COMPLETE INVARIANT FUZZY METRICS ON SEMIGROUPS AND GROUPS*

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Abstract In this paper, we study the Raǐkov completion of invariant fuzzy metric groups and complete fuzzy metric semigroups (in the sense of Kramosil and Michael). We establish that: (1) if \((G, M, *)\) is a fuzzy metric group such that \((M, *)\) is invariant, then the Raǐkov completion \(\rho G\) of \((G, \tau_M)\) is a fuzzy metric group \((\rho G, \tilde{M}, *)\) such that \((\tilde{M}, *)\) is invariant on \(\rho G\) and \(\tilde{M}(G \times G \times [0, \infty)) = M\); (2) if \((G, M, *)\) is a fuzzy metric semigroup such that \((M, *)\) is invariant, then a fuzzy metric completion \((\tilde{G}, \tilde{M}, *)\) of \((G, M, *)\) is a fuzzy metric semigroup and \((\tilde{M}, *)\) is invariant.

Keywords Fuzzy metric, topological group, topological semigroup, Raǐkov completion.


1. Introduction

In 1975, Kramosil and Michalek introduced a notion of metric fuzziness [10], which become an important tool in fuzzy normed spaces (among others, the interested reader can consult [2,4,9,11]). In fuzzy Topological Algebra, fuzzy metric topological groups are considered (see [8,12]). In [5], Gregori and Romaguera remove the symmetric condition in the definition of a fuzzy metric (in the sense of Kramosil and Michalek). This allows us to consider nonsymmetric structures which fit in the realm of fuzzy nonsymmetric topology. For example, fuzzy quasi-metric spaces and fuzzy quasi-normed spaces (see [2,6,7]). Recently, Sánchez and Sanchis [14] found sufficient conditions in order that a topological algebraic structure (in particular a nonsymmetric structure) become a stronger topological structure (in particular, a symmetric structure). For example:

**Theorem 1.1** ([14, Theorem 3.2]). If \((G, M, *)\) is a fuzzy pseudometric right topological group such that \((M, *)\) is left-invariant, then \((G, M, *)\) is a fuzzy topological group.

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*The author was supported by NSFC (Nos. 11601393, 11861018), the Natural Science Foundation of Guangdong Province under Grant (Nos. 2018A030313063 and 2021A1515010381), the Innovation Project of Department of Education of Guangdong Province(No. 2018KTSCX231) and the Jiangmen science and technology plan projects (No. 2020JC01039).
Recently, Sánchez and Sanchis studied complete invariant fuzzy metrics (in the sense of Kramosil and Michalek) on groups. They proved that:

**Theorem 1.2** ([13, Theorem 2.2]). If \((G, M, *)\) is a fuzzy metric group such that \((M, *)\) is invariant, then a fuzzy metric completion \((\tilde{G}, \tilde{M}, \tilde{*})\) of \((G, M, *)\) is a fuzzy metric group and \((M, *)\) is invariant.

In this paper, we consider the following two questions: (1) Let \((G, M, *)\) be a fuzzy metric group such that \((M, *)\) is invariant. Can the invariant fuzzy metric \((M, *)\) extend on \(\rho G\)? \(
\tilde{G}\) is the Raïkov completion of \((G, \tau_M)\), where the topology \(\tau_M\) is induced by the fuzzy metric \((M, *)\) on \(G\). (2) Does Theorem 1.2 hold for fuzzy metric semigroups?

We shall answer the two questions above. Firstly, some notations and definitions are stated.

Recall that a binary operation \(* : [0, 1] \times [0, 1] \rightarrow [0, 1]\) is a continuous \(t\)-norm [15] if * satisfies the following conditions: (i) * is associative and commutative; (ii) * is continuous; (iii) \(a * 1 = a\) for all \(a \in [0, 1]\) and (iv) \(a * b \leq c * d\) whenever \(a \leq c\) and \(b \leq d\), with \(a, b, c, d \in [0, 1]\).

Three paradigmatic examples of continuous \(t\)-norms are \(\wedge, \cdot\) and \(*_L\) (the Łukasiewicz \(t\)-norm), which are defined by \(a \wedge b = \min\{a, b\}, a \cdot b = ab\) and \(a *_L b = \max\{a + b - 1, 0\}\), respectively. It is well known that \(* \leq \wedge\) for every continuous \(t\)-norm *.

**Definition 1.1** ([10]). A fuzzy metric (in the sense of Kramosil and Michalek) on a set \(X\) is a pair \((M, *)\) such that \(M\) is a fuzzy set in \(X \times X \times [0, \infty)\) and * is a continuous \(t\)-norm satisfying for all \(x, y, z \in X\):

(i) \(M(x, y, 0) = 0\);
(ii) \(M(x, y, t) = 1\) for all \(t > 0\) if and only if \(x = y\);
(iii) \(M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)\) for all \(t, s > 0\);
(iv) \(M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]\) is a left continuous function;
(v) \(M(x, y, t) = M(y, x, t)\).

**Definition 1.2** ([14, Definition 2.4]). A fuzzy metric \((M, *)\) on a semigroup \(G\) is left-invariant (respectively, right-invariant) if \(M(x, y, t) = M(ax, ay, t)\) (respectively, \(M(x, y, t) = M(xa, ya, t)\)) whenever \(a, x, y \in G\) and \(t > 0\). We say that \((M, *)\) is invariant if it is both left-invariant and right-invariant.

By a fuzzy metric space we mean a triple \((X, M, *)\) such that \(X\) is a set and \((M, *)\) is a fuzzy metric on \(X\). Every fuzzy metric \((M, *)\) on a set \(X\) induces a topology \(\tau_M\) on \(X\), which has as a base the family of open sets of the form \(\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}\), where \(B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}\) for all \(x \in X, \varepsilon \in (0, 1), t > 0\).

By a fuzzy metric group (resp., fuzzy metric semigroup) we mean a 4-tuple \((G, M, *)\) such that \((G, M, *)\) is a fuzzy metric space and \((G, \tau_M)\) is a topological group (resp., topological semigroup).

A sequence \((x_n)_{n \in \mathbb{N}}\) in a fuzzy metric space \((X, M, *)\) is said to be a Cauchy sequence provided that for each \(\varepsilon \in (0, 1)\) and \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(M(x_n, x_m, t) > 1 - \varepsilon\) for every \(n, m \geq n_0\). A fuzzy metric space \((X, M, *)\) where every Cauchy sequence converges are called completeness.
2. Main results

A filter on a set $X$ is a family $\eta$ of non-empty subsets of $X$ satisfying the next two conditions: (i) If $U$ and $V$ are in $\eta$, then $U \cap V$ is also in $\eta$; (ii) If $U \in \eta$ and $U \subseteq W \subseteq X$, then $W \in \eta$.

Let $G$ be a topological group with the identity $e$. A filter $\eta$ of a topological group $G$ is said to be a Cauchy filter if for every open neighbourhood $V$ of $e$ in $G$, there exist $a, b \in G$ and $A, B \in \eta$ such that $A \subseteq aV$ and $B \subseteq Vb$. A topological group $G$ such that every Cauchy filter on $G$ converges is called Raïkov complete. Next we shall investigate the Raïkov complete of the group topologies induced by $G$.

**Proposition 2.1.** Let $(G, M, \ast)$ be a fuzzy metric group. If $(G, M, \ast)$ is complete, then $(G, \tau_M)$ is Raïkov complete.

**Proof.** Suppose that $(G, M, \ast)$ is complete. Take an arbitrary Cauchy filter $\eta$ in $G$. Then for each $n \in \mathbb{N}$ there are $F'_n \in \eta$ and $x_n \in G$ such that $F'_n \subseteq B_M(x_n, \frac{1}{n}, \frac{1}{n})$. Put $F_n = \bigcap_{i=n}^{\infty} F'_i$. Clearly, $F_n \subseteq B_M(x_n, \frac{1}{n}, \frac{1}{n})$ holds for each $n \in \mathbb{N}$. Take $y_n \in F_n$ for each $n \in \mathbb{N}$. Then the sequence $\{y_n\}$ is a Cauchy sequence in $(G, M, \ast)$.

In fact, for each $n \in \mathbb{N}$, since $t$-norm $\ast$ is continuous, there is $n_0 \in \mathbb{N}$ such that $(1 - \frac{1}{n_0}) \ast (1 - \frac{1}{n_0}) > (1 - \frac{1}{n})$ and $\frac{1}{n_0} \leq \frac{1}{2n}$. Clearly, $y_i, y_j \in F_{n_0} \subseteq B_M(x_{n_0}, \frac{1}{n_0}, \frac{1}{n_0})$ whenever $i, j \in \mathbb{N}$ and $i, j > n_0$. Thus

$$M(y_i, y_j, \frac{1}{n}) \geq M(y_i, x_{n_0}, \frac{1}{2n}) \ast M(x_{n_0}, y_j, \frac{1}{2n})$$

$$\geq M(y_i, x_{n_0}, \frac{1}{n_0}) \ast M(x_{n_0}, y_j, \frac{1}{n_0}) \geq (1 - \frac{1}{n_0}) \ast (1 - \frac{1}{n_0}) > (1 - \frac{1}{n})$$

whenever $i, j > n_0$. This show that $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Since $(G, M, \ast)$ is complete, $\{y_n\}_{n \in \mathbb{N}}$ converges to some $y \in G$. We shall show that the Cauchy filter $\eta$ converges to $y$, which implies that $(G, \tau_M)$ is Raïkov complete.

Take any open neighbourhood $V$ of $y$. Without loss of generality, we assume that $V = B_M(y, \frac{1}{n}, \frac{1}{n})$. Since $t$-norm $\ast$ is continuous, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \leq \frac{1}{2n}$ and $(1 - \frac{1}{n}) \ast (1 - \frac{1}{n}) > (1 - \frac{1}{n_0})$. Note that $\{y_i\}_{i \in \mathbb{N}}$ converges to $y$, then there is $n' \in \mathbb{N}$ such that $\frac{1}{2n} \geq \frac{1}{n_0}, (1 - \frac{1}{n}) \ast (1 - \frac{1}{n}) > (1 - \frac{1}{n_0})$ and $M(y, y_{n'}, \frac{1}{n_0}) \geq (1 - \frac{1}{n_0}) \ast (1 - \frac{1}{n_0}) > (1 - \frac{1}{n_0})$.

Then for each $x \in F_{n'}$, noting that $y_{n'} \in F_{n'} \subseteq B_M(x_{n'}, \frac{1}{n_0}, \frac{1}{n_0})$, we have

$$M(y, x, \frac{1}{n}) \geq M(y, y_{n'}, \frac{1}{2n}) \ast M(y_{n'}, x, \frac{1}{2n}) \geq M(y, y_{n'}, \frac{1}{n_0}) \ast M(y_{n'}, x, \frac{2}{n'})$$

$$\geq M(y, y_{n'}, \frac{1}{n_0}) \ast M(y_{n'}, x, \frac{1}{n'}) \ast M(x_{n'}, x, \frac{1}{n'})$$

$$\geq (1 - \frac{1}{n_0}) \ast (1 - \frac{1}{n'}) \ast (1 - \frac{1}{n}) \geq (1 - \frac{1}{n_0}) \ast (1 - \frac{1}{n_0}) \geq (1 - \frac{1}{n}).$$

This implies that $x \in B_M(y, \frac{1}{n}, \frac{1}{n'})$, i.e., $F_{n'} \subseteq B_M(y, \frac{1}{n}, \frac{1}{n'})$. Clearly, $F_{n'} \in \eta$, thus we have proved that $\eta$ converges to $y$. $\square$

**Theorem 2.1.** If $(G, M, \ast)$ is a fuzzy metric group such that $(M, \ast)$ is invariant, then the Raïkov completion $\bar{G}$ of $(G, \tau_M)$ is a fuzzy metric group $(\bar{G}, \bar{M}, \ast)$ such that $(\bar{M}, \ast)$ is invariant on $\bar{G}$ and $\bar{M}_{(G \times G \times [0, \infty)} = M$. 

Proof. Let \( (\hat{G}, \hat{M}, \ast) \) be a fuzzy metric completion of \((G, M, \ast)\). Then according to Theorem 1.2, \((\hat{G}, \hat{M}, \ast)\) is a fuzzy metric group \((\hat{G}, \hat{M}, \ast)\) satisfying: \((\hat{M}, \ast)\) is invariant on \(\hat{G}\) and \(\hat{M}_{|G \times G \times [0, \infty)} = M\). Then according to Proposition 2.1 it follows that \((\hat{G}, \tau_{\hat{M}})\) is Raïkov complete and \(G\) is a dense subgroup in \(\hat{G}\). Then according to [3, Theorem 3.6.14] there is a topological isomorphism \(\varphi : \varrho G \to \hat{G}\) such that \(\varphi(g) = g\) for each \(g \in G\). Thus we can define \(\hat{M} : \varrho G \times \varrho G \times [0, \infty) \to [0, 1]\) as following: \(\hat{M}(x, y, t) = \hat{M}(\varphi(x), \varphi(y), t)\) for each \((x, y, t) \in \varrho G \times \varrho G \times [0, \infty)\). One easily show that fuzzy metric \((\hat{M}, \ast)\) is required. This completes the proof.

\[ \square \]

**Proposition 2.2.** Let \((G, M, \ast)\) be a fuzzy metric group with \((M, \ast)\) being invariant. If \((G, \tau_{\hat{M}})\) is Raïkov complete, then \((G, M, \ast)\) is complete.

Proof. Suppose that \((G, \tau_{\hat{M}})\) is Raïkov complete. Take arbitrary Cauchy sequence \(\{x_n\}_{n \in \mathbb{N}}\) of \((G, M, \ast)\). Put \(\eta = \{A \subseteq G : F_n \subseteq A\) for some \(F_n\}, \) where \(F_n = \{x_i : i \geq n\}\) for each \(n \in \mathbb{N}\). Now we shall prove that \(\eta\) is a Cauchy filter of \(G\). Take any \(B_M(e, \frac{1}{n}, \frac{1}{n})\), where \(e\) is the identity of \(G\). Since \(\{x_i : i \in \mathbb{N}\}\) is a Cauchy sequence, there is \(n_0 \in \mathbb{N}\) such that \(M(x_k, x_m, \frac{1}{n}) > 1 - \frac{1}{n}\) whenever \(k, m \geq n_0\). This implies that \(x_k \in B(x_{n_0}, \frac{1}{n}, \frac{1}{n})\) whenever \(k \geq n_0\). Hence, \(F_{n_0} \subseteq B(x_{n_0}, \frac{1}{n}, \frac{1}{n})\). Noting that \(M\) is invariant, so \(B_M(x_{n_0}, \frac{1}{n}, \frac{1}{n}) = x_{n_0}B_M(e, \frac{1}{n}, \frac{1}{n}) = B_M(e, \frac{1}{n}, \frac{1}{n})x_{n_0}\), so \(F_{n_0} \subseteq x_{n_0}B_M(e, \frac{1}{n}, \frac{1}{n})\) and \(F_{n_0} \subseteq B_M(e, \frac{1}{n}, \frac{1}{n})x_{n_0}\). This implies that \(\eta\) is a Cauchy filter. Since \((G, \tau_{\hat{M}})\) is Raïkov complete, the Cauchy filter \(\eta\) converges to a point \(g\) in \(G\). Then one can easily show that the Cauchy sequence \(\{x_n\}_{n \in \mathbb{N}}\) converges to \(g\). This implies that \((G, M, \ast)\) is complete.

By Theorem 2.1 and Proposition 2.2 we have the following:

**Corollary 2.1.** Let \((G, M, \ast)\) be a fuzzy metric group such that \((M, \ast)\) is invariant on \(G\). Then \((G, M, \ast)\) is complete if and only if \((G, \tau_{\hat{M}})\) is Raïkov complete.

Since every Abelian fuzzy metric group \((G, M, \ast)\) satisfies that \((M, \ast)\) is invariant on \(G\), by Corollary 2.1 we have the following:

**Corollary 2.2.** Every Abelian fuzzy metric group \((G, M, \ast)\) is complete if and only if \((G, \tau_{\hat{M}})\) is Raïkov complete.

Next we shall show that Theorem 1.2 holds for semigroups.

**Theorem 2.2.** If \((G, M, \ast)\) is a fuzzy metric semigroup such that \((M, \ast)\) is invariant, then a fuzzy metric completion \((\hat{G}, \hat{M}, \ast)\) of \((G, M, \ast)\) is a fuzzy metric semigroup and \((\hat{M}, \ast)\) is invariant.

Proof. Firstly, we shall prove the following Claim 1.

**Claim 1.** Let \((a_n)_{n}\) and \((b_n)_{n}\) be Cauchy sequences in \(G\). Then \((a_nb_n)_{n}\) is also a Cauchy sequence in \(G\).

Fix \(\epsilon \in (0, 1)\) and \(t > 0\). Since \(\ast\) is a continuous \(t\)-norm, there is \(s > 0\) such that \((1 - s) \ast (1 - s) > 1 - \epsilon\). Observing \((a_n)_{n}\) and \((b_n)_{n}\) are Cauchy sequences in \(G\), so there exists \(n_0 \in \mathbb{N}\) such that \(M(a_n, a_m, \frac{1}{2}) > 1 - s\) and \(M(b_n, b_m, \frac{1}{2}) > 1 - s\) whenever \(n, m > n_0\). Since \(M\) is invariant in \(G\), we have

\[
M(a_n b_n, a_m b_m, t) \geq M(a_n b_n, a_m b_n, \frac{t}{2}) \ast M(a_m b_n, a_m b_n, \frac{t}{2}) = M(a_n, a_m, \frac{t}{2}) \ast M(b_n, b_m, \frac{t}{2}) > (1 - s) \ast (1 - s) > 1 - \epsilon
\]
whenever \( n, m > n_0 \).

This implies that \((a_n, b_n)_n\) is a Cauchy sequence in \( G \).

Now we define a binary operation \( \cdot \) on \( \widetilde{G} \) as follows: \((a_n)_n \cdot (b_n)_n = (a_n b_n)_n\) for each pair Cauchy sequences \((a_n)_n\) and \((b_n)_n\) in \( G \). Let us show that \( \cdot \) is well defined. According to Claim 1 it is enough to show the following Claim 2.

**Claim 2.** Let \((a_n)_n, (b_n)_n\) and \((a_n')_n, (b_n')_n\) be Cauchy sequences in \( G \) such that \( \widetilde{M}((a_n)_n, (a_n')_n, t) = \lim_{n \to \infty} M(a_n, a_n', t) = 1 \) and \( \widetilde{M}((b_n)_n, (b_n')_n, t) = \lim_{n \to \infty} M(b_n, b_n', t) = 1 \) for all \( t > 0 \). Then \( \widetilde{M}((a_n)_n, (a_n')_n, (b_n)_n, (b_n')_n, t) = \lim_{n \to \infty} M(a_n, a_n', b_n, b_n', t) = 1 \) for all \( t > 0 \).

Fix \( t > 0 \), taking \( \varepsilon \in (0, 1) \). Then there exists \( \varepsilon' \in (0, 1) \) such that \((1 - \varepsilon') * (1 - \varepsilon') > 1 - \varepsilon \). For \( \varepsilon' \), since \( \lim_{n \to \infty} M(a_n, a_n', t) = 1 \) and \( \lim_{n \to \infty} M(b_n, b_n', t) = 1 \), there is \( n_0 \in \mathbb{N} \) such that \( M(b_n, b_n', t) > 1 - \varepsilon \) whenever \( n > n_0 \).

Observing \((M, \cdot)\) is invariant, so we have:

\[
M(a_n, a_n', b_n, b_n', t) = M(a_n, a_n', b_n, b_n', t) > (1 - \varepsilon') * (1 - \varepsilon') > 1 - \varepsilon
\]

whenever \( n > n_0 \). This implies that \( \lim_{n \to \infty} M(a_n, a_n', b_n, b_n', t) = 1 \) for all \( t > 0 \). Thus, the binary operation \( \cdot \) is well defined.

Since \( G \) is a semigroup, one can easily show that \(((a_n)_n \cdot (b_n)_n) \cdot (c_n)_n = (a_n)_n \cdot (b_n)_n \cdot (c_n)_n\). Thus \((\widetilde{G}, \cdot)\) is a semigroup. Now, we shall show that \((\widetilde{M}, \cdot)\) is invariant on \((\widetilde{G}, \cdot)\).

Since \((M, \cdot)\) is invariant, we have:

\[
\widetilde{M}((a_n)_n, (b_n)_n, (c_n)_n, (d_n)_n, t) = \lim_{n \to \infty} M(a_n, b_n, c_n, d_n, t) = \widetilde{M}((a_n)_n, (b_n)_n, t) \cdot \widetilde{M}((c_n)_n, (d_n)_n, t)
\]

This implies that \((\widetilde{M}, \cdot)\) is left invariant on \((\widetilde{G}, \cdot)\). Similarly, one can show that \((\widetilde{M}, \cdot)\) is right invariant on \((\widetilde{G}, \cdot)\). Thus \((\widetilde{M}, \cdot)\) is invariant.

Finally, we shall show that \((\widetilde{G}, \widetilde{M}, \cdot)\) is a fuzzy metric semigroup. Let \((a_n)_n\) and \((b_n)_n\) be Cauchy sequences in \( G \). Take any open neighborhood \( U \) of \((a_n)_n \cdot (b_n)_n\). Then there exists \( \varepsilon \in (0, 1) \) and \( t > 0 \) such that \( B_{\widetilde{M}}((a_n)_n \cdot (b_n)_n, \varepsilon, t) \subseteq U \). Since \( t \)-norm is continuous, there exists \( \varepsilon' \in (0, 1) \) such that \((1 - \varepsilon') * (1 - \varepsilon') > 1 - \varepsilon \). Now we claim that:

\[
B_{\widetilde{M}}((a_n)_n, \varepsilon, t) \cdot B_{\widetilde{M}}((b_n)_n, \varepsilon', t) \subseteq B_{\widetilde{M}}((a_n)_n \cdot (b_n)_n, \varepsilon, t).
\]

This implies that the binary operation \( \cdot \) is joint continuous on \((\widetilde{G}, \widetilde{M}, \cdot)\). Thus, \((\widetilde{G}, \widetilde{M}, \cdot)\) is a fuzzy metric semigroup.

In fact, take any \((c_n)_n \in B_{\widetilde{M}}((a_n)_n, \varepsilon', \frac{t}{2})\) and \((c_n')_n \in B_{\widetilde{M}}((b_n)_n, \varepsilon', \frac{t}{2})\). Then:

\[
\widetilde{M}((a_n)_n \cdot (b_n)_n, (c_n)_n \cdot (c_n')_n, t) = \lim_{n \to \infty} M(a_n b_n, c_n c_n', t) \cdot \lim_{n \to \infty} M(a_n b_n, c_n c_n', t)
\]

\[
= \lim_{n \to \infty} M(a_n, c_n, \frac{t}{2}) \cdot M(b_n, c_n', \frac{t}{2}) \cdot \lim_{n \to \infty} M(a_n, c_n, \frac{t}{2}) \cdot M(b_n, c_n', \frac{t}{2})
\]

\[
> (1 - \varepsilon') * (1 - \varepsilon') > 1 - \varepsilon.
\]

This completes the proof. \(\square\)
References


