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POSITIVE AND SIGN-CHANGING SOLUTIONS FOR THE FRACTIONAL KIRCHHOFF EQUATION WITH CRITICAL GROWTH*

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Abstract We are interested in the existence of positive and sign-changing solutions for a fractional Kirchhoff equation. Under some mild conditions on the potentials V and h, using variational methods, we prove the existence of positive ground state solutions and least energy sign-changing solutions.

Keywords Fractional Kirchhoff equation, variational method, positive solution, sign-changing solution.

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1. Introduction and main results

The paper is to study the following fractional Kirchhoff equation

$$\left(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^s u+V(x)u=h(x)|u|^{p-2}u+|u|^{2_s^*-2}u \text{ in } \mathbb{R}^3, \quad (1.1)$$

where $a,b>0,\ s\in(\frac{3}{4},1)$ and $p\in(4,2_s^*)$ with $2_s^*=\frac{6}{3-2s}$. The potential functions $V(x)$ and $h(x)$ can be nonconstant, indefinite in sign and nonradial. Specifically, if we denote $V^-(x):=\max\{-V(x),0\}$, conditions are as follows:

(V_1) $V^-\in L^{\frac{2_s^*}{2_s^*-2}}(\mathbb{R}^3),\ \int_{\mathbb{R}^3}|V^-(x)|^{\frac{2_s^*}{2_s^*-2}}dx<S^{\frac{2_s^*}{2_s^*-2}}$, where S denotes the best Sobolev constant:

$$S:=\inf_{u\in D^{s,2}(\mathbb{R}^3)\setminus\{0\}}\frac{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx}{(\int_{\mathbb{R}^3}|u|^{2_s^*}dx)^{\frac{2}{2_s^*}}};$$

(V_2) There exist $\gamma>0,\ C_v>0$, such that

$$V(x)\le V_\infty-C_ve^{-\gamma|x|} \text{ for a.e. } x\in\mathbb{R}^3, \quad \text{where } V_\infty:=\lim_{|x|\to+\infty}V(x)>0;$$

(h) $h(x)\in C(\mathbb{R}^3)$, there exist $\theta>0,\ C_h>0$, such that

$$h(x)\ge h_\infty-C_he^{-\theta|x|} \text{ for a.e. } x\in\mathbb{R}^3, \quad \text{where } h_\infty:=\lim_{|x|\to+\infty}h(x)>0.$$

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The fractional Laplacian $(-\Delta)^s$ is a nonlocal operator which is defined by

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y = C_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y,$$

where $u \in S(\mathbb{R}^N)$, which stands for the Schwartz space of rapidly decaying C^∞ functions. $P.V.$ denotes the Cauchy principle value and $C_{N,s}$ denotes a normalization constant. This operator arises in the description of various phenomena in applied sciences, such as phase transitions, materials science, conservation laws, minimal surfaces, water waves, optimization, plasma physics and so on, see [5] and references therein for more detailed introduction.

When $s = 1$, problem 1.1 is related to the classical Kirchhoff problem:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

For the existence of sign-changing solutions to Kirchhoff problem like (1.2), we refer to [1, 17, 20] and references therein. For the critical situation, Xu and Chen [21] proved the existence of positive and sign-changing solutions with variational method.

When $a = 1$ and $b = 0$, problem (1.1) is related to the usual fractional Schrödinger problem:

$$(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

As we know, the path integral over Lévy-like quantum mechanics paths allows one to develop a generalization of quantum mechanics; namely, if the path integral over Brownian trajectories leads to the classical Schrödinger equation, then the path integral over Lévy trajectories leads to the fractional Schrödinger equation. The fractional Schrödinger equation is a fundamental equation in the study of particles on stochastic fields modeled by Lévy processes, which occur widely in physics, chemistry and biology. Therefore, the fractional Schrödinger problem like (1.3) has been extensively investigated. Concerning the existence of sign-changing solutions for it, we refer to [11, 13, 18]. Li etc [13] showed that problem (1.3) with $f(x,u) = f(u)$ has a positive ground state solution and a sign-changing solution. When $f(x,u) = |u|^{p-1}u$ where $p \in (1, 2_s^* - 1)$, Wang and Zhou [18] obtained a radial sign-changing solution. For the critical situation, [11] proved the existence of infinitely many non-radial sign-changing solutions.

The usual fractional Kirchhoff problem is as follows:

$$\left(a + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right) (-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

Recently, Fiscella and Valdinoci [7] first proposed a stationary Kirchhoff model involving the fractional Laplacian. Then, many papers have been devoted to studying the existence of solutions for fractional Kirchhoff like equation (1.4), see [6, 9, 15, 16, 22] and the references therein. We must point out that there are a few results on the existence of sign-changing solutions, see [2, 4, 10, 12]. Cheng and Gao [2] used the constraint variational method and quantitative deformation lemma to obtain a least energy nodal solution. Chen etc [4] studied the existence and asymptotic behavior of sign-changing solutions in low dimensions. Luo etc [12] proved a ground state sign-changing solution in bounded domains. The recent one, Isernia [10] used a minimization argument and a quantitative deformation lemma to establish the existence of least energy sign-changing solutions.

To the authors' knowledge, there is no result on the existence of least energy sign-changing solutions for problem (1.1). One of the main difficulties is the presence of nonlocal term $(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx)(-\triangle)^s u$, the other is the critical term that makes the problem complicated due to the lack of compactness.

In order to prove the existence of positive solutions: Firstly, through the Ekeland Variational Principle, we get a Palais-Smale sequence on the Nehari manifold. Later to overcome the problem of the lack of compactness, we use some comparison arguments about the minimax level of energy functional and that of the limit problem, here the conditions $(V_2),(h)$ play the important role. On the other hand, the existence of sign-changing solutions is usually studied on the Nodal manifold. People used to use the method of a finite dimensional space to approximate infinite dimensional space or the quantitative deformation lemma to prove it. However in our paper, we try to seek a minimizer of the energy functional over a manifold $\mathcal{N}^*$, which is a variant of Nodal Nehari manifold. Setting $\|u^{\pm}\|^2=\int_{\mathbb{R}^3}\left(a|(-\Delta)^{\frac{s}{2}}u^{\pm}|^2+V(x)(u^{\pm})^2\right)dx$, the manifold $\mathcal{N}^*$ is as follows:

$$\mathcal{N}^*=\left\{u\in X\backslash\{0\}: f\left(u^+\right)=f\left(u^-\right)=1\right\},$$

where

$$f(u^+)=\frac{\int_{\mathbb{R}^3}h(x)|u^+|^p\mathrm{d}x+\int_{\mathbb{R}^3}|u^+|^{2_s^*}\mathrm{d}x}{\|u^+\|^2+b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u^+|^2\mathrm{d}x},\quad \text{with } u^+:=\max\{u,0\},$$

$$f(u^-)=\frac{\int_{\mathbb{R}^3}h(x)|u^-|^p\mathrm{d}x+\int_{\mathbb{R}^3}|u^-|^{2_s^*}\mathrm{d}x}{\|u^-\|^2+b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u^-|^2\mathrm{d}x},\quad \text{with } u^-:=\min\{u,0\}.$$

Then we show the minimum of the energy functional on the manifold $\mathcal{N}^*$ is the sign-changing solution of problem (1.1).

From the above arguments, to overcome the lack of compactness, we are going to consider the limit problem of (1.1), namely

$$\left(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^s u+V_\infty u=h_\infty|u|^{p-2}u+|u|^{2_s^*-2}u. \tag{1.5}$$

Argued as in [9], we can easily prove the limit problem (1.5) has a positive ground state solution w. Thus if one set $\alpha:=\left(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}w|^2dx\right)^{\frac{1}{2s}}$, we can state the main results of this paper.

Theorem 1.1. *Assume* $(V_1)-(V_2)$, (h) *hold and* $s\in(\frac{3}{4},1)$, $p\in(4,2_s^*)$. *If* $\gamma<\theta<\frac{p\sqrt{V_\infty}}{\alpha}$, *the problem* (1.1) *possesses a positive ground state solution.*

Theorem 1.2. *Assume* $(V_1)-(V_2)$, (h) *hold and* $s\in(\frac{3}{4},1)$, $p\in(4,2_s^*)$. *If* $\gamma<\min\{\theta,\frac{\sqrt{V_\infty}}{\alpha}\}$, $\theta<\frac{p\sqrt{V_\infty}}{\alpha}$, *the problem* (1.1) *possesses a least energy sign-changing solution.*

Remark 1.1. The conditions on the exponent γ and θ are of technical nature, which will appear when trying to localize the minimax level (of the energy functional) in the correct compactness range. Moreover, for sign-changing solution, it needs a stronger restrictions on the exponent γ.

Notations:

• $L^p(\mathbb{R}^3), p\in[1,+\infty)$ is the Lebesgue space with the norm $\|u\|_p=(\int_{\mathbb{R}^3}|u|^p dx)^{\frac{1}{p}}$.
• $D^{s,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ endowed with the norm $\|u\|^2_{D^{s,2}}=\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx$.
• $H^s(\mathbb{R}^3)$ is the usual fractional Sobolev space endowed with the nature norm

$$\|u\|^2_{H^s(\mathbb{R}^3)}=\int_{\mathbb{R}^3}\left(|(-\Delta)^{\frac{s}{2}}u|^2+|u|^2\right)dx.$$

• In this paper, because of the presence of potential V, we denote the fractional Sobolev space for problem (1.1) as follows

$$X=\{u\in H^s(\mathbb{R}^3):\int_{\mathbb{R}^3}\left(a|(-\Delta)^{\frac{s}{2}}u|^2+V(x)u^2\right)dx<\infty\},$$

defined the norm in X by

$$\|u\|^2=\int_{\mathbb{R}^3}\left(a|(-\Delta)^{\frac{s}{2}}u|^2+V(x)u^2\right)dx.$$

By [13], we know X is continuously embedded into $L^p(\mathbb{R}^3)$ for $p\in[2,2_s^*]$.
• $C,\ C_i$ denote various positive constants, which may vary from line to line.

2. Positive solution

The energy functional associated with problem (1.1) is defined by

$$I(u)=\frac{1}{2}\|u\|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)^2-\frac{1}{p}\int_{\mathbb{R}^3}h(x)|u|^pdx-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx,$$

obviously, $I\in C^1(X,\mathbb{R})$ and the critical points of I are the weak solutions of problem (1.1).

Thought this paper, we denote

$$\|u\|^2_\infty:=\int_{\mathbb{R}^3}\left(a|(-\Delta)^{\frac{s}{2}}u|^2+V_\infty u^2\right)dx,$$

the energy functional associated with limit problem (1.5) is given by

$$I_\infty(u)=\frac{1}{2}\|u\|^2_\infty+\frac{b}{4}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)^2-\frac{1}{p}\int_{\mathbb{R}^3}h_\infty|u|^pdx-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx.$$

The functional $I(u)$ and $I_\infty(u)$ are respectively restricted on the following manifold $\mathcal{N},\mathcal{N}_\infty$:

$$\mathcal{N}=\{u\in X\backslash\{0\}:\langle I'(u),u\rangle=0\},\ m=\inf_{u\in\mathcal{N}}I(u).$$
$$\mathcal{N}_\infty=\{u\in X\backslash\{0\}:\langle I'_\infty(u),u\rangle=0\},\ m_\infty=\inf_{u\in\mathcal{N}_\infty}I(u).$$

Lemma 2.1. *The limit problem* (1.5) *has a positive solution $w\in X$ such that $I_\infty(w)=m_\infty$. Moreover, if we set $\alpha:=\left(a+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}w|^2dx\right)^{\frac{1}{2s}}$, then for any $\delta\in(0,\sqrt{V_\infty})$, there exists $C=C(\delta)>0$ such that*

$$w(x)\le Ce^{-\frac{\delta}{\alpha}|x|},\quad \forall\ x\in\mathbb{R}^3.$$

Proof. By (h), the existence of w is similar to section 2 in [9]. It's easy to see $\alpha^{2s} = a + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w|^2 dx$. Because the integral range of $b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w|^2 dx$ is $\mathbb{R}^3$, we know $b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w|^2 dx$ is a constant no matter w is $w(x)$ or $w(\alpha x)$. For any $x \in \mathbb{R}^3$, let $v(x) := w(\alpha x)$, there holds

$$\begin{aligned}(-\Delta)^s v(x) &= \alpha^{2s}(-\Delta)^s w(\alpha x) \\ &= h_\infty |w(\alpha x)|^{p-2} w(\alpha x) + |w(\alpha x)|^{2_s^*-2} w(\alpha x) - V_\infty w(\alpha x).\end{aligned}$$

Thus $(-\Delta)^s v + V_\infty v = h_\infty |v|^{p-2} v + |v|^{2_s^*-2} v$. By [8], we can get $v(x) \in L^\infty(\mathbb{R}^3)$ and $v(x) = w(\alpha x) \to 0$ as $|x| \to \infty$. So for any $0 < \delta < \sqrt{V_\infty}$, there exists $R := R(\delta) > 0$, such that for $|x| \geq R$, we have $V_\infty - h_\infty |v|^{p-2} - |v|^{2_s^*-2} \geq \delta^2$. Then $(-\triangle)^s v + \delta^2 v \leq 0$ for $|x| \geq R$. And there exists $M = M(\delta) > 0$, such that $v(x) \leq M$ for $|x| = R$. Let $\overline{v}(x) = M \cdot e^{-\delta(|x|-R)}$, a direct calculation can drive that $(-\triangle)^s \overline{v} + \delta^2 \overline{v} \geq 0$ for $x \neq 0$. The Maximum Principle implies that $v(x) \leq M \cdot e^{-\delta(|x|-R)}$ for $|x| \geq R$, thus $w(x) \leq Ce^{-\frac{\delta}{\alpha}|x|}$. □

Lemma 2.2. *$\mathcal{N}$ is nonempty and it's a C^1 manifold. Moreover, $m = \inf_{u \in \mathcal{N}} I(u) > 0$.*

Proof. For $n \in \mathbb{N}$, we define $w_n(x) := w(x - x_n)$, where w is given by Lemma 2.1 and $x_n := (0, 0, n)$. Since w is a positive solution, we have $\int_{\mathbb{R}^3} |w_n|^{2_s^*} dx > 0$. So $I(tw_n) > 0$ for $t > 0$ small and $I(tw_n) < 0$ for t large. $I(tw_n)$ achieves its maximum at some $t_n > 0$, thus $\langle I'(t_n w_n), (t_n w_n)\rangle = 0$, which implies $t_n w_n \in \mathcal{N} \neq \emptyset$. For any $u \in \mathcal{N}$,

$$\langle I'(u), u\rangle = \|u\|^2 + b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2 - \int_{\mathbb{R}^3} h(x)|u|^p dx - \int_{\mathbb{R}^3} |u|^{2_s^*} dx = 0.$$

By (h) and Sobolev embedding theorems, it follows that

$$\|u\|^2 \leq \int_{\mathbb{R}^3} h(x)|u|^p dx + \int_{\mathbb{R}^3} |u|^{2_s^*} dx \leq C\|u\|^p + S^{-\frac{2_s^*}{2}} \|u\|^{2_s^*},$$

since $p,\ 2_s^* > 2$, there exists $\varrho > 0$ such that

$$\|u\|^2 \geq \varrho > 0, \quad \forall\ u \in \mathcal{N}. \tag{2.1}$$

From (h), for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$, such that for any $u \in \mathcal{N}$, we have

$$\|u\|^2 \leq \int_{\mathbb{R}^3} h(x)|u|^p dx + \int_{\mathbb{R}^3} |u|^{2_s^*} dx \leq \varepsilon \int_{\mathbb{R}^3} |u|^2 dx + C(\varepsilon) \int_{\mathbb{R}^3} |u|^{2_s^*} dx, \tag{2.2}$$

set $\varepsilon < \frac{1}{2}$, by Sobolev embedding theorem, one can conclude that there exists $C(S) > 0$, such that

$$\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \leq C(S) \int_{\mathbb{R}^3} |u|^{2_s^*} dx, \tag{2.3}$$

hence $\int_{\mathbb{R}^3} |u|^{2_s^*} dx > C(S)^{\frac{\frac{2_s^*}{2}}{2-2_s^*}} > 0, \quad \forall\ u \in \mathcal{N}$.

If we define $J : X \to \mathbb{R}$ where $J(u) := \langle I'(u), u\rangle$, thus

$$\langle J'(u), u\rangle = (2-p)\|u\|^2 + (4-p)b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^2$$

$$+ (p - 2_s^*) \int_{\mathbb{R}^3} |u|^{2_s^*} dx < 0, \tag{2.4}$$

it follows from the Implicit Function Theorem that $\mathcal{N}$ is a C^1 manifold. Finally, if $u \in \mathcal{N}$, from (2.1) we have

$$I(u) = I(u) - \frac{1}{p}\langle I'(u), u\rangle \geq (\frac{1}{2} - \frac{1}{p})\|u\|^2 > 0, \tag{2.5}$$

so $m = \inf_{u \in \mathcal{N}} I(u) > 0$. The lemma is proved. □

Lemma 2.3. *$m < m_\infty$.*

Proof. Let w_n and t_n be defined as in the proof of Lemma 2.2. Since $t_n > 0$ and $t_n w_n \in \mathcal{N}$, there holds

$$\begin{aligned} &t_n^{-2}\|w_n\|^2 + b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx\right)^2 \\ &= t_n^{p-4} \int_{\mathbb{R}^3} h(x)|w_n|^p dx + t_n^{2_s^*-4} \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx, \end{aligned} \tag{2.6}$$

which implies that $\{t_n\}$ is bounded. Otherwise if $\{t_n\} \to \infty$, the left-hand side of (2.6) is bounded, the right-hand side is unbounded, which is a contradiction. Thus there exists $t_0 \geq 0$, such that $t_n \to t_0$, as $n \to \infty$. By (2.1), we have $0 < C \leq \|t_n w_n\|^2 = t_n^2\|w_n\|^2$, it's easy to see $t_0 > 0$.

Now we notice that

$$\begin{aligned} m \leq I(t_n w_n) &= I_\infty(t_n w_n) + \frac{t_n^2}{2}\int_{\mathbb{R}^3}(V(x) - V_\infty) w_n^2 dx + \frac{t_n^p}{p}\int_{\mathbb{R}^3}(h_\infty - h(x))\, w_n^p dx \\ &:= I_\infty(t_n w_n) + \frac{t_n^2}{2} A_n + \frac{t_n^p}{p} D_n. \end{aligned} \tag{2.7}$$

From Lemma 2.1, one has $w(x) \leq Ce^{-\frac{\delta}{\alpha}|x|}$. Using $|x + x_n| \leq |x| + n$, we infer from (V_2) that

$$\begin{aligned} A_n &\leq \int_{\mathbb{R}^3} (-C_v e^{-\gamma|x|}) w_n^2 dx = -C_v \int_{\mathbb{R}^3} (e^{-\gamma|x+x_n|}) w^2 dx \\ &\leq -C_v e^{-\gamma n} \int_{\mathbb{R}^3} (e^{-\gamma|x|}) w^2 dx \leq -Ce^{-\gamma n}. \end{aligned} \tag{2.8}$$

Since $\theta < \frac{p\sqrt{V_\infty}}{\alpha}$, we pick $\delta \in (\frac{\alpha\theta}{p}, \sqrt{V_\infty})$ satisfying that $\theta < \frac{p\delta}{\alpha}$, then together with (h), $n - |x| \leq |x + x_n|$ we have

$$\begin{aligned} D_n &\leq \int_{\mathbb{R}^3} (C_h e^{-\theta|x|}) w_n^p dx = C_h \int_{\mathbb{R}^3} (e^{-\theta|x+x_n|}) w^p dx \\ &\leq C_h e^{-\theta n} \int_{\mathbb{R}^3} e^{(\theta - \frac{p\delta}{\alpha})|x|} dx \leq Ce^{-\theta n}. \end{aligned} \tag{2.9}$$

On the other hand let $g(t_n) = I_\infty(t_n w)$, where $\{t_n\}$ is bounded. Note that $g(t_n)$ has a unique critical point corresponding to its maximum. Since $g'(1) = 0$, this critical point must be achieved at $t_n = 1$, thus $\max_{t_n \ge 0} g(t_n) = \max_{t_n \ge 0} I_\infty(t_n w) = I_\infty(w)$. From Lemma 2.1, we have

$$I_\infty(t_n w_n) = I_\infty(t_n w) \le I_\infty(w) = m_\infty. \tag{2.10}$$

As a consequence, by (2.7)-(2.10),

$$\begin{aligned} m &\le m_\infty - \frac{Ct_n^2}{2} e^{-\gamma n} + \frac{Ct_n^p}{p} e^{-\theta n} \\ &= m_\infty + e^{-\gamma n} \left(-\frac{Ct_n^2}{2} + \frac{Ct_n^p}{p} e^{(\gamma - \theta) n} \right). \end{aligned}$$

Recalling that $t_n \to t_0$ and $\gamma < \theta$, one can easily check that $m < m_\infty + o_n(1)$. Thus there exists $n_0 > 0$, when $n \ge n_0$, we have $m < m_\infty$. □

Proof of Theorem 1.1. The Ekeland Variational Principle provides $\{u_n\} \subset \mathcal{N}$ and $\{\lambda_n\} \subset \mathbb{R}$ such that $I(u_n) \to m$, $I'(u_n) + \lambda_n J'(u_n) \to 0$ with $J(u_n) = \langle I'(u_n), u_n \rangle$. Using (2.4), a standard argument shows that $I'(u_n) \to 0$, thus $\{u_n\}$ is a Palais-Smale sequence of I. Moreover it follows from (2.5) that $\{u_n\}$ is bounded. Hence, along a subsequence still denoted by $\{u_n\}$, $u_n \rightharpoonup u_0$ in X. One can easily deduce that $I'(u_0) = 0$.

We claim that $u_0 \ne 0$. Suppose for the contradiction that $u_0 \equiv 0$, thus $u_n \rightharpoonup 0$ in X and $u_n \to 0$ in $L^q_{loc}(\mathbb{R}^N)$ for any $q \in [2, 2_s^*)$, then $\|u_n\|_\infty^2 = \|u_n\|^2 + o_n(1)$. Since $\langle I'(u_n), u_n \rangle = 0$, by (2.9) one has

$$\|u_n\|_\infty^2 + b(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx)^2 = \int_{\mathbb{R}^3} h_\infty |u_n|^p dx + \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx + o_n(1). \tag{2.11}$$

Similar to Lemma 2.2, there exists $t_n > 0$ such that $t_n u_n \in \mathcal{N}_\infty$, namely

$$t_n^2 \|u_n\|_\infty^2 + b t_n^4 (\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx)^2 = t_n^p \int_{\mathbb{R}^3} h_\infty |u_n|^p dx + t_n^{2_s^*} \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx. \tag{2.12}$$

From (2.11)-(2.12), we have

$$(t_n^2 - t_n^p) \|u_n\|_\infty^2 + b(t_n^4 - t_n^p)(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx)^2 + (t_n^p - t_n^{2_s^*}) \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx = o_n(1),$$

one can easily check that $t_n \to 1$ as $n \to +\infty$, it follows that

$$\begin{aligned} I_\infty(t_n u_n) &= I_\infty(t_n u_n) - \frac{1}{p} \langle I'_\infty(t_n u_n), t_n u_n \rangle \\ &= (\frac{1}{2} - \frac{1}{p}) t_n^2 \|u_n\|_\infty^2 + (\frac{1}{4} - \frac{1}{p}) b t_n^p \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \right)^2 \\ &\quad + (\frac{1}{p} - \frac{1}{2_s^*}) t_n^{2_s^*} \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \\ &= (\frac{1}{2} - \frac{1}{p}) \|u_n\|^2 + (\frac{1}{4} - \frac{1}{p}) b \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \right)^2 \\ &\quad + (\frac{1}{p} - \frac{1}{2_s^*}) \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx + o_n(1) \end{aligned}$$

$$= I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle + o_n(1)$$
$$= I(u_n) + o_n(1).$$

On the other hand, by $t_n u_n \in \mathcal{N}_\infty$, we have $m_\infty \le I_\infty(t_n u_n)$, so

$$m_\infty \le I_\infty(t_n u_n) = I(u_n) + o_n(1) = m + o_n(1),$$

which contradicts Lemma 2.3, thus $u_0 \neq 0$. What's more

$$\begin{aligned} m &\le I(u_0) - \frac{1}{p}\langle I'(u_0), u_0\rangle \\ &= (\frac{1}{2} - \frac{1}{p})\|u_0\|^2 + (\frac{1}{4} - \frac{1}{p})b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_0|^2 dx\right)^2 + (\frac{1}{p} - \frac{1}{2_s^*})\int_{\mathbb{R}^3} |u_0|^{2_s^*} dx \\ &\le \liminf_{n\to\infty}\left(I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle\right) = m, \end{aligned}$$

obviously $I(u_0) = m$, which shows u_0 is a ground state solution.

Considering $\widetilde{u} = |u_0|$, we can easily conclude that $I(\widetilde{u}) = I(u_0) = m$ and $\widetilde{u} \in \mathcal{N}$. Hence $I'(\widetilde{u}) = \lambda J'(\widetilde{u})$ for some $\lambda \in \mathbb{R}$, where $J(\widetilde{u}) = \langle I'(\widetilde{u}), \widetilde{u}\rangle$. By (2.4), we have $\langle J'(\widetilde{u}), \widetilde{u}\rangle < 0$ and $\langle I'(\widetilde{u}), \widetilde{u}\rangle = 0$, it follows that $\lambda = 0$. Thus $\widetilde{u} \ge 0$ is a nonnegative ground state solution of problem (1.1). By the strong maximum principle we see that $\widetilde{u}$ is a positive solution of problem (1.1). □

Remark 2.1. If $\widetilde{u}$ is the positive solution given by Theorem 1.1, as in the proof of Lemma 2.1, we can verify that, for any $\mu > 0$, there exists $C = C(\mu) > 0$, such that $\widetilde{u}(x) \le Ce^{-\mu|x|}$, $\forall\, x \in \mathbb{R}^3$.

3. Sign-changing solution

In this section, we consider the existence of sign-changing solutions for problem (1.1). Define the functional $f(u^+), f(u^-)$ on X by

$$f(u^+) = \frac{\int_{\mathbb{R}^3} h(x)|u^+|^p \mathrm{d}x + \int_{\mathbb{R}^3} |u^+|^{2_s^*}\mathrm{d}x}{\|u^+\|^2 + b\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^+|^2 \mathrm{d}x}, \quad \text{where } u^+ := \max\{u, 0\},$$

$$f(u^-) = \frac{\int_{\mathbb{R}^3} h(x)|u^-|^p \mathrm{d}x + \int_{\mathbb{R}^3} |u^-|^{2_s^*}\mathrm{d}x}{\|u^-\|^2 + b\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^-|^2 \mathrm{d}x}, \quad \text{where } u^- := \min\{u, 0\}.$$

Then we define

$$\mathcal{N}^* = \left\{u \in X\backslash\{0\} : f\left(u^+\right) = f\left(u^-\right) = 1\right\},$$
$$U = \left\{u \in X\backslash\{0\} : \left|f\left(u^\pm\right) - 1\right| < \frac{1}{2}\right\}.$$

Lemma 3.1 (**Miranda Theorem** [14]). *Let $G = \{x \in \mathbb{R}^n : |x_i| < L,\ for\ 1 \le i \le n\}$ and suppose that the mapping $F = (f_1, f_2, ..., f_n) : \overline{G} \to \mathbb{R}^n$ is continuous on the closure $\overline{G}$ of G such that $F(x) \neq \theta = (0, 0, ..., 0)$ for x on the boundary ∂G of G, and*

(i) $f_i(x_1, x_2, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \ge 0$ *for* $1 \le i \le n$ *;*

(ii) $f_i(x_1, x_2, \ldots, x_{i-1}, +L, x_{i+1}, \ldots, x_n) \le 0$ *for* $1 \le i \le n$.

Then $F(x) = \theta$ *has a solution in* G.

Lemma 3.2. *For* $u \in U$*, there exists* $C > 0$*, such that* $\int_{\mathbb{R}^3} |u^{\pm}|^{2_s^*} \mathrm{d}x \ge C > 0$.

Proof. For $u \in U$, we have

$$\begin{aligned}\frac{1}{2}\|u^{\pm}\|^2 &\le \frac{1}{2}\|u^{\pm}\|^2 + \frac{1}{2}b\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u^{\pm}|^2 dx\right)^2 \\ &< \int_{\mathbb{R}^3} h(x)|u^{\pm}|^p dx + \int_{\mathbb{R}^3} |u^{\pm}|^{2_s^*} dx.\end{aligned}$$

Then similar to (2.2)-(2.3), we can derive that there exists $C > 0$, such that $\int_{\mathbb{R}^3} |u^{\pm}|^{2_s^*} \mathrm{d}x \ge C > 0, \ \ \forall\, u \in U$. □

Lemma 3.3. *Let* $u \in \mathcal{N}^*$ *and define* $h_u(t, s) := I(tu^+ + su^-)$*, where* $t, s \ge 0$*, then* h_u *attains its maximum at the point* $(1, 1) \in \mathbb{R}^2$.

Proof. For $u \in \mathcal{N}^*$, we have

$$\begin{aligned}h_u(t, s) = &\frac{1}{2}\|tu^+ + su^-\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}(tu^+ + su^-)|^2 \mathrm{d}x\right)^2 \\ &-\frac{1}{p}\int_{\mathbb{R}3} h(x)|tu^+ + su^-|^p \mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}3} |tu^+ + su^-|^{2_s^*}\mathrm{d}x,\end{aligned}$$

because $\lim\limits_{(t,s)\to+\infty} h_u(t, s) = -\infty$, it follows that the maximum is attained at some point $(t_0, s_0) \in [0, +\infty)^2$.

Claim 1. $s_0, t_0 > 0$.

By contradiction, we assume that $s_0 = 0$. However, since $h_u(0, 0) = 0$ and $h_u(t, s)$ reaches its maximum at the point (t_0, s_0), we have $t_0 > 0$. Furthermore, one can conclude that $I(su^-) > 0$ for $s > 0$ small, thus

$$\begin{aligned}h_{\mathrm{u}}(t_0, 0) &= I\left(t_0 u^+\right) \\ &< I\left(t_0 u^+\right) + I\left(su^-\right) + b\ t_0^2 s^2 \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^+|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^-|^2 \mathrm{d}x \\ &= h_{\mathrm{u}}(t_0, s),\end{aligned}$$

but $h_u(t, s)$ reaches its maximum at point (t_0, s_0), it follows that $s_0 > 0$. A similar argument shows that $t_0 > 0$.

Claim 2. $s_0, t_0 \in (0, 1]$.

Since the case $s_0 \le 1$ is analogous to $t_0 \le 1$, without loss of generality, we just need to prove that $t_0 \le 1$. Recalling that $I(tu^+ + su^-)$ reaches its maximum at the point (t_0, s_0), thus we have $\langle I'(t_0u^+ + s_0u^-), t_0u^+\rangle = 0$. Suppose that $s_0 \le t_0$, then

$$\begin{aligned}&t_0^2\|u^+\|^2 + bt_0^4 \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^+|^2 \mathrm{d}x \\ \ge &t_0^2\|u^+\|^2 + bt_0^4 \left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^+|^2 \mathrm{d}x\right)^2 \\ &+ bt_0^2 s_0^2 \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^-|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^+|^2 \mathrm{d}x\end{aligned}$$

$$=t_0^p \int_{\mathbb{R}3} h(x)|u^+|^p \mathrm{d}x + t_0^{2_s^*} \int_{\mathbb{R}3} |u^+|^{2_s^*} \mathrm{d}x, \tag{3.1}$$

which implies that

$$\begin{aligned} & t_0^{-2}\|u^+\|^2 + b\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^+|^2 \mathrm{d}x \\ \geq\ & t_0^{p-4} \int_{\mathbb{R}3} h(x)|u^+|^p \mathrm{d}x + t_0^{2_s^*-4} \int_{\mathbb{R}3} |u^+|^{2_s^*} \mathrm{d}x. \end{aligned} \tag{3.2}$$

Furthermore, for $u \in \mathcal{N}^*$, we have

$$\|u^+\|^2 + b\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u^+|^2 \mathrm{d}x = \int_{\mathbb{R}3} h(x)|u^+|^p \mathrm{d}x + \int_{\mathbb{R}3} |u^+|^{2_s^*} \mathrm{d}x. \tag{3.3}$$

By (3.2)-(3.3), we get

$$(t_0^{-2} - 1)\|u^+\|^2 \geq (t_0^{p-4} - 1)\int_{\mathbb{R}3} h(x)|u^+|^p \mathrm{d}x + (t_0^{2_s^*-4} - 1)\int_{\mathbb{R}3} |u^+|^{2_s^*} \mathrm{d}x, \tag{3.4}$$

thus $t_0 \leq 1$. Otherwise if $t_0 > 1$, we must have $\|u^+\|^2 \geq \int_{\mathbb{R}3} h(x)|u^+|^p \mathrm{d}x + \int_{\mathbb{R}3} |u^+|^{2_s^*} \mathrm{d}x$, which contradicts with (3.3).

For the case $t_0 \leq s_0$, it is sufficient to use $\langle I'(t_0u^+ + s_0u^-), s_0u^-\rangle = 0$, and similar to the above discussion, we have $s_0 \leq 1$.

Claim 3. h_u does not attain its maximum in $(0,1]^2 \setminus \{(1,1)\}$.

If $s_0 < 1$ or $t_0 < 1$, we have

$$\begin{aligned} h_u(t_0, s_0) &= I(t_0u^+ + s_0u^-) - \frac{1}{p}\left\langle I'(t_0u^+ + s_0u^-), (t_0u^+ + s_0u^-)\right\rangle \\ &= (\frac{1}{2} - \frac{1}{p})\left(t_0^2\|u^+\|^2 + s_0^2\|u^-\|^2\right) \\ &\quad + (\frac{1}{4} - \frac{1}{p})b\left(\int_{\mathbb{R}^3} \left|(-\triangle)^{\frac{s}{2}}(t_0u^+ + s_0u^-)\right|^2 \mathrm{d}x\right)^2 \\ &\quad + (\frac{1}{p} - \frac{1}{2_s^*})\int_{\mathbb{R}^3} \left|t_0u^+ + s_0u^-\right|^{2_s^*} dx \\ &< h_u(1,1), \end{aligned}$$

which is absurd. □

Following the idea of [3], we give some definitions. Denote P the cone of non-negative functions in X. Let $Q = [0,1] \times [0,1]$. Define

$$\begin{aligned} \Sigma := \{\sigma \in C(Q, X);\ & \sigma(t,0) = 0, \sigma(0,s) \in P, \sigma(1,s) \in -P, \\ & I(\sigma(t,1)) \leq 0, f(\sigma(t,1)) \geq 2, \forall\ t, s \in [0,1]\}. \end{aligned}$$

Choose $u \in X$ such that $u^{\pm} \neq 0$. Let $\sigma(t,s) = ks(1-t)u^+ + kstu^-$, where $k > 0$, $t, s \in [0,1]$. It is easy to check that $\sigma \in \Sigma$ for $k > 0$ large enough.

Lemma 3.4. $\inf\limits_{u \in \mathcal{N}^*} I(u) = \inf\limits_{\sigma \in \Sigma} \sup\limits_{u \in \sigma(Q)} I(u)$.

Proof. From the definition of Σ, for any $\sigma \in \Sigma$, $t \in [0,1]$, we have

$$f\left(\sigma^+(t,0)\right) + f\left(\sigma^-(t,0)\right) - 2 = -2 < 0,$$
$$f\left(\sigma^+(t,1)\right) + f\left(\sigma^-(t,1)\right) - 2 = f\left(\sigma(t,1)\right) - 2 \geq 0.$$

On the other hand, for any $\sigma \in \Sigma$, $s \in [0,1]$, we have

$$f\left(\sigma^+(0,s)\right) - f\left(\sigma^-(0,s)\right) = f\left(\sigma^+(0,s)\right) \geq 0,$$
$$f\left(\sigma^+(1,s)\right) - f\left(\sigma^-(1,s)\right) = -f\left(\sigma^-(1,s)\right) \leq 0.$$

Then from Miranda theorem in [14], we conclude that for any $\sigma \in \Sigma$, there exists $(\bar{t}, \bar{s}) \in Q$ such that

$$f\left(\sigma^+(\bar{t},\bar{s})\right) - f\left(\sigma^-(\bar{t},\bar{s})\right) = 0 = f\left(\sigma^+(\bar{t},\bar{s})\right) + f\left(\sigma^-(\bar{t},\bar{s})\right) - 2,$$

thus $f\left(\sigma^+(\bar{t},\bar{s})\right) = f\left(\sigma^-(\bar{t},\bar{s})\right) = 1$, which is $\sigma(\bar{t},\bar{s}) \in \mathcal{N}^*$. So there holds

$$\inf_{\sigma\in\Sigma} \sup_{u\in\sigma(Q)} I(u) \geq \inf_{u\in\mathcal{N}^*} I(u). \tag{3.5}$$

From Lemma 3.3, we know $I(tu^+ + su^-)$ attains its maximum at the point $(t,s) = (1,1)$, thus for every $u \in \mathcal{N}^*$, we have

$$\begin{aligned} I(u) = I(u^+ + u^-) &\geq \sup_{\alpha,\beta\geq 0} I\left(\alpha u^+ + \beta u^-\right) \\ &\geq \sup_{u\in\sigma(Q)} I(u) \geq \inf_{\sigma\in\Sigma} \sup_{u\in\sigma(Q)} I(u), \end{aligned}$$

which implies that

$$\inf_{\sigma\in\Sigma} \sup_{u\in\sigma(Q)} I(u) \leq \inf_{u\in\mathcal{N}^*} I(u). \tag{3.6}$$

From (3.5)-(3.6), Lemma 3.4 holds. □

Lemma 3.5. *There is a sequence* $\{u_n\} \subset U$ *such that* $I(u_n) \to c^* = \inf_{u\in\mathcal{N}^*} I(u)$ *and* $I'(u_n) \to 0$.

Proof. Define $c^* = \inf_{u\in\mathcal{N}^*} I(u)$, consider a minimizing sequence $\{\overline{u}_n\} \subset \mathcal{N}^*$ and choose $\overline{\sigma}_n \in \Sigma$ such that $\overline{\sigma}_n(Q) \subset \left\{\alpha_n \overline{u}_n^+ + \beta_n \overline{u}_n^-\right\}$ where $\alpha_n, \beta_n \in [0,1]$, then by Lemma 3.4 we have

$$\lim_{n\to\infty} \max_{u\in\overline{\sigma}_n(Q)} I(u) = \lim_{n\to\infty} I\left(\overline{u}_n\right) = c^*. \tag{3.7}$$

By [3], we can derive that there exists $\{u_n\} \subset X$ such that

$$I\left(u_n\right) \to c^*, \quad I'\left(u_n\right) \to 0 \quad \text{and} \quad \operatorname{dist}\left(u_n, \overline{\sigma}_n(Q)\right) \to 0, \tag{3.8}$$

we just need to prove $\{u_n\} \subset U$ for n large enough. By (3.7)-(3.8), there exists a sequence $\{v_n\}$, where $v_n = \alpha_n \overline{u}_n^+ + \beta_n \overline{u}_n^- \in \overline{\sigma}_n(Q)$, such that

$$I(v_n) \to c^*, \ \|v_n - u_n\| \to 0. \tag{3.9}$$

From Lemma 3.2, for any $\overline{u}_n \in \mathcal{N}^* \subset U$, $\int_{\mathbb{R}^3} \left|\overline{u}_n^{\pm}\right|^{2_s^*} dx \geq C > 0$, then by Sobolev embedding theorem, one has

$$I(\overline{u}_n^{\pm}) = I(\overline{u}_n^{\pm}) - \frac{1}{p}\langle I'(\overline{u}_n^{\pm}), \overline{u}_n^{\pm}\rangle \geq (\frac{1}{2} - \frac{1}{p})\|\overline{u}_n^{\pm}\|^2 \geq (\frac{1}{2} - \frac{1}{p})S^2 C^{\frac{4}{2_s^*}} > 0.$$

By (3.7), without loss of generality, we may assume that $I\left(\overline{u}_n^{+}\right) \to c_1^* > 0$, $I\left(\overline{u}_n^{-}\right) \to c_2^* > 0$. Thanks to $\overline{u}_n \in \mathcal{N}^*$, we can easily conclude that

$$\begin{aligned}
&I\left(\overline{u}_n^{+}\right) \geq I\left(\alpha_n \overline{u}_n^{+}\right) = I\left(v_n^{+}\right), \quad I\left(\overline{u}_n^{-}\right) \geq I\left(\beta_n \overline{u}_n^{-}\right) = I\left(v_n^{-}\right), \\
&b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \overline{u}_n^{+}|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \overline{u}_n^{-}|^2 dx \geq \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \alpha_n \overline{u}_n^{+}|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \beta_n \overline{u}_n^{-}|^2 dx \\
&\qquad = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^{+}|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^{-}|^2 dx.
\end{aligned}$$

Furthermore

$$\begin{aligned}
c^* &= \lim_{n\to\infty} I\left(\overline{u}_n\right) \\
&= \lim_{n\to\infty} \left[I\left(\overline{u}_n^{+}\right) + I\left(\overline{u}_n^{-}\right) + \frac{b}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \overline{u}_n^{+}|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \overline{u}_n^{-}|^2 dx\right] \\
&\geq \lim_{n\to\infty} \left[I\left(v_n^{+}\right) + I\left(v_n^{-}\right) + \frac{b}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^{+}|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^{-}|^2 dx\right] \\
&= \lim_{n\to\infty} I\left(v_n\right) = c^*,
\end{aligned}$$

so we have

$$\begin{aligned}
&\lim_{n\to\infty} I\left(v_n^{+}\right) = \lim_{n\to\infty} I\left(\overline{u}_n^{+}\right) = c_1^*, \quad \lim_{n\to\infty} I\left(v_n^{-}\right) = \lim_{n\to\infty} I\left(\overline{u}_n^{-}\right) = c_2^*, \\
&b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \overline{u}_n^{+}|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \overline{u}_n^{-}|^2 dx = b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^{+}|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n^{-}|^2 dx.
\end{aligned}$$

By (3.9), $\|v_n^{\pm} - u_n^{\pm}\| \to 0$, so $\lim_{n\to\infty} I\left(u_n^{+}\right) = c_1^* > 0$ and $\lim_{n\to\infty} I\left(u_n^{-}\right) = c_2^* > 0$, which implies $u_n^{\pm} \neq 0$. Moreover $I'(u_n) \to 0$, we have $\langle I'(u_n), u_n^{\pm}\rangle = 0$, thus $\{u_n\} \subset U$ for n large enough. □

Lemma 3.6. *Let $\{u_n\} \subset U$ be a sequence such that $\|u_n\|$ is bounded, $I(u_n) \to c^*$ and $I'(u_n) \to 0$. There exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$. We can assume that $u_n \rightharpoonup u$ weakly in X, thus $I'(u) = 0$. Set $v_n = u_n - u$, we have*

(i) $c^* \geq I(u) + I_\infty(v_n) + o_n(1)$,

(ii) $\langle I'_\infty(v_n), v_n\rangle \leq o_n(1)$.

Proof. For $v_n = u_n - u$, there hold $v_n \rightharpoonup 0$ weakly in X and $v_n \to 0$ in $L^p_{loc}(\mathbb{R}^3)$, thus $\|v_n\|_\infty^2 = \|v_n\|^2 + o_n(1)$, then by the Brezis-Lieb Lemma in [19], we get

$$\begin{aligned}
&\|v_n\|_\infty^2 = \|v_n\|^2 + o_n(1) = \|u_n\|^2 - \|u\|^2 + o_n(1), \\
&\int_{\mathbb{R}^3} |v_n|^{2_s^*} dx = \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx - \int_{\mathbb{R}^3} |u|^{2_s^*} dx + o_n(1), \\
&\int_{\mathbb{R}^3} h(x)|v_n|^p \mathrm{d}x = \int_{\mathbb{R}^3} h(x)|u_n|^p \mathrm{d}x - \int_{\mathbb{R}^3} h(x)|u|^p \mathrm{d}x + o_n(1).
\end{aligned} \tag{3.10}$$

Combining (h) with the third equality of (3.10), we have

$$\int_{\mathbb{R}^3} h_\infty |v_n|^p \mathrm{d}x = \int_{\mathbb{R}^3} h(x)|u_n|^p \mathrm{d}x - \int_{\mathbb{R}^3} h(x)|u|^p \mathrm{d}x + o_n(1). \tag{3.11}$$

However, let $\|u\|^2_{D^{s,2}} := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx$, there holds

$$\begin{aligned}\|u_n\|^4_{D^{s,2}} &= \left(\|u\|^2_{D^{s,2}} + \|v_n\|^2_{D^{s,2}} + o_n(1)\right)^2 \\ &\geq \|u\|^4_{D^{s,2}} + \|v_n\|^4_{D^{s,2}} + o_n(1).\end{aligned} \tag{3.12}$$

Combining (3.10)-(3.12), it's easy to see

$$c^* \geq I(u) + I_\infty(v_n) + o_n(1).$$

On the other hand, from $u_n \rightharpoonup u$, we have $\langle u_n, u\rangle_{D^{s,2}} \to \|u\|^2_{D^{s,2}}$. By the Brezis-Lieb Lemma in [19], we get $\|v_n\|^2_{D^{s,2}} = \|u_n\|^2_{D^{s,2}} - \|u\|^2_{D^{s,2}} + o_n(1)$.
Thus

$$\begin{aligned}&\|v_n\|^4_{D^{s,2}} - \left(\|u_n\|^4_{D^{s,2}} - \|u\|^4_{D^{s,2}}\right) \\ =& \left(\|u_n\|^2_{D^{s,2}} + \|u\|^2_{D^{s,2}} - 2\langle u_n, u\rangle_{D^{s,2}}\right)^2 - \|u_n\|^4_{D^{s,2}} + \|u\|^4_{D^{s,2}} \\ =& 2\|u\|^4_{D^{s,2}} - 2\|u_n\|^2_{D^{s,2}}\|u\|^2_{D^{s,2}} + o_n(1) \\ =& 2\|u\|^4_{D^{s,2}} - 2\left(\|v_n\|^2_{D^{s,2}} + \|u\|^2_{D^{s,2}}\right)\|u\|^2_{D^{s,2}} + o_n(1) \\ =& -2\|v_n\|^2_{D^{s,2}}\|u\|^2_{D^{s,2}} + o_n(1) \\ \leq& o_n(1).\end{aligned} \tag{3.13}$$

By (3.10)-(3.11) and (3.13), there holds

$$\langle I'_\infty(v_n), v_n\rangle - (\langle I'(u_n), u_n\rangle - \langle I'(u), u\rangle) \leq o_n(1),$$

together with $\langle I'(u_n), u_n\rangle = 0$, $\langle I'(u), u\rangle = 0$, we have $\langle I'_\infty(v_n), v_n\rangle \leq o_n(1)$. □

Lemma 3.7. *If the sequence $\{u_n\} \subset U$ satisfies that $\|u_n\|$ is bounded, $I(u_n) \to c^* \in (0, m + m_\infty)$ and $I'(u_n) \to 0$, then $u_n \to u$ in X.*

Proof. Since $\|u_n\|$ is bounded, there holds $u_n \rightharpoonup u$ in X and $I'(u) = 0$. Set $v_n = u_n - u$, from Lemma 3.6, we have

$$c^* \geq I(u) + I_\infty(v_n) + o_n(1), \tag{3.14}$$

$$\langle I'_\infty(v_n), v_n\rangle \leq o_n(1). \tag{3.15}$$

If $v_n \to 0$ strongly in X, then Lemma 3.7 holds. Now we consider v_n converges weakly (and not strongly) to 0 in X. Then either v_n^+ converges weakly (and not strongly) to 0 in X, or v_n^- converges weakly (and not strongly) to 0 in X. We will consider three cases as follows.

Case 1. v_n^+ converges weakly (and not strongly) to 0 in X, $v_n^- \to 0$ strongly in X.

We claim that $u_n \rightharpoonup u \neq 0$ weakly in X. By contradiction, if $u = 0$, then $u_n^- = v_n^- \to 0$ strongly in X, from Lemma 3.2, it's a contradiction with $\int_{\mathbb{R}^3} |u_n^-|^{2_s^*} \mathrm{d}x \geq C > 0$. So $u_n \rightharpoonup u \neq 0$ weakly in X.

Note that $v_n^+ \rightharpoonup 0$, there holds $\|v_{\mathrm{n}}^+\|^2_\infty = \|v_{\mathrm{n}}^+\|^2 + o_n(1)$, then by (3.15), we have

$$\left\|v_{\mathrm{n}}^+\right\|^2 + b\left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v_{\mathrm{n}}^+|^2 \mathrm{d}x\right)^2 \leq \int_{\mathbb{R}^3} h_\infty \left|v_n^+\right|^p \mathrm{d}x + \int_{\mathbb{R}^3} \left|v_n^+\right|^{2_s^*} \mathrm{d}x + o_n(1). \tag{3.16}$$

Similar to Lemma 2.2, there exists $t_n \in (0, +\infty)$ such that $t_n v_n^+ \in \mathcal{N}_\infty$, namely

$$t_n^2 \|v_{\mathrm{n}}^+\|^2 + bt_n^4 \left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v_{\mathrm{n}}^+|^2 \mathrm{d}x \right)^2 = t_n^p \int_{\mathbb{R}^3} h_\infty \left| v_n^+ \right|^p \mathrm{d}x + t_n^{2_s^*} \int_{\mathbb{R}^3} \left| v_n^+ \right|^{2_s^*} \mathrm{d}x. \tag{3.17}$$

By (3.16)-(3.17), one has

$$\begin{aligned} &\left(t_n^2 - 1\right) b \left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v_{\mathrm{n}}^+|^2 \mathrm{d}x \right)^2 \\ \geq &\left(t_n^{p-2} - 1\right) \int_{\mathbb{R}^3} h_\infty \left| v_n^+ \right|^p \mathrm{d}x + \left(t_n^{2_s^* - 2} - 1 \right) \int_{\mathbb{R}^3} \left| v_n^+ \right|^{2_s^*} \mathrm{d}x. \end{aligned} \tag{3.18}$$

Since v_n^+ converges weakly (and not strongly) to 0 in $L^p(\mathbb{R}^3)$, and by (h) we can derive that $\lim_{n\to\infty} \int_{\mathbb{R}^3} h_\infty |v_n^+|^p \,\mathrm{d}x > 0$. Then from (3.17), it's easy to get

$$t_n^2 \|v_{\mathrm{n}}^+\|^2 + bt_n^4 \left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v_{\mathrm{n}}^+|^2 \mathrm{d}x \right)^2 \geq t_n{}^{2_s^*} \int_{\mathbb{R}^3} \left| v_n^+ \right|^{2_s^*} \mathrm{d}x, \tag{3.19}$$

Combining (3.16) with (h), we know for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\begin{aligned} \left\| v_{\mathrm{n}}^+ \right\|^2 + b \left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v_{\mathrm{n}}^+|^2 \mathrm{d}x \right)^2 &\leq \int_{\mathbb{R}^3} h_\infty \left| v_n^+ \right|^p + \int_{\mathbb{R}^3} \left| v_n^+ \right|^{2_s^*} \mathrm{d}x + o_n(1) \\ &\leq \int_{\mathbb{R}^3} h(x) \left| v_n^+ \right|^p + \int_{\mathbb{R}^3} \left| v_n^+ \right|^{2_s^*} \mathrm{d}x + o_n(1) \\ &\leq \varepsilon \int_{\mathbb{R}^3} \left| v_n^+ \right|^2 + C(\varepsilon) \int_{\mathbb{R}^3} \left| v_n^+ \right|^{2_s^*} \mathrm{d}x + o_n(1). \end{aligned} \tag{3.20}$$

In view of (3.19)-(3.20), one can derive that $\{t_n\}$ is bounded. Otherwise if $\{t_n\}$ is unbounded, from (3.19), we must have $\|v_{\mathrm{n}}^+\|^2 + b \left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v_{\mathrm{n}}^+|^2 \mathrm{d}x \right)^2 > \int_{\mathbb{R}^3} |v_n^+|^{2_s^*} \mathrm{d}x$, which contradicts with (3.20). So there exists $t_1 \geq 0$, such that $t_n \to t_1$ as $n \to \infty$. If $t_1 > 1$, by (3.18) there holds

$$\lim_{n\to\infty} b \left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v_{\mathrm{n}}^+|^2 \mathrm{d}x \right)^2 > \lim_{n\to\infty} \int_{\mathbb{R}^3} h_\infty \left| v_n^+ \right|^p \mathrm{d}x + \lim_{n\to\infty} \int_{\mathbb{R}^3} \left| v_n^+ \right|^{2_s^*} \mathrm{d}x,$$

which contradicts with (3.16), so $t_n \to t_1 \leq 1$. On the other hand, let $s \in (0, +\infty)$, it's easy to see $I_\infty(sv_n^+) > 0$ for $s > 0$ small, $I_\infty(sv_n^+) < 0$ for s large. Thus $I_\infty(sv_n^+)$ achieves its maximum at some $s > 0$. Since $\langle I'_\infty(t_n v_n^+),\ t_n v_n^+ \rangle = 0$, we know the maximum must be achieved at $s = t_n$, which implies $\max_{s\in(0,+\infty)} I_\infty(sv_n^+) = I_\infty(t_n v_n^+)$. Moveover

$$\begin{aligned} I_\infty(t_n v_n^+) =& I_\infty(t_n v_n^+) - \frac{1}{p} \left\langle I'_\infty(t_n v_n^+), (t_n v_n^+) \right\rangle \\ =& (\frac{t_n^2}{2} - \frac{t_n^p}{p}) \left\| v_{\mathrm{n}}^+ \right\|^2 + b(\frac{t_n^4}{4} - \frac{t_n^p}{p}) \left(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v_{\mathrm{n}}^+|^2 \mathrm{d}x \right)^2 \\ &+ (\frac{t_n^{2_s^*}}{2_s^*} - \frac{t_n^p}{p}) \int_{\mathbb{R}^3} \left| v_n^+ \right|^{2_s^*} \mathrm{d}x, \end{aligned}$$

by $t_n \in (0,1]$, we can easily get $\max\limits_{t_n\in(0,1]} I_\infty(t_n v_n^+) = I_\infty(v_n^+)$. Thus by (3.14) one has

$$\begin{aligned} c^* &\geq I(u) + I_\infty(v_n^+) + o_n(1) \\ &\geq I(u) + I_\infty(t_n v_n^+) + o_n(1) \\ &\geq m + m_\infty + o_n(1), \end{aligned}$$

which is a contradiction to $c^* < m + m_\infty$.

Case 2. v_n^- converges weakly (and not strongly) to 0 in X, $v_n^+ \to 0$ strongly in X.

The proof is similar to Case 1.

Case 3. v_n^+ converges weakly (and not strongly) to 0 in X, v_n^- converges weakly (and not strongly) to 0 in X.

Similar to the proof of Case 1, we can derive that there exists $t_n^+, s_n^- \in (0,+\infty)$ such that $t_n^+ v_n^+, s_n^- v_n^- \in \mathcal{N}_\infty$ and $t_n^+ \to t^+ \leq 1, s_n^- \to s^- \leq 1$. Then by (3.14) and $m < m_\infty$ in Lemma 2.3, we have

$$\begin{aligned} c^* &> I_\infty(v_n) + o_n(1) \\ &> I_\infty(v_n^+) + I_\infty(v_n^-) + o_n(1) \\ &\geq I_\infty(t_n^+ v_n^+) + I_\infty(s_n^- v_n^-) + o_n(1) \\ &\geq 2m_\infty + o_n(1) \\ &> m + m_\infty + o_n(1), \end{aligned}$$

which is a contradiction to $c^* < m + m_\infty$. □

Proof of Theorem 1.2. From Lemma 3.5, there is a sequence $\{u_n\} \subset U$ such that $I(u_n) \to c^* = \inf\limits_{u\in\mathcal{N}^*} I(u)$ and $I'(u_n) \to 0$. Then similar to (2.5), we can get $\|u_n\|$ is bounded. From Lemma 3.2, for $\{u_n\} \subset U$, we have $\int_{\mathbb{R}^3} |u_n^\pm|^{2_s^*}\, dx \geq C > 0$. Thus if $u_n \to u$ in X, then $I'(u) = 0$, $\int_{\mathbb{R}^3} |u^\pm|^{2_s^*}\, dx \geq C > 0$, which implies that u is a sign-changing solution of (1.1). Now we prove $u_n \to u$ in X. From Lemma 3.7, we only need to prove that $c^* < m + m_\infty$.

Let $\widetilde{u}$ be the positive solution given by Theorem 1.1, and w_n was defined in Lemma 2.2. For any $n \in \mathbb{N}, x \in \mathbb{R}^3$ and $(t,s) \in [\frac{1}{2}, 2]^2$, we define $\psi_n(x) := t\widetilde{u}(x) + sw_n(x)$. We claim that there exists $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$ and $(t,s) \in [\frac{1}{2}, 2]^2$, $I(\psi_n) < m + m_\infty$.

$$I(t\widetilde{u} + sw_n) = I(t\widetilde{u}) + I_\infty(sw_n) + A_n + B_n + C_n + D_n + E_n + F_n. \tag{3.21}$$

From Lemma 2.3, we have

$$\begin{aligned} A_n &= \frac{1}{2}s^2 \int_{\mathbb{R}^3} (V(x) - V_\infty)\, w_n^2 dx \leq -C_1 e^{-\gamma n}, \\ D_n &= \frac{1}{p}s^p \int_{\mathbb{R}^3} (h_\infty - h(x))\, w_n^p dx \leq C_2 e^{-\theta n}. \end{aligned}$$

From Lemma 2.1 and Remark 2.1, there hold $\widetilde{u}(x) \leq Ce^{-\mu x}$, $w(x) \leq Ce^{-\frac{\delta}{\alpha}x}$. Since $\gamma < \frac{\sqrt{V_\infty}}{\alpha}$, let $\mu \in (\gamma, \frac{\sqrt{V_\infty}}{\alpha})$, similar to (2.8)-(2.9) we can get

$$E_n = -\frac{1}{p}\int_{\mathbb{R}^3} h(x)\left(|\psi_n|^p - |t\widetilde{u}|^p - |sw_n|^p\right) dx \leq -C_3 e^{-\mu n},$$

$$F_n = -\frac{1}{2_s^*}\int_{\mathbb{R}^3}\left(|\psi_n|^{2_s^*} - |t\widetilde{u}|^{2_s^*} - |sw_n|^{2_s^*}\right)dx \le -C_4 e^{-\mu n}.$$

Note that $\langle I'(\widetilde{u}), w_n) = 0$, there holds

$$\begin{aligned}B_n &= st\int_{\mathbb{R}^3} a|(-\triangle)^{\frac{s}{2}}\widetilde{u}|\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}w_n|dx + st\int_{\mathbb{R}^3}V(x)\widetilde{u}w_n\ dx\\ &= st(\int_{\mathbb{R}^3}h(x)|\widetilde{u}|^{p-1}w_n dx + \int_{\mathbb{R}^3}|\widetilde{u}|^{2_s^*-1}w_n dx\\ &\quad -b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}\widetilde{u}|^2\mathrm{d}x\cdot\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}\widetilde{u}|\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}w_n|)\\ &\le C_5 e^{-\mu n}.\end{aligned}$$

For the convenience, let $\|w_n\|^2_{D^{s,2}} := \int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}w_n|dx$, by $\langle I'_\infty(w_n), w_n\rangle = 0$, we have

$$a\|w_n\|^2_{D^{s,2}} + b\|w_n\|^4_{D^{s,2}} = \int_{\mathbb{R}^3}h_\infty|w_n|^p dx + \int_{\mathbb{R}^3}|w_n|^{2_s^*}dx - \int_{\mathbb{R}^3}V_\infty|w_n|^2 dx < Ce^{-\delta n},$$

thus $G_n := \|w_n\|^2_{D^{s,2}} \le \frac{-a+\sqrt{a^2+4Ce^{-\delta n}}}{2b} \to 0\ (n\to\infty)$, then

$$\begin{aligned}C_n = \frac{b}{4}[\ &4t^2s^2\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}\widetilde{u}|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}w_n|^2\mathrm{d}x\\ &+2t^2s^2(\int_{\mathbb{R}^3}(-\triangle)^{\frac{s}{2}}\widetilde{u}\mathrm{d}x\int_{\mathbb{R}^3}(-\triangle)^{\frac{s}{2}}w_n\mathrm{d}x)^2\\ &+4t^3s\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}\widetilde{u}|^2\mathrm{d}x\int_{\mathbb{R}^3}(-\triangle)^{\frac{s}{2}}\widetilde{u}\mathrm{d}x\int_{\mathbb{R}^3}(-\triangle)^{\frac{s}{2}}w_n\mathrm{d}x\\ &+4ts^3\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}w_n|^2\mathrm{d}x\int_{\mathbb{R}^3}(-\triangle)^{\frac{s}{2}}\widetilde{u}\mathrm{d}x\int_{\mathbb{R}^3}(-\triangle)^{\frac{s}{2}}w_n\mathrm{d}x]\\ \le\ &C_6G_n + C_7e^{-\mu n}.\end{aligned}$$

All the above inequalities can be replaced in (3.21) to provide

$$\begin{aligned}I(\psi_n) &\le m + m_\infty - C_1e^{-rn} + C_2e^{-\theta n}\\ &\quad -C_3e^{-\mu n} - C_4e^{-\mu n} + C_5e^{-\mu n} + C_6G_n + C_7e^{-\mu n}.\end{aligned}$$

So we obtain the inequality $I(\psi_n) \le m + m_\infty + o_n(1)$.

In view of the claim, to prove that $c^* < m + m_\infty$, it is sufficient to obtain $(t_0, s_0) \in [\frac{1}{2}, 2]^2$ such that $t_0\widetilde{u}(x) + s_0w_n(x) \in \mathcal{N}^*$. With this purpose, we define

$$h^\pm(t,s,n) := \left\langle I'(t\widetilde{u} + sw_n), (t\widetilde{u} + sw_n)^\pm\right\rangle.$$

Since $w_n \rightharpoonup 0$ weakly in X, and w is a solution of the limit problem, we can use (V_2), (h) to conclude that

$$\begin{aligned}h^-(0,2,n) &= 2^2\|w_n\|^2 + 2^4b\left(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}w_n|^2\right)^2 - 2^p\int_{\mathbb{R}^3}h(x)w_n^p\mathrm{d}x - 2^{2_s^*}\int_{\mathbb{R}^3}w_n^{2_s^*}\mathrm{d}x\\ &= (2^2 - 2^p)\|w\|^2_\infty + b(2^4 - 2^p)\left(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}w|^2\right)^2\end{aligned}$$

$$+(2^p - 2^{2_s^*})\int_{\mathbb{R}^3} w^{2_s^*}\mathrm{d}x + o_n(1),$$

so $h^-(0,2,n) < 0$ for n large. The same argument provides $h^-(0,\frac{1}{2},n) > 0$. Moreover by $\langle I'(\widetilde{u}),\widetilde{u}\rangle = 0$, we can conclude that

$$h^+(\frac{1}{2},0,n) > 0, \quad h^+(2,0,n) < 0.$$

From Lemma 3.1(Miranda Theorem [14]), there exists $(t_0,s_0) \in [\frac{1}{2},2]^2$ such that $h^{\pm}(t_0,s_0,n) = 0$ for n large, which is equivalent to $t_0\widetilde{u}(x) + s_0 w_n(x) \in \mathcal{N}^*$. □

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