omega\widetilde{g}_2,\quad k=0,1,2,\cdots, \tag{3.8}$$

where

$$\widetilde{L}_{\omega\gamma 2}=\begin{bmatrix}(1-\omega)I_p+\omega B_2^* & -\omega U_2^*\\ \omega\left(\gamma-1\right)\widetilde{L}_2-\omega\gamma\widetilde{L}_2B_2^* & \left(1-\omega\right)I_q+\omega\widetilde{C}_2+\omega\gamma\widetilde{L}_2U_2^*\end{bmatrix} \tag{3.9}$$

is the iteration matrix with $B_2^* = B - S_2\left(I_p - B\right), U_2^* = \left(I_p + S_2\right)U,\ \widetilde{L}_2 = (I_q + V_2)L,\ \widetilde{C}_2 = C - V_2(I_q - C),\ \widetilde{g}_2 = \begin{bmatrix} I_p & 0 \\ -\gamma L & I_q \end{bmatrix} \widetilde{P}_2 f$ is the corresponding known vector. Zhou *et al.* in [17] also proposed the preconditioner

$$\overline{P}_2 = \begin{bmatrix} I_p & 0 \\ K_2 & I_q \end{bmatrix},$$

where K_2 is defined as follows:

(1). when $q < p$,

$$K_2 = \begin{bmatrix} -l_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -l_{22} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -l_{qq} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

(2). when $q = p$,

$$K_2 = \begin{bmatrix} -l_{11} & 0 & \cdots & 0 \\ 0 & -l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -l_{qq} \end{bmatrix},$$

(3). when $q > p$,

$$K_2 = \begin{bmatrix} -l_{11} & 0 & \cdots & 0 \\ 0 & -l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -l_{pp} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The preconditioned GAOR method with preconditioner $\overline{P}_2$ for solving linear systems (1.1) can be defined by

$$y^{(k+1)} = \overline{L}_{\omega\gamma 2} y^{(k)} + \omega \overline{g}_2, \quad k = 0, 1, 2, \cdots, \tag{3.10}$$

where

$$\overline{L}_{\omega\gamma 2} = \begin{bmatrix} (1-\omega) I_p + \omega B & -\omega U \\ \omega\left(\gamma - 1\right)\overline{L}_2 - \omega\gamma\overline{L}_2 B & \left(1-\omega\right) I_q + \omega\overline{C}_2 + \omega\gamma\overline{L}_2 U \end{bmatrix} \tag{3.11}$$

is the iteration matrix with $\overline{L}_2 = L + K_2(I_p - B)$ and $\overline{C}_2 = C - K_2U$, and $\overline{g}_2 = \begin{bmatrix} I_p & 0 \\ -\gamma\overline{L}_2 & I_q \end{bmatrix} \overline{P}_2 f$. In view of the preconditioner $\widetilde{P}_2$ and $\overline{P}_2$, our second preconditioner has the form

$$\widehat{P}_2 = \begin{bmatrix} I_p + S_2 & 0 \\ K_2 & I_q + V_2 \end{bmatrix}, \tag{3.12}$$

where S_2, V_2 and K_2 are defined as above. Now the preconditioned matrix $\widehat{P}_2H$ can be expressed as

$$\widehat{P}_2H = \begin{bmatrix} I_p - \widehat{B}_2 & \widehat{U}_2 \\ \widehat{L}_2 & I_q - \widehat{C}_2 \end{bmatrix},$$

where $\widehat{B}_2 = B - S_2(I_p - B)$, $\widehat{U}_2 = (I_p + S_2)U$, $\widehat{L}_2 = (I_q + V_2)L + K_2(I_p - B)$ and $\widehat{C}_2 = C - V_2(I_q - C) - K_2U$. Applying the GAOR method to the preconditioned linear system (1.4) with the preconditioners $\widehat{P}_2$, we have the following new preconditioned GAOR method

$$y^{(k+1)} = \widehat{L}_{\omega\gamma 2}y^{(k)} + \omega\widehat{g}_2, \quad k = 0, 1, 2, \cdots, \tag{3.13}$$

where

$$\widehat{L}_{\omega\gamma 2} = \begin{bmatrix} (1-\omega)I_p + \omega\widehat{B}_2 & -\omega\widehat{U}_2 \\ \omega(\gamma-1)\widehat{L}_2 - \omega\gamma\widehat{L}_2\widehat{B}_2 & (1-\omega)I_q + \omega\widehat{C}_2 + \omega\gamma\widehat{L}_2\widehat{U}_2 \end{bmatrix}, \tag{3.14}$$

is the iteration matrix and $\widehat{g}_2 = \begin{bmatrix} I_p & 0 \\ -\gamma\widehat{L}_2 & I_q \end{bmatrix} \widehat{P}_2 f$ is the corresponding known vector.

4. Comparison results

In this section, some comparison theorems are established to demonstrate the efficiency of the proposed preconditioners $\widehat{P}_1$ and $\widehat{P}_2$ theoretically.

Firstly, the comparison results about the convergent rates of the preconditioned GAOR methods defined by (3.6) and (3.13) with that of the GAOR method defined by (1.2) are given. Comparing $\rho(\widehat{L}_{\omega\gamma 1})$ with $\rho(L_{\omega\gamma})$, we have following comparison result.

Theorem 4.1. *Let $L_{\omega\gamma}$ and $\widehat{L}_{\omega\gamma 1}$ be the iteration matrices of the GAOR method (1.2) and the preconditioned GAOR method (3.6), respectively. Assume that the matrix H in Equation (1.1) is irreducible with $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $0 < \omega \leq 1$, $0 \leq \gamma < 1$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \cdots, p-1\}$, $0 < \beta_{i+1} < \frac{1}{1-b_{ii}}$ whenever $0 \leq b_{ii} < 1$, or $\beta_i > 0$ whenever $b_{ii} \geq 1$ for $i \in \{1, 2, \cdots, p-1\}$; and $c_{j+1,j} > 0$ for some $j \in \{1, 2, \cdots, q-1\}$, $0 < \tau_{j+1} < \frac{1}{1-c_{jj}}$ whenever $0 \leq c_{jj} < 1$, or $\tau_j > 0$ whenever $c_{jj} \geq 1$ for $j \in \{1, 2, \cdots, q-1\}$; and $l_{q1} < 0$, $\theta > 0$ whenever $b_{11} \geq 1$, or $0 < \theta < 1 - b_{11}$ whenever $0 \leq b_{11} < 1$, then*

$$\rho(\widehat{L}_{\omega\gamma 1}) < \rho(L_{\omega\gamma}), \ \ if \quad \rho(L_{\omega\gamma}) < 1.$$

Proof. By assumptions, it is easy to verify that $L_{\omega\gamma}$ and $\widehat{L}_{\omega\gamma 1}$ are nonnegative and irreducible matrices. From Lemma 2.1, there is a positive vector x such that

$$L_{\omega\gamma}x = \lambda x, \tag{4.1}$$

where $\lambda = \rho(L_{\omega\gamma})$ and $\lambda \neq 1$. For otherwise, the matrix H is singular. Moreover, it holds that

$$\omega H x = \begin{bmatrix} I_p & 0 \\ \gamma L & I_q \end{bmatrix} (I_n - L_{\omega\gamma})x = (1-\lambda)\begin{bmatrix} I_p & 0 \\ \gamma L & I_q \end{bmatrix} x. \tag{4.2}$$

From (4.1) and (4.2), we can deduce that

$$\begin{aligned}
&\widehat{L}_{\omega\gamma 1}x - \lambda x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma\widehat{L}_1 & I_q \end{bmatrix}^{-1} \left\{ (1-\omega)I_n + (\omega-\gamma)\begin{bmatrix} 0 & 0 \\ -\widehat{L}_1 & 0 \end{bmatrix} + \omega \begin{bmatrix} \widehat{B}_1 & -\widehat{U}_1 \\ 0 & \widehat{C}_1 \end{bmatrix} \right\} x - \lambda x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma\widehat{L}_1 & I_q \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\omega I_p + \omega\widehat{B}_1 & -\omega\widehat{U}_1 \\ -\omega\widehat{L}_1 & -\omega I_q + \omega\widehat{C}_1 \end{bmatrix} + (1-\lambda)\begin{bmatrix} I_p & 0 \\ \gamma\widehat{L}_1 & I_q \end{bmatrix} \right\} x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma\widehat{L}_1 & I_q \end{bmatrix}^{-1} \left\{ -\omega H + \begin{bmatrix} -\omega S_1(I_p - B) & -\omega S_1 U \\ -\omega V_1 L - \omega K_1(I_p - B) & -\omega V_1(I_q - C) - \omega K_1 U \end{bmatrix} \right. \\
&\quad \left. + (1-\lambda)\begin{bmatrix} I_p & 0 \\ \gamma\widehat{L}_1 & I_q \end{bmatrix} \right\} x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma\widehat{L}_1 & I_q \end{bmatrix}^{-1} \left\{ \begin{bmatrix} S_1 & 0 \\ K_1 & V_1 \end{bmatrix} (-\omega H) + (1-\lambda)\begin{bmatrix} I_p & 0 \\ \gamma(\widehat{L}_1 - L) & I_q \end{bmatrix} \right\} x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma\widehat{L}_1 & I_q \end{bmatrix}^{-1} \left\{ (\lambda-1)\begin{bmatrix} S_1 & 0 \\ K_1 & V_1 \end{bmatrix} \begin{bmatrix} I_p & 0 \\ \gamma L & I_q \end{bmatrix} + (1-\lambda)\begin{bmatrix} 0 & 0 \\ \gamma V_1 L + \gamma K_1(I_p - B) & 0 \end{bmatrix} \right\} x \\
&= \begin{bmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{bmatrix} \left\{ (\lambda-1)\begin{bmatrix} S_1 & 0 \\ K_1 & V_1 \end{bmatrix} \begin{bmatrix} I_p & 0 \\ \gamma L & I_q \end{bmatrix} - (\lambda-1)\begin{bmatrix} 0 & 0 \\ \gamma V_1 L + \gamma K_1(I_p - B) & 0 \end{bmatrix} \right\} x \\
&= \begin{bmatrix} I_p & 0 \\ -\gamma\widehat{L}_1 & I_q \end{bmatrix} \left\{ (\lambda-1)\begin{bmatrix} S_1 & 0 \\ K_1 - \gamma K_1(I_p - B) & V_1 \end{bmatrix} \right\} x \\
&= (\lambda-1)\begin{bmatrix} S_1 & 0 \\ (1-\gamma)K_1 + \gamma K_1 B - \gamma L S_1 - \gamma V_1 L S_1 - \gamma K_1(I_p - B)S_1 & V_1 \end{bmatrix} x.
\end{aligned}$$

Since $0 < \omega \leq 1$, $0 \leq \gamma < 1$, K_1, V_1 and S_1 are nonnegative and nonzero matrices, we have

$$\begin{bmatrix} S_1 & 0 \\ (1-\gamma)K_1 + \gamma K_1 B - \gamma L S_1 - \gamma V_1 L S_1 - \gamma K_1(I_p - B)S_1 & V_1 \end{bmatrix} x \geq 0$$

and

$$\begin{bmatrix} S_1 & 0 \\ (1-\gamma)K_1 + \gamma K_1 B - \gamma L S_1 - \gamma V_1 L S_1 - \gamma K_1(I_p - B)S_1 & V_1 \end{bmatrix} x \neq 0.$$

If $\lambda < 1$, then $\widehat{L}_{\omega\gamma 1}x - \lambda x \leq 0$ and $\widehat{L}_{\omega\gamma 1}x - \lambda x \neq 0$, Lemma 2.2 gives $\rho(\widehat{L}_{\omega\gamma 1}) < \rho(L_{\omega\gamma}) < 1$. □

Similarly, comparing $\rho(\widehat{L}_{\omega\gamma 2})$ with $\rho(L_{\omega\gamma})$, we have the following comparison result.

Theorem 4.2. *Let $L_{\omega\gamma}$ and $\widehat{L}_{\omega\gamma 2}$ be the iteration matrices of the GAOR method (1.2) and the preconditioned GAOR method (3.13), respectively. Assume that the matrix H in Equation (1.1) is irreducible with $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $0 < \omega \leq 1$, $0 \leq \gamma < 1$, $b_{i,i+1} > 0$ for some $i \in \{1, \cdots, p-1\}$, and $c_{j,j+1} > 0$ for some $j \in \{1, \cdots, q-1\}$, and $l_{ii} < 0$ for $i \in \{1, 2, \cdots, p\}$, then*

$$\rho(\widehat{L}_{\omega\gamma 2}) < \rho(L_{\omega\gamma}), \ if \quad \rho(L_{\omega\gamma}) < 1.$$

Secondly, we will compare the convergent rate of the preconditioned GAOR method (3.6) with that of the preconditioned GAOR method (3.1), the convergent rate of the preconditioned GAOR method (3.13) with that of the preconditioned GAOR method (3.8), respectively. Comparing $\rho(\widehat{L}_{\omega\gamma 1})$ with $\rho(\widetilde{L}_{\omega\gamma 1})$, we have the following theorem.

Theorem 4.3. *Let $\widetilde{L}_{\omega\gamma 1}$ and $\widehat{L}_{\omega\gamma 1}$ be the iteration matrices of the preconditioned GAOR methods (3.1) and (3.6), respectively. Assume that the matrix H in Equation (1.1) is irreducible with $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $0 < \omega \leq 1$, $0 \leq \gamma < 1$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \cdots, p-1\}$, $0 < \beta_{i+1} < \frac{1}{1-b_{ii}}$ whenever $0 \leq b_{ii} < 1$, or $\beta_i > 0$ whenever $b_{ii} \geq 1$ for $i \in \{1, 2, \cdots, p-1\}$; and $c_{j+1,j} > 0$ for some $j \in \{1, 2, \cdots, q-1\}$, $0 < \tau_{j+1} < \frac{1}{1-c_{jj}}$ whenever $0 \leq c_{jj} < 1$, or $\tau_j > 0$ whenever $c_{jj} \geq 1$ for $j \in \{1, 2, \cdots, q-1\}$; and $l_{q1} < 0$, $\theta > 0$ whenever $b_{11} \geq 1$, then*

$$\rho(\widehat{L}_{\omega\gamma 1}) < \rho(\widetilde{L}_{\omega\gamma 1}), \ if \quad \rho(\widetilde{L}_{\omega\gamma 1}) < 1.$$

Proof. By assumptions, it is easy to show that $\widetilde{L}_{\omega\gamma 1}$ and $\widehat{L}_{\omega\gamma 1}$ are irreducible and non-negative. It follows from Lemma 2.1 that there is a positive vector x such that

$$\widetilde{L}_{\omega\gamma 1}x = \xi x, \tag{4.3}$$

where $\xi = \rho(\widetilde{L}_{\omega\gamma 1})$ and $\xi \neq 1$. Moreover, note that $K_1 S_1 = 0$, so it hold that

$$\omega \widetilde{P}_1 H x = \begin{bmatrix} I_p & 0 \\ \gamma L_1^* & I_q \end{bmatrix} (I_n - \widetilde{L}_{\omega\gamma 1})x = (1-\xi) \begin{bmatrix} I_p & 0 \\ \gamma L_1^* & I_q \end{bmatrix} x \tag{4.4}$$

and

$$\begin{bmatrix} -\omega I_p + \omega \widehat{B}_1 & -\omega \widehat{U}_1 \\ -\omega \widehat{L}_1 & -\omega I_q + \omega \widehat{C}_1 \end{bmatrix} = \begin{bmatrix} -I_p & 0 \\ -K_1 & -I_q \end{bmatrix} \omega \widetilde{P}_1 H. \tag{4.5}$$

From (4.3), (4.4) and (4.5), we can deduce that

$$\begin{aligned}
&\widehat{L}_{\omega\gamma 1} x - \xi x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_1 & I_q \end{bmatrix}^{-1} \left\{ (1-\omega) I_n + (\omega - \gamma) \begin{bmatrix} 0 & 0 \\ -\widehat{L}_1 & 0 \end{bmatrix} + \omega \begin{bmatrix} \widehat{B}_1 & -\widehat{U}_1 \\ 0 & \widehat{C}_1 \end{bmatrix} \right\} x - \xi x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_1 & I_q \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\omega I_p + \omega \widehat{B}_1 & -\omega \widehat{U}_1 \\ -\omega \widehat{L}_1 & -\omega I_q + \omega \widehat{C}_1 \end{bmatrix} + (1-\xi) \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_1 & I_q \end{bmatrix} \right\} x \\
&= (1-\xi) \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_1 & I_q \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ (\gamma-1)K_1 - \gamma K_1 B & 0 \end{bmatrix} x \\
&= (1-\xi) \begin{bmatrix} 0 & 0 \\ (\gamma-1)K_1 - \gamma K_1 B & 0 \end{bmatrix} x.
\end{aligned}$$

By the assumptions, we have

$$\begin{bmatrix} 0 & 0 \\ (\gamma-1)K_1 - \gamma K_1 B & 0 \end{bmatrix} x \le 0 \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ (\gamma-1)K_1 - \gamma K_1 B & 0 \end{bmatrix} x \ne 0.$$

If $\xi < 1$, then $\widehat{L}_{\omega\gamma 1} x - \xi x \le 0$ and $\widehat{L}_{\omega\gamma 1} x - \xi x \ne 0$, Lemma 2.2 gives $\rho(\widehat{L}_{\omega\gamma 1}) < \rho(\widetilde{L}_{\omega\gamma 1}) < 1$. □

Comparing $\rho(\widehat{L}_{\omega\gamma 2})$ with $\rho(\widetilde{L}_{\omega\gamma 2})$, we have the following theorem.

Theorem 4.4. *Let $\widetilde{L}_{\omega\gamma 2}$ and $\widehat{L}_{\omega\gamma 2}$ be the iteration matrices of the preconditioned GAOR methods (3.8) and (3.13), respectively. Assume that the matrix H in Equation (1.1) is irreducible with $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 < \omega \le 1$, $0 \le \gamma < 1$, $b_{i,i+1} > 0$ for some $i \in \{1, \cdots, p-1\}$, and $0 \le b_{11}$, $b_{pp} < 1$ and $b_{ii} = 1$ for $i \in \{2, \cdots, p-1\}$, and $c_{j,j+1} > 0$ for some $j \in \{1, \cdots, q-1\}$, and $l_{ii} < 0$ for $i \in \{1, 2, \cdots, p\}$, then*

$$\rho(\widehat{L}_{\omega\gamma 2}) < \rho(\widetilde{L}_{\omega\gamma 2}), \ if \quad \rho(\widetilde{L}_{\omega\gamma 2}) < 1.$$

Proof. We can see that $\widetilde{L}_{\omega\gamma 2}$ and $\widehat{L}_{\omega\gamma 2}$ are irreducible and non-negative matrices under the assumptions. Let positive vector x be the eigenvector of $\widetilde{L}_{\omega\gamma 2}$ corresponding to the eigenvalue $\nu = \rho(\widetilde{L}_{\omega\gamma 1})$, then we have

$$\widetilde{L}_{\omega\gamma 2} x = \nu x.$$

Moreover,it hold that

$$\omega \widetilde{P}_2 H x = \begin{bmatrix} I_p & 0 \\ \gamma L_2^* & I_q \end{bmatrix} (I_n - \widetilde{L}_{\omega\gamma 2}) x = (1-\nu) \begin{bmatrix} I_p & 0 \\ \gamma L_2^* & I_q \end{bmatrix} x$$

and

$$\begin{bmatrix} -\omega I_p + \omega \widehat{B}_2 & -\omega \widehat{U}_2 \\ -\omega \widehat{L}_2 & -\omega I_q + \omega \widehat{C}_2 \end{bmatrix} = \begin{bmatrix} -I_p & 0 \\ -K_2 & -I_q \end{bmatrix} \omega \widetilde{P}_2 H + \omega \begin{bmatrix} 0 & 0 \\ K_2 S_2 (I_p - B) & K_2 S_2 U \end{bmatrix}.$$

Note that $\begin{bmatrix} 0 & 0 \\ K_2 S_2 (I_p - B) & K_2 S_2 U \end{bmatrix} \leq 0$ under the assumptions, so similar to the proof of Theorem 4.3, we can deduce that

$$\begin{aligned} &\widehat{L}_{\omega\gamma 2} x - \nu x \\ &= \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_2 & I_q \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\omega I_p + \omega \widehat{B}_2 & -\omega \widehat{U}_2 \\ -\omega \widehat{L}_2 & -\omega I_q + \omega \widehat{C}_2 \end{bmatrix} + (1-\nu) \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_2 & I_q \end{bmatrix} \right\} x \\ &\leq (1-\nu) \begin{bmatrix} 0 & 0 \\ (\gamma - 1) K_2 - \gamma K_2 B & 0 \end{bmatrix} x \\ &\leq 0. \end{aligned}$$

If $\nu < 1$, then $\widehat{L}_{\omega\gamma 2} x - \nu x \leq 0$ and $\widehat{L}_{\omega\gamma 2} x - \nu x \neq 0$, Lemma 2.2 gives $\rho(\widehat{L}_{\omega\gamma 2}) < \rho(\widetilde{L}_{\omega\gamma 2}) < 1$. □

Finally, we will show that the proposed preconditioner $\widehat{P}_1$ is better than the preconditioner $\overline{P}_1$ considered in [6], and the proposed preconditioner $\widehat{P}_2$ is better than the preconditioner $\overline{P}_2$ considered in [17]. Comparing the spectral radius of the matrix $\widehat{L}_{\omega\gamma 1}$ with that of the matrix $\overline{L}_{\omega\gamma 1}$, we have the following comparison theorem.

Theorem 4.5. *Let $\overline{L}_{\omega\gamma 1}$ and $\widehat{L}_{\omega\gamma 1}$ be the iteration matrices of the GAOR methods (3.3) and (3.6), respectively. Assume that the matrix H in Equation (1.1) is irreducible with $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $0 < \omega \leq 1$, $0 \leq \gamma < 1$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \cdots, p-1\}$, $0 < \beta_{i+1} < \frac{1}{1-b_{ii}}$ whenever $0 \leq b_{ii} < 1$, or $\beta_i > 0$ whenever $b_{ii} \geq 1$ for $i \in \{1, 2, \cdots, p-1\}$; and $c_{j+1,j} > 0$ for some $j \in \{1, 2, \cdots, q-1\}$, $0 < \tau_{j+1} < \frac{1}{1-c_{jj}}$ whenever $0 \leq c_{jj} < 1$, or $\tau_j > 0$ whenever $c_{jj} \geq 1$ for $j \in \{1, 2, \cdots, q-1\}$; and $l_{q1} < 0$, $\theta > 0$ whenever $b_{11} \geq 1$, or $0 < \theta < 1 - b_{11}$ whenever $0 \leq b_{11} < 1$ for $j \in \{1, 2, \cdots, q-1\}$, then*

$$\rho(\widehat{L}_{\omega\gamma 1}) < \rho(\overline{L}_{\omega\gamma 1}), \ if \quad \rho(\overline{L}_{\omega\gamma 1}) < 1.$$

Proof. By assumptions, it is easy to show that $\overline{L}_{\omega\gamma 1}$ and $\widehat{L}_{\omega\gamma 1}$ are irreducible and non-negative. From Lemma 2.1, there is a positive vector x such that

$$\overline{L}_{\omega\gamma 1} x = \mu x, \tag{4.6}$$

where $\mu = \rho(\overline{L}_{\omega\gamma 1})$ and $\mu \neq 1$. Moreover, it holds that

$$\omega \overline{P}_1 H x = \begin{bmatrix} I_p & 0 \\ \gamma \overline{L}_1 & I_q \end{bmatrix} (I_n - \overline{L}_{\omega\gamma 1}) x = (1-\mu) \begin{bmatrix} I_p & 0 \\ \gamma \overline{L}_1 & I_q \end{bmatrix} x. \tag{4.7}$$

From (4.6) and (4.7), we can deduce that

$$\begin{aligned}
&\widehat{L}_{\omega\gamma 1} x - \mu x \\
&= (\widehat{L}_{\omega\gamma 1} - \overline{L}_{\omega\gamma 1}) x \\
&= \begin{bmatrix} \omega(\widehat{B}_1 - B_1^*) & -\omega(\widehat{U}_1 - U_1^*) \\ \omega(\gamma-1)(\widehat{L}_1 - \overline{L}_1) - \omega\gamma(\widehat{L}_1 \widehat{B}_1 - \overline{L}_1 B_1^*) & \omega(\widehat{C}_1 - \overline{C}_1) + \omega\gamma(\widehat{L}_1 \widehat{U}_1 - \overline{L}_1 U_1^*) \end{bmatrix} x \\
&= \begin{bmatrix} 0 & 0 \\ \omega\gamma V_1 L(I_p - B) - \omega V_1 L + \omega\gamma V_1 L S_1 (I_p - B) & -\omega V_1 (I_q - C) + \omega\gamma V_1 L U + \omega\gamma V_1 L S_1 U \end{bmatrix} x \\
&= \begin{bmatrix} 0 & 0 \\ \gamma V_1 L (I_p + S_1) & -V_1 \end{bmatrix} \begin{bmatrix} \omega(I_P - B) & \omega U \\ \omega L & \omega(I_q - C) \end{bmatrix} x \\
&= \begin{bmatrix} 0 & 0 \\ \gamma V_1 L (I_p + S_1) & -V_1 \end{bmatrix} \overline{P}_1^{-1} \omega \overline{P}_1 H x \\
&= (1-\mu) \begin{bmatrix} 0 & 0 \\ \gamma V_1 L (I_p + S_1) & -V_1 \end{bmatrix} \begin{bmatrix} (I_P + S_1)^{-1} & 0 \\ -K_1 (I_P + S_1)^{-1} & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ \gamma \overline{L}_1 & I_q \end{bmatrix} x \\
&= (\mu - 1) \begin{bmatrix} 0 & 0 \\ \gamma V_1 K_1 (I_p - B) - V_1 K_1 (I_P + S_1)^{-1} & V_1 \end{bmatrix} x.
\end{aligned}$$

By the assumptions, $0 < \omega \leq 1$, $0 \leq \gamma < 1$, K_1, V_1 and S_1 are nonnegative and nonzero matrices. Moreover, note that $V_1 K_1 = 0$, thus $\gamma V_1 K_1 (I_p - B) - V_1 K_1 (I_P + S_1)^{-1} = 0$, we have

$$\begin{bmatrix} 0 & 0 \\ 0 & V_1 \end{bmatrix} x \geq 0 \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & V_1 \end{bmatrix} x \neq 0.$$

If $\mu < 1$, then $\widehat{L}_{\omega\gamma 1} x - \mu x \leq 0$ and $\widehat{L}_{\omega\gamma 1} x - \mu x \neq 0$, hence, Lemma 2.2 gives $\rho(\widehat{L}_{\omega\gamma 1}) < \rho(\overline{L}_{\omega\gamma 1}) < 1$. □

Comparing $\rho(\widehat{L}_{\omega\gamma 2})$ with $\rho(\overline{L}_{\omega\gamma 2})$, we can deduce the following comparison result.

Theorem 4.6. *Let $\overline{L}_{\omega\gamma 2}$ and $\widehat{L}_{\omega\gamma 2}$ be the iteration matrices of the GAOR methods (3.10) and (3.13), respectively. Assume that the matrix H in Equation (1.1) is irreducible with $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $0 < \omega \leq 1$, $0 \leq \gamma < 1$, $b_{i,i+1} > 0$*

for some $i \in \{1, \cdots, p-1\}$, $b_{ii} \geq 1$ *for* $i = 2, \cdots, p$, *and* $c_{j,j+1} > 0$ *for some* $j \in \{1, \cdots, q-1\}$, *and* $l_{ii} < 0$ *for* $i \in \{1, 2, \cdots, p\}$, *then*

$$\rho(\widehat{L}_{\omega\gamma 2}) < \rho(\overline{L}_{\omega\gamma 2}), \ \ if \quad \rho(\overline{L}_{\omega\gamma 2}) < 1.$$

Proof. Let

$$\bar{P}_2 = \begin{bmatrix} I_p + S_2 & 0 \\ K_2 & I_q \end{bmatrix}$$

and

$$\bar{L}_{\omega\gamma 2} = \begin{bmatrix} (1-\omega)I_p + \omega B_2^* & -\omega U_2^* \\ \omega(\gamma-1)\overline{L}_2 - \omega\gamma\overline{L}_2 B_2^* & (1-\omega)I_q + \omega\overline{C}_2 + \omega\gamma\overline{L}_2 U_2^* \end{bmatrix}.$$

It is easy to show that $\bar{L}_{\omega\gamma 2}$, $\overline{L}_{\omega\gamma 2}$ and $\widehat{L}_{\omega\gamma 2}$ are irreducible and non-negative under the assumptions.

Firstly, let us show that the inequality $\rho(\bar{L}_{\omega\gamma 2}) < \rho(\overline{L}_{\omega\gamma 2})$ holds if $\rho(\overline{L}_{\omega\gamma 2}) < 1$. It follows from Lemma 2.1 that there is a positive vector x such that

$$\overline{L}_{\omega\gamma 2} x = \eta x, \tag{4.8}$$

where $\eta = \rho(\overline{L}_{\omega\gamma 2})$ and $\eta \neq 1$. Moreover, it holds that

$$\omega\overline{P}_2 H x = \begin{bmatrix} I_p & 0 \\ \gamma\overline{L}_2 & I_q \end{bmatrix} (I_n - \overline{L}_{\omega\gamma 2})x = (1-\eta)\begin{bmatrix} I_p & 0 \\ \gamma\overline{L}_2 & I_q \end{bmatrix} x. \tag{4.9}$$

From (4.8) and (4.9), we get that

$$\begin{aligned}
&\bar{L}_{\omega\gamma 2}x - \eta x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma\overline{L}_2 & I_q \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\omega I_p + \omega\ B_2^* & -\omega U_2^* \\ -\omega\overline{L}_2 & -\omega I_q + \omega\overline{C}_2 \end{bmatrix} + (1-\eta)\begin{bmatrix} I_p & 0 \\ \gamma\overline{L}_2 & I_q \end{bmatrix} \right\} x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma\overline{L}_2 & I_q \end{bmatrix}^{-1} \left\{ -\omega\overline{P}_2 H + \begin{bmatrix} S_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\omega I_p + \omega B & -\omega U \\ 0 & 0 \end{bmatrix} + (1-\eta)\begin{bmatrix} I_p & 0 \\ \gamma\overline{L}_2 & I_q \end{bmatrix} \right\} x \\
&= \begin{bmatrix} I_p & 0 \\ \gamma\overline{L}_2 & I_q \end{bmatrix}^{-1} \begin{bmatrix} S_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\omega I_p + \omega B & -\omega U \\ -\omega\overline{L}_2 & -\omega I_q + \omega\overline{C}_2 \end{bmatrix} x \\
&= (1-\eta)\begin{bmatrix} -S_2 & 0 \\ \gamma\overline{L}_2 S_2 & 0 \end{bmatrix} x.
\end{aligned}$$

Under the assumptions, we know that $\begin{bmatrix} -S_2 & 0 \\ \gamma\overline{L}_2 S_2 & 0 \end{bmatrix} x \leq 0$ and $\begin{bmatrix} -S_2 & 0 \\ \gamma\overline{L}_2 S_2 & 0 \end{bmatrix} x \neq 0$, so when $\eta < 1$, from Lemma 2.2, we have

$$\rho(\bar{L}_{\omega\gamma 2}) < \rho(\overline{L}_{\omega\gamma 2}). \tag{4.10}$$

Secondly, let us show that the inequality $\rho(\widehat{L}_{\omega\gamma 2}) < \rho(\bar{L}_{\omega\gamma 2})$ holds if $\rho(\bar{L}_{\omega\gamma 2}) < 1$. By Lemma 2.1, there is a positive vector z such that

$$\bar{L}_{\omega\gamma 2} z = \zeta z, \tag{4.11}$$

where $\zeta = \rho(\bar{L}_{\omega\gamma 2})$ and $\zeta \neq 1$. Moreover, it holds that

$$\omega \bar{P}_2 H z = \begin{bmatrix} I_p & 0 \\ \gamma \bar{L}_2 & I_q \end{bmatrix} (I_n - \bar{L}_{\omega\gamma 2}) z = (1-\zeta) \begin{bmatrix} I_p & 0 \\ \gamma \bar{L}_2 & I_q \end{bmatrix} z. \tag{4.12}$$

Note that $\begin{bmatrix} 0 & 0 \\ V_2 K_2 (I_p - B) & V_2 K_2 U \end{bmatrix} \leq 0$, from (4.11) and (4.12), we can deduce that

$$\begin{aligned}
&\widehat{L}_{\omega\gamma 2} z - \zeta z \\
&= \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_2 & I_q \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\omega I_p + \omega \widehat{B}_2 & -\omega \widehat{U}_2 \\ -\omega \widehat{L}_2 & -\omega I_q + \omega \widehat{C}_2 \end{bmatrix} + (1-\zeta) \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_2 & I_q \end{bmatrix} \right\} z \\
&= \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_2 & I_q \end{bmatrix}^{-1} \left\{ -\omega \bar{P}_2 H + \omega \begin{bmatrix} 0 & 0 \\ -V_2 L & -V_2 (I_q - C) \end{bmatrix} + (1-\zeta) \begin{bmatrix} I_p & 0 \\ \gamma \widehat{L}_2 & I_q \end{bmatrix} \right\} z \\
&= \begin{bmatrix} I_p & 0 \\ -\gamma \widehat{L}_2 & I_q \end{bmatrix} \left\{ (1-\zeta) \begin{bmatrix} 0 & 0 \\ -\gamma V_2 K_2 (I_p - B) & -V_2 \end{bmatrix} + \omega \begin{bmatrix} 0 & 0 \\ V_2 K_2 (I_p - B) & V_2 K_2 U \end{bmatrix} \right\} z \\
&\leq (1-\zeta) \begin{bmatrix} I_p & 0 \\ -\gamma \widehat{L}_2 & I_q \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\gamma V_2 K_2 (I_p - B) & -V_2 \end{bmatrix} y \\
&= (1-\zeta) \begin{bmatrix} 0 & 0 \\ -\gamma V_2 K_2 (I_p - B) & -V_2 \end{bmatrix} z.
\end{aligned}$$

Under the assumptions, we have

$$\begin{bmatrix} 0 & 0 \\ -\gamma V_2 K_2 (I_p - B) & -V_2 \end{bmatrix} z \leq 0 \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ -\gamma V_2 K_2 (I_p - B) & -V_2 \end{bmatrix} z \neq 0,$$

so when $\zeta < 1$, Lemma 2.2 gives

$$\rho(\widehat{L}_{\omega\gamma 2}) < \rho(\bar{L}_{\omega\gamma 2}). \tag{4.13}$$

Finally, combining the inequalities (4.10) and (4.13), we get that $\rho(\widehat{L}_{\omega\gamma 2}) < \rho(\overline{L}_{\omega\gamma 2})$ if $\rho(\overline{L}_{\omega\gamma 2}) < 1$. □

The comparison results in Theorems 4.1–4.6 show the effectiveness of the proposed preconditioners $\widehat{P}_1$ and $\widehat{P}_2$ in this paper. More precisely, Theorems 4.1–4.2 illustrate that the proposed preconditioned GAOR methods are superior to the original GAOR method, Theorems 4.3–4.6 indicate that the proposed preconditioners $\widehat{P}_1$ and $\widehat{P}_2$ are more efficient than the corresponding preconditioners in [6, 16, 17].

5. Numerical example

In this section, an example with numerical experiments is given to confirm the theoretical results.

Example 5.1. The coefficient matrix H in Equation (1.1) is given by

$$H = \begin{bmatrix} I_p - B & U \\ L & I_q - C \end{bmatrix}$$

where $B = (b_{ij}) \in R^{p\times p}$, $C = (c_{ij}) \in R^{q\times q}$, $L = (l_{ij}) \in R^{q\times p}$, and $U = (u_{ij}) \in R^{p\times q}$ with

$$\begin{aligned}
b_{ii} &= \frac{1}{10(i+1)}, \quad 1 \le i \le p, \\
b_{ij} &= \frac{1}{30} - \frac{1}{30j+i}, \quad 1 \le i < j \le p, \\
b_{ij} &= \frac{1}{30} - \frac{1}{30(i-j+1)+i}, \quad 1 \le j < i \le p, \\
c_{ii} &= \frac{1}{10(p+i+1)}, \quad 1 \le i \le q, \\
c_{ij} &= \frac{1}{30} - \frac{1}{30(p+j)+p+i}, \quad 1 \le i < j \le q, \\
c_{ij} &= \frac{1}{30} - \frac{1}{30(i-j+1)+p+i}, \quad 1 \le j < i \le q, \\
l_{ij} &= \frac{1}{30(p+i-j+1)+p+i} - \frac{1}{30}, \quad 1 \le i \le q,\ 1 \le j \le p, \\
u_{ij} &= \frac{1}{30(p+j)+i} - \frac{1}{30}, \quad 1 \le i \le p,\ 1 \le j \le q.
\end{aligned}$$

Table 1 displays the spectral radii of the corresponding iteration matrices with some randomly chosen parameters ω, γ, p and q. The randomly chosen parameters $\beta_i = \tau_i = \theta = 0.5$ satisfy the conditions of Theorems 4.1–4.6, all computations were obtained with the help of MATLAB 7.

Table 1. Spectral radii of GAOR and preconditioned GAOR iteration matrices

n	5	10	15	20	25	30
p	3	5	8	10	12	16
ω	0.6	0.85	0.9	0.95	0.5	0.6
γ	0.8	0.95	0.7	0.85	0.8	0.9
$\rho(L_{\omega\gamma})$	0.45736791	0.30767282	0.4169924	0.51628091	0.83506446	0.91224842
$\rho(\widetilde{L}_{\omega\gamma 1})$	0.4543861	0.30184813	0.41173503	0.51123732	0.83324585	0.91118683
$\rho(\widetilde{L}_{\omega\gamma 2})$	0.45033466	0.29148393	0.40154745	0.5011917	0.82960536	0.90906632
$\rho(\overline{L}_{\omega\gamma 1})$	0.45072737	0.30252013	0.41227847	0.51285524	0.83395616	0.91157453
$\rho(\overline{L}_{\omega\gamma 2})$	0.45350997	0.30543023	0.41504625	0.51520332	0.83475041	0.91210154
$\rho(\widehat{L}_{\omega\gamma 1})$	0.44987408	0.29940948	0.40969232	0.51011698	0.83292189	0.91103589
$\rho(\widehat{L}_{\omega\gamma 2})$	0.44763716	0.28740331	0.39595058	0.4967042	0.82799786	0.90816138

From Table 1, we can see that these numerical results are consistent with the conclusions of Theorems 4.1–4.6. It should be remarked that in Theorem 4.3 and 4.6, the conditions of $b_{ii} = 1$ and $b_{ii} \geq 1$ such that the corresponding comparison results hold, however, from the numerical results, we can see that the corresponding comparison results still hold even without these conditions.

6. Conclusions

In this paper, new preconditioned GAOR methods are proposed for solving a class of 2×2 block structure linear systems. Comparison results show that the convergence rates of the new preconditioned GAOR methods are better than those of the preconditioned GAOR methods in the previous literatures whenever these methods are convergent. A numerical example is given to confirm our theoretical results.

Acknowledgements

The authors would like to thank the editors and reviewers for their valuable comments, which greatly improved the readability of this paper.

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