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THE PRECONDITIONED GAOR METHODS FOR GENERALIZED LEAST SQUARES PROBLEMS

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Abstract In this paper, we present some preconditioned generalized AOR (denoted by GAOR) methods for solving generalized least squares problems. We also compare the spectral radii of the iteration matrices of the proposed preconditioned and original methods. Finally, numerical experiments are provided to confirm the theoretical results.

Keywords Generalized least squares problem, GAOR method, preconditioned, convergence analysis.

MSC(2010) 65F10.

1. Introduction

Given the matrix $A \in \mathbb{R}^{n\times n}$, the vector $b \in \mathbb{R}^n$ and the symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$, the generalized least squares problem is as

$$\min_{x\in\mathbb{R}^n} \quad (Ax-b)^{\mathrm{T}} W^{-1}(Ax-b). \tag{1.1}$$

According to the Karush-Kuhn-Tucker (KKT) condition [5,6], the optimal solution of the problem (1.1) will satisfy the following equation

$$(W^{-\frac{1}{2}}A)^{\mathrm{T}} W^{-\frac{1}{2}}(Ax-b) = 0.$$

Let $r = W^{-\frac{1}{2}}(Ax-b)$, then we can obtain the linear system

$$\begin{pmatrix} I & -W^{-\frac{1}{2}}A \\ A^{\mathrm{T}}W^{-\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} -W^{-\frac{1}{2}}b \\ 0 \end{pmatrix}. \tag{1.2}$$

The generalized least squares problems arise in many scientific and engineering applications and have received comprehensive study. The parameter estimation in mathematical modelling [13,14] is a typical source of the generalized least squares problems. More about the generalized least squares problems, it can see [3,4,7,8,10–12,15,16] and the reference therein.

To solve the generalized least squares problem (1.1), one is to solve the nonsingular linear system as

$$\mathcal{H}y = f, \tag{1.3}$$

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where

$$\mathcal{H} = \begin{pmatrix} I - B & U \\ L & I - C \end{pmatrix}$$

is an invertible matrix with $p + q = n$ and

$$B = (b_{ij}) \in \mathbb{R}^{p\times p}, \quad C = (c_{ij}) \in \mathbb{R}^{q\times q}, \quad L = (l_{ij}) \in \mathbb{R}^{q\times p}, \quad U = (u_{ij}) \in \mathbb{R}^{p\times q}.$$

We remark that the linear system (1.2) is a special case of (1.3) with

$$B = 0,\ C = I,\ U = -W^{-\frac{1}{2}}A,\ L = A^{\mathrm{T}}W^{-\frac{1}{2}},\ f = \begin{pmatrix} -W^{-\frac{1}{2}}b \\ 0 \end{pmatrix}.$$

In order to solve the linear system $Ax = b$ with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, Hadjidimos in [2] proposed the accelerated overrelaxation (AOR) method. Based on the AOR method, Yuan and Jin [14] in 1999 proposed the generalized AOR (GAOR) method for solving the linear system (1.3), which splits the matrix $\mathcal{H}$ as

$$\mathcal{H} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix} - \begin{pmatrix} B & -U \\ 0 & C \end{pmatrix}.$$

Then, for $\omega \neq 0$, the GAOR method can be defined by

$$y^{(k+1)} = \mathcal{L}_{\tau,\omega} y^{(k)} + \omega g, \quad k = 0, 1, 2, \cdots, \tag{1.4}$$

where

$$\begin{aligned} \mathcal{L}_{\tau,\omega} &= \begin{pmatrix} I & 0 \\ \tau L & I \end{pmatrix}^{-1} \left[(1-\omega)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + (\omega - \tau)\begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix} + \omega \begin{pmatrix} B & -U \\ 0 & C \end{pmatrix} \right] \\ &= \begin{pmatrix} (1-\omega)I + \omega B & -\omega U \\ \omega(\tau - 1)L - \omega\tau LB & (1-\omega)I + \omega C + \omega\tau LU \end{pmatrix} \end{aligned} \tag{1.5}$$

is the iteration matrix and

$$g = \begin{pmatrix} I & 0 \\ -\tau L & I \end{pmatrix} f.$$

The spectral radius of the iteration matrix $\mathcal{L}_{\tau,\omega}$ is smaller, the convergence rate is faster. To decrease the spectral radius of $\mathcal{L}_{\tau,\omega}$, an effective method is to precondition the linear system (1.3), namely,

$$\mathcal{P}\mathcal{H}y = \mathcal{P}f,$$

where $\mathcal{P}$ is a nonsingular matrix and called a preconditioner. Moreover, if we express $\mathcal{P}\mathcal{H}$ as

$$\mathcal{P}\mathcal{H} = \begin{pmatrix} I - B^* & U^* \\ L^* & I - C^* \end{pmatrix},$$

then the preconditioned GAOR method can be defined by

$$y^{(k+1)} = \mathcal{L}^*_{\tau,\omega} y^{(k)} + \omega g^*, \quad k = 0, 1, 2, \cdots, \tag{1.6}$$

where

$$\mathcal{L}^*_{\tau,\omega} = \begin{pmatrix} (1-\omega)I + \omega B^* & -\omega U^* \\ \omega(\tau-1)L^* - \omega\tau L^* B^* & (1-\omega)I + \omega C^* + \omega\tau L^* U^* \end{pmatrix} \tag{1.7}$$

and

$$g^* = \begin{pmatrix} I & 0 \\ -\tau L^* & I \end{pmatrix} \mathcal{P} f.$$

For more details, one can see [3,4].

In [16], Zhou et al. gave some preconditioners $\widehat{\mathcal{P}}_i$ for the GAOR method of the form

$$\widehat{\mathcal{P}}_i = \begin{pmatrix} I & 0 \\ K_i & I \end{pmatrix}, \quad i = 1, 2, 3, \tag{1.8}$$

where $K_i \in \mathbb{R}^{q\times p}$ $(i = 1, 2, 3)$ are as follows:

$$K_1 = \begin{pmatrix} 0 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \vdots & \vdots \quad\ \vdots \\ -\frac{l_{q1}}{\mu} & 0 \cdots 0 \end{pmatrix} (\mu > 0) \tag{1.9}$$

and

$$K_2 = \begin{pmatrix} -l_{11} & 0 \cdots 0 \\ -l_{21} & 0 \cdots 0 \\ \vdots & \vdots \quad\ \vdots \\ -l_{q1} & 0 \cdots 0 \end{pmatrix}. \tag{1.10}$$

If $q < p$, then

$$K_3 = \begin{pmatrix} -l_{11} & 0 & \cdots & 0 & 0 \cdots 0 \\ 0 & -l_{22} & \cdots & 0 & 0 \cdots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \quad\ \vdots \\ 0 & 0 & \cdots & -l_{qq} & 0 \cdots 0 \end{pmatrix}. \tag{1.11}$$

If $q = p$, then

$$K_3 = \begin{pmatrix} -l_{11} & 0 & \cdots & 0 \\ 0 & -l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -l_{qq} \end{pmatrix}. \tag{1.12}$$

If $q > p$, then

$$K_3 = \begin{pmatrix} -l_{11} & 0 & \cdots & 0 \\ 0 & -l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -l_{pp} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{1.13}$$

Then $\widehat{\mathcal{P}}_i\mathcal{H}$ can be expressed as

$$\widehat{\mathcal{P}}_i\mathcal{H} = \begin{pmatrix} I-B & U \\ \widehat{L}_i & I-\widehat{C}_i \end{pmatrix}, \quad i = 1,2,3,$$

where $\widehat{L}_i = L + K_i(I-B)$ and $\widehat{C}_i = C - K_iU$.

In [11], Wang et al. proposed another type preconditioners for the GAOR method of the following form

$$\mathcal{P}_i = \begin{pmatrix} I+S_i & 0 \\ 0 & I \end{pmatrix}, \quad i = 1,2,3, \tag{1.14}$$

where $S_i \in \mathbb{R}^{p\times p}$ $(i = 1,2,3)$ are as follows:

$$S_1 = \begin{pmatrix} 0 & b_{12} & \cdots & 0 & 0 \\ b_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & b_{p-1,p} \\ 0 & 0 & \cdots & b_{p,p-1} & 0 \end{pmatrix}, \tag{1.15}$$

$$S_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_{21} & 0 & \cdots & 0 & 0 \\ 0 & b_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{p,p-1} & 0 \end{pmatrix}, \tag{1.16}$$

$$S_3 = \begin{pmatrix} 0 & b_{12} & 0 & \cdots & 0 \\ 0 & 0 & b_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{p-1,p} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{1.17}$$

Then $\mathcal{P}_i\mathcal{H}$ can be expressed as

$$\mathcal{P}_i\mathcal{H} = \begin{pmatrix} I - B_i & U_i \\ L & I - C \end{pmatrix}, \quad i = 1, 2, 3,$$

where $B_i = B - S_i(I - B)$ and $U_i = (I + S_i)U$.

Based on [11] and [16], Huang et al. [4] presented new preconditioners $\mathcal{P}_i^*$ of the following form

$$\mathcal{P}_i^* = \begin{pmatrix} I + S_i & 0 \\ 0 & I + V_i \end{pmatrix}, \quad i = 1, 2, 3, \tag{1.18}$$

where S_i $(i = 1, 2, 3)$ are defined as (1.15), (1.16) and (1.17), and

$$V_1 = \begin{pmatrix} 0 & c_{12} & \cdots & 0 & 0 \\ c_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & c_{q-1,q} \\ 0 & 0 & \cdots & c_{q,q-1} & 0 \end{pmatrix}, \tag{1.19}$$

$$V_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_{21} & 0 & \cdots & 0 & 0 \\ 0 & c_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{q,q-1} & 0 \end{pmatrix}, \tag{1.20}$$

$$V_3 = \begin{pmatrix} 0 & c_{12} & 0 & \cdots & 0 \\ 0 & 0 & c_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{q-1,q} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{1.21}$$

Then $\mathcal{P}_i^*\mathcal{H}$ can be expressed as

$$\mathcal{P}_i^*\mathcal{H} = \begin{pmatrix} I - B_i^* & U_i^* \\ L_i^* & I - C_i^* \end{pmatrix}, \quad i = 1, 2, 3,$$

where $B_i^* = B - S_i(I - B)$, $U_i^* = (I + S_i)U$, $L_i^* = (I + V_i)L$ and $C_i^* = C - V_i(I - C)$.

The preconditioned GAOR methods for solving $\mathcal{P}_i^*\mathcal{H}y = \mathcal{P}_i^*f$ are

$$y^{(k+1)} = \mathcal{L}_{\tau,\omega}^{*(i)} y^{(k)} + \omega g_i^*, \quad k = 0, 1, 2, \cdots, \tag{1.22}$$

where

$$\mathcal{L}_{\tau,\omega}^{*(i)} = \begin{pmatrix} (1-\omega)I + \omega B_i^* & -\omega U_i^* \\ \omega(\tau - 1)L_i^* - \omega\tau L_i^* B_i^* & (1-\omega)I + \omega C_i^* + \omega\tau L_i^* U_i^* \end{pmatrix} \tag{1.23}$$

and

$$g_i^* = \begin{pmatrix} I & 0 \\ -\tau L_i^* & I \end{pmatrix} \mathcal{P}_i^* f$$

for $i = 1, 2, 3$.

For the preconditioners $\mathcal{P}_i^* (i = 1, 2, 3)$, Huang et al. [4] gave the following comparison results.

Lemma 1.1 ([4], Theorem 3.1). *Let $\mathcal{L}_{\tau,\omega}$ and $\mathcal{L}_{\tau,\omega}^{*(1)}$ be the iteration matrices defined by* (1.5) *and* (1.23)*, respectively. Assume that the matrix $\mathcal{H}$ in* (1.3) *is irreducible with $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \dots, p-1\}$, $b_{ii} > 0$ whenever $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for $i \in \{1, 2, \dots, p-1\}$; and $c_{i,i+1} > 0$, $c_{i+1,i} > 0$ for some $i \in \{1, 2, \dots, q-1\}$, $c_{ii} > 0$ whenever $c_{i,i+1} > 0$, $c_{i+1,i} > 0$ for $i \in \{1, 2, \dots, q-1\}$, $0 < \omega \le 1$, $0 \le \tau < 1$, then*

(a) $\rho(\mathcal{L}_{\tau,\omega}^{*(1)}) < \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) < 1$;

(b) $\rho(\mathcal{L}_{\tau,\omega}^{*(1)}) > \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) > 1$.

Lemma 1.2 ([4], Theorem 3.2). *Let $\mathcal{L}_{\tau,\omega}$ and $\mathcal{L}_{\tau,\omega}^{*(2)}$ be the iteration matrices defined by* (1.5) *and* (1.23)*, respectively. Assume that the matrix $\mathcal{H}$ in* (1.3) *is irreducible with $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \dots, p-1\}$, $b_{ii} > 0$ whenever $b_{i+1,i} > 0$ for $i \in \{1, 2, \dots, p-1\}$; and $c_{i+1,i} > 0$ for some $i \in \{1, 2, \dots, q-1\}$, $c_{ii} > 0$ whenever $c_{i+1,i} > 0$ for $i \in \{1, 2, \dots, q-1\}$, $0 < \omega \le 1$, $0 \le \tau < 1$, then*

(a) $\rho(\mathcal{L}_{\tau,\omega}^{*(2)}) < \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) < 1$;

(b) $\rho(\mathcal{L}_{\tau,\omega}^{*(2)}) > \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) > 1$.

Lemma 1.3 ([4], Theorem 3.3). *Let $\mathcal{L}_{\tau,\omega}$ and $\mathcal{L}_{\tau,\omega}^{*(3)}$ be the iteration matrices defined by* (1.5) *and* (1.23)*, respectively. Assume that the matrix $\mathcal{H}$ in* (1.3) *is irreducible with $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $b_{i,i+1} > 0$ for some $i \in \{1, 2, \dots, p-1\}$, $b_{ii} > 0$ whenever $b_{i,i+1} > 0$ for $i \in \{1, 2, \dots, p-1\}$; and $c_{i,i+1} > 0$ for some $i \in \{1, 2, \dots, q-1\}$, $c_{ii} > 0$ whenever $c_{i,i+1} > 0$ for $i \in \{1, 2, \dots, q-1\}$, $0 < \omega \le 1$, $0 \le \tau < 1$, then*

(a) $\rho(\mathcal{L}_{\tau,\omega}^{*(3)}) < \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) < 1$;

(b) $\rho(\mathcal{L}_{\tau,\omega}^{*(3)}) > \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) > 1$.

Comparison results show that the convergence rates of the preconditioned GAOR methods defined by (1.22) are better than the preconditioned GAOR methods with the preconditioners (1.14) proposed by Wang et al. in [11] whenever these methods are convergent.

In addition, the authors of [4] also considered the preconditioners $\widetilde{P}_i$ of form

$$\widetilde{P}_i = \begin{pmatrix} I + W_i & 0 \\ K_i & I \end{pmatrix}, \quad i = 1, 2, 3, \tag{1.24}$$

where K_i $(i = 1, 2, 3)$ are defined as (1.9), (1.10), (1.11), (1.12) and (1.13), and

$$W_1 = \begin{pmatrix} 0 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \vdots & \vdots \quad\; \vdots \\ \frac{b_{p1}}{\nu} & 0 \cdots 0 \end{pmatrix} (\nu > 0), \tag{1.25}$$

$$W_2 = \begin{pmatrix} 0 & 0 \cdots 0 \\ b_{21} & 0 \cdots 0 \\ \vdots & \vdots \quad\; \vdots \\ b_{p1} & 0 \cdots 0 \end{pmatrix}, \tag{1.26}$$

$$W_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_{21} & 0 & \cdots & 0 & 0 \\ 0 & b_{32} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{p,p-1} & 0 \end{pmatrix}. \tag{1.27}$$

Then $\widetilde{P}_i\mathcal{H}$ can be expressed as

$$\widetilde{P}_i\mathcal{H} = \begin{pmatrix} I - \widetilde{B}_i & \widetilde{U}_i \\ \widetilde{L}_i & I - \widetilde{C}_i \end{pmatrix}, \quad i = 1, 2, 3,$$

where $\widetilde{B}_i = B - W_i(I - B)$, $\widetilde{U}_i = (I + W_i)U$, $\widetilde{L}_i = L + K_i(I - B)$ and $\widetilde{C}_i = C - K_iU$.

The preconditioned GAOR methods for solving $\widetilde{\mathcal{P}}_i\mathcal{H}y = \widetilde{\mathcal{P}}_i f$ are

$$y^{(k+1)} = \widetilde{\mathcal{L}}^{(i)}_{\tau,\omega} y^{(k)} + \omega \widetilde{g}_i, \quad k = 0, 1, 2, \cdots, \tag{1.28}$$

where

$$\widetilde{\mathcal{L}}^{(i)}_{\tau,\omega} = \begin{pmatrix} (1-\omega)I + \omega\widetilde{B}_i & -\omega\widetilde{U}_i \\ \omega(\tau-1)\widetilde{L}_i - \omega\tau\widetilde{L}_i\widetilde{B}_i & (1-\omega)I + \omega\widetilde{C}_i + \omega\tau\widetilde{L}_i\widetilde{U}_i \end{pmatrix} \tag{1.29}$$

and

$$\widetilde{g}_i = \begin{pmatrix} I & 0 \\ -\tau\widetilde{L}_i & I \end{pmatrix} \widetilde{\mathcal{P}}_i f$$

for $i = 1, 2, 3$.

For the preconditioners $\widetilde{P}_i (i = 1, 2, 3)$, the following comparison results are presented in [4].

Lemma 1.4 ([4], Theorem 3.11)**.** *Let* $\mathcal{L}_{\tau,\omega}$ *and* $\widetilde{L}^{(1)}_{\tau,\omega}$ *be the iteration matrices defined by* (1.5) *and* (1.29)*, respectively. Assume that the matrix* $\mathcal{H}$ *in* (1.3) *is irreducible with* $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $b_{p,1} > 0$, $l_{n-p,1} > 0$, $\alpha > 1 - b_{11}$, $\beta > 0$, $\beta > 1 - b_{11}$, *and* $0 < \omega \leq 1$, $0 \leq \tau < 1$, *then*

(*a*) $\rho(\widetilde{L}^{(1)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) < 1$;

(*b*) $\rho(\widetilde{L}^{(1)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) > 1$.

Lemma 1.5 ([4], Theorem 3.12)**.** *Let* $\mathcal{L}_{\tau,\omega}$ *and* $\widetilde{L}^{(2)}_{\tau,\omega}$ *be the iteration matrices defined by* (1.5) *and* (1.29)*, respectively. Assume that the matrix* $\mathcal{H}$ *in* (1.3) *is irreducible with* $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $b_{11} > 0$, $b_{i,1} > 0$ *for some* $i \in \{2, \ldots, p\}$, *and* $l_{i,1} < 0$ *for some* $i \in \{1, 2, \ldots, n-p\}$, $0 < \omega \leq 1$, $0 \leq \tau < 1$, *then*

(*a*) $\rho(\widetilde{L}^{(2)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) < 1$;

(*b*) $\rho(\widetilde{L}^{(2)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) > 1$.

Lemma 1.6 ([4], Theorem 3.13)**.** *Let* $\mathcal{L}_{\tau,\omega}$ *and* $\widetilde{L}^{(3)}_{\tau,\omega}$ *be the iteration matrices defined by* (1.5) *and* (1.29)*, respectively. Assume that the matrix* $\mathcal{H}$ *in* (1.3) *is irreducible with* $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $b_{i+1,i} > 0$ *for some* $i \in \{1, 2, \ldots, p-1\}$, $b_{ii} > 0$ *whenever* $b_{i+1,i} > 0$ *for* $i \in \{1, 2, \ldots, p-1\}$; *and* $l_{i,i} < 0$ *for some* $i \in \{1, 2, \ldots, n-p\}$, $b_{ii} > 0$ *whenever* $l_{i,i} < 0$ *for* $i \in \{1, 2, \ldots, n-p\}$, $0 < \omega \leq 1$, $0 \leq \tau < 1$, *then*

(*a*) $\rho(\widetilde{L}^{(3)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) < 1$;

(*b*) $\rho(\widetilde{L}^{(3)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) > 1$.

Comparison results show that the convergence rates of the preconditioned GAOR methods defined by (1.28) are better than the preconditioned GAOR methods with the preconditioners (1.8) proposed by Zhou et al. in [16] whenever these methods are convergent.

Recently, Huang et al. [3] proposed two preconditioners $\overrightarrow{\mathcal{P}}_i$ of the forms

$$\overrightarrow{\mathcal{P}}_i = \begin{pmatrix} I + \overrightarrow{W}_i & 0 \\ \overrightarrow{K}_i & I \end{pmatrix}, \quad i = 1, 2, \tag{1.30}$$

where $\overrightarrow{K}_i$ is as follows:

$$\overrightarrow{K}_1 = \begin{pmatrix} -\mu_1 l_{11} & 0 \cdots & 0 \\ -\mu_2 l_{21} & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \\ -\mu_q l_{q1} & 0 \cdots & 0 \end{pmatrix}.$$

If $q < p$, then

$$\overrightarrow{K}_2 = \begin{pmatrix} -\nu_1 l_{11} & 0 & \cdots & 0 & 0 \cdots 0 \\ 0 & -\nu_2 l_{22} & \cdots & 0 & 0 \cdots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \quad \vdots \\ 0 & 0 & \cdots & -\nu_q l_{qq} & 0 \cdots 0 \end{pmatrix}.$$

If $q = p$, then

$$\overrightarrow{K}_2 = \begin{pmatrix} -\nu_1 l_{11} & 0 & \cdots & 0 \\ 0 & -\nu_2 l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\nu_q l_{qq} \end{pmatrix}.$$

If $q > p$, then

$$\overrightarrow{K}_2 = \begin{pmatrix} -\nu_1 l_{11} & 0 & \cdots & 0 \\ 0 & -\nu_2 l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\nu_q l_{qq} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

And

$$\overrightarrow{W}_1 = \begin{pmatrix} 0 & 0 \cdots 0 \\ \gamma_2 b_{21} & 0 \cdots 0 \\ \vdots & \vdots \quad \vdots \\ \gamma_p b_{p1} & 0 \cdots 0 \end{pmatrix}, \quad \overrightarrow{W}_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \delta_2 b_{21} & 0 & \cdots & 0 & 0 \\ 0 & \delta_3 b_{32} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_p b_{p,p-1} & 0 \end{pmatrix}.$$

The preconditioned GAOR methods for solving $\overrightarrow{\mathcal{P}}_i \mathcal{H} y = \overrightarrow{\mathcal{P}}_i f$ are

$$y^{(k+1)} = \overrightarrow{\mathcal{L}}^{(i)}_{\tau,\omega} y^{(k)} + \omega \overrightarrow{g}_i, \quad k = 0, 1, 2, \cdots, \tag{1.31}$$

where

$$\overrightarrow{\mathcal{L}}^{(i)}_{\tau,\omega} = \begin{pmatrix} (1-\omega)I + \omega \overrightarrow{B}_i & -\omega \overrightarrow{U}_i \\ \omega(\tau - 1)\overrightarrow{L}_i - \omega\tau \overrightarrow{L}_i \overrightarrow{B}_i & (1-\omega)I + \omega \overrightarrow{C}_i + \omega\tau \overrightarrow{L}_i \overrightarrow{U}_i \end{pmatrix} \tag{1.32}$$

and

$$\overrightarrow{g}_i = \begin{pmatrix} I & 0 \\ -\tau \overrightarrow{L}_i & I \end{pmatrix} \overrightarrow{\mathcal{P}}_i f$$

for $i = 1, 2$.

For the preconditioners $\overrightarrow{\mathcal{P}}_i (i = 1, 2)$, Huang et al. [3] gave the following comparison results.

Lemma 1.7 ([3], Theorem 1)**.** *Let* $\mathcal{L}_{\tau,\omega}$ *and* $\overrightarrow{\mathcal{L}}^{(1)}_{\tau,\omega}$ *be the iteration matrices defined by* (1.5) *and* (1.32)*, respectively. Assume that the matrix* $\mathcal{H}$ *in* (1.3) *is irreducible with* $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 < \omega \le 1$, $0 \le \tau < 1$, $b_{i1} > 0$ *for some* $i = 2, 3, \ldots, p$, $l_{i1} < 0$ *for some* $i = 1, 2, \ldots, q$, $0 < \max_{2 \le i \le p} \gamma_i < \dfrac{1}{1 - b_{11}}$ *and* $0 < \max_{1 \le i \le q} \mu_i < \dfrac{1}{1 - b_{11}}$ *whenever* $0 \le b_{11} < 1$, *or* $\gamma_i > 0$ *for* $i = 2, 3, \ldots, p$ *and* $\mu_i > 0$ *for* $i = 1, 2, \ldots, q$ *whenever* $b_{11} \ge 1$, *then*

(a) $\rho(\overrightarrow{\mathcal{L}}^{(1)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) < 1$;

(b) $\rho(\overrightarrow{\mathcal{L}}^{(1)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) > 1$.

Lemma 1.8 ([3], Theorem 2)**.** *Let* $\mathcal{L}_{\tau,\omega}$ *and* $\overrightarrow{\mathcal{L}}^{(2)}_{\tau,\omega}$ *be the iteration matrices defined by* (1.5) *and* (1.32)*, respectively. Assume that the matrix* $\mathcal{H}$ *in* (1.3) *is irreducible with* $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 < \omega \le 1$, $0 \le \tau < 1$, $b_{i+1,i} > 0$ *for some* $i = 2, 3, \ldots, p$, $l_{ii} < 0$ *for some* $i = 1, 2, \ldots, q$, $\delta_i > 0$ *whenever* $b_{i-1,i-1} \ge 1$, *or* $0 < \delta_i < \dfrac{1}{1 - b_{i-1,i-1}}$ *whenever* $0 \le b_{i-1,i-1} < 1$ *for* $i = 2, 3, \ldots, p$, *and* $\nu_i < \dfrac{1}{1 - b_{ii}}$ *whenever* $0 \le b_{ii} < 1$, *or* $\nu_i > 0$ *whenever* $b_{ii} \ge 1$ *for* $i = 1, 2, \ldots, q$, *then*

(a) $\rho(\overrightarrow{\mathcal{L}}^{(2)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) < 1$;

(b) $\rho(\overrightarrow{\mathcal{L}}^{(2)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega})$ *if* $\rho(\mathcal{L}_{\tau,\omega}) > 1$.

Comparison results show that the convergence rates of the preconditioned GAOR methods (1.31) are better than the preconditioned GAOR methods presented in [8] by Shen et al. whenever these methods are convergent.

In addition, the authors of [3] also considered other two preconditioners $\overleftarrow{\mathcal{P}}_i$ of the forms

$$\overleftarrow{\mathcal{P}}_i = \begin{pmatrix} I + \overleftarrow{S}_i & 0 \\ 0 & I + \overleftarrow{V}_i \end{pmatrix}, \quad i = 1, 2, \tag{1.33}$$

where

$$\overleftarrow{S}_1 = \begin{pmatrix} 0 & \alpha_2 b_{12} \cdots & 0 & 0 \\ \beta_2 b_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & \alpha_p b_{p-1,p} \\ 0 & 0 & \cdots & \beta_p b_{p,p-1} & 0 \end{pmatrix}, \quad \overleftarrow{S}_2 = \begin{pmatrix} 0 & \alpha_2 b_{12} & \cdots & \alpha_p b_{1p} \\ \beta_2 b_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \beta_p b_{p1} & 0 & \cdots & 0 \end{pmatrix},$$

$$\overleftarrow{V}_1 = \begin{pmatrix} 0 & \tau_2 c_{12} \cdots & 0 & 0 \\ \sigma_2 c_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & \tau_q c_{q-1,q} \\ 0 & 0 & \cdots & \sigma_q c_{q,q-1} & 0 \end{pmatrix}, \quad \overleftarrow{V}_2 = \begin{pmatrix} 0 & \tau_2 c_{12} & \cdots & \tau_p c_{1p} \\ \sigma_2 c_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \sigma_q c_{q1} & 0 & \cdots & 0 \end{pmatrix}.$$

In [3], Huang et al. also gave some comparison results for the preconditioners $\overleftarrow{\mathcal{P}}_i (i = 1, 2)$. See Theorems 5 and 6 in [3]. Comparison results show that the convergence rates of the preconditioned GAOR methods with the preconditioner (1.33) are better than the preconditioned GAOR methods presented in [10] by Wang et al. whenever these methods are convergent.

Several kinds of preconditioners for the GAOR method for solving the linear system (1.1) has been proposed in the literatures, see for example [7]. Inspired by the above facts, we will give some generalized preconditioners to establish some new preconditioned GAOR methods, and study their convergence rates. Finally, one numerical example is given to verify the theoretical results.

At the end of this section, we give some notations and results, which will be used in next section.

Notation. For a vector $x \in \mathbb{R}^n$, $x \geq 0$ $(x > 0)$ denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ $(x > y)$ means that $x - y \geq 0$ $(x - y > 0)$. These definitions carry immediately over to the matrices. For a square matrix A, $\rho(A)$ denotes the spectral radius of A, and A is called irreducible if the directed graph of A is strongly connected [9].

Lemma 1.9. ([4,9]) *Let $A \geq 0$ be an irreducible matrix. Then the following results hold.*

(1) *A has a positive eigenvalue equal to $\rho(A)$.*

(2) *A has an eigenvector $x > 0$ corresponding to $\rho(A)$.*

(3) *$\rho(A)$ is a simple eigenvalue of A.*

Lemma 1.10. ([1,4]) *Let $A \geq 0$ be a matrix. Then the following results hold.*

(1) *If $Ax \geq \beta x$ for a vector $x \geq 0$ and $x \neq 0$, then $\rho(A) \geq \beta$.*

(2) *If $Ax \leq \alpha x$ for a vector $x > 0$, then $\rho(A) \leq \alpha$.*

(3) *Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$, $\alpha x \neq Ax$, $Ax \neq \beta x$ for a vector $x \geq 0$ and $x \neq 0$, then $\alpha < x < \beta$.*

2. Generalized preconditioned GAOR methods

In the following, we propose three generalized preconditioners $\overline{\mathcal{P}}_i$ of the following forms

$$\overline{\mathcal{P}}_i = \begin{pmatrix} I + \alpha S_i + \beta W_i & 0 \\ \gamma K_i & I + \delta V_i \end{pmatrix}, \quad i = 1, 2, 3, \tag{2.1}$$

where the parameters α, β, γ and δ are nonnegative real numbers and satisfy

$$\alpha + \beta = 1, \quad \gamma + \delta = 1,$$

and S_i, W_i, K_i, V_i $(i = 1, 2, 3)$ are defined as Section 1.

Remark 2.1. It can see that

- when $\alpha = 1, \beta = 0, \gamma = 0$ and $\delta = 1$, the preconditioners (2.1) reduce to (1.18);
- when $\alpha = 0, \beta = 1, \gamma = 1$ and $\delta = 0$, the preconditioners (2.1) reduce to (1.24).

It follows (2.1) that $\overline{\mathcal{P}}_i\mathcal{H}$ can be expressed as

$$
\begin{aligned}
\overline{\mathcal{P}}_i\mathcal{H} &= \begin{pmatrix} I + \alpha S_i + \beta W_i & 0 \\ \gamma K_i & I + \delta V_i \end{pmatrix} \begin{pmatrix} I - B & U \\ L & I - C \end{pmatrix} \\
&= \begin{pmatrix} I - \big[B - \alpha S_i(I - B) - \beta W_i(I - B)\big] & \big(I + \alpha S_i + \beta W_i\big)U \\ \gamma K_i(I - B) + (I + \delta V_i)L & I - \big[C - \delta V_i(I - C) - \gamma K_i U\big] \end{pmatrix} \\
&:= \begin{pmatrix} I - \overline{B}_i & \overline{U}_i \\ \overline{L}_i & I - \overline{C}_i \end{pmatrix}. \qquad (2.2)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\overline{B}_1 &= B - \alpha S_1(I - B) - \beta W_1(I - B) = \alpha\big[B - S_1(I - B)\big] + \beta\big[B - W_1(I - B)\big] \\
&= \alpha \begin{pmatrix} b_{11} + b_{12}b_{21} & b_{12}b_{22} & \cdots & b_{1p} + b_{12}b_{2p} \\ b_{21}b_{11} + b_{23}b_{31} & b_{22} + b_{21}b_{12} + b_{23}b_{32} & \cdots & b_{2p} + b_{21}b_{1p} + b_{23}b_{3p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p-1,1} + b_{p-1,p-2}b_{p-2,1} + b_{p-1,p}b_{p1} & b_{p-1,2} + b_{p-1,p-2}b_{p-2,2} + b_{p-1,p}b_{p2} & \cdots & b_{p-1,p-2}b_{p-2,p} + b_{p-1,p}b_{pp} \\ b_{p1} + b_{p,p-1}b_{p-1,1} & b_{p2} + b_{p,p-1}b_{p-1,2} & \cdots & b_{pp} + b_{p,p-1}b_{p-1,p} \end{pmatrix} \\
&+ \beta \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p-1,1} & b_{p-1,2} & \cdots & b_{p-1,p} \\ (1 - \frac{1 - b_{11}}{\nu})b_{p1} & b_{p2} + \frac{b_{p1}b_{12}}{\nu} & \cdots & b_{pp} + \frac{b_{p1}b_{1p}}{\nu} \end{pmatrix}, \\
\overline{B}_2 &= B - \alpha S_2(I - B) - \beta W_2(I - B) = \alpha\big[B - S_2(I - B)\big] + \beta\big[B - W_2(I - B)\big] \\
&= \alpha \begin{pmatrix} b_{11} & \cdots & b_{1,p-1} & b_{1p} \\ b_{21}b_{11} & \cdots & b_{2,p-1} + b_{21}b_{1,p-1} & b_{2p} + b_{21}b_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1} + b_{p,p-1}b_{p-1,1} & \cdots & b_{p,p-1}b_{p-1,p-1} & b_{pp} + b_{p,p-1}b_{p-1,p} \end{pmatrix}
\end{aligned}
$$

$$+\beta\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21}b_{11} & b_{22}+b_{21}b_{12} & \cdots & b_{2p}+b_{21}b_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1}b_{11} & b_{p2}+b_{p1}b_{12} & \cdots & b_{pp}+b_{p1}b_{1p} \end{pmatrix},$$

$$\overline{B}_3 = B - \alpha S_3(I-B) - \beta W_3(I-B) = \alpha\big[B - S_3(I-B)\big] + \beta\big[B - W_3(I-B)\big]$$

$$= \alpha\begin{pmatrix} b_{11}+b_{21}b_{12} & \cdots & b_{1,p-1}+b_{2,p-1}b_{12} & b_{1p}+b_{2p}b_{12} \\ b_{21}+b_{31}b_{23} & \cdots & b_{2,p-1}+b_{3,p-1}b_{23} & b_{2p}+b_{3p}b_{23} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1} & \cdots & b_{p,p-1} & b_{pp} \end{pmatrix}$$

$$+\beta\begin{pmatrix} b_{11} & \cdots & b_{1,p-1} & b_{1p} \\ b_{21}b_{11} & \cdots & b_{2,p-1}+b_{21}b_{1,p-1} & b_{2p}+b_{21}b_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1}+b_{p,p-1}b_{p-1,1} & \cdots & b_{p,p-1}b_{p-1,p-1} & b_{pp}+b_{p,p-1}b_{p-1,p} \end{pmatrix}.$$

In addition, we have

$$K_1(I-B) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ \dfrac{-l_{q1}(1-b_{11})}{\mu} & \dfrac{l_{q1}b_{12}}{\mu} & \cdots & \dfrac{l_{q1}b_{1p}}{\mu} \end{pmatrix},$$

$$K_2(I-B) = \begin{pmatrix} -l_{11}(1-b_{11}) & l_{11}b_{12} & \cdots & l_{11}b_{1p} \\ -l_{21}(1-b_{11}) & l_{21}b_{12} & \cdots & l_{21}b_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ -l_{q1}(1-b_{11}) & l_{q1}b_{12} & \cdots & l_{q1}b_{1p} \end{pmatrix}.$$

And if $q < p$, then

$$K_3(I-B) = \begin{pmatrix} -l_{11}(1-b_{11}) & l_{11}b_{12} & \cdots & l_{11}b_{1q} & l_{11}b_{1,q+1} & \cdots & l_{11}b_{1p} \\ l_{22}b_{21} & -l_{22}(1-b_{22}) & \cdots & l_{22}b_{2q} & l_{22}b_{2,q+1} & \cdots & l_{22}b_{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ l_{qq}b_{q1} & l_{qq}b_{q2} & \cdots & -l_{qq}(1-b_{qq}) & l_{qq}b_{q,q+1} & \cdots & l_{qq}b_{qp} \end{pmatrix}.$$

If $q = p$, then

$$K_3(I-B) = \begin{pmatrix} -l_{11}(1-b_{11}) & l_{11}b_{12} & \cdots & l_{11}b_{1p} \\ l_{22}b_{21} & -l_{22}(1-b_{22}) & \cdots & l_{22}b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ l_{pp}b_{p1} & l_{pp}b_{p2} & \cdots & -l_{pp}(1-b_{pp}) \end{pmatrix}.$$

If $q > p$, then

$$K_3(I-B) = \begin{pmatrix} -l_{11}(1-b_{11}) & l_{11}b_{12} & \cdots & l_{11}b_{1p} \\ l_{22}b_{21} & -l_{22}(1-b_{22}) & \cdots & l_{22}b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ l_{pp}b_{p1} & l_{pp}b_{p2} & \cdots & -l_{pp}(1-b_{pp}) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

And we can obtain that

$$\begin{aligned} & C - V_1(I-C) \\ = & \begin{pmatrix} c_{11}+c_{12}c_{21} & c_{12}c_{22} & \cdots & c_{1q}+c_{12}c_{2q} \\ c_{21}c_{11}+c_{23}c_{31} & c_{22}+c_{21}c_{12}+c_{23}c_{32} & \cdots & c_{2q}+c_{21}c_{1q}+c_{23}c_{3q} \\ \vdots & \vdots & \vdots & \vdots \\ c_{q-1,1}+c_{q-1,q-2}c_{q-2,1}+c_{q-1,q}c_{q1} & c_{q-1,2}+c_{q-1,q-2}c_{q-2,2}+c_{q-1,q}c_{q2} & \cdots & c_{q-1,q-2}c_{q-2,q}+c_{q-1,q}c_{qq} \\ c_{q1}+c_{q,q-1}c_{q-1,1} & c_{q2}+c_{q,q-1}c_{q-1,2} & \cdots & c_{qq}+c_{q,q-1}c_{q-1,q} \end{pmatrix}, \end{aligned}$$

$$C - V_2(I-C) = \begin{pmatrix} c_{11} & \cdots & c_{1,q-1} & c_{1q} \\ c_{21}c_{11} & \cdots & c_{2,q-1}+c_{21}c_{1,q-1} & c_{2q}+c_{21}c_{1q} \\ \vdots & \vdots & \vdots & \vdots \\ c_{q1}+c_{q,q-1}c_{q-1,1} & \cdots & c_{q,q-1}c_{q-1,q-1} & c_{qq}+c_{q,q-1}c_{q-1,q} \end{pmatrix},$$

$$C - V_3(I-C) = \begin{pmatrix} c_{11}+c_{21}c_{12} & \cdots & c_{1,q-1}+c_{2,q-1}c_{12} & c_{1q}+c_{2q}c_{12} \\ c_{21}+c_{31}c_{23} & \cdots & c_{2,q-1}+c_{3,q-1}c_{23} & c_{2q}+c_{3q}c_{23} \\ \vdots & \vdots & \vdots & \vdots \\ c_{q1} & \cdots & c_{q,q-1} & c_{qq} \end{pmatrix}.$$

The preconditioned GAOR methods for solving $\overline{\mathcal{P}}_i\mathcal{H}y = \overline{\mathcal{P}}_i f$ are

$$y^{(k+1)} = \overline{\mathcal{L}}^{(i)}_{\tau,\omega} y^{(k)} + \omega \overline{g}^{(i)}, \quad k = 0, 1, 2, \cdots, \tag{2.3}$$

where $i = 1, 2, 3$,

$$\overline{\mathcal{L}}^{(i)}_{\tau,\omega} = \begin{pmatrix} (1-\omega)I + \omega\overline{B}_i & -\omega\overline{U}_i \\ \omega(\tau-1)\overline{L}_i - \omega\tau\overline{L}_i\overline{B}_i & (1-\omega)I + \omega\overline{C}_i + \omega\tau\overline{L}_i\overline{U}_i \end{pmatrix} \tag{2.4}$$

and

$$\overline{g}^{(i)} = \begin{pmatrix} I & 0 \\ -\tau\overline{L}_i & I \end{pmatrix} \overline{\mathcal{P}}_i f.$$

Now, we compare the convergence rate of the generalized preconditioned GAOR method defined by (2.3) with that of the GAOR method defined by (1.4).

Theorem 2.1. *Let $\mathcal{L}_{\tau,\omega}$ and $\overline{\mathcal{L}}^{(1)}_{\tau,\omega}$ be the iteration matrices defined by* (1.5) *and* (2.4)*, respectively. Assume that the matrix $\mathcal{H}$ in* (1.3) *is irreducible with $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \dots, p-1\}$, $b_{ii} > 0$ whenever $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for $i \in \{1, 2, \dots, p-1\}$, $b_{p1} > 0$. Assume that the numbers $\tau, \omega, \nu, \mu, \alpha, \beta, \gamma, \delta$ satisfy*

(1) $0 \le \tau < 1$, $0 < \omega \le 1$ *and* $\nu, \mu > 0$;

(2) $\alpha, \beta \ge 0$, $\alpha + \beta = 1$ *and*

$$\big(b_{p,p-1}b_{p-1,1} + \frac{1-b_{11}}{\nu}b_{p1}\big)\beta \le b_{p1} + b_{p,p-1}b_{p-1,1};$$

(3) $\gamma, \delta \ge 0$, $\gamma + \delta = 1$ *and*

$$\big[\frac{l_{q1}}{\mu}(1-b_{11}) + c_{q,q-1}l_{q-1,1}\big]\gamma \ge c_{q,q-1}l_{q-1,1} + l_{q1}.$$

Then, we have

$$\rho(\overline{\mathcal{L}}^{(1)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega}) < 1 \quad \textit{or} \quad \rho(\overline{\mathcal{L}}^{(1)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega}) > 1.$$

Proof. By direct operation, we have

$$\mathcal{L}_{\tau,\omega} = \begin{pmatrix} (1-\omega)I + \omega B & -\omega U \\ -\omega(1-\tau)L & (1-\omega)I + \omega C \end{pmatrix} + \omega\tau \begin{pmatrix} 0 & 0 \\ -LB & LU \end{pmatrix}. \tag{2.5}$$

Since $0 < \omega \le 1$, $0 \le \tau < 1$, $L \le 0$, $U \le 0$, $B \ge 0$ and $C \ge 0$, we obtain

$$\begin{pmatrix} 0 & 0 \\ -LB & LU \end{pmatrix} \ge 0,$$

and $\mathcal{L}_{\tau,\omega}$ is nonnegative. Since $\mathcal{H}$ is irreducible, from (2.5), it is easy to see that the matrix $\mathcal{L}_{\tau,\omega}$ is nonnegative and irreducible.

Obviously, by $B \ge 0$ and $U \le 0$, we can get

$$\overline{U}_1 = \big(I + \alpha S_1 + \beta W_1\big)U \le 0.$$

And it is easy to see that $C - V_1(I - C) \ge 0$ and $C - K_1U \ge 0$, then

$$\overline{C}_1 = C - \delta V_1(I - C) - \gamma K_1 U = \delta[C - V_1(I - C)] + \gamma[C - K_1U] \ge 0.$$

Notice that $B \geq 0$, $\nu > 0$, $\alpha + \beta = 1$ and $\big(b_{p,p-1}b_{p-1,1} + \frac{1-b_{11}}{\nu}b_{p1}\big)\beta \leq b_{p1} + b_{p,p-1}b_{p-1,1}$, we have

$$\overline{B}_1 \geq 0.$$

In addition, as $L \leq 0$, $\mu > 0$, $\gamma + \delta = 1$ and $\big[\frac{l_{q1}}{\mu}(1 - b_{11}) + c_{q,q-1}l_{q-1,1}\big]\gamma \geq c_{q,q-1}l_{q-1,1} + l_{q1}$, we can obtain

$$\overline{L}_1 = \gamma K_1(I - B) + (I + \delta V_1)L \leq 0.$$

Hence, similarly, we can prove that the matrix $\overline{\mathcal{L}}^{(1)}_{\tau,\omega}$ is a nonnegative and irreducible matrix.

By Lemma 1.9, there exists a positive vector x such that

$$\mathcal{L}_{\tau,\omega}x = \lambda x, \tag{2.6}$$

where $\lambda = \rho(\mathcal{L}_{\tau,\omega})$. Since the matrix $\mathcal{H}$ is nonsingular, $\lambda \neq 1$. Hence, $\lambda > 1$ or $\lambda < 1$. From (2.6), we have

$$\left[(1-\omega)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + (\omega - \tau)\begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix} + \omega\begin{pmatrix} B & -U \\ 0 & C \end{pmatrix}\right]x = \lambda\begin{pmatrix} I & 0 \\ \tau L & I \end{pmatrix}x \tag{2.7}$$

and

$$\omega\mathcal{H}x = \begin{pmatrix} I & 0 \\ \tau L & I \end{pmatrix}(I - \mathcal{L}_{\tau,\omega})x = (1-\lambda)\begin{pmatrix} I & 0 \\ \tau L & I \end{pmatrix}x. \tag{2.8}$$

From (2.2), (2.4), (2.7) and (2.8), we have

$$\begin{aligned}
&\overline{\mathcal{L}}^{(1)}_{\tau,\omega}x - \lambda x \\
&= \begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}^{-1}\left[(1-\omega)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + (\omega-\tau)\begin{pmatrix} 0 & 0 \\ -\overline{L}_1 & 0 \end{pmatrix} + \omega\begin{pmatrix} \overline{B}_1 & -\overline{U}_1 \\ 0 & \overline{C}_1 \end{pmatrix}\right]x - \lambda x \\
&= \begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}^{-1}\left[(1-\omega)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + (\omega-\tau)\begin{pmatrix} 0 & 0 \\ -\overline{L}_1 & 0 \end{pmatrix} + \omega\begin{pmatrix} \overline{B}_1 & -\overline{U}_1 \\ 0 & \overline{C}_1 \end{pmatrix} - \lambda\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right]x \\
&= \begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}^{-1}\left[-\omega\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \omega\begin{pmatrix} 0 & 0 \\ -\overline{L}_1 & 0 \end{pmatrix} + \omega\begin{pmatrix} \overline{B}_1 & -\overline{U}_1 \\ 0 & \overline{C}_1 \end{pmatrix} + (1-\lambda)\begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}\right]x \\
&= \begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}^{-1}\left[\begin{pmatrix} -\omega I + \omega\overline{B}_1 & -\omega\overline{U}_1 \\ -\omega\overline{L}_1 & -\omega I + \omega\overline{C}_1 \end{pmatrix} + (1-\lambda)\begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}\right]x \\
&= \begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}^{-1}\left[\begin{pmatrix} -\omega(\alpha S_1 + \beta W_1)(I - B) & -\omega\big(\alpha S_1 + \beta W_1\big)U \\ -\omega[\gamma K_1(I - B) + \delta V_1 L] & -\omega[\delta V_1(I - C) + \gamma K_1 U] \end{pmatrix} - \omega\mathcal{H}\right. \\
&\quad \left. + (1-\lambda)\begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}\right]x
\end{aligned}$$

$$= \begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}^{-1} \left[\begin{pmatrix} -\omega(\alpha S_1 + \beta W_1)(I-B) & -\omega\big(\alpha S_1 + \beta W_1\big)U \\ -\omega[\gamma K_1(I-B) + \delta V_1 L] & -\omega[\delta V_1(I-C) + \gamma K_1 U] \end{pmatrix} \right.$$

$$\left. + (1-\lambda) \begin{pmatrix} 0 & 0 \\ \tau(\overline{L}_1 - L) & 0 \end{pmatrix} \right] x$$

$$= \begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}^{-1} \left[\begin{pmatrix} \alpha S_1 + \beta W_1 & 0 \\ \gamma K_1 & \delta V_1 \end{pmatrix} \begin{pmatrix} -\omega(I-B) & -\omega U \\ -\omega L & -\omega(I-C) \end{pmatrix} \right.$$

$$\left. + (1-\lambda) \begin{pmatrix} 0 & 0 \\ \tau(\overline{L}_1 - L) & 0 \end{pmatrix} \right] x$$

$$= \begin{pmatrix} I & 0 \\ \tau\overline{L}_1 & I \end{pmatrix}^{-1} \left[\begin{pmatrix} \alpha S_1 + \beta W_1 & 0 \\ \gamma K_1 & \delta V_1 \end{pmatrix} (\lambda - 1) \begin{pmatrix} I & 0 \\ \tau L & I \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 & 0 \\ \tau(\overline{L}_1 - L) & 0 \end{pmatrix} \right] x$$

$$= (\lambda - 1) \begin{pmatrix} I & 0 \\ -\tau\overline{L}_1 & I \end{pmatrix} \begin{pmatrix} \alpha S_1 + \beta W_1 & 0 \\ \gamma K_1 - \tau\gamma K_1(I-B) & \delta V_1 \end{pmatrix} x$$

$$= (\lambda - 1) \begin{pmatrix} I & 0 \\ -\tau\overline{L}_1 & I \end{pmatrix} \begin{pmatrix} \alpha S_1 + \beta W_1 & 0 \\ \gamma K_1[(1-\tau)I + \tau B] & \delta V_1 \end{pmatrix} x. \tag{2.9}$$

Since $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \ldots, p-1\}$, $b_{p1} > 0$ and $\nu > 0$, we obtain that S_1 and W_1 are nonnegative matrices and $S_1 \neq 0$, $W_1 \neq 0$. Note that $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$, then $\alpha S_1 + \beta W_1$ is nonnegative and $\alpha S_1 + \beta W_1 \neq 0$. Moreover, by assumptions, $\overline{L}_1 \leq 0$ and $\gamma K_1[(1-\tau)I + \tau B] \geq 0$. So we have

$$\begin{pmatrix} I & 0 \\ -\tau\overline{L}_1 & I \end{pmatrix} \begin{pmatrix} \alpha S_1 + \beta W_1 & 0 \\ \gamma K_1[(1-\tau)I + \tau B] & \delta V_1 \end{pmatrix} x \geq 0$$

and

$$\begin{pmatrix} I & 0 \\ -\tau\overline{L}_1 & I \end{pmatrix} \begin{pmatrix} \alpha S_1 + \beta W_1 & 0 \\ \gamma K_1[(1-\tau)I + \tau B] & \delta V_1 \end{pmatrix} x \neq 0.$$

Hence, we have the following results.

Case I: If $\lambda < 1$, from (2.9), then $\overline{\mathcal{L}}^{(1)}_{\tau,\omega} x \leq \lambda x$ and $\overline{\mathcal{L}}^{(1)}_{\tau,\omega} x \neq \lambda x$. By Lemma 1.10, we get $\rho(\overline{\mathcal{L}}^{(1)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega})$.

Case II: If $\lambda > 1$, from (2.9), then $\overline{\mathcal{L}}^{(1)}_{\tau,\omega} x \geq \lambda x$ and $\overline{\mathcal{L}}^{(1)}_{\tau,\omega} x \neq \lambda x$. By Lemma 1.10, we get $\rho(\overline{\mathcal{L}}^{(1)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega})$. □

In a manner similar to that done for Theorem 2.1, we can obtain the following comparison theorems for $\overline{\mathcal{L}}^{(2)}_{\tau,\omega}$ and $\overline{\mathcal{L}}^{(3)}_{\tau,\omega}$. The only difference in the proof is that some assumptions are changed so that $S_i \neq 0, W_i \neq 0$ and $\overline{\mathcal{L}}^{(i)}_{\tau,\omega}$ is irreducible for $i = 2, 3$.

Theorem 2.2. *Let $\mathcal{L}_{\tau,\omega}$ and $\overline{\mathcal{L}}^{(2)}_{\tau,\omega}$ be the iteration matrices defined by* (1.5) *and* (2.4)*, respectively. Assume that the matrix $\mathcal{H}$ in* (1.3) *is irreducible with $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \ldots, p-1\}$, $b_{ii} > 0$ whenever $b_{i+1,i} > 0$ for $i \in \{1, 2, \ldots, p-1\}$; and $b_{i1} > 0$ for some $i \in \{2, 3, \ldots, p-1\}$. Assume that the numbers $\tau, \omega, \alpha, \beta, \gamma, \delta$ satisfy*

(1) $0 \leq \tau < 1,\ 0 < \omega \leq 1$;
(2) $\alpha, \beta \geq 0,\ \alpha + \beta = 1$;
(3) $\gamma, \delta \geq 0,\ \gamma + \delta = 1,\ -\gamma l_{11}(1 - b_{11}) + l_{11} \leq 0$,

$$-\gamma l_{i+1,1}(1 - b_{11}) + \delta c_{i+1,i} l_{i1} + l_{i+1,1} \leq 0 \quad \textit{for} \quad i = 1, 2, \ldots, q-1.$$

Then, we have

$$\rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega}) < 1 \quad \textit{or} \quad \rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega}) > 1.$$

Theorem 2.3. *Let $\mathcal{L}_{\tau,\omega}$ and $\overline{\mathcal{L}}^{(3)}_{\tau,\omega}$ be the iteration matrices defined by* (1.5) *and* (2.4)*, respectively. Assume that the matrix $\mathcal{H}$ in* (1.3) *is irreducible with $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for some $i \in \{1, 2, \ldots, p-1\}$, $b_{ii} > 0$ whenever $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for $i \in \{1, 2, \ldots, p-1\}$. Assume that the numbers $\tau, \omega, \alpha, \beta, \gamma, \delta$ satisfy*

(1) $0 \leq \tau < 1,\ 0 < \omega \leq 1$;
(2) $\alpha, \beta \geq 0,\ \alpha + \beta = 1$;
(3) $\gamma, \delta \geq 0,\ \gamma + \delta = 1$,

- *if $q \leq p$, $-\gamma l_{ii}(1 - b_{ii}) + l_{ii} + \delta c_{i,i+1} l_{i+1,i} \leq 0$, $i = 1, 2, \ldots, q-1$ and $-\gamma l_{qq}(1 - b_{qq}) + l_{qq} \leq 0$;*
- *if $q > p$, $-\gamma l_{ii}(1 - b_{ii}) + l_{ii} + \delta c_{i,i+1} l_{i+1,i} \leq 0$, $i = 1, 2, \ldots, p$.*

Then, we have

$$\rho(\overline{\mathcal{L}}^{(3)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega}) < 1 \quad \textit{or} \quad \rho(\overline{\mathcal{L}}^{(3)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega}) > 1.$$

From Remark 2.1 as well as Theorems 2.1, 2.2 and 2.3, it indicates that the preconditioned GAOR methods defined by (2.3) under the appropriate parameters may have the better numerical performance than the preconditioned GAOR methods defined by (1.22) and (1.28) proposed in [4] whenever these methods are convergent. And it is worth mentioning that the comparison results of [4] show that the convergence rates of the preconditioned GAOR methods (1.22) and (1.28) are better than those of the preconditioned GAOR methods with the preconditioners (1.14) and (1.8) whenever these methods are convergent, respectively. Hence, the proposed preconditioners $\overline{\mathcal{P}}_i (i = 1, 2, 3)$ (2.1) would be also better than (1.8) and (1.14).

3. Numerical experiments

In this section, we give one numerical example to illustrate the theoretical results in Section 2.

Example 3.1. This example is a numerical example introduced in [16]. The coefficient matrix $\mathcal{H}$ in (1.3) is given by

$$\mathcal{H} = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix},$$

where $p+q=n$, $B=(b_{ij})\in\mathbb{R}^{p\times p}$, $C=(c_{ij})\in\mathbb{R}^{q\times q}$, $L=(l_{ij})\in\mathbb{R}^{q\times p}$ and $U=(u_{ij})\in\mathbb{R}^{p\times q}$ with

- $b_{ii} = \dfrac{1}{10(i+1)}, \quad 1\le i\le p,$
- $b_{ij} = \dfrac{1}{30} - \dfrac{1}{30j+i}, \quad 1\le i<j\le p,$
- $b_{ij} = \dfrac{1}{30} - \dfrac{1}{30(i-j+1)+i}, \quad 1\le j<i\le p,$
- $c_{ii} = \dfrac{1}{10(p+i+1)}, \quad 1\le i\le q,$
- $c_{ij} = \dfrac{1}{30} - \dfrac{1}{30(p+j)+p+i}, \quad 1\le i<j\le q,$
- $c_{ij} = \dfrac{1}{30} - \dfrac{1}{30(i-j+1)+p+i}, \quad 1\le j<i\le q,$
- $l_{ij} = \dfrac{1}{30(p+i-j+1)+p+i} - \dfrac{1}{30}, \quad 1\le i\le q, \quad 1\le j\le p,$
- $u_{ij} = \dfrac{1}{30(p+j)+i} - \dfrac{1}{30}, \quad 1\le i\le p, \quad 1\le j\le q.$

For Example 3.1, the numerical results are presented in Tables 1–6. The symbols ‡ and § on these tables correspond to the values of $\rho(\mathcal{L}^{*(i)}_{\tau,\omega})$ and $\rho(\widetilde{\mathcal{L}}^{(i)}_{\tau,\omega})$, where $\mathcal{L}^{*(i)}_{\tau,\omega}$ and $\widetilde{\mathcal{L}}^{(i)}_{\tau,\omega}$ are defined in (1.23) and (1.29), respectively.

From Tables 1–6, we can see that

- $\rho(\overline{\mathcal{L}}^{(i)}_{\tau,\omega}) < \rho(\mathcal{L}_{\tau,\omega})$ for $i=1,2,3$ when $\rho(\mathcal{L}_{\tau,\omega})=0.1877<1$;
- $\rho(\overline{\mathcal{L}}^{(i)}_{\tau,\omega}) > \rho(\mathcal{L}_{\tau,\omega})$ for $i=1,2,3$ when $\rho(\mathcal{L}_{\tau,\omega})=1.2679>1$;

under the chosen parameters $\tau,\omega,\mu,\nu,\alpha,\beta(=1-\alpha),\gamma,\delta(=1-\gamma)$. These numerical results are in accordance with the theoretical results given in Theorems 2.1, 2.2 and 2.3.

In addition, from Table 2, it can be seen that

- when $\alpha=0$, $\gamma=0$, $\rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega})=0.1690$, which is the minimal value in Table 2;
- when $\alpha=1$, $\gamma=1$, $\rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega})=0.1752$, which is the maximal value in Table 2;
- when $\alpha=1$ and $\gamma=0$, the preconditioner (2.1) reduces to (1.18) with

$$\rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega}) = \rho(\mathcal{L}^{*(2)}_{\tau,\omega}) = 0.1737;$$

- when $\alpha = 0$ and $\gamma = 1$, the preconditioner (2.1) reduces to (1.24) with

$$\rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega}) = \rho(\widetilde{\mathcal{L}}^{(2)}_{\tau,\omega}) = 0.1710.$$

And from Table 5, it can be seen that

- when $\alpha = 0$, $\gamma = 0$, $\rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega}) = 1.2754$, which is the maximal value in Table 5;
- when $\alpha = 1$, $\gamma = 1$, $\rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega}) = 1.2716$, which is the minimal value in Table 5;
- when $\alpha = 1$ and $\gamma = 0$, the preconditioner (2.1) reduces to (1.18) with

$$\rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega}) = \rho(\mathcal{L}^{*(2)}_{\tau,\omega}) = 1.2746;$$

- when $\alpha = 0$ and $\gamma = 1$, the preconditioners (2.1) reduces to (1.24) with

$$\rho(\overline{\mathcal{L}}^{(2)}_{\tau,\omega}) = \rho(\widetilde{\mathcal{L}}^{(2)}_{\tau,\omega}) = 1.2724.$$

Table 1. Spectral radii of $\overline{\mathcal{L}}^{(1)}_{\tau,\omega}$ with $n = 10$, $p = 5$, $\tau = 0.99$, $\omega = 0.99$, $\mu = \nu = 3$ ($\rho(\mathcal{L}_{\tau,\omega}) = 0.1877$)

γ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\alpha = 0.0$	0.1719	0.1731	0.1744	0.1757	0.1770	0.1784	0.1799	0.1814	0.1830	0.1847	**0.1864**§
$\alpha = 0.1$	0.1701	0.1713	0.1726	0.1739	0.1753	0.1768	0.1783	0.1798	0.1815	0.1831	0.1849
$\alpha = 0.2$	0.1683	0.1695	0.1708	0.1722	0.1736	0.1751	0.1767	0.1783	0.1800	0.1817	0.1835
$\alpha = 0.3$	0.1665	0.1678	0.1691	0.1705	0.1720	0.1735	0.1751	0.1768	0.1785	0.1803	0.1821
$\alpha = 0.4$	0.1647	0.1661	0.1675	0.1689	0.1704	0.1720	0.1736	0.1753	0.1771	0.1789	0.1808
$\alpha = 0.5$	0.1630	0.1644	0.1658	0.1673	0.1689	0.1705	0.1722	0.1739	0.1757	0.1775	0.1795
$\alpha = 0.6$	0.1614	0.1628	0.1642	0.1658	0.1674	0.1690	0.1707	0.1725	0.1743	0.1762	0.1782
$\alpha = 0.7$	0.1597	0.1612	0.1627	0.1643	0.1659	0.1676	0.1693	0.1712	0.1730	0.1750	0.1770
$\alpha = 0.8$	0.1582	0.1597	0.1612	0.1628	0.1645	0.1662	0.1680	0.1698	0.1718	0.1737	0.1758
$\alpha = 0.9$	0.1566	0.1582	0.1597	0.1614	0.1631	0.1649	0.1667	0.1686	0.1705	0.1726	0.1746
$\alpha = 1.0$	**0.1551**‡	0.1567	0.1583	0.1600	0.1617	0.1635	0.1654	0.1674	0.1693	0.1714	0.1735

Table 2. Spectral radii of $\overline{\mathcal{L}}^{(2)}_{\tau,\omega}$ with $n = 10$, $p = 5$, $\tau = 0.99$, $\omega = 0.99$ ($\rho(\mathcal{L}_{\tau,\omega}) = 0.1877$)

γ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\alpha = 0.0$	**0.1690**	0.1691	0.1693	0.1695	0.1697	0.1699	0.1701	0.1703	0.1705	0.1707	0.1710§
$\alpha = 0.1$	0.1695	0.1697	0.1698	0.1700	0.1702	0.1704	0.1706	0.1708	0.1710	0.1712	0.1714
$\alpha = 0.2$	0.1700	0.1702	0.1703	0.1705	0.1707	0.1709	0.1711	0.1713	0.1715	0.1717	0.1719
$\alpha = 0.3$	0.1705	0.1707	0.1708	0.1710	0.1712	0.1714	0.1715	0.1717	0.1719	0.1721	0.1723
$\alpha = 0.4$	0.1710	0.1711	0.1713	0.1715	0.1716	0.1718	0.1720	0.1722	0.1724	0.1726	0.1728
$\alpha = 0.5$	0.1715	0.1716	0.1718	0.1719	0.1721	0.1723	0.1725	0.1726	0.1728	0.1730	0.1732
$\alpha = 0.6$	0.1719	0.1721	0.1722	0.1724	0.1726	0.1727	0.1729	0.1731	0.1732	0.1734	0.1736
$\alpha = 0.7$	0.1724	0.1725	0.1727	0.1728	0.1730	0.1732	0.1733	0.1735	0.1737	0.1739	0.1740
$\alpha = 0.8$	0.1728	0.1730	0.1731	0.1733	0.1734	0.1736	0.1737	0.1739	0.1741	0.1743	0.1744
$\alpha = 0.9$	0.1733	0.1734	0.1735	0.1737	0.1739	0.1740	0.1742	0.1743	0.1745	0.1747	0.1748
$\alpha = 1.0$	0.1737‡	0.1738	0.1740	0.1741	0.1743	0.1744	0.1746	0.1747	0.1749	0.1751	**0.1752**

4. Conclusions

In this paper, we propose a new type of preconditioners for solving the generalized least squares problems and study the convergence rates of the new preconditioned GAOR methods.

When the GAOR method (1.4) is convergent, the new preconditioned GAOR methods (2.3) have the better convergence rates than the GAOR method (1.4)

Table 3. Spectral radii of $\overline{\mathcal{L}}^{(3)}_{\tau,\omega}$ with $n = 10$, $p = 5$, $\tau = 0.99$, $\omega = 0.99$ ($\rho(\mathcal{L}_{\tau,\omega}) = 0.1877$)

γ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\alpha = 0.0$	0.1704	0.1709	0.1715	0.1722	0.1728	0.1735	0.1741	0.1748	0.1756	0.1763	**0.1770**§
$\alpha = 0.1$	0.1702	0.1708	0.1714	0.1720	0.1727	0.1733	0.1740	0.1747	0.1754	0.1761	0.1769
$\alpha = 0.2$	0.1700	0.1706	0.1712	0.1719	0.1725	0.1732	0.1738	0.1745	0.1753	0.1760	0.1768
$\alpha = 0.3$	0.1698	0.1704	0.1710	0.1717	0.1723	0.1730	0.1737	0.1744	0.1751	0.1759	0.1766
$\alpha = 0.4$	0.1696	0.1702	0.1709	0.1715	0.1722	0.1728	0.1735	0.1742	0.1749	0.1757	0.1765
$\alpha = 0.5$	0.1695	0.1701	0.1707	0.1713	0.1720	0.1726	0.1733	0.1740	0.1748	0.1755	0.1763
$\alpha = 0.6$	0.1692	0.1698	0.1705	0.1711	0.1718	0.1725	0.1732	0.1739	0.1746	0.1754	0.1761
$\alpha = 0.7$	0.1690	0.1696	0.1703	0.1709	0.1716	0.1723	0.1730	0.1737	0.1744	0.1752	0.1760
$\alpha = 0.8$	0.1688	0.1694	0.1700	0.1707	0.1714	0.1721	0.1728	0.1735	0.1742	0.1750	0.1758
$\alpha = 0.9$	0.1686	0.1692	0.1698	0.1705	0.1712	0.1718	0.1726	0.1733	0.1740	0.1748	0.1756
$\alpha = 1.0$	**0.1683**‡	0.1689	0.1696	0.1702	0.1709	0.1716	0.1723	0.1731	0.1738	0.1746	0.1754

Table 4. Spectral radii of $\overline{\mathcal{L}}^{(1)}_{\tau,\omega}$ with $n = 40$, $p = 10$, $\tau = 0.99$, $\omega = 0.99$, $\mu = \nu = 1$ ($\rho(\mathcal{L}_{\tau,\omega}) = 1.2679$)

γ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\alpha = 0.0$	1.2810	1.2798	1.2785	1.2772	1.2759	1.2747	1.2734	1.2721	1.2708	1.2696	**1.2683**§
$\alpha = 0.1$	1.2814	1.2801	1.2789	1.2776	1.2763	1.2750	1.2737	1.2725	1.2712	1.2699	1.2686
$\alpha = 0.2$	1.2818	1.2805	1.2792	1.2779	1.2767	1.2754	1.2741	1.2728	1.2715	1.2702	1.2690
$\alpha = 0.3$	1.2821	1.2809	1.2796	1.2783	1.2770	1.2757	1.2744	1.2732	1.2719	1.2706	1.2693
$\alpha = 0.4$	1.2825	1.2812	1.2799	1.2786	1.2774	1.2761	1.2748	1.2735	1.2722	1.2709	1.2696
$\alpha = 0.5$	1.2829	1.2816	1.2803	1.2790	1.2777	1.2764	1.2751	1.2738	1.2725	1.2712	1.2700
$\alpha = 0.6$	1.2832	1.2819	1.2806	1.2793	1.2781	1.2768	1.2755	1.2742	1.2729	1.2716	1.2703
$\alpha = 0.7$	1.2836	1.2823	1.2810	1.2797	1.2784	1.2771	1.2758	1.2745	1.2732	1.2719	1.2706
$\alpha = 0.8$	1.2839	1.2826	1.2813	1.2800	1.2787	1.2774	1.2761	1.2748	1.2735	1.2722	1.2709
$\alpha = 0.9$	1.2843	1.2830	1.2817	1.2804	1.2791	1.2778	1.2765	1.2752	1.2739	1.2726	1.2713
$\alpha = 1.0$	**1.2846**‡	1.2833	1.2820	1.2807	1.2794	1.2781	1.2768	1.2755	1.2742	1.2729	1.2716

Table 5. Spectral radii of $\overline{\mathcal{L}}^{(2)}_{\tau,\omega}$ with $n = 40$, $p = 10$, $\tau = 0.99$, $\omega = 0.99$ ($\rho(\mathcal{L}_{\tau,\omega}) = 1.2679$)

γ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\alpha = 0.0$	**1.2754**	1.2751	1.2748	1.2745	1.2742	1.2739	1.2736	1.2733	1.2730	1.2727	1.2724§
$\alpha = 0.1$	1.2753	1.2750	1.2748	1.2745	1.2742	1.2738	1.2735	1.2732	1.2729	1.2726	1.2723
$\alpha = 0.2$	1.2753	1.2750	1.2747	1.2744	1.2741	1.2738	1.2735	1.2732	1.2729	1.2726	1.2723
$\alpha = 0.3$	1.2752	1.2749	1.2746	1.2743	1.2740	1.2737	1.2734	1.2731	1.2728	1.2725	1.2722
$\alpha = 0.4$	1.2751	1.2748	1.2745	1.2742	1.2739	1.2736	1.2733	1.2730	1.2727	1.2724	1.2721
$\alpha = 0.5$	1.2750	1.2747	1.2744	1.2741	1.2738	1.2735	1.2732	1.2729	1.2726	1.2723	1.2720
$\alpha = 0.6$	1.2750	1.2747	1.2744	1.2741	1.2738	1.2735	1.2732	1.2729	1.2726	1.2723	1.2720
$\alpha = 0.7$	1.2749	1.2746	1.2743	1.2740	1.2737	1.2734	1.2731	1.2728	1.2725	1.2722	1.2719
$\alpha = 0.8$	1.2748	1.2745	1.2742	1.2739	1.2736	1.2733	1.2730	1.2727	1.2724	1.2721	1.2718
$\alpha = 0.9$	1.2747	1.2744	1.2741	1.2738	1.2735	1.2732	1.2729	1.2726	1.2723	1.2720	1.2717
$\alpha = 1.0$	1.2746‡	1.2744	1.2741	1.2738	1.2735	1.2732	1.2729	1.2726	1.2723	1.2720	**1.2716**

Table 6. Spectral radii of $\overline{\mathcal{L}}^{(3)}_{\tau,\omega}$ with $n = 40$, $p = 10$, $\tau = 0.99$, $\omega = 0.99$ ($\rho(\mathcal{L}_{\tau,\omega}) = 1.2679$)

γ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\alpha = 0.0$	1.2771	1.2764	1.2757	1.2750	1.2743	1.2736	1.2729	1.2721	1.2714	1.2707	**1.2700**§
$\alpha = 0.1$	1.2772	1.2765	1.2758	1.2751	1.2744	1.2736	1.2729	1.2722	1.2715	1.2708	1.2701
$\alpha = 0.2$	1.2773	1.2766	1.2758	1.2751	1.2744	1.2737	1.2730	1.2723	1.2716	1.2708	1.2701
$\alpha = 0.3$	1.2773	1.2766	1.2759	1.2752	1.2745	1.2738	1.2731	1.2723	1.2716	1.2709	1.2702
$\alpha = 0.4$	1.2774	1.2767	1.2760	1.2753	1.2746	1.2738	1.2731	1.2724	1.2717	1.2710	1.2703
$\alpha = 0.5$	1.2775	1.2768	1.2761	1.2753	1.2746	1.2739	1.2732	1.2725	1.2718	1.2710	1.2703
$\alpha = 0.6$	1.2775	1.2768	1.2761	1.2754	1.2747	1.2740	1.2733	1.2725	1.2718	1.2711	1.2704
$\alpha = 0.7$	1.2776	1.2769	1.2762	1.2755	1.2748	1.2740	1.2733	1.2726	1.2719	1.2712	1.2705
$\alpha = 0.8$	1.2777	1.2770	1.2763	1.2755	1.2748	1.2741	1.2734	1.2727	1.2720	1.2712	1.2705
$\alpha = 0.9$	1.2778	1.2770	1.2763	1.2756	1.2749	1.2742	1.2735	1.2727	1.2720	1.2713	1.2706
$\alpha = 1.0$	**1.2778**‡	1.2771	1.2764	1.2757	1.2750	1.2742	1.2735	1.2728	1.2721	1.2714	1.2707

under some conditions. The numerical results given in Section 3 are consistent with the theoretical results, that are Theorems 2.1, 2.2 and 2.3, which are exhibited in Section 2.

Moreover, we can choose the appropriate parameters such that the preconditioners $\overline{\mathcal{P}}_i(i=1,2,3)$ (2.1) have the better numerical performance than the preconditioners $\mathcal{P}_i^*(i=1,2,3)$ (1.18) and $\widetilde{\mathcal{P}}_i(i=1,2,3)$ (1.24) proposed in [4], while the corresponding generalized preconditioned GAOR methods are convergent.

In fact, the convergence region of the preconditioned GAOR method depends on the involved relaxed parameters. For the new proposed preconditioners $\overline{\mathcal{P}}_i(i=1,2,3)$ (2.1), we just consider six parameters τ, ω, μ, ν, α, γ with $\beta = 1-\alpha$ and $\delta = 1-\gamma$. However, for the preconditioners (1.30) and (1.33) proposed in [3], the convergence region of the preconditioned GAOR methods rely on the relaxed parameters τ, ω, $\mu_i(1 \le i \le q)$, $\nu_i(1 \le i \le q)$, $\gamma_i(2 \le i \le p)$, $\delta_i(2 \le i \le p)$ and $\alpha_i(2 \le i \le p)$, $\beta_i(2 \le i \le p)$, $\tau_i(2 \le i \le q)$, $\sigma_i(2 \le i \le q)$. Admittedly, as the numbers of the parameters increase, it becomes more and more difficult to choose the optimal parameters.

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References

[1] A. Berman and R. J. Plemmoms, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, PA, 1994.

[2] A. Hadjidimos, *Accelerated overrelaxation method*, Math. Comput., 1978, 32, 149–157.

[3] Z. Huang, L. Wang, Z. Xu and J. Cui, *Some new preconditioned generalized AOR methods for solving weighted linear least squares problems*, Computat. Appl. Math., 2018, 37(1), 415–438.

[4] Z. Huang, Z. Xu, Q. Lu and J. Cui, *Some new preconditioned generalized AOR methods for generalized least-squares problems*, Appl. Math. Comput., 2015, 269, 87–104.

[5] W. Karush, Minima of Functions of Several Variables with Inequalities as Side Constraints. M.Sc. Dissertation. Dept. of Mathematics, Univ. of Chicago, Chicago, Illinois, 1939.

[6] H. W. Kuhn and A. W. Tucker, "Nonlinear programming". Proceedings of 2nd Berkeley Symposium. Berkeley: University of California Press, 1951, 481-492.

[7] S. Miao, Y. Luo and G. Wang, *Two new preconditioned GAOR methods for weighted linear least squares problems*, Appl. Math. Comput., 2018, 324, 93–104.

[8] H. Shen, X. Shao, L. Wang and T. Zhang, *Preconditioned iterative methods for solving weghted linear least squares problems*, Appl. Math. Mech. Engl. Ed., 2012, 33(3), 375–384.

[9] R. S. Varga, *Matrix Iterative Analysis*, Springer, Berlin, 2000.

[10] G. Wang, Y. Du and F. Tan, *Comparison results on preconditioned GAOR methods for weghted linear least squares problems*, J. Appl. Math., Volume 2012, Article ID 563586, 9 pages.

[11] G. Wang, T. Wang and F. Tan, *Some results on preconditioned GAOR methods*, Appl. Math. Comput., 2013, 219, 5811–5816.

[12] G. Wang, H. Wen, L. Li and X. Li, *Convergence of GAOR method for doubly diagonally dominantmatrices*, Appl. Math. Comput., 2011, 217, 7509–7514.

[13] J. Yuan, *Numerical methods for generalized least squares problems*, J. Comput. Appl. Math., 1996, 66, 571–584.

[14] J. Yuan and X. Jin, *Convergence of the generalized AOR method*, Appl. Math. Comput., 1999, 99, 35–46.

[15] J. Zhao, C. Li, F. Wang and Y. Li, *Some new preconditioned generalized AOR methods for generalized least-squares problems*, Int. J. Comput. Math., 2014, 91, 1370–1383.

[16] X. Zhou, Y. Song, L. Wang and Q. Liu, *Preconditioned GAOR methods for solving weighted linear least squares problems*, J. Comput. Appl. Math., 2009, 224, 242–249.