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CRANK-NICOLSON DIFFERENCE SCHEME FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH THE RIESZ SPACE FRACTIONAL DERIVATIVE*

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Abstract This paper studied the Crank-Nicolson(CN) difference scheme for the derivative nonlinear Schrödinger equation with the Riesz space fractional derivative, which generalized the classical Schrödinger equation that was used as a model in quantum mechanics. The existence of this difference solution is proved by the Brouwer fixed point theorem. Since the difference solution of the equation satisfies the mass conservation law, the corresponding convergence is also investigated in the L_2 norm, which turns out to be the second order accuracy in both temporal and space directions. Especially when the fractional order equals to two, all those results are in accordance with the conclusions for the difference solution developed for the non-fractional derivative Schrödinger equation. Finally, some numerical examples are carried out and further verified the theoretical results.

Keywords Derivative Schrödinger equation, Riesz space fractional derivative, Crank-Nicolson scheme, convergence.

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1. Introduction

It is well known that the Schrödinger equation (SE) is one of the most important equations in quantum mechanics, and the standard Schrödinger equation was derived by R. P. Feynman and A. R. Hibbs from the path integrals over Brownian paths [5]. In 1970s, another kind of Schrödinger wave equation, which was called derivative nonlinear Schrödinger equation (DNLSE) as the form

$$\mathrm{i}u_t + u_{xx} - \mathrm{i}(|u|^2u)_x = 0, \tag{1.1}$$

was derived for studying the propagation of the circular polarised nonlinear Alfvén waves in magnetized plasma with a constant magnetic field [19,20,24]. Both the classical Schrödinger equation and the DNLSE (1.1) have been extensively studied with

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respect to the mathematical theory and physical applications. In 2000, N. Laskin generalized the classical Schrödinger equation to the fractional Schrödinger equation(FSE), which involves the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}(1<\alpha\leq 2)$ by replacing the Brownian trajectories in Feynman path integrals by the Lévy flights [16, 17]. This generalization extends the standard quantum mechanics to the fractional ones. Since the fractional quantum mechanics play a very vital role in the quantum phenomena, it is natural to generalize the derivative nonlinear Schrödinger equation into fractional case. Thus in this paper, we are going to consider the following fractional derivative nonlinear Schrödinger equation(FDNLSE)

$$\mathrm{i}u_t-(-\Delta)^{\frac{\alpha}{2}}u-\mathrm{i}(|u|^2u)_x=0, \tag{1.2}$$

where $1<\alpha\leq 2$, and $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian which is defined as a pseudo-differential operator with the symbol $-|\xi|^{\alpha}$

$$(-\Delta)^{\frac{\alpha}{2}}u(x,t)=\mathcal{F}^{-1}(|\xi|^{\alpha}\hat{u}(\xi,t)),$$

where $\mathcal{F}(\cdot)$ denotes the usual Fourier transform.

When $\alpha=2$ and in one dimension, the FDNLSE (1.2) is reduced to the usual DNLSE (1.1). There are various papers devoted to investigate the mathematical theoretical properties, exact and numerical solutions for the DNLSE (1.1). D. J. Kaup and A. C. Newell [15] showed that the equation (1.1) was completely integrable, and they succeeded to apply the inverse scattering techniques to obtain one-soliton as well as the infinite family of conservation laws. Some local and global well-posedness results in different function spaces have been also obtained by using mass and energy conservation laws, the proper gauge transformations and other methods, as seen in [10–12, 21, 28] and reference therein. For the numerical solutions of the DNLSE (1.1), M. S. Ismail and T. R. Taha introduced a finite difference method for the numerical simulation, which was second-order in space and conserved the energy exactly [14]. T. C. Wang *et al*. proposed some symplectic and conservative difference schemes for the coupled nonlinear Schrödinger system, which were also second order convergence [32,33]. For some other numerical results, we refer readers to [4, 27] and reference therein.

In the fractional case, some researches about the fractional Schrödinger equation, which possesses the term $|u|^2u$ but the derivative term $(|u|^2u)_x$ have been processed by many researchers, such as the global existence [7, 8], complex dynamic behavior [40], ground state solution [6] and so on. From the numerical point of view, there are still several methods that have been developed to solve the fractional Schrödinger equation with the term $|u|^2u$. D. L. Wang *et al*. proposed some nonlinear and linearized difference schemes [29–31], P. D. Wang and C. M. Huang constructed an energy conservative nonlinear difference scheme and a linearized difference scheme to solve the equation numerically, respectively [34, 35]. Some other techniques, such as the finite element methods [18], collocation method [2], compact difference method [42] have been developed in the literature. For more numerical methods and simulations, one can see [23, 36, 39, 41] and reference therein.

Although these above mentioned methods are interesting and instructive, they were proposed to handle the fractional Schrödinger equation with the normal term $|u|^2u$. However, to our best knowledge, there are very few works concerning on the fractional derivative nonlinear Schrödinger equation (1.2), which indicates that the equation has the derivative term $(|u|^2u)_x$, even the basic mathematical results.

Since the emergence of the derivative term, it is different from the usual fractional one and will cause much more difficulties in handling this nonlinear term. Thus in this paper, we are going to study the fractional derivative nonlinear Schrödinger equation (1.2) mathematically. As a start point, we will investigate the equation from the numerical point. More specifically speaking, we consider the following FDNLSE with the Riesz space fractional derivative ($1 < \alpha \le 2$)

$$\mathrm{i}u_t - (-\Delta)^{\frac{\alpha}{2}} u - \mathrm{i}(|u|^2 u)_x = 0, \quad a < x < b, \quad 0 < t \le T, \tag{1.3}$$

with the initial condition

$$u(x, 0) = u_0(x), \quad a < x < b, \tag{1.4}$$

and the Dirichlet boundary condition

$$u(a, t) = u(b, t) = 0, \quad 0 \le t \le T, \tag{1.5}$$

where $u = u(x, t)$ is the complex function on $(x, t) \in [a, b] \times [0, T]$, $T \ge 0$. The initial condition $u_0(x)$ is a given smooth function vanishing at the end points $x = a$ and $x = b$.

The rest of paper is organized as follows. In Section 2, we briefly give some notations and preliminaries. In Section 3, some Crank-Nicolson(CN) difference scheme for the FDNLSE is proposed, and the existence of this difference solution is proved by the Brouwer fixed point theorem. The convergence of the CN scheme is also investigated in the L_2 norm by the use of the mass conservation law and some delicate estimates. Furthermore, the uniqueness of the difference solution is also presented. In Section 4, two numerical examples are present and verified the correction of the theoretical analysis. In the last Section 5, we make some conclusions.

2. Notations and preliminaries

In this section, we will give out some notations and preliminaries for the fractional derivative and difference scheme. First of all, it is well known that the fractional Laplace operator $-(-\Delta)^{\frac{\alpha}{2}}$ is equivalent to the Riesz fractional derivative operator $\frac{\partial^\alpha}{\partial|x|^\alpha}$ under homogeneous Dirichlet boundary conditions [25]. And the Riesz fractional derivative for $1 < \alpha \le 2$ is defined as [26]

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha} = -\frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{-\infty}^{\infty} |x-\xi|^{1-\alpha} u(\xi, t) d\xi, \tag{2.1}$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Gamma function.

There are several numerical methods to approximate the Riesz fractional derivative, such as the standard and shifted Grünwald formula approximations [37], matrix transform method (MTM) [13], finite element method [38] and so on. Here we mainly adopt the fractional centered difference proposed by M. D. Ortigueira [22]. Omitting the time variable and denoting $u(x) = u(x, t)$, the fractional centered difference for $\alpha > -1$ is defined as

$$\Delta_h^\alpha u(x) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\alpha/2 - k + 1)\Gamma(\alpha/2 + k + 1)} u(x - kh). \tag{2.2}$$

And it is also shown that for $1 < \alpha \leq 2$, there holds

$$\frac{\partial^\alpha u(x)}{\partial |x|^\alpha} = -\lim_{h\to 0}\frac{\Delta_h^\alpha u(x)}{h^\alpha} = -\lim_{h\to 0}\sum_{k=-\infty}^{+\infty}\frac{c_k}{h^\alpha}u(x-kh), \tag{2.3}$$

where $c_k = \frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)}$ $(k = 0, \pm 1, \pm 2, \cdots)$ are the coefficients. First for the coefficients c_k, we have the following properties.

Lemma 2.1 ([3]). *Let c_k be the coefficients of the centered finite difference approximation (2.3) for $k = 0, \pm 1, \pm 2, \cdots$ and $\alpha > -1$. Then*

$$c_0 \geq 0, \quad c_{-k} = c_k \leq 0, \text{ for all } |k| \geq 1. \tag{2.4}$$

Lemma 2.2. *Let $u \in C^5(\mathbb{R})$ and all derivatives up to order five belong to $L_1(\mathbb{R})$, and the fractional centered difference $\Delta_h^\alpha u(x)$ be given in (2.2). Then for $1 < \alpha \leq 2$, we have*

$$-h^{-\alpha}\Delta_h^\alpha u(x) = \frac{\partial^\alpha u(x)}{\partial |x|^\alpha} + O(h^2), \tag{2.5}$$

as $h \to 0$ and $\frac{\partial^\alpha u(x)}{\partial |x|^\alpha}$ is the Riesz fractional derivative.

Proof. This lemma can be proved by using the Fourier transform. For detailed proof, we refer readers to [3]. □

Remark 2.1. If $\alpha = 2$, there yields $c_0 = 2, c_{-1} = c_1 = -1$ and $c_k = 0, (k = \pm 2, \pm 3, \cdots)$, and the difference (2.2) coincides with the classical centered second difference estimator for the second derivative.

For the difference scheme, we introduce some notations. Let J, N be any positive integers, and $h = \frac{b-a}{J}, \tau = \frac{T}{N}$. Define $\Omega_h = \{x_j = jh; j = 0, 1, \cdots, J\}, \Omega_\tau = \{t_n = n\tau; n = 0, 1, \cdots, N\}$, and $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Let $\omega = \{w_j^n; j = 0, 1, \cdots, J, n = 0, 1, \cdots, N\}$ be a discrete function on $\Omega_{h\tau}$ and $\Omega_{h,0} = \{w | w = (w_0, w_1, \cdots, w_J), w_0 = w_J = 0\}$ be the complex grid function space on Ω_h and $\Omega^n = (w_0^n, w_1^n, \cdots, w_J^n)$. Meanwhile, one denotes that

$$w_j^{n+\frac{1}{2}} = \frac{w_j^{n+1} + w_j^n}{2}, \qquad (w_j^n)_x = \frac{w_{j+1}^n - w_j^n}{2h}, \qquad (w_j^n)_{\hat{x}} = \frac{w_{j+1}^n - w_{j-1}^n}{2h},$$

$$(w_j^n)_t = \frac{w_j^{n+1} - w_j^n}{\tau}, \qquad \langle w^n, v^n\rangle = h\sum_{j=1}^{J-1} w_j^n \cdot \overline{v_j^n}, \quad \|w^n\|^2 = \langle w^n, w^n\rangle,$$

$$\|w_x^n\|^2 = h\sum_{j=1}^{J-1}|(w_j^n)_x|^2, \quad \|w_{\hat{x}}^n\|^2 = h\sum_{j=1}^{J-1}|(w_j^n)_{\hat{x}}|^2, \quad \|w^n\|_\infty = \max_{0\leq j\leq J}|w_j^n|.$$

Without any ambiguity, we denote generic positive constants by C_k, C and so on, which may have different values in different occurrences.

In what follows, we will use the following inequalities and lemmas.

Lemma 2.3 (Sobolev's estimate, [43]). *For any discrete function $\{u_j^n | j = 0, 1, \cdots, J\}$ on the finite interval $[x_L, x_R]$, there is the inequality*

$$\|u^n\|_\infty \leq C_0\|u^n\|^{\frac{1}{2}}(\|u_x^n\| + \|u^n\|)^{\frac{1}{2}},$$

or

$$\|u^n\|_\infty \le \varepsilon \|u_x^n\| + C(\varepsilon)\|u^n\|,$$

where C_0, ε and $C(\varepsilon)$ are three constants independent of function $\{u_j^n\}$ and the step length h. ε can be any small and $C(\varepsilon)$ is a constant dependent on ε.

Lemma 2.4 ([27]). *For any complex functions U, V, u and v, we have*

$$\left||U|^2V - |u|^2v\right| \le \left(\max\{|U|, |V|, |u|, |v|\}\right)^2 \cdot \left(2|U-u| + |V-v|\right). \tag{2.6}$$

Proof. By direct calculation, one has

$$\begin{aligned}\left||U|^2V - |u|^2v\right| &= \left||U|^2(V-v) + (|U|^2 - |u|^2)v\right| \\ &= \left||U|^2(V-v) + [U(\overline{U} - \overline{u}) + (U-u)\right| \\ &= |U|^2|V-v| + |Uv||\overline{U} - \overline{u}| + |\overline{u}v||U-u| \\ &\le \left(\max\{|U|, |V|, |u|, |v|\}\right)^2 \cdot \left(2|U-u| + |V-v|\right),\end{aligned}$$

which concludes (2.6). □

Lemma 2.5 ([9]). *Suppose that $a > 0, b > 0, c > 0, b^2 - 4ac > 0$ and $-az^2 + bz - c \le 0$, then there holds*

$$z \le \frac{2c}{b}, \qquad or \qquad z \ge \frac{b}{a} - \frac{2c}{b}.$$

Lemma 2.6 (Brouwer fixed point theorem, [1]). *Let $H(, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space, $\|\cdot\|$ the associated norm, and $g: H \to H$ be continuous. Assume moreover that*

$$\exists \alpha > 0,\ \forall z \in H,\ \|z\| = \alpha,\ \mathrm{Re}(g(z), z) \ge 0.$$

Then, there exists a $z^* \in H$ such that $g(z^*) = 0$ and $\|z^*\| \le \alpha$.

3. Crank-Nicolson difference scheme

In this section, we will propose some Crank-Nicolson(CN) difference scheme for the FDNLSE (1.3)-(1.5), and analyze its existence, convergence and uniqueness. Firstly, the equation (1.3) can be rewritten as

$$u_t + \mathrm{i}(-\Delta)^{\frac{\alpha}{2}}u - (|u|^2)_x u - |u|^2 u_x = 0. \tag{3.1}$$

Let $u_j^n = u(x_j, t_n)$ be the true solution of $u(x,t)$ at $x = x_j, t = t_n$, and U_j^n be the numerical approximation of $u(x_j, t_n)$. Then combining (2.3), we adopt the following CN difference scheme for the FDNLSE (1.3)-(1.5)

$$\begin{aligned}&(U_j^n)_t + \frac{\mathrm{i}}{h^\alpha}\sum_{l=1}^{J-1} c_{j-l}U_l^{n+\frac{1}{2}} - (|U_j^{n+\frac{1}{2}}|^2)_{\hat{x}}U_j^{n+\frac{1}{2}} \\ &- \frac{\overline{U_{j+1}^{n+\frac{1}{2}}} + \overline{U_{j-1}^{n+\frac{1}{2}}}}{2}U_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\hat{x}} = 0,\end{aligned} \tag{3.2}$$

$$U_j^0 = u_0(x_j), \quad U_0^n = U_J^n = 0, \tag{3.3}$$

where $1 \le j \le J-1$, and $1 \le n \le N-1$.

In what follows, we will discuss the existence, convergence and uniqueness of the CN difference scheme (3.2) and (3.3).

3.1. Existence

Before proving the existence of the numerical solution for the CN difference scheme (3.2) and (3.3), we have the following lemma for the fractional derivative part.

Lemma 3.1. *For the functions U_j^n, there holds*

$$\mathrm{Im}\left(\sum_{j=1}^{J-1}\sum_{l=1}^{J-1} c_{j-l} U_l^{n+\frac{1}{2}} \overline{U_j^{n+\frac{1}{2}}}\right) = 0, \tag{3.4}$$

where Im *means taking the imaginary part.*

Proof. By direct calculation, we have

$$\begin{aligned}
&\sum_{j=1}^{J-1}\sum_{l=1}^{J-1} c_{j-l} U_l^{n+\frac{1}{2}} \overline{U_j^{n+\frac{1}{2}}}\\
=&c_0 \sum_{j=1}^{J-1} U_j^{n+\frac{1}{2}} \overline{U_j^{n+\frac{1}{2}}} + \sum_{l=1}^{J-2}\sum_{j=l+1}^{J-1}\left(c_{j-l} U_l^{n+\frac{1}{2}} \overline{U_j^{n+\frac{1}{2}}} + c_{l-j} U_j^{n+\frac{1}{2}} \overline{U_l^{n+\frac{1}{2}}}\right) \qquad (3.5)\\
=&\frac{c_0}{h}\|U^{n+\frac{1}{2}}\|^2 + 2\mathrm{Re}\sum_{l=1}^{J-2}\sum_{j=l+1}^{J-1}\left(c_{j-l} U_l^{n+\frac{1}{2}} \overline{U_j^{n+\frac{1}{2}}}\right),
\end{aligned}$$

where we used the results of $c_{l-j} = c_{j-l}$ from Lemma 2.1 and $\overline{U_l^{n+\frac{1}{2}} \overline{U_j^{n+\frac{1}{2}}}} = U_j^{n+\frac{1}{2}} \overline{U_l^{n+\frac{1}{2}}}$. Thus (3.5) implies the result (3.4). □

Now we have the following existence result for the numerical solutions.

Theorem 3.1 (Existence)**.** *Let $Z_\Delta = \{s | s \in \Omega_{h,0}\}$, then the solution U^n of the difference scheme (3.2) and (3.3) exists, and $U^n \in Z_\Delta$.*

Proof. Here we mainly employ the Brouwer fixed point theorem and induction argument to prove this theorem. First from the original problem (1.3)-(1.5), it is easy to find that $U^0 \in Z_\Delta$ exists and satisfies the difference scheme. Now assume that there exist $U^0, U^1, \cdots, U^n \in Z_\Delta$ which satisfy the difference scheme (3.2)(3.3) for $n \le N-1$, we need to prove that there exists $U^{n+1} \in Z_\Delta$ which also satisfies the difference scheme.

Since $U_j^{n+1} = 2U_j^{n+\frac{1}{2}} - U_j^n$, and for fixed n, we can rewrite (3.2) as the following form

$$\begin{aligned}
U_j^{n+\frac{1}{2}} = U_j^n - \frac{\mathrm{i}\tau}{2h^\alpha}\sum_{l=1}^{J-1} c_{j-l} U_l^{n+\frac{1}{2}} + \frac{\tau}{2}(|U_j^{n+\frac{1}{2}}|^2)_{\hat{x}} U_j^{n+\frac{1}{2}}\\
+ \frac{\tau}{2}\frac{\overline{U_{j+1}^{n+\frac{1}{2}}} + \overline{U_{j-1}^{n+\frac{1}{2}}}}{2} U_j^{n+\frac{1}{2}} (U_j^{n+\frac{1}{2}})_{\hat{x}}.
\end{aligned} \tag{3.6}$$

Let $s_j = U_j^{n+\frac{1}{2}}$ and define the mapping $s_j \to \omega(s)_j$ on Z_Δ by

$$\omega(s)_j = s_j - U_j^n + \frac{\mathrm{i}\tau}{2h^\alpha}\sum_{l=1}^{J-1} c_{j-l} s_l - \frac{\tau}{2}(|s_j|^2)_{\hat{x}} s_j - \frac{\tau}{2}\frac{\overline{s_{j+1}} + \overline{s_{j-1}}}{2} s_j (s_j)_{\hat{x}}. \tag{3.7}$$

Multiplying the equation (3.7) by $\overline{s_j}h$, summing from $j=1$ to $J-1$ and taking the real part, we can obtain

$$\begin{aligned}\operatorname{Re}\left(h\sum_{j=1}^{J-1}\omega(s)_j\overline{s_j}\right) &= \operatorname{Re}\left(h\sum_{j=1}^{J-1}s_j\overline{s_j}\right) - \operatorname{Re}\left(h\sum_{j=1}^{J-1}U_j^n\overline{s_j}\right)\\ &\quad + \operatorname{Re}\left(h\sum_{j=1}^{J-1}\frac{\mathrm{i}\tau}{2h^\alpha}\sum_{l=1}^{J-1}c_{j-l}s_l\overline{s_j}\right)\\ &\quad - \operatorname{Re}\left(h\sum_{j=1}^{J-1}\frac{\tau}{2}(|s_j|^2)_{\hat{x}}s_j\overline{s_j}\right)\\ &\quad - \operatorname{Re}\left(h\sum_{j=1}^{J-1}\frac{\tau}{2}\frac{\overline{s_{j+1}}+\overline{s_{j-1}}}{2}s_j(s_j)_{\hat{x}}\overline{s_j}\right).\end{aligned} \tag{3.8}$$

From Lemma 3.1, there holds

$$\operatorname{Re}\left(h\sum_{j=1}^{J-1}\frac{\mathrm{i}\tau}{2h^\alpha}\sum_{l=1}^{J-1}c_{j-l}s_l\overline{s_j}\right) = 0. \tag{3.9}$$

And by direct computations, we have

$$\operatorname{Re}\left(h\sum_{j=1}^{J-1}\frac{\tau}{2}(|s_j|^2)_{\hat{x}}s_j\overline{s_j}\right) = \operatorname{Re}\left(\frac{h\tau}{2}\sum_{j=1}^{J-1}\frac{|s_{j+1}|^2-|s_{j-1}|^2}{2h}|s_j|^2\right) = 0, \tag{3.10}$$

and

$$\begin{aligned}\operatorname{Re}\left(h\sum_{j=1}^{J-1}\frac{\tau}{2}\frac{\overline{s_{j+1}}+\overline{s_{j-1}}}{2}s_j(s_j)_{\hat{x}}\overline{s_j}\right) &= \operatorname{Re}\left(\frac{h\tau}{2}\sum_{j=1}^{J-1}\frac{\overline{s_{j+1}}+\overline{s_{j-1}}}{2}\frac{s_{j+1}+s_{j-1}}{2h}|s_j|^2\right)\\ &= 0.\end{aligned} \tag{3.11}$$

Combining (3.8)-(3.11) together, we have

$$\begin{aligned}\operatorname{Re}\langle\omega(s),s\rangle &= \|s\|^2 - \operatorname{Re}\left(h\sum_{j=1}^{J-1}U_j^n\overline{s_j}\right)\\ &\geq \|s\|^2 - \frac{1}{2}(\|U^n\|^2+\|s\|^2)\\ &= \frac{1}{2}(\|s\|^2-\|U^n\|^2).\end{aligned} \tag{3.12}$$

Now taking $\alpha=\sqrt{1+\|U^n\|^2}$, for $\forall s: \|s\|=\alpha$, we can have $\operatorname{Re}\langle\omega(s),s\rangle \geq \frac{1}{2}$. Thus by the Brouwer fixed point theorem in Lemma 2.6, there exists an element $s^* \in Z_\Delta$ such that $\omega(s^*)=0$. Let $U^{n+1}=2s^*-U^n$, thus one can obtain that $U^{n+1}\in Z_\Delta$ is the solution of the scheme (3.2)(3.3). The proof of Theorem 3.1 is completed. □

3.2. Convergence

In this subsection, we use some important inequalities and the induction argument to prove the second-order convergence of the difference solution. First for the difference solution of CN scheme (3.2)(3.3), we have the following priori estimates, which show that the CN scheme is conservative.

Lemma 3.2. *The CN scheme (3.2)(3.3) is conservative in the sense*

$$Q^n = Q^{n-1} = \cdots = Q^0, \tag{3.13}$$

where $Q^l = \|U^l\|^2, l = 0, 1, 2, \cdots, n,\ 0 \le n \le N$. *Furthermore, if* $u_0 \in L_2([a, b])$, *then the numerical solution of (3.2) and (3.3) is bounded, i.e., there exists some constant* $C_b > 0$, *such that*

$$\|U^n\| \le C_b, \quad n = 0, 1, 2, \cdots, N. \tag{3.14}$$

Proof. Multiplying the equation (3.2) by $\overline{U_j^{n+\frac{1}{2}}}h$, summing from $j = 1$ to $J-1$ and taking the real part, we can obtain

$$\begin{aligned}
&\mathrm{Re}\left(h\sum_{j=1}^{J-1}\frac{U_j^{n+1}-U_j^n}{\tau}\frac{\overline{U_j^{n+1}}+\overline{U_j^n}}{2}\right)+\mathrm{Re}\left(\mathrm{i}h^{1-\alpha}\sum_{j=1}^{J-1}\sum_{l=1}^{J-1}c_{j-l}U_l^{n+\frac{1}{2}}\overline{U_j^{n+\frac{1}{2}}}\right)\\
=&\mathrm{Re}\left(h\sum_{j=1}^{J-1}\frac{|U_{j+1}^{n+\frac{1}{2}}|^2-|U_{j-1}^{n+\frac{1}{2}}|^2}{2h}U_j^{n+\frac{1}{2}}\overline{U_j^{n+\frac{1}{2}}}\right)\\
&+\mathrm{Re}\left(h\sum_{j=1}^{J-1}\frac{\overline{U_{j+1}^{n+\frac{1}{2}}}+\overline{U_{j-1}^{n+\frac{1}{2}}}}{2}U_j^{n+\frac{1}{2}}\frac{U_{j+1}^{n+\frac{1}{2}}-U_{j-1}^{n+\frac{1}{2}}}{2h}\overline{U_j^{n+\frac{1}{2}}}\right).
\end{aligned} \tag{3.15}$$

Since $\mathrm{Re}\left(U_j^{n+1}\overline{U_j^n}-U_j^n\overline{U_j^{n+1}}\right)=0$, we have

$$\begin{aligned}
\mathrm{Re}\left(h\sum_{j=1}^{J-1}\frac{U_j^{n+1}-U_j^n}{\tau}\frac{\overline{U_j^{n+1}}+\overline{U_j^n}}{2}\right)&=\frac{h}{2\tau}\left(\sum_{j=1}^{J-1}U_j^{n+1}\overline{U_j^{n+1}}-U_j^n\overline{U_j^n}\right)\\
&=\frac{1}{2\tau}\left(\|U^{n+1}\|^2-\|U^n\|^2\right).
\end{aligned} \tag{3.16}$$

From Lemma 3.1, there holds

$$\mathrm{Re}\left(\mathrm{i}h^{1-\alpha}\sum_{j=1}^{J-1}\sum_{l=1}^{J-1}c_{j-l}U_l^{n+\frac{1}{2}}\overline{U_j^{n+\frac{1}{2}}}\right)=0. \tag{3.17}$$

For the last two terms on the right hand of (3.15), and noticing the boundary condition, we have

$$\begin{aligned}
&\mathrm{Re}\left(h\sum_{j=1}^{J-1}\frac{|U_{j+1}^{n+\frac{1}{2}}|^2-|U_{j-1}^{n+\frac{1}{2}}|^2}{2h}U_j^{n+\frac{1}{2}}\overline{U_j^{n+\frac{1}{2}}}\right)\\
=&\mathrm{Re}\left(\frac{1}{2}\sum_{j=1}^{J-1}\left(|U_{j+1}^{n+\frac{1}{2}}|^2|U_j^{n+\frac{1}{2}}|^2-|U_{j-1}^{n+\frac{1}{2}}|^2|U_j^{n+\frac{1}{2}}|^2\right)\right)\\
=&0,
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&\mathrm{Re}\left(h\sum_{j=1}^{J-1}\frac{\overline{U_{j+1}^{n+\frac12}}+\overline{U_{j-1}^{n+\frac12}}}{2}U_j^{n+\frac12}\,\frac{U_{j+1}^{n+\frac12}-U_{j-1}^{n+\frac12}}{2h}\overline{U_j^{n+\frac12}}\right)\\&=\frac14\mathrm{Re}\left(\sum_{j=1}^{J-1}\left(|U_{j+1}^{n+\frac12}|^2-\overline{U_{j+1}^{n+\frac12}}U_{j-1}^{n+\frac12}+\overline{U_{j-1}^{n+\frac12}}U_{j+1}^{n+\frac12}-|U_{j-1}^{n+\frac12}|^2\right)|U_j^{n+\frac12}|^2\right)\\&=\frac14\sum_{j=1}^{J-1}|U_{j+1}^{n+\frac12}|^2|U_j^{n+\frac12}|^2-\frac14\sum_{j=1}^{J-1}|U_{j-1}^{n+\frac12}|^2|U_j^{n+\frac12}|^2\\&=0.\end{aligned}\tag{3.19}$$

Combining (3.15)–(3.19) together, there yields

$$\|U^{n+1}\|^2=\|U^n\|^2. \tag{3.20}$$

Thus this completes the proof of Lemma 3.2. □

The convergence result for the scheme (3.2) and (3.3) can be stated as

Theorem 3.2. *Suppose that the original problem (1.3)–(1.5) has a smooth solution $u(x,t)$, and if $h,\tau,\frac{\tau}{h}$ and $\frac{\tau}{h^2}$ are small enough. Then the numerical solution U^n of the CN difference scheme (3.2) and (3.3) is convergent to the true solution $u(x,t)$ with the error $O(\tau^2+h^2)$ in the L_2 norm.*

Proof. Let $u_j^n=u(x_j,t_n)$ and define the truncation errors of the scheme (3.2)(3.3) as

$$\begin{aligned}\xi_j^n=(u_j^n)_t+\frac{\mathrm{i}}{h^\alpha}\sum_{l=1}^{J-1}c_{j-l}u_l^{n+\frac12}-(|u_j^{n+\frac12}|^2)_{\hat x}u_j^{n+\frac12}\\-\frac{\overline{u_{j+1}^{n+\frac12}}+\overline{u_{j-1}^{n+\frac12}}}{2}u_j^{n+\frac12}(u_j^{n+\frac12})_{\hat x}.\end{aligned}\tag{3.21}$$

Then from Lemma 2.2 and the Taylor's expansion, there exists a constant $C_\xi>0$ such that

$$|\xi_j^n|\le C_\xi(\tau^2+h^2). \tag{3.22}$$

Now one lets $e_j^n=u_j^n-U_j^n$ and combines (3.2)(3.21) together to have that

$$\xi_j^n=(e_j^n)_t+\frac{\mathrm{i}}{h^\alpha}\sum_{l=1}^{J-1}c_{j-l}e_l^{n+\frac12}+F_j^{n+\frac12}+G_j^{n+\frac12}, \tag{3.23}$$

where

$$\begin{aligned}F_j^{n+\frac12}&=(|U_j^{n+\frac12}|^2)_{\hat x}U_j^{n+\frac12}-(|u_j^{n+\frac12}|^2)_{\hat x}u_j^{n+\frac12},\\G_j^{n+\frac12}&=\frac{\overline{U_{j+1}^{n+\frac12}}+\overline{U_{j-1}^{n+\frac12}}}{2}U_j^{n+\frac12}(U_j^{n+\frac12})_{\hat x}-\frac{\overline{u_{j+1}^{n+\frac12}}+\overline{u_{j-1}^{n+\frac12}}}{2}u_j^{n+\frac12}(u_j^{n+\frac12})_{\hat x}.\end{aligned}\tag{3.24}$$

Now taking the real part of the inner product of (3.23) with $e_j^{n+\frac12}$ and noticing

Lemma 3.1, we have

$$\begin{aligned}\mathrm{Re}\left\langle \xi_j^n, e_j^{n+\frac{1}{2}}\right\rangle &= \mathrm{Re}\left\langle (e_j^n)_t + \frac{\mathrm{i}}{h^\alpha}\sum_{l=1}^{J-1} c_{j-l}e_l^{n+\frac{1}{2}} + F_j^{n+\frac{1}{2}} + G_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\right\rangle \\ &= \frac{1}{2\tau}\left(\|e^{n+1}\|^2 - \|e^n\|^2\right) + \mathrm{Re}\left\langle F_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\right\rangle \\ &\quad + \mathrm{Re}\left\langle G_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\right\rangle.\end{aligned} \tag{3.25}$$

This implies

$$\begin{aligned}\frac{1}{\tau}\left(\|e^{n+1}\|^2 - \|e^n\|^2\right) &= 2\mathrm{Re}\left\langle \xi_j^n, e_j^{n+\frac{1}{2}}\right\rangle - 2\mathrm{Re}\left\langle F_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\right\rangle \\ &\quad - 2\mathrm{Re}\left\langle G_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\right\rangle \\ &:= I_1 + I_2 + I_3,\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}|I_1| &= \left|2\mathrm{Re}\left\langle \xi_j^n, e_j^{n+\frac{1}{2}}\right\rangle\right| \le \|\xi^n\|^2 + \|e^{n+\frac{1}{2}}\|^2 \\ &\le \|\xi^n\|^2 + \frac{1}{2}(\|e^{n+1}\|^2 + \|e^n\|^2),\end{aligned} \tag{3.27}$$

$$\begin{aligned}|I_2| &= \left|-2\mathrm{Re}\left\langle F_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\right\rangle\right| \\ &= 2\left|h\sum_{j=1}^{J-1}\frac{\left(|U_{j+1}^{n+\frac{1}{2}}|^2 - |U_{j-1}^{n+\frac{1}{2}}|^2\right)U_j^{n+\frac{1}{2}} - \left(|u_{j+1}^{n+\frac{1}{2}}|^2 - |u_{j-1}^{n+\frac{1}{2}}|^2\right)u_j^{n+\frac{1}{2}}}{2h}\overline{e_j^{n+\frac{1}{2}}}\right| \\ &\le 2\left|h\sum_{j=1}^{J-1}\frac{(|U_{j+1}^{n+\frac{1}{2}}|^2 - |U_{j-1}^{n+\frac{1}{2}}|^2)e_j^{n+\frac{1}{2}}}{2h}\overline{e_j^{n+\frac{1}{2}}}\right| \\ &\quad + 2\left|h\sum_{j=1}^{J-1}\frac{(U_{j+1}^{n+\frac{1}{2}}\overline{e_{j+1}^{n+\frac{1}{2}}} + \overline{u_{j+1}^{n+\frac{1}{2}}}e_{j+1}^{n+\frac{1}{2}})u_j^{n+\frac{1}{2}}}{2h}\overline{e_j^{n+\frac{1}{2}}}\right| \\ &\quad + 2\left|h\sum_{j=1}^{J-1}\frac{(U_{j-1}^{n+\frac{1}{2}}\overline{e_{j-1}^{n+\frac{1}{2}}} + \overline{u_{j-1}^{n+\frac{1}{2}}}e_{j-1}^{n+\frac{1}{2}})u_j^{n+\frac{1}{2}}}{2h}\overline{e_j^{n+\frac{1}{2}}}\right|,\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}I_3 &= \left|-2\mathrm{Re}\left\langle G_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\right\rangle\right| \\ &= 2\left|h\sum_{j=1}^{J-1}\frac{\overline{U_{j+1}^{n+\frac{1}{2}}} + \overline{U_{j-1}^{n+\frac{1}{2}}}}{2}U_j^{n+\frac{1}{2}}\frac{U_{j+1}^{n+\frac{1}{2}} - U_{j-1}^{n+\frac{1}{2}}}{2h} - \frac{\overline{u_{j+1}^{n+\frac{1}{2}}} + \overline{u_{j-1}^{n+\frac{1}{2}}}}{2}u_j^{n+\frac{1}{2}}\frac{u_{j+1}^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}}}{2h}\overline{e_j^{n+\frac{1}{2}}}\right| \\ &= 2\left|h\sum_{j=1}^{J-1}\frac{(\overline{U_{j+1}^{n+\frac{1}{2}}} + \overline{U_{j-1}^{n+\frac{1}{2}}})(U_{j+1}^{n+\frac{1}{2}} - U_{j-1}^{n+\frac{1}{2}})U_j^{n+\frac{1}{2}} - (\overline{u_{j+1}^{n+\frac{1}{2}}} + \overline{u_{j-1}^{n+\frac{1}{2}}})(u_{j+1}^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}})u_j^{n+\frac{1}{2}}}{4h}\overline{e_j^{n+\frac{1}{2}}}\right|\end{aligned}$$

$$\le 2\left|h\sum_{j=1}^{J-1}\frac{\left(\overline{U_{j+1}^{n+\frac{1}{2}}}+\overline{U_{j-1}^{n+\frac{1}{2}}}\right)\left(U_{j+1}^{n+\frac{1}{2}}-U_{j-1}^{n+\frac{1}{2}}\right)e_j^{n+\frac{1}{2}}}{4h}\overline{e_j^{n+\frac{1}{2}}}\right|$$
$$+2\left|h\sum_{j=1}^{J-1}\frac{\left(\left|U_{j+1}^{n+\frac{1}{2}}\right|+\left|u_{j-1}^{n+\frac{1}{2}}\right|\right)u_j^{n+\frac{1}{2}}\overline{e_{j+1}^{n+\frac{1}{2}}}}{4h}\overline{e_j^{n+\frac{1}{2}}}\right|$$
$$+2\left|h\sum_{j=1}^{J-1}\frac{\left(\left|\overline{u_{j+1}^{n+\frac{1}{2}}}\right|+\left|\overline{u_{j-1}^{n+\frac{1}{2}}}\right|\right)u_j^{n+\frac{1}{2}}e_{j+1}^{n+\frac{1}{2}}}{4h}\overline{e_j^{n+\frac{1}{2}}}\right|$$
$$+2\left|h\sum_{j=1}^{J-1}\frac{\left(\left|U_{j+1}^{n+\frac{1}{2}}\right|+\left|u_{j-1}^{n+\frac{1}{2}}\right|\right)u_j^{n+\frac{1}{2}}\overline{e_{j-1}^{n+\frac{1}{2}}}}{4h}\overline{e_j^{n+\frac{1}{2}}}\right|$$
$$+2\left|h\sum_{j=1}^{J-1}\frac{\left(\left|\overline{U_{j+1}^{n+\frac{1}{2}}}\right|+\left|\overline{U_{j-1}^{n+\frac{1}{2}}}\right|\right)u_j^{n+\frac{1}{2}}e_{j-1}^{n+\frac{1}{2}}}{4h}\overline{e_j^{n+\frac{1}{2}}}\right|. \tag{3.29}$$

Now we need to estimate I_2 and I_3, respectively. First according to the assumptions of the theorem and Lemma 2.3, we have $\|u^n\|_\infty \le C_u$, and

$$\begin{aligned}\|e^n\|_\infty &\le C_0\|e^n\|^{\frac{1}{2}}(\|e_x^n\|+\|e^n\|)^{\frac{1}{2}}\\ &\le C_0\|e^n\|^{\frac{1}{2}}\left(\frac{2}{h}\|e^n\|+\|e^n\|\right)^{\frac{1}{2}}\\ &\le C_0\sqrt{\frac{2}{h}+1}\|e^n\|.\end{aligned}$$

Thus there is

$$\|U^n\|_\infty \le \|u^n\|_\infty+\|e^n\|_\infty \le C_u+C_0\sqrt{\frac{2}{h}+1}\|e^n\|, \tag{3.30}$$
$$\begin{aligned}\|U^{n+\frac{1}{2}}\|_\infty &\le \frac{1}{2}\left(\|U^{n+1}\|_\infty+\|U^n\|_\infty\right)\\ &\le C_u+\frac{C_0}{2}\sqrt{\frac{2}{h}+1}\left(\|e^n\|+\|e^{n+1}\|\right),\end{aligned} \tag{3.31}$$

and thus

$$\begin{aligned}\|U^{n+\frac{1}{2}}\|_\infty^2 &\le 2\left[C_u^2+\frac{C_0^2(2+h)}{4h}\left(\|e^n\|+\|e^{n+1}\|\right)^2\right]\\ &\le 2C_u^2+C_0^2\left(1+\frac{2}{h}\right)\left(\|e^n\|^2+\|e^{n+1}\|^2\right).\end{aligned} \tag{3.32}$$

Now combining (3.31) and (3.32), we have the following estimates

$$|I_2| \le \frac{2}{h}\|U^{n+\frac{1}{2}}\|_\infty^2\|e^{n+\frac{1}{2}}\|^2$$

$$
\begin{aligned}
&+\frac{1}{h}\left(\|U^{n+\frac12}\|_\infty\|u^{n+\frac12}\|_\infty+\|u^{n+\frac12}\|_\infty^2\right)\|e^{n+\frac12}\|^2\\
&+\frac{1}{h}\left(\|u^{n+\frac12}\|_\infty\|U^{n+\frac12}\|_\infty+\|u^{n+\frac12}\|_\infty^2\right)\|e^{n+\frac12}\|^2\\
\le&\frac{1}{h}\left(\|e^n\|^2+\|e^{n+1}\|^2\right)\left(\|U^{n+\frac12}\|_\infty^2+\|U^{n+\frac12}\|_\infty\|u^{n+\frac12}\|_\infty\right)\\
\le&\frac{1}{h}\left(\|e^n\|^2+\|e^{n+1}\|^2\right)\left(\frac32\|U^{n+\frac12}\|_\infty^2+\frac12\|u^{n+\frac12}\|_\infty^2\right),
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
|I_3|\le&\frac{1}{2h}4\|U^{n+\frac12}\|_\infty^2\|e^{n+\frac12}\|^2\\
&+\frac{1}{2h}\left(\|U^{n+\frac12}\|_\infty\|u^{n+\frac12}\|_\infty+\|u^{n+\frac12}\|_\infty^2\right)\|e^{n+\frac12}\|^2\\
&+\frac{1}{2h}\left(2\|u^{n+\frac12}\|_\infty^2\right)\|e^{n+\frac12}\|^2\\
&+\frac{1}{2h}\left(\|U^{n+\frac12}\|_\infty\|u^{n+\frac12}\|_\infty+\|u^{n+\frac12}\|_\infty^2\right)\|e^{n+\frac12}\|^2\\
&+\frac{1}{2h}\left(2\|U^{n+\frac12}\|_\infty\|u^{n+\frac12}\|_\infty\right)\|e^{n+\frac12}\|^2\\
\le&\frac{3}{2h}\left(\|e^n\|^2+\|e^{n+1}\|^2\right)\left(\|U^{n+\frac12}\|_\infty^2+\|u^{n+\frac12}\|_\infty^2\right),
\end{aligned}
\tag{3.34}
$$

and thus

$$
\begin{aligned}
|I_2+I_3|\le&\frac{1}{h}\left(\|e^n\|^2+\|e^{n+1}\|^2\right)\left(3\|U^{n+\frac12}\|_\infty^2+2\|u^{n+\frac12}\|_\infty^2\right)\\
\le&\frac{8C_u^2+3C_0^2\left(1+\frac{2}{h}\right)\left(\|e^n\|^2+\|e^{n+1}\|^2\right)}{h}\left(\|e^n\|^2+\|e^{n+1}\|^2\right).
\end{aligned}
\tag{3.35}
$$

From this inequality, we know that there exist two positive constants C_1 and C_2 independent of h,τ such that

$$
\begin{aligned}
|I_2+I_3|\le&(C_1h^{-1}+C_2h^{-2}\|e^n\|^2)\|e^n\|^2\\
&+(C_1h^{-1}+C_2h^{-2}\|e^{n+1}\|^2)\|e^{n+1}\|^2.
\end{aligned}
\tag{3.36}
$$

Together with (3.26) and (3.27), we obtain

$$
\begin{aligned}
\|e^{n+1}\|^2-\|e^n\|^2\le&\frac12(\|e^n\|^2+\|e^{n+1}\|^2)\tau+\|\xi^n\|^2\tau\\
&+(C_1h^{-1}+C_2h^{-2}\|e^n\|^2)\tau\|e^n\|^2\\
&+(C_1h^{-1}+C_2h^{-2}\|e^{n+1}\|^2)\tau\|e^{n+1}\|^2\\
\le&(\frac12+C_1h^{-1}+C_2h^{-2}\|e^{n+1}\|^2)\tau\|e^{n+1}\|^2\\
&+(\frac12+C_1h^{-1}+C_2h^{-2}\|e^n\|^2)\tau\|e^n\|^2+\|\xi^n\|^2\tau.
\end{aligned}
\tag{3.37}
$$

For convenience sake, we denote $W^n=\|e^n\|^2$, and (3.37) can be rewritten as

$$
\begin{aligned}
&\left[1-\left(\frac12+C_1h^{-1}\right)\tau-C_2\tau h^{-2}W^{n+1}\right]W^{n+1}\\
\le&\left[1+\left(\frac12+C_1h^{-1}\right)\tau+C_2\tau h^{-2}W^n\right]W^n+\|\xi^n\|^2\tau.
\end{aligned}
\tag{3.38}
$$

If one sets $Y^n = \left[1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau - C_2\tau h^{-2} W^n\right] W^n$, then

$$Y^{n+1} = \left[1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau - C_2\tau h^{-2} W^{n+1}\right] W^{n+1}, \tag{3.39}$$

and (3.38) will be equivalent to the form

$$Y^{n+1} \leq \frac{1 + \left(\frac{1}{2} + C_1 h^{-1}\right)\tau + C_2\tau h^{-2} W^n}{1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau - C_2\tau h^{-2} W^n} Y^n + \|\xi^n\|^2 \tau. \tag{3.40}$$

Again from (3.39) (3.40), we have

$$- C_2\tau h^{-2}(W^{n+1})^2 + \left[1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau\right] W^{n+1} - C(W^n) \leq 0, \tag{3.41}$$

where

$$C(W^n) = \frac{1 + \left(\frac{1}{2} + C_1 h^{-1}\right)\tau + C_2\tau h^{-2} W^n}{1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau - C_2\tau h^{-2} W^n} Y^n + \|\xi^n\|^2 \tau. \tag{3.42}$$

According Lemma 3.2, we know the boundedness of W^n and Y^n. Meanwhile, one can take $h, \tau, \frac{\tau}{h}$ and $\frac{\tau}{h^2}$ small enough, such that

$$0 < C_2\tau h^{-2} < \frac{1}{2C_b^2}, \quad 1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau > \frac{1}{2} > 0, \quad C(W^n) > 0, \tag{3.43}$$

and

$$\left[1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau\right]^2 - 4C_2\tau h^{-2} \cdot C(W^n) > 0. \tag{3.44}$$

Then from Lemma 2.5, we have

$$W^{n+1} \geq \frac{1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau}{C_2\tau h^{-2}} - \frac{2}{1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau} \cdot C(W^n), \tag{3.45}$$

or

$$W^{n+1} \leq \frac{2}{1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau} \cdot C(W^n). \tag{3.46}$$

In what follows, we will use (3.40)(3.46) and the induction argument to prove the following estimates hold

$$Y^n \leq C(\tau^2 + h^2)^2, \quad W^n \leq C(\tau^2 + h^2)^2, \tag{3.47}$$

for some constants. For $n = 0$, there is $W^0 = 0, Y^0 = 0$, and it satisfies (3.47) obviously. In fact, inequality (3.45) does not hold for any n. For example, if (3.45) holds for $n = 0$, then $C(W^0) = 0$ and for $n = 1$, since $h, \tau, \frac{\tau}{h}$ and $\frac{\tau}{h^2}$ are taken small enough, such that (3.43) satisfy, there arrives at

$$W^1 \geq \frac{1 - \left(\frac{1}{2} + C_1 h^{-1}\right)\tau}{C_2\tau h^{-2}} \geq \frac{1}{2C_2\tau h^{-2}} > 4C_b^2, \tag{3.48}$$

which is contradiction to

$$W^n = \|e^n\|^2 = \|u^n - U^n\|^2 \leq 4C_b^2. \tag{3.49}$$

Actually for $n=1$, from (3.40)(3.46), we have

$$\begin{aligned} &Y^1 \le \tau\|\xi^0\|^2 \le \tau C_\xi^2(\tau^2+h^2)^2,\\ &W^1 \le \frac{2C(W^n)}{1-\left(\frac{1}{2}+C_1h^{-1}\right)\tau} \le 4\tau\|\xi^0\|^2 \le 4\tau C_\xi^2(\tau^2+h^2)^2. \end{aligned} \tag{3.50}$$

Then we suppose that (3.47) holds for all $n=k((k+1)\tau \le T)$, then we can determine some C_* to be a sufficiently large constant independent of h and τ, and $\frac{\tau}{h^2}$ be sufficiently small such that

$$\frac{1+\left(\frac{1}{2}+C_1h^{-1}\right)\tau + C_2\tau h^{-2}W^n}{1-\left(\frac{1}{2}+C_1h^{-1}\right)\tau - C_2\tau h^{-2}W^n} \le 1+C_*\tau. \tag{3.51}$$

Now combining (3.50), we suppose that for all the $s \le k$, there holds

$$Y^s \le \frac{C_\xi^2}{C_*}\left[(1+C_*\tau)^s-1\right](\tau^2+h^2)^2, \tag{3.52}$$

and

$$W^s \le \frac{4C_\xi^2}{C_*}e^{C_*T}(\tau^2+h^2)^2. \tag{3.53}$$

Then for $n=k+1$ and from (3.40) (3.46) and (3.51), we have

$$\begin{aligned} Y^{k+1} &\le (1+C_*\tau)Y^k + \tau C_\xi^2(\tau^2+h^2)^2\\ &\le (1+C_*\tau)\left[\frac{C_\xi^2}{C_*}\left((1+C_*\tau)^k-1\right)(\tau^2+h^2)^2\right] + \tau C_\xi^2(\tau^2+h^2)^2\\ &= \frac{C_\xi^2}{C_*}\left[(1+C_*\tau)^{k+1}-(1+C_*\tau)+C_*\tau\right](\tau^2+h^2)^2\\ &= \frac{C_\xi^2}{C_*}\left[(1+C_*\tau)^{k+1}-1\right](\tau^2+h^2)^2, \end{aligned} \tag{3.54}$$

and

$$\begin{aligned} W^{k+1} &\le \frac{2}{1-\left(\frac{1}{2}+C_1h^{-1}\right)\tau}\left[\frac{1+\left(\frac{1}{2}+C_1h^{-1}\right)\tau + C_2\tau h^{-2}W^k}{1-\left(\frac{1}{2}+C_1h^{-1}\right)\tau - C_2\tau h^{-2}W^k}Y^k + \|\xi^n\|^2\tau\right]\\ &\le 4\left[(1+C_*\tau)Y^k + \|\xi^n\|^2\tau\right]\\ &\le 4\left[(1+C_*\tau)\left(\frac{C_\xi^2}{C_*}\left((1+C_*\tau)^k-1\right)(\tau^2+h^2)^2\right) + \tau C_\xi^2(\tau^2+h^2)^2\right]\\ &\le \frac{4C_\xi^2}{C_*}\left[(1+C_*\tau)^{k+1}-1\right](\tau^2+h^2)^2\\ &\le \frac{4C_\xi^2}{C_*}\left[\left(1+C_*\cdot\frac{T}{k+1}\right)^{\frac{(k+1)C_*T}{C_*T}}\right](\tau^2+h^2)^2\\ &\le \frac{4C_\xi^2}{C_*}e^{C_*T}(\tau^2+h^2)^2, \end{aligned} \tag{3.55}$$

where the following inequality

$$\left(1 + C_* \cdot \frac{T}{k+1}\right)^{\frac{(k+1)C_*T}{C_*T}} \leq e^{C_*T}, \tag{3.56}$$

for all $(k+1)\tau \leq T$ is applied. Thus the estimates (3.47) can be obtained for all n. The proof of Theorem 3.2 is completed. □

3.3. Uniqueness

Finally, based on the results of the existence and convergence, we also have the following uniqueness result.

Theorem 3.3. *Under the conditions of Theorem 3.2, the difference solution of the CN difference scheme (3.2) and (3.3) is unique.*

Proof. Assume U^n and V^n both satisfy the CN difference scheme (3.2) and (3.3). Then for $Y^n = U^n - V^n$, we have

$$(Y_j^n)_t + \frac{\mathrm{i}}{h^\alpha}\sum_{l=1}^{J-1} c_{j-l} Y_l^{n+\frac{1}{2}} - F_j^{n+\frac{1}{2}}(U_j^n, V_j^n) - G_j^{n+\frac{1}{2}}(U_j^n, V_j^n) = 0, \tag{3.57}$$

$$Y_j^0 = 0, \quad Y_0^n = Y_J^n = 0, \quad 1 \leq j \leq J, \quad 0 \leq n \leq N. \tag{3.58}$$

where

$$\begin{aligned} F_j^{n+\frac{1}{2}}(U_j^n, V_j^n) &= (|U_j^{n+\frac{1}{2}}|^2)_{\hat{x}} U_j^{n+\frac{1}{2}} - (|V_j^{n+\frac{1}{2}}|^2)_{\hat{x}} V_j^{n+\frac{1}{2}}, \\ G_j^{n+\frac{1}{2}}(U_j^n, V_j^n) &= \frac{\overline{U_{j+1}^{n+\frac{1}{2}}} + \overline{U_{j-1}^{n+\frac{1}{2}}}}{2} U_j^{n+\frac{1}{2}} (U_j^{n+\frac{1}{2}})_{\hat{x}} - \frac{\overline{V_{j+1}^{n+\frac{1}{2}}} + \overline{V_{j-1}^{n+\frac{1}{2}}}}{2} V_j^{n+\frac{1}{2}} (V_j^{n+\frac{1}{2}})_{\hat{x}}. \end{aligned} \tag{3.59}$$

Similar to the proof of Theorem 3.2, as $h \to 0$ and $\tau \to 0$, we have

$$\|Y^n\| = 0, \tag{3.60}$$

which means the uniqueness. This completes the proof of Theorem 3.3. □

4. Examples and numerical results

In this section, we compute two numerical examples to demonstrate the effectiveness of the CN difference scheme (3.2) and (3.3).

Example 4.1. Let $\alpha = 2$, then the fractional derivative nonlinear Schrödinger equation (1.3)-(1.5) reduce to the usual derivative nonlinear Schrödinger equation (1.1)

$$\mathrm{i}u_t + u_{xx} - \mathrm{i}(|u|^2 u)_x = 0, \tag{4.1}$$

with the initial condition

$$u(x, 0) = u_0(x), \tag{4.2}$$

and the Dirichlet boundary condition

$$u(a, t) = u(b, t) = 0. \tag{4.3}$$

Here we take the initial value as $u_0(x) = 2\sqrt{\text{sech}(2x)}\exp\left[\frac{3}{2}\arctan(\sinh(2x))\mathrm{i}\right]$, and the exact solution for this initial value problem (4.1) and (4.2) is given by

$$u(x,t) = 2\sqrt{\text{sech}(2x)}\exp\left[\frac{3}{2}\arctan(\sinh(2x))\mathrm{i} + \mathrm{i}t\right]. \tag{4.4}$$

The initial-boundary value problem (4.1)–(4.3) can be considered to the initial value problem (4.1) and (4.2) for $a \ll 0$ and $b \gg 0$. Since the initial value $u_0(x)$ exponentially decays to zero with the variable x away from the origin, thus it is reasonable to consider that the wave function is negligible outside the interval $[a,b]$, and we can set $u(a,t) = u(b,t) = 0$ for $a \ll 0$ and $b \gg 0$. In this example, we chose $a = -20$ and $b = 20$.

Table 1. Errors between the difference solution and true solution of Example 4.1 at $t = 2$.

h	τ	λ	$\|u^N - U^N\|$	Order	$\|u^N - U^N\|_\infty$	Order
0.1	0.1	10	2.178783e-01	-	1.573651e-01	-
0.05	0.05	20	5.428458e-02	2.00341	3.960803e-02	1.993252
0.025	0.025	40	1.355145e-02	2.00145	9.907701e-03	1.999427
0.0125	0.0125	80	3.386133e-03	2.00051	2.476947e-03	1.999991

Table 2. Discrete mass Q^n in Example 4.1 at different time with $h = \tau = 0.02$.

t	Q^n	$(Q^n - Q^0)/Q^0$
0	6.283185307179577	0
0.1	6.283185395698440	1.408821465908791e-08
0.2	6.283185575083924	4.263830119556149e-08
0.3	6.283185725425165	6.656585281784965e-08
0.4	6.283185833555792	8.377537662344859e-08
0.5	6.283185909172271	9.581011308330088e-08
0.6	6.283185961712505	1.041721509555845e-07
0.7	6.283185997937546	1.099375452538331e-07
0.8	6.283186022469789	1.138419731184578e-07
0.9	6.283186038484496	1.163907928827964e-07
1.0	6.283186048204716	1.179378139730703e-07

Table 1 gives some errors between the difference solution and true solution (4.4) of Example 4.1 with different mesh ratios $\lambda = \frac{\tau}{h^2}$, which verifies the second order convergence and good stability of the numerical solutions. The conservative law of discrete mass computed by the CN difference scheme can be also seen in Table 2, which also shows that the scheme converses the discrete masses very well.

Example 4.2. When $1 < \alpha < 2$, we also consider the FDNLSE (1.3)–(1.5) with the same initial value

$$u_0(x) = 2\sqrt{\text{sech}(2x)}\exp\left[\frac{3}{2}\arctan(\sinh(2x))\mathrm{i}\right],$$

and truncate the problem in $[-20, 20]$, which implies $u(-20,t) = u(20,t) = 0$. Fig.1–Fig.4 present the numerical solutions for different values of order α, where we take $h = \tau = 0.05$. As we can find that, the order of α will affect the shape of the solution both in the height and width. When α becomes larger, the modulus value at the center is larger, and the shape will change more quickly. When α tends to 2, the numerical solutions of the fractional equation are convergent to the solutions of the usual non-fractional equation, as the case in Example 4.1. Meanwhile, we observe that the shape of the solutions is more smoother as the values of order α is larger, as seen in Fig.5, which represents the solutions of the FDNLSE at $t = 2$ with different values of α.

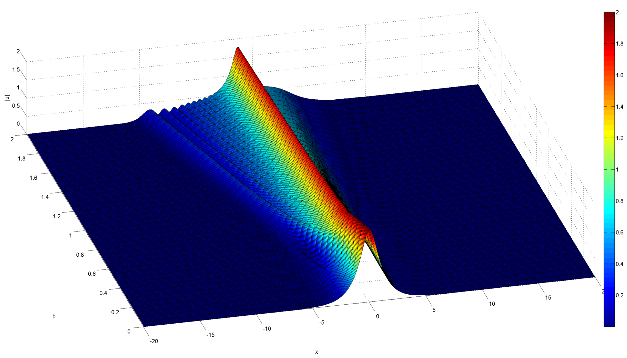

Figure 1. Numerical solutions for $\alpha = 1.7$

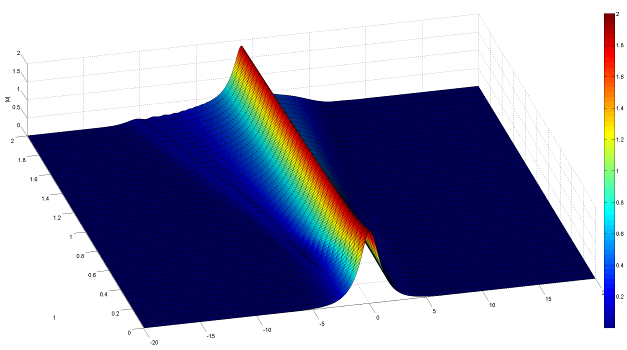

Figure 2. Numerical solutions for $\alpha = 1.8$

Similarly, for the CN difference scheme (3.2)(3.3) of the FDNLSE (1.3)–(1.5) with $1 < \alpha < 2$, it still preserve the discrete masses, as expected. Table 3 shows the errors of $(Q^n - Q^0)/Q^0$ for different α at some different time.

5. Conclusions

In this paper, we proposed and analyzed a conservative difference scheme for the derivative nonlinear Schrödinger equation with the Riesz space fractional derivative, which can be used as important model in plasma physics. On one hand, the term $(|u|^2u)_x$ brings more difficulties in the theoretical analysis and numerical

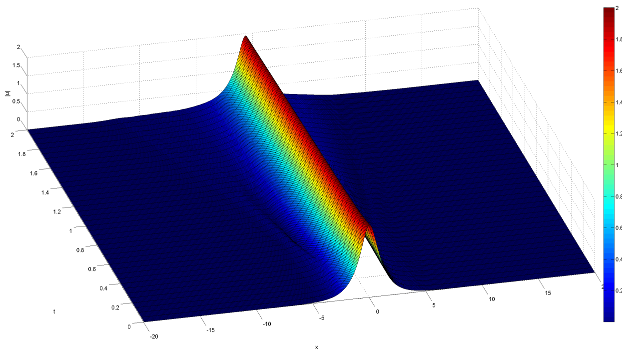

Figure 3. Numerical solutions for $\alpha = 1.9$

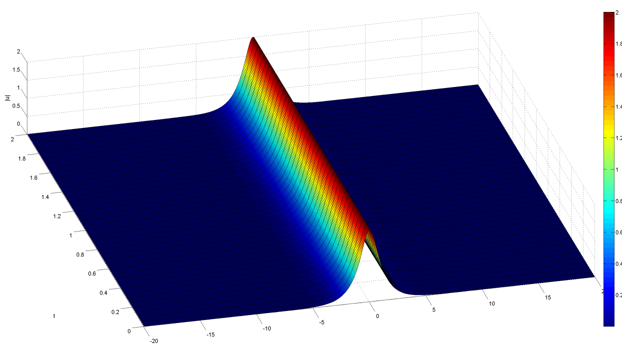

Figure 4. Numerical solutions for $\alpha = 2$

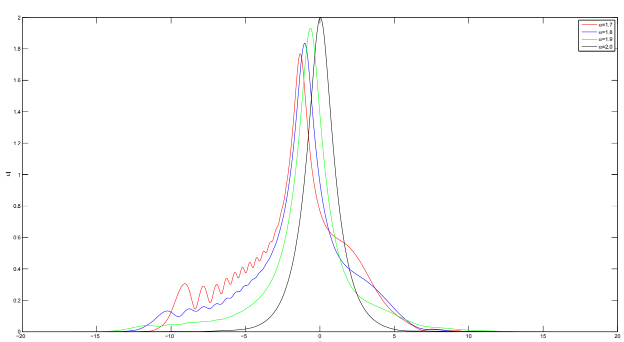

Figure 5. Numerical solutions for different values of α at $t = 2$.

Table 3. The errors of $\left|\frac{Q^n - Q^0}{Q^0}\right|$ for different α at different time with $h = \tau = 0.05$.

α	$t = 0.1$	$t = 0.5$	$t = 1$	$t = 1.5$	$t = 2$
1.7	6.5544e-04	8.7366e-04	6.8926e-04	5.7651e-04	5.1118e-04
1.8	1.0333e-04	6.1432e-05	1.0323e-04	1.1785e-04	1.0821e-04
1.9	3.4014e-05	2.7388e-05	4.9055e-05	5.4610e-05	5.1194e-05

simulations, compared to the usual Schrödinger equation. Thus we proposed the Crank-Nicolson difference schemes and handled the nonlinear term skillfully. On the other hand, there are many methods to approximate the Riesz space-fractional derivative, which may lead to different accuracy. Fortunately, we adopt the fractional centered difference operator, as defined in (2.2) to get the approximation. Furthermore, there holds the conservation law, as seen in Lemma 3.1, it plays an important role in the theoretical analysis for the derivative nonlinear Schrödinger equation, and the symmetry of the coefficients is also crucial for the numerical computations. Based on the discrete conservation laws and some delicate priori estimates, the optimal convergence rate for the CN difference schemes at the order of $O(\tau^2 + h^2)$ is obtained, and some numerical tests are carried out, which confirmed our theoretical results.

References

[1] G. D. Akrivis, *Finite difference discretization of the cubic Schrödinger equation*, IMA J. Numer. Anal., 1993, 13(1), 115–124.

[2] A. H. Bhrawy and M. A. Abdelkawy, *A fully spectral collocation approximation for multi-dimensional fractional Schrödinger equations*, J. Comput. Phys., 2015, 294, 462–483.

[3] C. Çelik and M. Duman, *Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative*, J. Comput. Phys., 2012, 231(4), 1743–1750.

[4] A. Durán and J. M. Sanz-Serna, *The numerical integration of relative solutions. The nonlinear Schrödinger equation*, IMA J. Numer. Anal., 2000, 20(2), 235–261.

[5] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals*, McGraw-Hill, New York, 1965.

[6] R. L. Frank and E. Lenzmann, *Uniqueness of non-linear ground states for fractional Laplacians in* $\mathbb{R}$, Acta Math., 2013, 210(2), 261–318.

[7] B. Guo, Y. Han and J. Xin, *Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation*, Appl. Math. Comput., 2008, 204(1), 468–477.

[8] B. Guo, X. Pu and F. Huang, *Fractional partial differential equations and their numerical solutions*, Beijing Science Press, 2015.

[9] B. Guo, *The convergence of numerical method for nonlinear Schrödinger equations*, J. Comput. Math., 1986, 4(2), 121–130.

[10] Z. Guo and Y. Wu, *Global well-posedness for the derivative nonlinear Schrödinger equation in* $H^{\frac{1}{2}}(\mathbb{R})$, Disc. Cont. Dyn. Sys., 2017, 37(1), 257–264.

[11] M. Hayashi and T. Ozawa, *Well-posedness for a generalized derivative nonlinear Schrödinger equation*, J. Diff. Eqs., 2016, 261(10), 5424–5445.

[12] S. Herr, *On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition*, Int. Math. Res. Not., 2006, 96763, 1–33.

[13] M. Ilic, F. Liu, I. Turner and V. Anh, *Numerical approximation of a fractional-in-space diffusion equation*, I., Fract. Calc. Appl. Anal., 2005, 8(3), 323–341.

[14] M. S. Ismail and T. R. Taha, *Numerical simulation of coupled nonlinear Schrödinger equation*, Math. Comput. Simul., 2001, 56(6), 547–562.

[15] D. J. Kaup and A. C. Newell, *An exact solution for a derivative nonlinear Schrödinger equation*, J. Math. Phys., 1978, 19(4), 798–801.

[16] N. Laskin, *Fractional quantum mechanics*, Phys. Rev. E, 2000, 62(3), 3135–3145.

[17] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A, 2000, 268(4-6), 298–305.

[18] M. Li, C. Huang and P. Wang, *Galerkin finite element method for nonlinear fractional Schrödinger equations*, Numer. Algor., 2017, 74(2), 499–525.

[19] K. Mio, T. Ogino, K. Minami and S. Takeda, *Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas*, J. Phys. Soc. Jpn., 1976, 41(1), 265–271.

[20] E. Mjϕlhus, *On the modulational instability of hydromagnetic waves parallel to the magnetic field*, J. Plasma Phys., 1976, 16(3), 321–334.

[21] R. Mosincat, *Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in* $H^{\frac{1}{2}}$, J. Diff. Eqs., 2017, 263(8), 4658–4722.

[22] M. D. Ortigueira, *Riesz potential operators and inverses via fractional centred derivatives*, Int. J. Math. Math. Sci., 2006, 48391, 1–12.

[23] K. M. Owolabi and A. Atangana, *Numerical solution of fractional-in-space nonlinear Schrödinger equation with the Riesz fractional derivative*, Eur. Phys. J. Plus, 2016, 131(9), 335.

[24] A. Rogister, *Parallel propagation of nonlinear low-frequency waves in high-β plasma*, Phys. Fluids, 1971, 14(12), 2733–2739.

[25] A. I. Saichev and G. M. Zaslavsky, *Fractional kinetic equations: solutions and applications*, Chaos, 1997, 7(4), 753–764.

[26] S. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Transl. from the Russian. New York, Gordon and Breach, 1993.

[27] Z. Sun and D. Zhao, *On the* L_∞ *convergence of a difference scheme for coupled nonlinear Schrödinger equations*, Comput. Math. Appl., 2010, 59(10), 3286–3300.

[28] M. Tsutsumi and I. Fukuda, *On solutions of the derivative nonlinear Schrödinger equation: existence and uniqueness theorems*, Funkcial. Ekvac., 1980, 23(3), 259–277.

[29] D. Wang, A. Xiao and W. Yang, *Crank-Nicolson difference scheme for the coupled nonlinear Schrödinger equations with the Riesz space fractional derivative*, J. Comput. Phys., 2013, 242(6), 670–681.

[30] D. Wang, A. Xiao and W. Yang, *A linearly implicit conservative difference scheme for the space fractional coupled nonlinear Schrödinger equations*, J. Comput. Phys., 2014, 272, 644–655.

[31] D. Wang, A. Xiao and W. Yang, *Maximum-norm error analysis of a difference scheme for the space fractional CNLS*, Appl. Math. Comput., 2015, 257, 241–251.

[32] T. Wang, T. Nie and L. Zhang, *Analysis of a symplectic difference scheme for a coupled nonlinear Schrödinger system*, J. Comput. Appl. Math., 2009, 231(2), 745–759.

[33] T. Wang, B. Guo and L. Zhang, *New conservative difference schemes for a coupled nonlinear Schrödinger system*, Appl. Math. Comput., 2010, 217(4), 1604–1619.

[34] P. Wang and C. Huang, *An energy conservative difference scheme for the nonlinear fractional Schrödinger equations*, J. Comput. Phys., 2015, 293, 238–251.

[35] P. Wang and C. Huang, *A conservative linearized difference scheme for the nonlinear fractional Schrödinger equation*, Numer. Algor., 2015, 69(3), 625–641.

[36] P. Wang, C. Huang and L. Zhao, *Point-wise error estimate of a conservative difference scheme for the fractional Schrödinger equation*, J. Comput. Appl. Math., 2016, 306, 231–247.

[37] Q. Yang, F. Liu and I. Turner, *Numerical methods for fractional partial differential equations with Riesz space fractional derivatives*, Appl. Math. Model., 2010, 34(1), 200–218.

[38] H. Zhang, F. Liu and V. Anh, *Galerkin finite element approximations of symmetric space-fractional partial differential equations*, Appl. Math. Comput., 2010, 217(6), 2534–2545.

[39] H. Zhang, X. Jiang, C. Wang and S. Chen, *Crank-Nicolson Fourier spectral methods for the space fractional nonlinear Schrödinger equation and its parameter estimation*, Int. J. Comput. Math., 2019, 96(2), 238–263.

[40] Q. Zhang, X. Liu and M. R. Belić, *et al.*, *Propagation dynamics of a light beam in a fractional Schrödinger equation*, Phys. Rev. Lett., 2015, 115, 180403.

[41] R. Zhang, Y. Zhang and Z. Wang, *et al.*, *A conservative numerical method for the fractional nonlinear Schrödinger equation in two dimensions*, Sci. Sin. Math.,2019, 62(10), 1997–2014.

[42] X. Zhao, Z. Sun and Z. Hao, *A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schrödinger equation*, SIAM J. Sci. Comput., 2014, 36(6), A2865–A2886.

[43] Y. Zhou, *Application of discrete functional analysis to the finite difference methods*, International Academic Publishers, Beijing, 1990.