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NUMERICAL APPROXIMATION OF THE ELLIPTIC EIGENVALUE PROBLEM BY STABILIZED NONCONFORMING FINITE ELEMENT METHOD*

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Abstract In this paper, a stabilized nonconforming mixed finite element method is used to solve the elliptic eigenvalue problem. Firstly, the lower-equal order element is used to discretize the space combined with the stabilization term based on the velocity projection method, and the error analysis is given. Moreover, the upper and lower bounds of eigenvalues are obtained. Finally, numerical experiments are carried out to verify the effectiveness of the proposed method.

Keywords Elliptic eigenvalue problem, stabilized nonconforming method, error estimate, lower and upper bounds.

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1. Introduction

The eigenvalue problems of partial differential equations play an increasingly important role in many scientific fields, such as quantum mechanics, structural mechanics and fluid mechanics. Meanwhile, the numerical solution of eigenvalue problems has attracted more and more attention in recent decades. For example, the error analysis of finite element method (FEM) is given in [2, 3], and the upper and lower bounds of eigenvalues are given in [8, 9, 17]. The literature [15] proposed multi-level correction method for eigenvalue problem. The convergence analysis of mixed FEM for eigenvalue problems is presented in [19]. The literature [7] proposed the accelerated two-grid stabilization FEM to solve Stokes eigenvalue problem. The eigenvalue

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problem is discretized by weak Galerkin method in [27, 28].

Compared with the FEM, the mixed element method can approximate both the original function and the corresponding derivatives function and reduce the smoothness of the problem. The LBB condition plays an important role in the mixed elements methdos. It can guarantee the stability of numerical schemes. However, the LBB condition does not allow lower-equal order element interpolation. The lower-equal order elements are relatively simple and unified data structure. Recently, low-equal order elements combined with pressure projection stabilization terms [4, 12] have been widely used in computational fluid dynamics. The stabilization term does not require stabilization parameters and does not need to calculate any higher derivatives or boundary information.

Recently, a new mixed element scheme [23] based on low regularity of flux function is proposed to solve elliptic problems. Its characteristic is that the flux function space is square integrable, not classical divergence space. This variational formula makes it easy to select two finite element spatial functions and automatically meets the LBB condition. Subsequently, this method was further applied to different equations [13, 20, 21, 26]. Especially, the nonconforming mixed element based on the velocity projection stabilization term are used to solve the second-order elliptic problem in [10], and the corresponding error analysis is given. Compared with the conforming finite element, the non-conforming element has simple selection and compact support of the basis function. Moreover, the non-conforming element is easier to satisfy the discrete LBB condition, and it can relax the high-order continuity requirement. In the present study, the method is extended to solve elliptic eigenvalue problems.

One possible way for finding lower bounds of operators' eigenvalues is nonconforming finite element. Many literatures have shown that nonconforming elements can produce lower bounds from numerical point of view, such as the Wilson element [26], the nonconforming rotated Q_1 element [21], the Crouzeix-Raviart (CR) nonconforming linear element [6], and the enriched nonconforming rotated Q_1 element [13] for second order elliptic problem, the Adini element [11] and the Morley element [20] for fourth order elliptic problem, and the enriched Crouzeix-Raviart elements [8, 14, 16]. Especially, the literature [16] gives the lower-bound analysis of the Stokes eigenvalue problem by four kinds of nonconforming mixed FEM in detail. Compared with enriched CR elements, it can be found that the original CR elements can only produce lower bounds of the eigenvalues in singular cases. In the case of smoothness, original CR elements cannot produce lower bounds of the eigenvalues. But our approach has the advantages of the mixed finite element.

In this paper, nonconforming mixed FEM combined with the stablized term based on the velocity projection is studied. The paper is organized as follows. In Section 2, we will discuss the problem with some basic statements. Stabilized nonconforming mixed finite element scheme for elliptic eigenvalue problems is given in Section 3. In Section 4, we establish the error estimates of the nonconforming mixed scheme for eigenvalue problems. In Section 5, the lower bounds of the eigenvalues are derived. Section 6 gives some numerical results in agreement with our theoretical analysis, and the last section gives some concluding remarks.

2. Preliminaries

In this paper, we consider the following elliptic eigenvalue problems

$$\begin{aligned} -\Delta p &= \lambda p, \quad \text{in} \quad \Omega, \\ p &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\Omega \subset R^2$ is a bounded convex polygonal domain with a Lipschitz-continuous boundary $\partial\Omega$, $p(\mathbf{x})$ represents the eigenfunction and $\lambda \in R$ is the associated eigenvalue.

By the flux $\mathbf{u} = \nabla p$, the corresponding mixed variational formulas are given as follows: find $((\mathbf{u}, p), \lambda) \in (\mathbf{V} \times W) \times R$ and $\|p\|_0 = 1$ such that

$$(\mathbf{u}, \nabla q)_{L^2} = \lambda(p, q)_{L^2}, \qquad \forall q \in W, \tag{2.1}$$

$$(\mathbf{u}, \mathbf{v})_{L^2} - (\nabla p, \mathbf{v})_{L^2} = 0, \quad \forall \mathbf{v} \in \mathbf{V}. \tag{2.2}$$

Here $\mathbf{V} = [L^2(\Omega)]^2$ and $W = H_0^1(\Omega)$.

The spaces $[L^2(\Omega)]^m$ $(m = 1, 2)$ are equipped with the L^2-scalar product $(\cdot, \cdot)_{L^2}$ and L^2-norm $\|\cdot\|_0$. The corresponding norm and seminorm in $[H^k(\Omega)]^d$ are denoted by $\|\cdot\|_k$ and $|\cdot|_k$, respectively. The space W is equipped with the norm $\|\nabla \cdot\|_0$. Note that this norm is equivalent to norm $\|\cdot\|_1$ due to Poincaré inequality. Spaces consisting of vector-valued functions are denoted in boldface. For convenience, bilinear forms $a(\cdot, \cdot)$ and $d(\cdot, \cdot)$ on $\mathbf{V} \times \mathbf{V}$ and $\mathbf{V} \times W$ is defined as follows respectively,

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= (\mathbf{u}, \mathbf{v})_{L^2}, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, \\ d(\mathbf{v}, q) &= (\mathbf{v}, \nabla q)_{L^2}, \qquad \forall \mathbf{v} \in \mathbf{V}, \forall q \in W, \end{aligned}$$

and a generalized bilinear form $B((\cdot, \cdot), (\cdot, \cdot))$ on $(\mathbf{V} \times W) \times (\mathbf{V} \times W)$

$$B((\mathbf{u}, p), (\mathbf{v}, q)) = a(\mathbf{u}, \mathbf{v}) - d(\mathbf{v}, p) + d(\mathbf{u}, q), \quad \forall (\mathbf{u}, p), (\mathbf{v}, q) \in \mathbf{V} \times W.$$

Using the above bilinear form, the equivalent variational formulation of problem (2.1)–(2.2) reads as follows: find $(\mathbf{u}, p; \lambda) \in (\mathbf{V} \times W) \times R$ with $\|p\|_0 = 1$ such that

$$B((\mathbf{u}, p), (\mathbf{v}, q)) = \lambda(p, q)_{L^2}, \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times W. \tag{2.3}$$

Under the assumptions we have made, (2.3) has a countable sequence of real eigenvalues [2]

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

and the corresponding eigenfunctions

$$(\mathbf{u}_1, p_1), \ (\mathbf{u}_2, p_2), \ (\mathbf{u}_3, p_3), \ \cdots$$

with the property $(p_i, p_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker symbol.

Moreover, the bilinear form $d(\cdot, \cdot)$ satisfies the inf-sup condition, i.e. there exists a constant $\beta > 0$ independent of mesh size, such that (see e.g. [23] for its proof)

$$\inf_{w \in W} \sup_{\mathbf{v} \in \mathbf{V}} \frac{-(\mathbf{v}, \nabla w)_{L^2}}{\|\mathbf{v}\|_{\mathbf{V}} \|w\|_W} \ge \beta.$$

The positive constant c or C may change from place to place in this article, but it has nothing to do with the mesh parameter.

3. Stabilized nonconforming-mixed finite element method

Let T_h be a regular partition of Ω into triangles in the sense of Ciarlet [5]. Γ_h denote the set of all element sides in the mesh. We introduce the non-conforming Crouzeix-Raviart finite element space of piecewise linears for the velocity and the conforming finite element space of piecewise linear for pressure as follows:

$$\begin{aligned}
\mathbf{V}_h &= \{\mathbf{v} : \mathbf{v}_j = \mathbf{v}|T \in [P_1(T)]^2 : \int_e [\mathbf{v}] ds = 0, \forall e \in \Gamma_h\},\\
W_h &= \{w \in C^0(\bar{\Omega}) \cap W : w \in P_1(T),\ \forall T \in T_h\},
\end{aligned}$$

where $P_1(T)$ represents the space of linear functions on T and $[\mathbf{v}]$ denotes the jump across the edge for internal edges and $[\mathbf{v}] = \mathbf{v}$ for $e \cap \partial\Omega \neq \emptyset$. Based on the idea of [4, 10, 12], the velocity projection stabilization term is given as follows.

Let $\Pi : \mathbf{V} \to \mathbf{R}_0$ be the L^2- projection as follows:

$$\begin{aligned}
&(\mathbf{u}, \mathbf{v})_{L^2} = (\Pi\mathbf{u}, \mathbf{v})_{L^2}, && \forall \mathbf{u} \in \mathbf{V}, \mathbf{v} \in \mathbf{R}_0,\\
&\|\Pi\mathbf{u}\|_0 \le \|\mathbf{u}\|_0, && \forall \mathbf{u} \in \mathbf{V},\\
&\|\mathbf{u} - \Pi\mathbf{u}\|_0 \le ch\|\mathbf{u}\|_1, && \forall \mathbf{u} \in \mathbf{H}^1(\Omega),
\end{aligned} \tag{3.1}$$

where $\mathbf{R}_0 = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_T \in \mathbf{P}_0(T), \forall T \in T_h\}$. We introduce the velocity projection stabilization term

$$Q(\mathbf{u}, \mathbf{v}) = (\mathbf{u} - \Pi\mathbf{u}, \mathbf{v} - \Pi\mathbf{v})_{L^2}, \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}_h. \tag{3.2}$$

Obviously, the bilinear form $Q(\mathbf{u}, \mathbf{v})$ in (3.2) is a symmetric and semi-definite matrix generated on local set T.

The NCP_1-P_1 finite element pair defined by the spaces $\mathbf{V}_h \times W_h$ does not satisfy the discrete LBB condition in [10]. Thus,

$$\inf_{w_h \in W_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{-(\mathbf{v}_h, \nabla w_h)_{L^2}}{\|\mathbf{v}_h\|\mathbf{v}\|w_h\|_W} = 0.$$

For the stability of numerical schemes, we employ the local stabilized form based on the local polynomial velocity projection. The stabilized scheme is as follows: find $((\mathbf{u}_h, p_h), \lambda) \in (\mathbf{V}_h \times W_h) \times R$ and $\|p_h\|_0 = 1$ such that

$$(\mathbf{u}_h, \nabla q_h)_{L^2} = \lambda(p_h, q_h)_{L^2}, \qquad \forall q_h \in W_h, \tag{3.3}$$

$$(\mathbf{u}_h, \mathbf{v}_h)_{L^2} - (\nabla p_h, \mathbf{v}_h)_{L^2} + Q(\mathbf{u}_h, \mathbf{v}_h) = 0, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.4}$$

The bilinear form with the stabilized term are given as follows: $\forall (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times W_h$

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = a(\mathbf{u}_h, \mathbf{v}_h) - d(\mathbf{v}_h, p_h) + d(\mathbf{u}_h, q_h) + Q(\mathbf{u}_h, \mathbf{v}_h).$$

Next, we will give the continuity property and the weak coercivity property of the bilinear form $B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))$ for the stabilized nonconforming-mixed the above pair $\mathbf{V}_h \times W_h$ in [10].

Theorem 3.1. *For all* $(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times W_h$*, there exist positive constants* C *and* β_3 *independent of* h*, such that*

$$|B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))| \le C(\|\mathbf{u}_h\|_0 + \|p_h\|_1)(\|\mathbf{v}_h\|_0 + \|q_h\|_1) \tag{3.5}$$

and

$$\sup_{(\mathbf{v}_h, q_h)\in(\mathbf{V}_h, W_h)} \frac{|B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))|}{\|\mathbf{v}_h\|_0 + \|q_h\|_1} \ge \beta_3(\|\mathbf{u}_h\|_0 + \|p_h\|_1). \tag{3.6}$$

The discrete stabilized scheme reads as follows: find $(\mathbf{u}_h, p_h; \lambda_h) \in (\mathbf{V}_h \times W_h \backslash\{0\}) \times R$ with $\|p_h\|_0 = 1$ such that (3.3)-(3.4) is equivalent to

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \lambda_h (p_h, q_h)_{L^2}, \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times W_h, \tag{3.7}$$

where

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = B((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) + Q(\mathbf{u}_h, \mathbf{v}_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times W_h.$$

The continuity property (3.5) and the weak coercivity property (3.6) can guarante the well-posedness of the discrete weak form of (3.7).

Let $Y = L^2(\Omega)$. For all $f \in Y$, find $(\mathbf{u}_f, p_f) \in \mathbf{V} \times W$ such that

$$(\mathbf{u}_f, \nabla q)_{L^2} = (f, q)_{L^2}, \qquad \forall q \in W, \tag{3.8}$$

$$(\mathbf{u}_f, \mathbf{v})_{L^2} - (\nabla p_f, \mathbf{v})_{L^2} = 0, \quad \forall \mathbf{v} \in \mathbf{V}. \tag{3.9}$$

The corresponding discrete scheme is: find $(\mathbf{u}_{fh}, p_{fh}) \in \mathbf{V}_h \times W_h$ such that

$$(\mathbf{u}_{fh}, \nabla q_h)_{L^2} = (f, q_h)_{L^2}, \qquad \forall q_h \in W_h, \tag{3.10}$$

$$(\mathbf{u}_{fh}, \mathbf{v}_h)_{L^2} - (\nabla p_{fh}, \mathbf{v}_h)_{L^2} + Q(\mathbf{u}_h, \mathbf{v}_h) = 0, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.11}$$

Similarly to [19], the bounded linear operators are defined as $G : Y \to \mathbf{V},\ S : Y \to W$ such that the pair $Gf = \mathbf{u}_f$ and $Sf = p_f$ are respectively the solution to the elliptic equation (3.8)-(3.9) and $G_h : Y \to \mathbf{V}_h,\ S_h : Y \to W_h$ such that the pair $G_h f = \mathbf{u}_{fh}$ and $S_h f = p_{fh}$ are the discrete elliptic equation (3.10)-(3.11) with stabilized NCP_1-P_1 finite element scheme.

Lemma 3.1. *S and S_h are two selfadjoint operators.*

Proof. For $f \in L^2(\Omega)$, Eqs. (3.8)–(3.9) can be rewriten as follows:

$$(Gf, \nabla q)_{L^2} = (f, q)_{L^2}, \qquad \forall q \in W, \tag{3.12}$$

$$(Gf, \mathbf{v})_{L^2} - (\nabla Sf, \mathbf{v})_{L^2} = 0, \qquad \forall \mathbf{v} \in \mathbf{V}. \tag{3.13}$$

For $g \in L^2(\Omega)$, Eqs. (3.8)–(3.9) can also be rewriten as follows:

$$(Gg, \nabla q)_{L^2} = (g, q)_{L^2}, \qquad \forall q \in W, \tag{3.14}$$

$$(Gg, \mathbf{v})_{L^2} - (\nabla Sg, \mathbf{v})_{L^2} = 0, \qquad \forall \mathbf{v} \in \mathbf{V}. \tag{3.15}$$

In Eqs. (3.12)–(3.13), we take $q = Sg$, $\mathbf{v} = Gg$ and obtain

$$(Gf, \nabla Sg)_{L^2} = (f, Sg)_{L^2}, \tag{3.16}$$

$$(Gf, Gg)_{L^2} - (\nabla Sf, Gg)_{L^2} = 0. \tag{3.17}$$

Let $q = Sf$, $\mathbf{v} = Gf$, we have

$$(Gg, \nabla Sf)_{L^2} = (g, Sf)_{L^2}, \tag{3.18}$$

$$(Gg, Gf)_{L^2} - (\nabla Sg, Gf)_{L^2} = 0, v \in \mathbf{V}. \tag{3.19}$$

By Eqs. (3.16)–(3.19), we can obtain

$$(f, Sg) = (Gf, \nabla Sg) = (Gg, Gf) = (\nabla Sf, Gg) = (g, Sf). \tag{3.20}$$

In addition, the proof of S_h can be proven in a similar way and is omitted. □

Due to the selfadjoint operator S_h and the weak coercivity property (3.6), Eq. (3.7) has a finite sequence of real eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \lambda_{3,h} \leq \cdots \lambda_{N_h,h}$$

and the corresponding discrete eigenvectors

$$(\mathbf{u}_{1,h}, p_{1,h}),\ (\mathbf{u}_{2,h}, p_{2,h}),\ (\mathbf{u}_{3,h}, p_{3,h}),\ \cdots, (\mathbf{u}_{N_h,h}, p_{N_h,h})$$

with the property $(p_{i,h}, p_{j,h}) = \delta_{ij}, 1 \leq i, j \leq N_h$, where N_h is the dimension of W_h.

4. Error estimates for the eigenvalue problem

Since the convergence of the finite element approximation to the eigenvalue problem depends on the regularity of the original eigenvalue problem, here and hereafter, we assume the regularity of the eigenfunction $(p, \mathbf{u}) \in H^2(\Omega) \times (H^1(\Omega))^2$.

First, the error analysis of the discrete scheme is given. For all $f \in Y$, Eqs. (3.8)–(3.9) can be rewrite as follows:

$$B((\mathbf{u}_f, p_f), (\mathbf{v}, q)) = (f, q)_{L^2}, \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times W. \tag{4.1}$$

The regularity result in the convex domain from [23] shows that

$$\|\mathbf{u}_f\|_1 + \|p_f\|_2 \leq c\|f\|_0. \tag{4.2}$$

We can rewite discrete scheme (3.10)–(3.11) as follows:

$$B_h((\mathbf{u}_{fh}, p_{fh}), (\mathbf{v}_h, q_h)) = (f, q_h)_{L^2}, \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times W_h. \tag{4.3}$$

In order to obtain the error estimates, the approximation properties are defined as follows in [5]: for any $(\mathbf{v}, q) \in [H^1(\Omega)]^2 \times H^2(\Omega)$,

$$\|q - I_h q\|_0 + h(\|q - I_h q\|_1 + \|\mathbf{v} - J_h \mathbf{v}\|_0) \leq ch^2(\|q\|_2 + \|\mathbf{v}\|_1), \tag{4.4}$$

where the interpolation operator $I_h : H^2(\Omega) \cap W \to W_h$ satisfies

$$(q - I_h q, q_1)_{L^2} = 0, \quad q \in W, \quad q_1 \in W_h, \tag{4.5}$$

and the interpolation operator $J_h : H^1(\Omega)^2 \cap \mathbf{V} \to \mathbf{V}_h$ satisfies

$$\int_e (\mathbf{v} - J_h \mathbf{v}) ds = 0, \quad \forall e \in \Gamma_h.$$

Lemma 4.1. *For error estimate of the bounded linear operators, we have*

$$\|Sf - S_h f\|_0 + h(\|Sf - S_h f\|_1 + \|Gf - G_h f\|_0) \le ch^2 \|f\|_0.$$

Proof. By [22, Theorem 3], a priori error estimate for the elliptic equation problem based on the stabilized nonconforming mixed FEM is given as follows. For any $f \in L^2(\Omega)$, we have

$$\|Gf - G_h f\|_0 + \|Sf - S_h f\|_1 \le ch\|f\|_0. \tag{4.6}$$

To obtain the L^2-estimation for p, we use a standard duality argument. Define the dual problem: find $(\psi_1, \phi_1) \in (\mathbf{V}, W)$ such that

$$B((\psi_1, \phi_1); e, \eta) = (\eta, Sf - S_h f)_{L^2}, \quad \forall (e, \eta) \in (\mathbf{V}, W). \tag{4.7}$$

Then, from a priori estimate (4.2), we can get

$$\|\psi_1\|_1 + \|\phi_1\|_2 \le \|Sf - S_h f\|_0. \tag{4.8}$$

Since (Gf, Sf) and $(G_h f, S_h f)$ satisfy (4.1) and (4.3), subtracting (4.1) from (4.3), we can see that

$$B_h((Gf - G_h f, Sf - S_h f); (\psi_h, \phi_h)) = Q(Gf, \psi_h), \quad \forall (\psi_h, \phi_h) \in (\mathbf{V}_h, W_h). \tag{4.9}$$

Setting $e = Gf - G_h f$, $\eta = Sf - S_h f$ in (4.7) and taking $(\psi_h, \phi_h) = (J_h \psi_1, I_h \phi_1) \in (\mathbf{V}_h, W_h)$ in (4.9), then by use of the interpolation theory (4.4) and applying (3.6), (3.1), (4.6) and (4.8), we can obtain

$$\begin{aligned}
\|Sf - S_h f\|_0^2 &= B_h((\psi_1 - J_h\psi_1, \phi_1 - I_h\phi_1); (e, \eta)) + Q(Gf, J_h\psi_1) - Q(e, \psi_1) \\
&\le C(\|\psi_1 - J_h\psi_1\|_0 + \|\phi_1 - I_h\phi_1\|_1)(\|e\|_0 + \|\eta\|_1) + Ch^2\|Gf\|_1\|\psi_1\|_1 \\
&\le Ch(\|e\|_0 + \|\eta\|_1)(\|\psi_1\|_1 + \|\phi_1\|_2) + Ch^2(\|\psi_1\|_1 + \|\phi_1\|_2) \\
&\le Ch^2\|Sf - S_h f\|_0 \\
&\le Ch^4\|f\|_0,
\end{aligned}$$

which completes the proof. □

Next, we present the following error estimates of the eigenvalues and eigenfunctions for the eigenvalue problems.

Theorem 4.1. *Let $(\mathbf{u}_h, p_h; \lambda_h)$ be the i-th discrete solution of (3.7). Then there exists an i-th solution $(\mathbf{u}, p; \lambda)$ of (2.3) which satisfies the following error estimation:*

$$\begin{aligned}
&\|p - p_h\|_0 + h(\|\mathbf{u} - \mathbf{u}_h\|_0 + \|p - p_h\|_1) \le ch^2, \\
&|\lambda - \lambda_h| \le ch^2.
\end{aligned} \tag{4.10}$$

Proof. By Lemma 4.1, errors for the operator norm

$$\|G - G_h\|_0 \triangleq \sup_{f \in Y} \frac{\|Gf - G_h f\|_0}{\|f\|_0}, \quad \|S - S_h\|_1 \triangleq \sup_{f \in Y} \frac{\|Sf - S_h f\|_1}{\|f\|_0}$$

can be estimated by:

$$\|G - G_h\|_0 + \|S - S_h\|_1 \le ch.$$

Based on the abstract theory of [2] and [19], for the i-th discrete eigenpair $(\mathbf{u}_h, p_h; \lambda_h)$, there exists an i-th eigenpair $(\mathbf{u}, p; \lambda)$ such that

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_h\|_0+\|p-p_h\|_1 \le c(\|G-G_h\|_0+\|S-S_h\|_1) \le ch,\\ &\|\lambda-\lambda_h| \le c\|S-S_h\|_0 \le ch^2.\end{aligned}$$

Moreover, by $\|Sf-S_hf\|_0 \le ch^2\|f\|_0$ in Lemma 4.1, the standard argument (cf. page 448 in [19]) leads to

$$\|p-p_h\|_0 \le ch^2,$$

which completes the proof. □

Remark 4.1. Consider the following finite element spaces

$$\mathbf{K}_h=\{\mathbf{v}_h=(v_1,v_2)\in\mathbf{V}: v_i|_T\in P_1(T)\oplus\mathbf{span}\{\lambda^1\lambda^2\lambda^3\},\ \forall T\in T_h, i=1,2\},$$

where λ^i $(i=1,2,3)$ are the barycentric coordinates on T and the $P_1\oplus\mathbf{span}\{\lambda^1\lambda^2\lambda^3\}$ represents a space of linear functions enriched by a cubic bubble functions. The finite element space $\mathbf{K}_h\times W_h$ in [1] satisfies the LBB condition, and the corresponding error results are the same as follows:

$$\begin{aligned}&\|p-p_h\|_0+h(\|\mathbf{u}-\mathbf{u}_h\|_0+\|p-p_h\|_1)\le ch^2,\\ &|\lambda-\lambda_h|\le ch^2.\end{aligned}$$

The proof is similar to that of Theorem 4.1, and will be omitted here for the sake of brevity.

Remark 4.2. For the regularity of the eigenfunction $(p,\mathbf{u})\in H^{r+1}(\Omega)\times(H^r(\Omega))^2$ $(0<r\le 1)$, we can obtain the similar result. Numerical example in Section 6 shows the second order convergence in the L-shape domain.

5. Eigenvalue approximations from below

The lower bound of the eigenvalue for stabilized nonconforming scheme is given in this section. The basic expansion form of the eigenvalues is given as follows:

Lemma 5.1. *Suppose* $(\mathbf{u}, p; \lambda)$ *is the solution of the original problem (2.3),* $(\mathbf{u}_h, p_h; \lambda_h)\in(\mathbf{V}_h\times W_h)\times R$ *is the solution of the discrete problem (3.7), we have the following expansion*

$$\begin{aligned}\lambda-\lambda_h=&\|\mathbf{u}-\mathbf{u}_h\|_0^2+Q(\mathbf{u}_h,\mathbf{u}_h)+2a(\mathbf{u},\mathbf{u}_h)-2d(\mathbf{u}_h,q_h)\\ &-\lambda_h\|q_h-p_h\|_0^2+\lambda_h(\|q_h\|_0^2-\|p_h\|_0^2).\end{aligned}\tag{5.1}$$

Proof. In view of (2.3) and (3.7), we see that

$$\|\mathbf{u}\|_0^2=\lambda,\quad \|\mathbf{u}_h\|_0^2+Q(\mathbf{u}_h,\mathbf{u}_h)=\lambda_h.$$

Therefore, we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_0^2&=\|\mathbf{u}\|_0^2+\|\mathbf{u}_h\|_0^2-2a(\mathbf{u},\mathbf{u}_h)\\ &=\lambda+\lambda_h-Q(\mathbf{u}_h,\mathbf{u}_h)-2a(\mathbf{u},\mathbf{u}_h).\end{aligned}\tag{5.2}$$

Adding $-2d(\mathbf{u}_h, q_h)$ to both sides of (5.2) and using (3.3)

$$\begin{aligned}-2d(\mathbf{u}_h, q_h) + \|\mathbf{u} - \mathbf{u}_h\|_0^2 &= \lambda + \lambda_h - Q(\mathbf{u}_h, \mathbf{u}_h) - 2a(\mathbf{u}, \mathbf{u}_h) - 2\lambda_h(p_h, q_h)\\ &= \lambda + \lambda_h - Q(\mathbf{u}_h, \mathbf{u}_h) - 2a(\mathbf{u}, \mathbf{u}_h)\\ &\quad + \lambda_h\|q_h - p_h\|_0^2 - \lambda_h(\|q_h\|_0^2 - \|p_h\|_0^2) - 2\lambda_h.\end{aligned}$$

Thus

$$\begin{aligned}\lambda - \lambda_h = \|\mathbf{u} - \mathbf{u}_h\|_0^2 + Q(\mathbf{u}_h, \mathbf{u}_h) + 2a(\mathbf{u}, \mathbf{u}_h) - 2d(\mathbf{u}_h, q_h)\\ - \lambda_h\|q_h - p_h\|_0^2 + \lambda_h(\|q_h\|_0^2 - \|p_h\|_0^2).\end{aligned}$$

The proof is completed. □

Theorem 5.1. *Let an i-th* $(\mathbf{u}_i, p_i; \lambda_i) \in (H^1(\Omega) \times H^2(\Omega)) \times R$ *be i-th solution of (2.3). Assume that* $(\mathbf{u}_{i,h}, p_{i,h}; \lambda_{i,h}) \in (\mathbf{V}_h \times W_h) \times R$ *is the i-th numerical solution of scheme (3.7) and* $\|\mathbf{u}_i - \mathbf{u}_{i,h}\|_0^2 \geq ch^{2-2\varepsilon}$, *where* $\varepsilon > 0$ *can be made arbitrarily small. Then*

$$\lambda_{i,h} \leq \lambda_i$$

holds provided h is sufficiently small.

Proof. We choose $q_{i,h} = I_h p_i$ with (4.5) in (5.1). For the third and fourth term in (5.1), applying $\mathbf{u}_i = \nabla p_i$ and Green's formula and interpolation definition (4.5), it is easy to show that

$$2a(\mathbf{u}_i, \mathbf{u}_{i,h}) - 2d(\mathbf{u}_{i,h}, I_h p_i) = 2(\nabla p_i - \nabla I_h p_i, \mathbf{u}_{i,h})_{L^2} = 0.$$

For the fifth term in (5.1), using (4.4) and (4.10), we arrive at

$$\lambda_{i,h}\|I_h p_i - p_{i,h}\|_0^2 \leq \lambda_{i,h}(\|I_h p_i - p_i\|_0 + \|p_i - p_{i,h}\|_0)^2 \leq ch^4. \tag{5.3}$$

We define the piecewise constant projection operator with scalar function

$$P_0 w|T = \frac{1}{|T|}\int_T w dx dy, \quad \forall T \in T_h, \tag{5.4}$$

which leads to $\|w - P_0 w\| \leq ch|w|_1, \quad \forall w \in H^1(\Omega)$.

For the sixth term in (5.1), by (4.5), (5.3) and (5.4), it leads directly to

$$\begin{aligned}\lambda_{i,h}(\|I_h p_i\|_0^2 - \|p_{i,h}\|_0^2) &= \lambda_{i,h}(I_h p_i - p_{i,h}, I_h p_i + p_{i,h})_{L^2}\\ &= \lambda_{i,h}(I_h p_i - p_{i,h}, I_h p_i + p_{i,h} - P_0(I_h p_i + p_{i,h}))_{L^2}\\ &\leq ch\|I_h p_i - p_{i,h}\|_0 \leq ch^3.\end{aligned} \tag{5.5}$$

Moreover, utilizing (3.2), we have

$$Q(\mathbf{u}_{i,h}, \mathbf{u}_{i,h}) \leq \|\mathbf{u}_{i,h} - \Pi\mathbf{u}_{i,h}\|_0^2 \leq ch^2.$$

From (5.3) and (5.5) plus the saturation condition $\|\mathbf{u}_i - \mathbf{u}_{i,h}\|_0^2 \geq ch^{2-2\varepsilon}$, we can find the second term, the fifth term and the sixth term on the right-hand side of (5.1) is of higher order than the first term, namely

$$\lambda_i - \lambda_{i,h} = \|\mathbf{u}_i - \mathbf{u}_{i,h}\|_0^2 + O(h^2) \geq ch^{2-2\varepsilon} + O(h^2). \tag{5.6}$$

From (5.6), if h is small enough, we obtain

$$\lambda_{i,h} \leq \lambda_i.$$

□

Remark 5.1. The upper bound of the eigenvalues obtained by conforming element scheme (P_{1b}-P_1 pair and P_0-P_1 pair) is due to the minimum-maximum principle [18] for $\mathbf{K}_h \subset \mathbf{V}$ as follows

$$\lambda_i \leq \lambda_{i,h}.$$

6. Numerical results

In this section, we use the P_0-P_1 pair, P_{1b}-P_1 pair and NCP_1-P_1 pair methods to verify the numerical stability and accuracy and compare these three methods. For the stabilized nonconforming mixed scheme, the stabilization term [10] is expressed by the difference between two local Gauss integrals, which is rewritten as follows

$$Q(\mathbf{u}, \mathbf{v}) = \sum_{K \in K_h} (\int_{K,2} \mathbf{u} \cdot \mathbf{v} dxdy - \int_{K,1} \mathbf{u} \cdot \mathbf{v} dxdy), \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}_h,$$

where $\int_{K,i} g(x,y)dxdy$ represents the Gauss integral over the region over K, which is accurate for polynomials of degree $i, i = 1, 2$. For more information, please refer to the literature [12, 29].

To solve the eigenvalue problem, we denote by $\mathbf{U}$ the vector of the velocity and by P the vector of the pressure. It is easy to see that (3.7) can be written in matrix form

$$\begin{bmatrix} A+Q & B \\ -B^T & O \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ P \end{bmatrix} = \lambda_h \begin{bmatrix} O & O \\ O & E \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ P \end{bmatrix},$$

where the matrices A, B, Q and E are deduced in the usual manner, using the basis functions of $\mathbf{V}_h$ and W_h, from the bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$, $Q(\cdot,\cdot)$ and $(\cdot,\cdot)_{L^2}$, respectively, and B^T is the transpose of matrix B. The left coefficient matrix is solved by LU decomposition method with a fixed tolerance as 10^{-6}. The right coefficient matrix is solved by conjugate gradient method with a fixed tolerance as 10^{-6}. The inverse power method is used for solving generalized eigenvalue problem.

First, the computational region $\Omega = [0,1] \times [0,1]$ in R^2 are consider. We just consider the first eigenvalue of the elliptic eigenvalue problem, that is, the firsr eigenvalue $\lambda = 2\pi^2$. From Tables 1–3, it can be found that the convergence rates of the three methods are consistent with the theoretical analysis. The lower bounds of eigenvalues are obtained by nonconforming mixed finite element scheme. The error of the nonconforming element is smaller than that of the conforming element conforming P_0-P_1 version, because the degree of freedom of the nonconforming element is more than that of the P_0-P_1 version. Non-conforming element has the same accuracy as P_{1b}-P_1 method, but it takes less CPU-time because the degree of freedom of NCP_1-P_1 method is less than that of P_{1b}-P_1 version.

Secondly, we consider the first four eigenvalues in L-type calculation region $[0,1] \times [0,1/2] \bigcup [0,1/2] \times [1/2,1]$ in R^2. Because the exact solution is unknown, the exact solution is obtained by by the standard Galerkin method (P_2 element) computed on a very fine mesh (35198 triangle elements). Here, we take $\lambda_1 = 9.64094, \lambda_2 = 15.1973, \lambda_3 = 19.7392$ and $\lambda_4 = 29.5215$, as the exact eigenvalues. The notation N_{el} represents number of elements for triangulation.

Table 1. Results get from NCP_1-P_1 element methods on the unit square

$1/h$	λ_h	$\frac{\lvert\lambda-\lambda_h\rvert}{\lvert\lambda\rvert}$	Rate	CPU time
16	19.6640	3.812E-3		0.156
24	19.7043	1.767E-3	1.895	0.421
32	19.7192	1.014E-3	1.932	0.843
40	19.7262	6.562E-4	1.950	1.453
48	19.7301	4.590E-4	1.961	2.422
56	19.7325	3.389E-4	1.968	3.797
64	19.7341	2.604E-4	1.974	5.453

Table 2. Results get from the P_{1b}-P_1 element methods on the unit square

$1/h$	λ_h	$\frac{\lvert\lambda-\lambda_h\rvert}{\lvert\lambda\rvert}$	Rate	CPU time
16	19.8168	3.929E-3		0.203
24	19.7729	1.709E-3	2.053	0.469
32	19.758	9.513E-4	2.036	0.984
40	19.7511	6.052E-4	2.027	1.734
48	19.7475	4.187E-4	2.021	2.781
56	19.7453	3.068E-4	2.016	4.172
64	19.7438	2.345E-4	2.013	5.875

Table 3. Results get from the P_0-P_1 element methods on the unit square

$1/h$	λ_h	$\frac{\lvert\lambda-\lambda_h\rvert}{\lvert\lambda\rvert}$	Rate	CPU time
16	19.9298	9.655E-3		0.078
24	19.8238	4.287E-3	2.002	0.203
32	19.7868	2.411E-3	2.001	0.375
40	19.7697	1.543E-3	2.000	0.625
48	19.7604	1.072E-3	2.000	0.969
56	19.7547	7.874E-4	1.999	1.406
64	19.7511	6.029E-4	2.000	2.016

The convergence rates of the first eigenvalue in the L-shape domain are reported in Fig. 1. From Tables 4–6, the lower bounds of the exact solution is obtained by stabilized nonconforming element scheme, and the upper bounds of the eigenvalue is obtained by the conforming element scheme, which verifies the previous theoretical results. Moreover, the velocity streamlines and pressure level lines of numerical solutions of three schemes are presented in Figs. 2 and 3 by the conforming elements and stabilized nonconforming element with 11,104 triangle elements for the detail. On the same grids, though the stability of three schemes is obtained from both Figs 2 and 3, the conforming element (P_0-P_1) shows more oscillatory than the stabilized nonconforming element on the velocity streamlines.

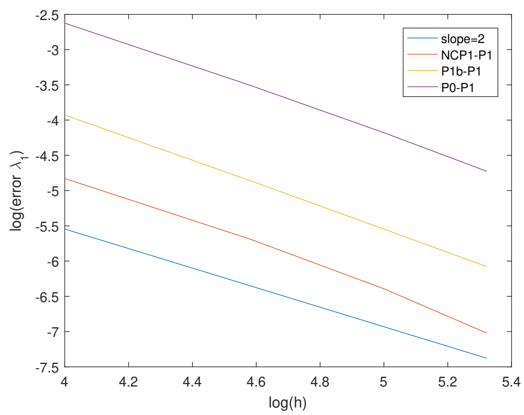


Figure 1. The convergence rate analysis of the stabilized mixed methods for the first eigenvalue in the L-shape domain

Table 4. Results get from the NCP_1-P_1 element methods on the L-shape domain

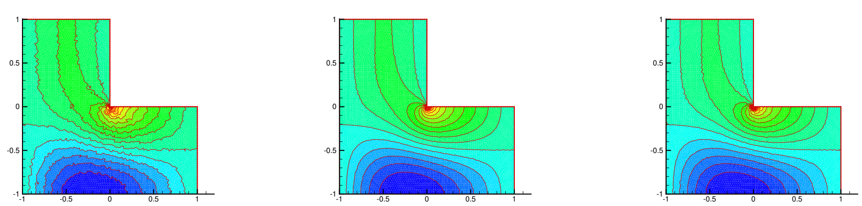
N_{el}	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
684	9.60571	15.1117	19.6086	29.2229
1548	9.62162	15.1608	19.6803	29.39
2682	9.62903	15.1746	19.7048	29.4451
4354	9.63325	15.1832	19.7168	29.4714
11104	9.63703	15.1916	19.7301	29.5018
16840	9.6382	15.1935	19.7335	29.5089
"Exact" solution trend	9.64094	15.1973	19.7392	29.5215
	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 5. Results get from the P_{1b}-P_1 element methods on the L-shape domain

N_{el}	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
684	9.70669	15.2545	19.8607	29.7774
1548	9.6752	15.2223	19.788	29.6272
2682	9.66236	15.212	19.7674	29.5838
4354	9.65572	15.2065	19.7568	29.5586
11104	9.64771	15.2009	19.7459	29.5363
16840	9.64569	15.1996	19.7437	29.5315
"Exact" solution trend	9.64094	15.1973	19.7392	29.5215
	↘	↘	↘	↘

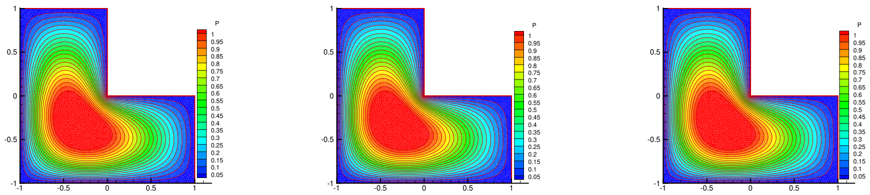
Table 6. Results get from the P_0-P_1 element methods on the L-shape domain

N_{el}	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
684	9.80312	15.3812	20.0791	30.2782
1548	9.72857	15.2786	19.8835	29.8411
2682	9.69618	15.2455	19.8228	29.7086
4354	9.67871	15.227	19.791	29.6342
11104	9.65888	15.209	19.7595	29.5665
16840	9.65366	15.2049	19.7526	29.5516
"Exact" solution trend	9.64094	15.1973	19.7392	29.5215
	↘	↘	↘	↘



(a) (b) (c)

Figure 2. Plot of the velocity streamlines at $Dof = 11104$: numerical solution of P_0-P_1 element (a), P_{1b}-P_1 element (b) and stabilized nonconforming element(c)



(a) (b) (c)

Figure 3. Plot of the pressure level lines at $Dof = 11104$: numerical solution of P_0-P_1 element (a), P_{1b}-P_1 element (b) and stabilized nonconforming element(c)

7. Conclusions

In this work, we used the lower-equal order nonconforming mixed FEM combined with the velocity projection stabilization term for the elliptic eigenvalue problem. Moreover, the error analysis of the mixed FEM scheme and the lower bounds of the eigenvalue are obtained. Numerical experiments show the effectiveness of our scheme. Obviously, this method can be extended to the case of three dimensions in the future.

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