EXISTENCE AND MULTIPLICITY OF POSITIVE PERIODIC SOLUTIONS FOR A CLASS OF SECOND ORDER DAMPED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

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Abstract By using the Krasnoselskii fixed point theorem, sufficient conditions are obtained for the existence and multiplicity of positive periodic solutions for a class of second order damped functional differential equations with multiple delays. Our results are a further expansion of the previous research results.

Keywords Periodic solutions, Green’s function, Krasnoselskii fixed point theorem.

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1. Introduction

For the following equation

\[ x'' = f(t, x(t)), \quad t \in \mathbb{R}, \]

where \( f \in C(\mathbb{R}/TZ, (0, +\infty)) \), there are many results [3, 18] on the periodic solution of this equation.

However, the systems controlled by feedback loops in engineering, predator-prey models in ecosystems [8, 12], and value laws in economics in real life all have the influence of delay factors, so the research on functional differential equations has already stepped into a climax period [1, 15, 17]. At the same time, many research methods have been considered, such as the upper and lower solutions method and monotone iterative technique [10, 16], fixed point theorems [11, 13, 21] and so on [5, 9, 14, 19, 20].

Jiang et al. [10] studied the following periodic problem

\[ -x'' = f(t, x(t), x(t - \tau(t))), \quad t \in \mathbb{R}, \]

where \( f \in C(\mathbb{R}^3, \mathbb{R}), \tau \in C(\mathbb{R}, [0, +\infty)) \), and they are \( T \)-periodic functions. They established the existence results of \( T \)-periodic solutions by using monotone iterative technique.

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However, for many problems in real life, we only need to consider the properties of its positive periodic solution. In [21], Wu obtained the existence and multiplicity of the solutions to the following equation

\[ x'' + a(t)x = \lambda f(t, x(t - \tau_0(t)), x(t - \tau_1(t)), ..., x(t - \tau_n(t))), \quad t \in \mathbb{R}, \]

where \( a \in C(\mathbb{R}/TZ, (0, +\infty)) \), \( f \in C((\mathbb{R}/TZ) \times [0, +\infty)^n, [0, +\infty)) \), \( \tau_i(t) \in C(\mathbb{R}/TZ, \mathbb{R}) \), and \( a(t) \) satisfies the condition that \( 0 < a(t) < \frac{\pi^2}{r^2} \) for every \( t \in \mathbb{R} \).

Li et al. studied the following equation in [13]

\[ x'' + a(t)x = f(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_n(t))), \quad t \in \mathbb{R}, \]

where \( a \in C(\mathbb{R}/TZ, (0, +\infty)) \), \( f \in C((\mathbb{R}/TZ) \times [0, +\infty)^n, [0, +\infty)) \), \( \tau_i(t) \in C(\mathbb{R}/TZ, [0, +\infty)) \), they obtained the existence of positive periodic solution by using the first eigenvalue corresponding to the relevant linear operator and fixed-point index theory in cones.

In [11], Kang et al. considered the following equation with damped term

\[ x'' + h(t)x' + a(t)x = \lambda g(t)f(t, x(t - \tau(t))), \quad t \in \mathbb{R}, \]

where \( h \in C(\mathbb{R}/TZ, [0, +\infty)) \), \( a \in C(\mathbb{R}/TZ, [0, +\infty)) \), \( f \in C((\mathbb{R}/TZ) \times \mathbb{R}, [0, +\infty)) \), \( \tau_i(t) \in C(\mathbb{R}/TZ, \mathbb{R}) \), \( g \in C(\mathbb{R}/TZ, [0, +\infty)) \). They obtained the existence and multiplicity of positive periodic solutions when the coefficients \( h(t), a(t) \) and \( g(t) \) satisfy \( \int_0^T h(\xi)d\xi > 0 \), \( \int_0^T a(\xi)d\xi > 0 \) and \( \int_0^T g(\xi)d\xi > 0 \), respectively, moreover, \( f \) is nondecreasing in the second variable.

Motivated by the above papers, in this paper, we study the existence, multiplicity of positive periodic solutions for the following equation

\[ x'' + h(t)x' + a(t)x = \lambda g(t)f(t, x(t - \tau_0(t)), x(t - \tau_1(t)), ..., x(t - \tau_n(t))), \quad (1.1) \]

where \( h \in C(\mathbb{R}/TZ, \mathbb{R}) \), \( a \in C(\mathbb{R}/TZ, \mathbb{R}) \), \( f \in C((\mathbb{R}/TZ) \times [0, +\infty)^n, [0, +\infty)) \) and \( f(t, x_0, x_1, ..., x_n) > 0 \) for \( (x_i \geq 0, 0 \leq i \leq n, (x_0, x_1, ..., x_n) \neq (0, 0, ..., 0)) \), \( \tau_i(t) \in C(\mathbb{R}/TZ, \mathbb{R}) \), \( g \in C(\mathbb{R}/TZ, [0, +\infty)) \) and \( \int_0^T g(\xi)d\xi > 0 \), \( \lambda > 0 \) is a parameter.

Three highlights should be pointed out. Firstly, compared with the equation studied in [13, 21], we add the damping term \( h(t)x' \). Secondly, different from [11], the equation we studied has multiple delays. Thirdly, we relax the restrictions for the coefficients \( h(t) \) and \( a(t) \) in [11].

2. Preliminaries

If the unique solution of linear equation

\[ x'' + h(t)x' + a(t)x = 0, \quad (2.1) \]

associated to periodic boundary conditions

\[ x(0) = x(T), \quad x'(0) = x'(T) \quad (2.2) \]
is trivial, then it is nonresonant. By Fredholm’s alternative theorem, we know that when \((2.1)-(2.2)\) is nonresonant,

\[
x'' + h(t) x' + a(t)x = l(t)
\]

has a unique solution and it can be expressed as

\[
x(t) = \int_0^T G(t, \xi)l(\xi)d\xi,
\]

where \(G(t, \xi)\) is the Green’s function of \((2.1)-(2.2)\).

Next we assume that:

(A0) The Green’s function \(G(t, \xi)\) of system \((2.1)-(2.2)\), is positive for all \((t, \xi) \in [0, T] \times [0, T]\).

In general, condition (A0) is difficult to establish. However, through the anti-maximum principle established by Hakl and Torres (see [7]), Chu, Fan and Torres obtained that (A0) is true in [2]. Describe the above criterion by defining the following function

\[
\sigma(h)(t) = \exp(\int_0^t h(\xi)d\xi),
\]

and

\[
\sigma_1(h)(t) = \sigma(h)(T) \int_0^t \sigma(h)(\xi)d\xi + \int_t^T \sigma(h)(\xi)d\xi.
\]

**Lemma 2.1** (Corollary 2.6, [7]). If \(a(t) \neq 0\) and the following two inequalities

\[
\int_0^T a(\xi)\sigma(h)(\xi)\sigma_1(-h)(\xi)d\xi \geq 0, \quad \text{(H1)}
\]

and

\[
\sup_{0 \leq t \leq T} \left\{ \int_t^{t+T} \sigma(-h)(\xi)d\xi \int_t^{t+T} [a(\xi)]_+ \sigma(h)(\xi)d\xi \right\} \leq 4 \quad \text{(H2)}
\]

are satisfied, where \([a(\xi)]_+ = \max\{a(\xi), 0\}\). Then (A0) holds.

When (A0) holds, we always denote

\[
A = \min_{0 \leq \xi, \tau \leq T} G(t, \xi), \quad B = \max_{0 \leq \xi, \tau \leq T} G(t, \xi), \quad \sigma = A/B. \quad \text{(2.4)}
\]

Obviously \(B > A > 0\) and \(0 < \sigma < 1\).

Then, let \(X = C[0, T], \left\| x \right\| = \max\{|x(t)| : x(t) \in X, \ t \in [0, T]\}\), and \(P = \{x(t) \in X : x(t) \geq \sigma \left\| x \right\|, \ t \in [0, T]\}\). Moreover, for \(r > 0\), let \(\Omega_r = \{x \in X, \left\| x \right\| < r\}\) and

\[
m(r) = \min\{f(t, x_0, x_1, ..., x_n) : 0 \leq t \leq T, \ \sigma r \leq x_i \leq r, \ 0 \leq i \leq n\};
\]

\[
M(r) = \max\{f(t, x_0, x_1, ..., x_n) : 0 \leq t \leq T, \ 0 \leq x_i \leq r, \ 0 \leq i \leq n\}.
\]

Define operator:

\[
Q_\lambda x(t) = \lambda \int_0^T G(t, \xi)g(\xi)f(\xi, x(\xi - \tau_0(\xi)), x(\xi - \tau_1(\xi)), ..., x(\xi - \tau_n(\xi)))d\xi.
\]

Therefore, the fixed point of the operator equation \(x = Q_\lambda x\) is the \(T\)-periodic solution of \((1.1)\).
Lemma 2.2. \( Q_\lambda : P \to P \) is completely continuous and \( Q_\lambda (P) \subset P \).

Proof. Since
\[
Q_\lambda x(t) \geq \lambda A \int_0^T g(\xi) f(x, x(\xi - \tau_0(\xi)), x(\xi - \tau_1(\xi)), \ldots, x(\xi - \tau_n(\xi))) d\xi,
\]
and
\[
\| Q_\lambda x(t) \| \leq \lambda B \int_0^T g(\xi) f(x, x(\xi - \tau_0(\xi)), x(\xi - \tau_1(\xi)), \ldots, x(\xi - \tau_n(\xi))) d\xi,
\]
therefore
\[
Q_\lambda x(t) \geq \lambda A \frac{\| Q_\lambda x(t) \|}{\lambda B} = \sigma \| Q_\lambda x(t) \|.
\]

Then, according to the Arscoli-Arzele theorem, \( Q_\lambda \) is completely continuous. The proof is completed.

Lemma 2.3. If \( x \in P \cap \partial \Omega_r \) for \( r > 0 \), then
\[
\lambda Am(r) \int_0^T g(\xi) d\xi \leq \| Q_\lambda x(t) \| \leq \lambda BM(r) \int_0^T g(\xi) d\xi.
\]

Proof. Since \( x \in P \cap \partial \Omega_r \), it is clear that \( \sigma r \leq x(t) \leq r \), that is
\[
Q_\lambda x(t) \geq \lambda A \int_0^T g(\xi) m(r) d\xi
\]
\[
= \lambda Am(r) \int_0^T g(\xi) d\xi,
\]
hence \( \| Q_\lambda x(t) \| \geq \lambda Am(r) \int_0^T g(\xi) d\xi \). And
\[
Q_\lambda x(t) \leq \lambda B \int_0^T g(\xi) M(r) d\xi
\]
\[
= \lambda BM(r) \int_0^T g(\xi) d\xi,
\]
thus \( \| Q_\lambda x(t) \| \leq \lambda BM(r) \int_0^T g(\xi) d\xi \). The proof is finished.

Lemma 2.4 ([4,6]). Let \( X \) be a Banach space and \( P \) be a close convex cone in \( X \). \( \Omega_1, \Omega_2 \) are bounded open subsets of \( X \), \( \emptyset \in \Omega_1, \overline{\Omega_1} \subset \Omega_2 \). \( Q : P \cap \overline{(\Omega_2 \setminus \Omega_1)} \to P \) is a completely continuous operator. Assume that \( Q \) satisfies one of the following conditions:

(i) \( \| Qx \| \geq \| x \| \) for \( x \in P \cap \partial \Omega_1 \), \( \| Qx \| \leq \| x \| \) for \( x \in P \cap \partial \Omega_2 \);

(ii) \( \| Qx \| \leq \| x \| \) for \( x \in P \cap \partial \Omega_1 \), \( \| Qx \| \geq \| x \| \) for \( x \in P \cap \partial \Omega_2 \).

Then \( Q \) has at least one fixed point in \( P \cap \overline{(\Omega_2 \setminus \Omega_1)} \).

3. Main results

Let \( x = (x_0, x_1, \ldots, x_n) \in [0, +\infty)^{n+1}, \ x \triangleq \max\{x_0, x_1, \ldots, x_n\} \).
Next, make the following assumptions about $f$:

$$f^0 = \limsup_{x \to 0^+} \max_{t \in [0, T]} f(t, x), \quad f_\infty = \liminf_{x \to +\infty} \min_{t \in [0, T]} f(t, x),$$

$$f_0 = \liminf_{x \to 0^+} \min_{t \in [0, T]} f(t, x), \quad f_\infty = \limsup_{x \to +\infty} \max_{t \in [0, T]} f(t, x).$$

Assume that:

- $j_0$ is the number of zeros in set $\{f^0, f_\infty\}$; $j_\infty$ is the number of infinities in set $\{f^0, f_\infty\}$;
- $j'_0$ is the number of zeros in set $\{f_0, f_\infty\}$; $j'_\infty$ is the number of infinities in set $\{f_0, f_\infty\}$.

**Theorem 3.1.** Suppose that (A0) holds.

1. If $j_0 = 1$ or 2, when $\lambda > \frac{1}{Am(1)f_\infty g(\xi)d\xi} > 0$, equation (1.1) has at least $j_0$ positive $T$-periodic solution(s).
2. If $j'_\infty = 1$ or 2, when $0 < \lambda < \frac{1}{BM(1)f_0 g(\xi)d\xi}$, equation (1.1) has at least $j'_\infty$ positive $T$-periodic solution(s).
3. If $j'_0 = 0$ or $j_\infty = 0$, there is no positive $T$-periodic solution to equation (1.1) for sufficiently large or sufficiently small $\lambda > 0$, respectively.

**Proof.** For $\phi \in P \cap \partial \Omega_r$, define

$$\Phi(t) = (\phi(t - \tau_0(t)), \phi(t - \tau_1(t)), ..., \phi(t - \tau_n(t)))$$

and $\overline{\Phi(t)} = \max_{0 \leq t \leq n} \{\phi(t - \tau_i(t))\}$.

1. Let $r_1 = 1$, by Lemma 2.3, we can obtain that there exists $\lambda_0 = \frac{1}{Am(1)f_\infty g(\xi)d\xi} > 0$, such that

$$\|Q_\lambda \phi\| \geq \lambda Am(1) \int_0^T g(\xi)d\xi > \|\phi\|, \quad \phi \in P \cap \partial \Omega_1, \quad \lambda > \lambda_0.$$

If $f^0 = 0$, then we have $f(t, x) \leq \varepsilon x$ for $0 < x \leq r_2$ and $t \in [0, T]$, where $\varepsilon > 0$ satisfies $\lambda \varepsilon B \int_0^T g(\xi)d\xi < 1$, and $0 < r_2 < r_1 = 1$, obviously, $\Omega_{r_2} \subset \Omega_1$.

Then $0 < \sigma r_2 = \sigma \|\phi\| \leq \overline{\Phi(t)} \leq \|\phi\| = r_2$, for all $\phi \in P \cap \partial \Omega_{r_2}$, $t \in [0, T]$, thus

$$f(t, \Phi(t)) \leq \varepsilon \overline{\Phi(t)}.$$

From the definition of $Q_\lambda$, for $\phi \in P \cap \partial \Omega_{r_2}$, we can obtain

$$\|Q_\lambda \phi\| \leq \lambda \varepsilon B \int_0^T g(\xi)\overline{\Phi(t)}d\xi$$

$$\leq \lambda \varepsilon B \|\phi\| \int_0^T g(\xi)d\xi < \|\phi\|.$$  \hfill (3.1)

Thus, by Lemma 2.4(ii), the operator $Q_\lambda$ has at least one fixed point in $P \cap (\bar{\Omega}_1 \setminus \Omega_{r_2})$.

If $f_\infty = 0$, then there exists $H > 0$, such that $f(t, x) \leq \varepsilon x$ for $x \geq H$ and $t \in [0, T]$, where $\varepsilon > 0$ still satisfies $\lambda \varepsilon B \int_0^T g(\xi)d\xi < 1$. Moreover, select $r_3 = \max\{2, \frac{H}{\varepsilon}\}$, obviously, $\Omega_1 \subset \Omega_{r_3}$. 

Then \( \Phi(t) \geq \sigma \parallel \phi \parallel = \sigma r_3 \geq H \), for all \( \phi \in P \cap \partial \Omega_{r_3}, t \in [0, T] \), thus

\[
f(t, \Phi(t)) \leq \varepsilon \Phi(t).
\]

Then for \( \phi \in P \cap \partial \Omega_{r_3} \), we can obtain

\[
\parallel Q\lambda \phi \parallel \leq \lambda \varepsilon B \parallel \phi \parallel \int_0^T g(\xi)d\xi < \parallel \phi \parallel.
\]

Thus, by Lemma 2.4(i), the operator \( Q\lambda \) has at least one fixed point in \( P \cap (\Omega_{r_3} \setminus \Omega_1) \).

Above all, if \( f^0 = 0 \) and \( f^\infty = 0 \), the operator \( Q\lambda \) has at least two fixed points in \( P \cap (\Omega_{r_3} \setminus \Omega_{r_2}) \), that is, (1.1) has at least two positive \( T \)-periodic solutions for \( \lambda > \lambda_0 \).

(2) Let \( r_1 = 1 \), by Lemma 2.3, we can obtain that there exists \( \lambda_0 = \frac{1}{BM(1)f_0^3} g(\xi)d\xi > 0 \), such that

\[
\parallel Q\lambda \phi \parallel \leq \lambda BM(1) \int_0^T g(\xi)d\xi < \parallel \phi \parallel, \quad \phi \in P \cap \partial \Omega_1, \quad 0 < \lambda < \lambda_0.
\]

If \( f_0 = \infty \), then we have \( f(t, x) \geq \eta \varpi \) for \( 0 < \varpi \leq r_2 \) and \( t \in [0, T] \), where \( \eta > 0 \) satisfies \( \lambda \eta \sigma A \int_0^\infty g(\xi)d\xi > 1 \), and \( 0 < r_2 < r_1 = 1 \), obviously, \( \Omega_{r_2} \subset \Omega_1 \).

Then \( 0 < \sigma r_2 = \sigma \parallel \phi \parallel \leq \Phi(t) \leq \parallel \phi \parallel = r_2 \), for all \( \phi \in P \cap \partial \Omega_{r_2}, t \in [0, T] \), thus

\[
f(t, \Phi(t)) \geq \eta \Phi(t).
\]

From the definition of \( Q\lambda \), for \( \phi \in P \cap \partial \Omega_{r_2} \), we can obtain

\[
\parallel Q\lambda \phi \parallel \geq \lambda \eta A \int_0^T g(\xi)\Phi(\xi)d\xi
\]

\[
\geq \lambda \eta \sigma A \parallel \phi \parallel \int_0^T g(\xi)d\xi > \parallel \phi \parallel.
\]

Thus, by Lemma 2.4(i), the operator \( Q\lambda \) has at least one fixed point in \( P \cap (\Omega_1 \setminus \Omega_{r_2}) \).

If \( f_\infty = \infty \), then there exists \( H' > 0 \), such that \( f(t, x) \geq \eta \varpi \) for \( \varpi \geq H' \) and \( t \in [0, T] \), where \( \eta > 0 \) still satisfies \( \lambda \eta \sigma A \int_0^\infty g(\xi)d\xi > 1 \). Moreover, select \( r_3 = \max\{2, H'\} \), obviously, \( \Omega_1 \subset \Omega_{r_2} \).

Then \( \Phi(t) \geq \sigma \parallel \phi \parallel \Phi_3 \geq H' \), for all \( \phi \in P \cap \partial \Omega_{r_3}, t \in [0, T] \), thus

\[
f(t, \Phi(t)) \geq \eta \Phi(t).
\]

Then for \( \phi \in P \cap \partial \Omega_{r_3} \), we can obtain

\[
\parallel Q\lambda \phi \parallel \geq \lambda \eta \sigma A \parallel \phi \parallel \int_0^T g(\xi)d\xi > \parallel \phi \parallel.
\]

Thus, by Lemma 2.4(ii), the operator \( Q\lambda \) has at least one fixed point in \( P \cap (\Omega_{r_3} \setminus \Omega_1) \).

Above all, if \( f_0 = \infty \) and \( f_\infty = \infty \), the operator \( Q\lambda \) has at least two fixed points \( P \cap (\Omega_{r_3} \setminus \Omega_{r_2}) \), that is, (1.1) has at least two positive \( T \)-periodic solutions for \( 0 < \lambda < \lambda_0 \).
(3) If \( j_0^* = 0 \), then \( f_0 > 0 \) and \( f_\infty > 0 \), that is, there exist positive constants \( \omega_1, \omega_2, r_1, r_2 \), where \( r_1 < r_2 \), such that
\[
\begin{align*}
  f(t, x) &\geq \omega_1 \bar{\tau}, \quad \bar{\tau} \in [0, r_1], \quad t \in [0, T]; \\
  f(t, x) &\geq \omega_2 \bar{\tau}, \quad \bar{\tau} \in [r_2, +\infty), \quad t \in [0, T].
\end{align*}
\]
Select \( c_1 = \min \{ \omega_1, \omega_2, \min \{ \frac{f(t, \bar{\tau})}{\bar{\tau}} : t \in [0, T], \bar{\tau} \in [r_1, r_2] \} \} \). Thus \( c_1 > 0 \), and
\[
  f(t, x) \geq c_1 \bar{\tau}, \quad \forall x \in [0, +\infty)^{n+1}, \quad t \in [0, T].
\]
Assume \( \varphi(t) \) is the fixed point of the operator \( Q_\lambda \), then \( Q_\lambda \varphi(t) = \varphi(t), t \in [0, T] \). Moreover, define \( \varphi' = (\varphi(t-\tau_0(t)), \varphi(t-\tau_1(t)), \ldots, \varphi(t-\tau_n(t))) \), thus \( f(t, \varphi') \geq c_1 \varphi' \).

On the other hand, there exists \( \lambda_0 = \frac{1}{c_1 \sigma A \int_0^T g(\xi) d\xi} \), such that
\[
  \| \varphi \| = \| Q_\lambda \varphi \| \geq \lambda c_1 \sigma A \| \varphi \| \int_0^T g(\xi) d\xi > \| \varphi \|,
\]
for \( \lambda > \lambda_0 \). This is contradictory.

If \( f_\infty = 0 \), then \( f_0 < 0 \) and \( f_\infty < 0 \), that is, there exist positive constants \( \xi_1, \xi_2, r_1, r_2 \), where \( r_1 < r_2 \), such that
\[
\begin{align*}
  f(t, x) &\leq \xi_1 \bar{\tau}, \quad \bar{\tau} \in [0, r_1], \quad t \in [0, T]; \\
  f(t, x) &\leq \xi_2 \bar{\tau}, \quad \bar{\tau} \in [r_2, +\infty), \quad t \in [0, T].
\end{align*}
\]
Select \( c_2 = \max \{ \xi_1, \xi_2, \max \{ \frac{f(t, \bar{\tau})}{\bar{\tau}} : t \in [0, T], \bar{\tau} \in [r_1, r_2] \} \} \). Thus \( c_2 > 0 \), and
\[
  f(t, x) \leq c_2 \bar{\tau}, \quad \forall x \in [0, +\infty)^{n+1}, \quad t \in [0, T].
\]
Assume \( \psi(t) \) is the fixed point of the operator \( Q_\lambda \), then \( Q_\lambda \psi(t) = \psi(t), t \in [0, T] \). Moreover, define \( \psi' = (\psi(t-\tau_0(t)), \psi(t-\tau_1(t)), \ldots, \psi(t-\tau_n(t))) \), thus \( f(t, \psi') \leq c_2 \psi' \).

On the other hand, there exists \( \lambda_0 = \frac{1}{c_2 B \int_0^T g(\xi) d\xi} \), such that
\[
  \| \psi \| = \| Q_\lambda \psi \| \leq \lambda c_2 B \| \psi \| \int_0^T g(\xi) d\xi < \| \psi \|,
\]
for \( 0 < \lambda < \lambda_0 \). This is also contradictory.

This proves the theorem. \( \square \)

**Corollary 3.1.** Suppose that (A0) holds.

1. If there exists a \( c_1 > 0 \) such that \( f(t, x) \geq c_1 \bar{\tau} \) for \( t \in [0, T], x \in [0, +\infty)^{n+1} \), when \( \lambda > \frac{1}{c_1 \sigma A \int_0^T g(\xi) d\xi} \), equation (1.1) has no positive T-periodic solution.

2. If there exists a \( c_2 > 0 \) such that \( f(t, x) \leq c_2 \bar{\tau} \) for \( t \in [0, T], x \in [0, +\infty)^{n+1} \), when \( 0 < \lambda < \frac{1}{c_2 B \int_0^T g(\xi) d\xi} \), equation (1.1) has no positive T-periodic solution.

**Theorem 3.2.** Suppose that (A0) holds and \( j_0 = j_0^* = j_\infty = j_\infty^* = 0 \).

1. If \( \int_0^T B < \int_0^T \sigma A \), when \( \frac{1}{\int_0^T \sigma A \int_0^T g(\xi) d\xi} \leq \lambda < \frac{1}{\int_0^T B \int_0^T g(\xi) d\xi} \), equation (1.1) has at least a positive T-periodic solution.

2. If \( \int_0^T \sigma A > \int_0^T B \), when \( \frac{1}{\int_0^T \sigma A \int_0^T g(\xi) d\xi} \leq \lambda < \frac{1}{\int_0^T B \int_0^T g(\xi) d\xi} \), equation (1.1) has at least a positive T-periodic solution.
Thus, we have

$$\Phi(t) = (\phi(t - \tau_0(t)), \phi(t - \tau_1(t)), ..., \phi(t - \tau_n(t)))$$

and $\Phi(t) = \max_{0 \leq i \leq n} \{\phi(t - \tau_i(t))\}$, for $t \in \mathbb{R}$.

(1) Assume $f^0 B < f^\infty A$, then $f^0 < f^\infty$, when $f^{0\sigma A} \int_0^T g(\xi) d\xi < \lambda < f^{\infty B} \int_0^T g(\xi) d\xi$, then there exists $0 < \varepsilon < f^\infty$, such that

$$\frac{1}{(f^\infty - \varepsilon) \sigma A} \int_0^T g(\xi) d\xi < \lambda < \frac{1}{(f^0 + \varepsilon) B} \int_0^T g(\xi) d\xi.$$ 

for the above $\varepsilon$, choose $r_1 > 0$, such that $f(t, x) \leq (f^0 + \varepsilon) \overline{x}$ for $x \in [0, r_1]$, $t \in [0, T]$. Thus, for all $\phi \in P \cap \partial \Omega_{r_1}$, we have $0 \leq \Phi(t) \leq r_1$, that is

$$f(t, \Phi(t)) \leq (f^0 + \varepsilon) \overline{\Phi(t)}.$$ 

Thus, we have

$$\| Q_\lambda \phi \| \leq \lambda (f^0 + \varepsilon) B \| \phi \| \int_0^T g(\xi) d\xi < \| \phi \|,$$ 

for all $\phi \in P \cap \partial \Omega_{r_1}$.

On the other hand, there exists $H_1 > 0$, such that $f(t, x) \geq (f^\infty - \varepsilon) \overline{x}$ for $\overline{x} \geq H_1$ and $t \in [0, T]$. Moreover, select $r_2 = \max\{2r_1, \frac{H_1}{\sigma}\}$, obviously, $\Omega_{r_1} \subset \Omega_{r_2}$.

Then $\Phi(t) \geq \sigma \| \phi \| = \sigma r_2 \geq H_1$, for all $\phi \in P \cap \partial \Omega_{r_2}$, $t \in [0, T]$. Thus

$$f(t, \Phi(t)) \geq (f^\infty - \varepsilon) \overline{\Phi(t)}.$$ 

Then, for $\phi \in P \cap \partial \Omega_{r_2}$, we can obtain

$$\| Q_\lambda \phi \| \geq \lambda (f^\infty - \varepsilon) A \| \phi \| \int_0^T g(\xi) d\xi > \| \phi \|.$$ 

Thus, by Lemma 2.4(ii), the operator $Q_\lambda$ has at least one fixed point in $P \cap (\overline{\Omega_{r_2}} \setminus \Omega_1)$, that is, (1.1) has at least a positive $T$-periodic solution for $\frac{1}{f^{\infty B} \int_0^T g(\xi) d\xi} < \lambda < \frac{1}{f^{0\sigma A} \int_0^T g(\xi) d\xi}$.

(2) Assume $f^\infty A > f^\infty B$, then $f^0 > f^\infty$, when $f^{0\sigma A} \int_0^T g(\xi) d\xi < \lambda < f^{\infty B} \int_0^T g(\xi) d\xi$, then there exists $0 < \varepsilon < f_0$, such that

$$\frac{1}{(f_0 - \varepsilon) \sigma A} \int_0^T g(\xi) d\xi < \lambda < \frac{1}{(f^\infty + \varepsilon) B} \int_0^T g(\xi) d\xi.$$ 

for the above $\varepsilon$, choose $r_1 > 0$, such that $f(t, x) \geq (f_0 - \varepsilon) \overline{x}$ for $x \in [0, r_1]$, $t \in [0, T]$. Thus, for all $\phi \in P \cap \partial \Omega_{r_1}$, we have $0 \leq \Phi(t) \leq r_1$, that is

$$f(t, \Phi(t)) \geq (f_0 - \varepsilon) \overline{\Phi(t)}.$$ 

Thus, we have

$$\| Q_\lambda \phi \| \geq \lambda \sigma (f_0 - \varepsilon) A \| \phi \| \int_0^T g(\xi) d\xi > \| \phi \|,$$ 

(3.4)
for all \( \phi \in P \cap \partial \Omega_{r_1} \).

On the other hand, there exists \( H_2 > 0 \), such that \( f(t, x) \leq (f^\infty + \varepsilon)T \) for \( x \geq H_2 \) and \( t \in [0, T] \). Moreover, select \( r_2 = \max \{2r_1, \frac{H_2}{\sigma} \} \), obviously, \( \Omega_{r_2} \subset \Omega_{r_2} \).

Then \( \Phi(t) \geq \sigma \| \phi \| \sigma r_2 \geq H_2 \), for all \( \phi \in P \cap \partial \Omega_{r_2}, t \in [0, T] \). Thus
\[
\int_0^T f(t, \Phi(t)) \leq (f^\infty + \varepsilon)\Phi(T).
\]

Then, for \( \phi \in P \cap \partial \Omega_{r_2} \), we can obtain
\[
\| Q_\lambda \phi \| \leq \lambda (f^\infty + \varepsilon)B \| \phi \| \int_0^T g(\xi)d\xi < \| \phi \| .
\]

Thus, by Lemma 2.4(i), the operator \( Q_\lambda \) has at least one fixed point in \( P \cap (\overline{\Omega_{r_2} \setminus \Omega_1}) \), which is the positive \( T \)-periodic solution of (1.1) for \( \frac{1}{\int_0^T g(\xi)d\xi} \leq \lambda < \frac{\lambda}{\int_0^\infty g(\xi)d\xi} \).

The proof is completed. \( \square \)

**Corollary 3.2.** Suppose \( h(t) \equiv 0, a(t) \neq 0 \), then (A0) holds if \( \int_0^T a(t)d\xi \geq 0 \) and \( \int_0^T [a(t)]_+ d\xi \leq \frac{T}{4} \).

4. Example

**Example 4.1.** Consider the following equations:
\[
\phi'' + 2\phi' + \phi = \lambda(1 + \sin 8t)\frac{2 + \cos 8t}{2 + \phi(t - \tau(t))}, \quad n > 0, \quad (4.1)
\]
where \( h(t) = 2, a(t) = 1, g(t) = 1 + \sin 8t, f(t, x) = \frac{2 + \cos 8t}{2 + x^n} \), obviously, they are all \( T = \frac{T}{4} \) periodic functions in \( t \), moreover, \( \tau(t) \) is an arbitrary \( \frac{T}{4} \)-periodic continuous function.

Through some calculations, the conditions of Lemma 2.1 are satisfied,
\[
A = \frac{\frac{T}{4}}{[\exp(\frac{T}{4}) - 1]^2}, \quad B = \frac{\frac{T}{4}\exp(\frac{T}{4})}{[\exp(\frac{T}{4}) - 1]^2}, \quad \sigma = \exp(-\frac{\pi}{2}),
\]
and
\[
\int_0^\frac{T}{4} g(\xi)d\xi = \int_0^\frac{T}{4} (1 + \sin 8\xi)d\xi = \frac{\pi}{4},
\]
\[
m(1) = \min \{ f(t, x), 0 \leq t \leq \frac{\pi}{4}, \exp(-\frac{\pi}{2}) \leq x \leq 1 \} = \min \{ \frac{2 + \cos 8t}{2 + x^n}, 0 \leq t \leq \frac{\pi}{4}, \exp(-\frac{\pi}{2}) \leq x \leq 1 \} = \frac{1}{3},
\]
\[
M(1) = \max \{ f(t, x), 0 \leq t \leq \frac{\pi}{4}, 0 \leq x \leq 1 \} = \max \{ \frac{2 + \cos 8t}{2 + x^n}, 0 \leq t \leq \frac{\pi}{4}, 0 \leq x \leq 1 \} = \frac{3}{2},
\]
Moreover,
\[
f^0 = \lim_{x \to 0^+} \sup \max_{t \in [0, \frac{T}{4}]} \frac{f(t, x)}{x} = \lim_{x \to 0^+} \sup \max_{t \in [0, \frac{T}{4}]} \frac{2 + \cos 8t}{x(2 + x^n)} = \infty,
\]
Periodic solutions of the equation

\[ f_\infty = \liminf_{x \to +\infty} \min_{t \in [0, \frac{\pi}{4}]} \frac{f(t, x)}{x} = \liminf_{x \to +\infty} \min_{t \in [0, \frac{\pi}{4}]} \frac{2 + \cos 8t}{x(2 + x^n)} = 0, \]

\[ f_0 = \liminf_{x \to 0^+} \min_{t \in [0, \frac{\pi}{4}]} \frac{f(t, x)}{x} = \liminf_{x \to 0^+} \min_{t \in [0, \frac{\pi}{4}]} \frac{2 + \cos 8t}{x(2 + x^n)} = \infty, \]

\[ f^\infty = \limsup_{x \to +\infty} \max_{t \in [0, \frac{\pi}{4}]} \frac{f(t, x)}{x} = \limsup_{x \to +\infty} \max_{t \in [0, \frac{\pi}{4}]} \frac{2 + \cos 8t}{x(2 + x^n)} = 0. \]

Thus, \( j_0 = 1, j'_\infty = 1 \), furthermore,

\[ \lambda_{01} = \frac{1}{A_{m(1)}} \int_0^\frac{\pi}{4} g(\xi) d\xi = \frac{48[\exp(\frac{\pi}{4})-1]^2}{\pi^2}, \quad \lambda_{02} = \frac{1}{B_{m(1)}} \int_0^\frac{\pi}{4} g(\xi) d\xi = \frac{32[\exp(\frac{\pi}{4})-1]^2}{3\pi^2 \exp(\frac{\pi}{4})}. \]

Therefore, by Theorem 3.1(1), Eq. (4.1) has at least a positive \( \frac{\pi}{4} \)-periodic solution for \( \lambda > \lambda_{01} = \frac{48[\exp(\frac{\pi}{4})-1]^2}{\pi^2} \), and by Theorem 3.1(2), Eq. (4.1) has at least a positive \( \frac{\pi}{4} \)-periodic solution for \( 0 < \lambda < \lambda_{02} = \frac{32[\exp(\frac{\pi}{4})-1]^2}{3\pi^2 \exp(\frac{\pi}{4})} \).

When \( n = 5, \tau = 0.7 \) and \( \lambda = 10 \), now \( \lambda > \lambda_{01} \), Figure 1 is the numerical simulation of Example 4.1.

**Figure 1.** The numerical simulation of Example 4.1.

**Example 4.2.** Now consider the following equations:

\[ \phi'' + 2\phi' + \phi = \lambda(1 + \sin 8t) \frac{\phi(t - \tau(t))^2(2 + \cos 8t)}{2 + \phi(t - \tau(t))^6}, \quad (4.2) \]

note that \( f(t, x) = \frac{x^2(2 + \cos 8t)}{2 + x^n} \), moreover, \( \tau(t) \) is still an arbitrary \( \frac{\pi}{4} \)-periodic continuous function.

Now the conditions of Lemma 2.1 are still satisfied,

\[ A = \frac{\frac{\pi}{4}}{[\exp(\frac{\pi}{4}) - 1]^2}, \quad B = \frac{\frac{\pi}{4} \exp(\frac{\pi}{4})}{[\exp(\frac{\pi}{4}) - 1]^2}, \quad \sigma = \exp(-\frac{\pi}{2}), \]

and

\[ \int_0^\frac{\pi}{4} g(\xi) d\xi = \int_0^\frac{\pi}{4} (1 + \sin 8\xi) d\xi = \frac{\pi}{4}. \]
\[ m(1) = \min \{ f(t, x), 0 \leq t \leq \frac{\pi}{4}, \exp(-\frac{\pi}{2}) \leq x \leq 1 \} \]
\[ = \min \{ \frac{x^2(2 + \cos 8t)}{2 + x^6}, 0 \leq t \leq \frac{\pi}{4}, \exp(-\frac{\pi}{2}) \leq x \leq 1 \} \]
\[ = \frac{1}{2} \exp(\pi) + \exp(-2\pi), \]

moreover,
\[ f^0 = \lim \sup_{x \to 0^+} \max_{t \in [0, \frac{\pi}{4}]} f(t, x) = \lim \sup_{x \to 0^+} \max_{t \in [0, \frac{\pi}{4}]} \frac{x^2(2 + \cos 8t)}{x(2 + x^6)} \]
\[ = \lim \sup_{x \to 0^+} \max_{t \in [0, \frac{\pi}{4}]} \frac{x(2 + \cos 8t)}{2 + x^6} = 0, \]
\[ f^\infty = \lim \sup_{x \to +\infty} \max_{t \in [0, \frac{\pi}{4}]} f(t, x) = \lim \sup_{x \to +\infty} \max_{t \in [0, \frac{\pi}{4}]} \frac{x^2(2 + \cos 8t)}{x(2 + x^6)} \]
\[ = \lim \sup_{x \to +\infty} \max_{t \in [0, \frac{\pi}{4}]} \frac{x(2 + \cos 8t)}{2 + x^6} = 0. \]

Thus, \( j_0 = 2, \)
\[ \lambda_{01} = \frac{1}{Am(1) \int_0^T g(\xi)d\xi} = \frac{16[\exp(\frac{\pi}{4}) - 1]^2[2\exp(\pi) + \exp(-2\pi)]}{\pi^2}. \]

Therefore, by Theorem 3.1(1), Eq.(4.2) has at least two positive \( \frac{\pi}{4} - \)periodic solutions for \( \lambda > \lambda_{01} = \frac{16[\exp(\frac{\pi}{4}) - 1]^2[2\exp(\pi) + \exp(-2\pi)]}{\pi^2}. \)

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**References**


Periodic solutions of the equation


