Journal of Applied Analysis and Computation
Volume 11, Number 3, June 2021, 1177–1190
Website:http://www.jaac-online.com
DOI:10.11948/20200030

GLOBAL WELL-POSEDNESS OF THE GENERALIZED ROTATING MAGNETOHYDRODYNAMICS EQUATIONS IN VARIABLE EXPONENT FOURIER-BESOV SPACES*

Muhammad Zainul Abidin[1,†] and Jiecheng Chen[1]

Abstract In this paper we study the three dimensional incompressible generalized rotating magnetohydrodynamics equations. By using littlewood-Paley decomposition, we obtain the global well-posedness result for small initial data belong to critical variable exponent Fourier-Besov spaces $\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}$. This paper extends some recent work about generalized Navier-Stokes equations.

Keywords Generalized rotating magnetohydrodynamics equations, global well-posedness, variable exponent Fourier-Besov spaces.

MSC(2010) 35Q30, 35Q86, 42B37.

1. Introduction

In this paper we consider the 3D incompressible generalized rotating magnetohydrodynamics (grMHD) equations:

$$\begin{cases} u_t + (u\cdot\nabla)u + \mu(-\Delta)^\alpha u - (B\cdot\nabla)B + \Omega e_3 \times u + \nabla P = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\ B_t + (u\cdot\nabla)B + \gamma(-\Delta)^\alpha B - (B\cdot\nabla)u = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\ divu = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^3\times\mathbb{R}_+, \end{cases} \tag{grMHD}$$

where $u=(u_1,u_2,u_3)$ is the velocity field of the fluid, $\frac{1}{2}<\alpha\le 1$, $\nabla=(\partial_{x_1},\partial_{x_2},\partial_{x_3})$, $divu=\partial_{x_1}u_1+\partial_{x_2}u_2+\partial_{x_3}u_3$, the operator $(-\Delta)^\alpha$ is the Fourier multiplier with symbol $|\xi|^{2\alpha}$, $P=p+\frac{1}{2}|B|^2$ in which p is the pressure and B is the magnetic field, μ is the viscosity coefficient, γ is the diffusion of magnetic field and $\Omega\in\mathbb{R}$ denotes twice the speed of rotation around the vertical unit vector $e_3=(0,0,1)$. For simplicity, we take $\mu=\gamma=1$ throughout the paper.

†The corresponding author.
Email: mzainulabidin@zjnu.edu.cn, zainbs359@gmail.com (M. Z. Abidin)
[1]College of Mathematics and Computer Science, Zhejiang Normal University, Jinhua 321004, China
*The authors were supported by Zhejiang Normal University Postdoctoral Research fund under grant No. ZC304020909 and NSF of China (No. 12071437, 10271437)

For $\alpha = 1$, mathematically, the grMHD equations explain why the earth has non-zero large scale magnetic field whose polarity turns out to invert over several hundred centuries. For more detailed explanation, one can refer to [7] and references therein.

When $\Omega = 0$, the grMHD equations reduce to the generalized MHD equations, which deals with magnetic properties of electrically conducting fluids. Duvaut and Lions [14] proved a global weak solution to MHD for initial data with finite energy. Since Baraka and Toumlilin [15] established the global well-posedness for generalized MHD equations with small initial data belonging to the critical Fourier-Besov-Morrey spaces, 3D MHD equation remains an outstanding mathematical problem whether there always exists a global smooth solution for smooth initial data. For more results in this direction, one can refer to [27, 37, 40] and references therein.

When $\alpha = 1$, $B = 0$ and $\Omega \neq 0$, Babin, Mahalov and Nicolaenko [3, 4] proved the global existence to the system grMHD with periodic initial velocity in the case $|\Omega|$ is enough large. Hieber and Shibata [19] proved the uniform global well-posedness for small initial data in $H^{\frac{1}{2}}(\mathbb{R}^3)^3$ where Chemin, Desjardins, Gallagher and Grenier [7] proved that there exist a unique solution in the case $|\Omega| > \Omega_0 > 0$ for some $\Omega_0 = \Omega_0(u_0)$, if u_0 is the given divergence free initial velocity belonging to $L^2(\mathbb{R}^2) + H^{\frac{1}{2}}(\mathbb{R}^3)$. Iwabuchi and Takada [21] proved the existence of global unique solutions to the Navier-Stokes equations with Coriolis force in Sobolev spaces $\dot{H}^s(\mathbb{R}^3)$ with $\frac{1}{2} < s < \frac{3}{4}$ if the speed of rotation Ω is sufficiently large. Moreover, Iwabuchi and Takada [22] also proved the global in time existence and the uniqueness of the mild solution for small initial data in Fourier-Besov spaces $\mathcal{F}\dot{\mathscr{B}}^{-1}_{1,2}(\mathbb{R}^3)$. Recently Wang and Wu [35] proved the global mild solution of the generalized Navier-Stokes equations with Coriolis force, if the initial data are in $\mathcal{X}^{1-2\alpha} := \{u \in \mathcal{D}'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{1-2\alpha}|\widehat{u}(\xi)|d\xi < +\infty\}$.

For $\Omega = 0$ and $B = 0$, the grMHD equations reduce to the fractional Navier-Stokes equations (FNS). Lions [30] proved the global existence of classical in 3D when $\alpha \geq \frac{5}{4}$(see also Wu [38] in n dimensions). Wu [39] studied the well-posedness for the important case $\alpha < \frac{5}{4}$ in $\mathscr{B}^{1-2\alpha+\frac{3}{p}}_{p,q}$. Dong and Li [13] established the optimal local smoothing estimates of solutions to (FNS) in Lebesgue spaces. Inspired by Xiao [41] in the classical case $\alpha = 1$, Li and Zhai [29] studied FNS in some critical Q-type sapces for $\alpha \in (\frac{1}{2}, 1)$ and Zhai [42] showed the well-posedness in $BMO^{1-2\alpha}$ for $\alpha \in (\frac{1}{2}, 1)$. Deng and Yao [11] studied FNS in Triebel-Lizorkin spaces $\dot{F}^{-\alpha}_{\beta,r}$ and obtained the well-posedness in $\dot{F}^{-\alpha}_{3/(\alpha-1),2}$ and ill-posedness in $\dot{F}^{-\alpha}_{3/(\alpha-1),r}(r > 2)$ in the case for $\alpha \in (1, \frac{5}{4})$. Recently Baraka and Toumlilin [16] studied FNS in the critical case for $\alpha > \frac{1}{2}$ when a small u_0 belongs to Fourier-Besov-Morrey space $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}(\mathbb{R}^3)$.

When $\alpha = 1$, $\Omega = 0$ and $B = 0$, the grMHD equations reduce to the classical Navier Stokes equations, which have been intensively studied. In this case, Kato and Fujita [18, 24] transformed the classical incompressible Navier–Stokes equations into an integral equation and proved its local existence in some Lebesgue and Sobolev spaces. Kato [23] proved that Navier Stokes equations are locally well-posed in $L^3(R^3)$ and globally well-posed if the initial data are small in the Lebesgue space $L^3(R^3)$. Koch and Tataru [25] studied the well-posedness for the Navier stokes equations in BMO^{-1}. However, the ill-posedness of Navier stokes equations in the largest critical space $\dot{B}^{-1}_{\infty,\infty}$ was proved by Bourgain and Pavlović [6]. Recently

Navier Stokes equations are studied in Fourier Besov spaces $\mathcal{F}\dot{\mathscr{B}}^{s}_{p,q}$ by many authors, such as [5, 26, 28].

In this paper we study the global well-posedness for grMHD equations with small data in the variable exponent Fourier-Besov spaces $\mathcal{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q}$. Spaces of variable integrability, also known as Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, have been widely used in harmonic analysis, see [9, 10]. Apart from theoretical consideration, the variable exponent function spaces have interesting applications in fluid dynamics [1, 33], image processing [8] and partial differential equations [17]. However, due to the special structure of the space with variable, one cannot apply it to the global well-posedness of Navier-Stokes equations. In this paper, by combining the proof of propositions of frequency space and the definition of variable exponent Fourier-Besov spaces, we show the global well-posedness of (grMHD) equations in variable exponent frequency spaces. We avoid the discussion of the value of Ω because our work space is in the frequency space rather than the physical space. In fact, the value of Ω cannot be large in some physical models.

2. Preliminaries

Let $\mathcal{P}_0$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ such that

$$p_- = essinf_{x\in\mathbb{R}^n} p(x) \ , \ p_+ = esssupp(x).$$

For $p \in \mathcal{P}_0(\mathbb{R}^n)$, let $L^p(\cdot)(\mathbb{R}^n)$ be the set of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\varrho_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} (\frac{|f(x)|}{\lambda})^{p(x)} dx.$$

The infimum of such λ is denoted by $\|f\|_{L^{p(\cdot)}}$.The set $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a quasi Banach function space when equipped with Luxemburge-Nakano norm $\|f\|_{L^{p(\cdot)}}$ ([31, 34]), where

$$\|f\|_{L^{p(\cdot)}} = inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1\}.$$

We postulate the following standard conditions to ensure that the Hardy-Maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$:
1) p is said to satisfy the Locally log-Hölder's continuous condition if there exists a positive constant $C_{log}(p)$ such that $|p(x) - p(y)| \leq \frac{C_{log}(p)}{log(e+|x-y|^{-1})}$, (for all $x, y \in \mathbb{R}^n, x \neq y$);
2) p is said to satisfy the Globally-log-Hölder's continuous condition if there exists a positive constant $C_{log}(p)$ and p_∞, such that $|p(x) - p_\infty| \leq \frac{C_{log}(p)}{log(e+|x|)}$, (for all $x \in \mathbb{R}^n$).
We use $C_{log}(\mathbb{R}^n)$ as the set of all real valued functions $p : \mathbb{R}^n \to \mathbb{R}$ satisfying (1) and (2).

Let us recall the littlewood-Paley (or dyadic) decomposition. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. Choose two non negative radial functions χ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\text{supp } \chi \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \frac{4}{3}\},$$

$$\text{supp } \varphi \subset \{\xi \in \mathbb{R}^n : \frac{4}{3} \le |\xi| \le \frac{8}{3}\},$$
$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \xi \in \mathbb{R}^n,$$
$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \xi \in \mathbb{R}^n/\{0\},$$

and denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and $h_j = \mathcal{F}^{-1}\varphi_j$, we define the frequency localization operator as follows

$$\Delta_j u := \mathcal{F}^{-1}\varphi_j \mathcal{F} u = \int_{\mathbb{R}^n} h_j(y)u(x-y)dy, \forall j \in \mathbb{Z},$$
$$S_j u = \sum_{k\le j-1} \Delta_k u.$$

Informally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$, while S_j is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. One can easily obtain that

$$\Delta_j\Delta_k \equiv 0 \text{ if } |j-k| \ge 2 \quad \text{and} \quad \Delta(S_{k-1}u\Delta_k u) \equiv 0 \text{ if } |j-k| \ge 5.$$

Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ we use $\ell^{q(\cdot)}(L^{p(\cdot)})$ to denote the space consisting of all sequences $\{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|\{g_j\}_{j\in\mathbb{Z}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = inf\{\mu > 0, \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{\frac{f_j}{\mu}\}_{j\in\mathbb{Z}}) \le 1\} \le \infty,$$

where

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}}) = \sum_{k\in\mathbb{Z}} inf\{\lambda > 0 : \int_{\mathbb{R}^n} (\frac{|f_j(x)|}{\lambda^{\frac{1}{q(x)}}})^{p(x)}dx \le 1\}.$$

Since we assume that $q_+ < \infty$, $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}}) = \sum_{j\in\mathbb{Z}} \||f_j|^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}$ holds.

Definition 2.1. let $p(\cdot), q(\cdot) \in C^{log}\mathbb{R}^n \cap \mathcal{P}_0(\mathbb{R}^n)$ and $s(\cdot) \in C^{log}(\mathbb{R}^n)$. The homogeneous Besov space with variable exponents $\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)} = \{f \in \mathcal{S}' : \|f\|_{\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} < \infty\},$$
$$\|f\|_{\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\Delta_j f\}_{j\in\mathbb{Z}}\|_{\ell^{q(\cdot)}L^{p(\cdot)}} < \infty,$$

where $\mathcal{S}'$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : (D^\alpha \widehat{f})(0) = 0, \forall\alpha\}$.

For $T > 0$ and $\rho \in [1,\infty]$, we denote by $L^\rho(0,T,\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})$ the set of all tempered distribution u satisfying

$$\|u\|_{L^\rho(0,T,\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})} := \|(\sum_{j=0}^{\infty} \|2^{js(\cdot)}\Delta_j u\|^r_{L^{p(\cdot)}})^{\frac{1}{r}}\|_{L^\rho_T} < \infty.$$

The mixed $\widetilde{L}^\rho(0,T,\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})$ is the set of all tempered distribution u satisfying

$$\|u\|_{\widetilde{L}^\rho(0,T,\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})} := (\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\Delta_j u\|^r_{L^\rho_T L^{p(\cdot)}})^{\frac{1}{r}} < \infty.$$

For simplicity, we denote

$$L^\rho_T\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r} := L^\rho(0,T,\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r}) \text{ and } \widetilde{L}^\rho\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r} := \widetilde{L}^\rho(0,T,\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r}).$$

By virtue of the Minkowski inequality, we have

$$\|u\|_{\widetilde{L}^\rho_T\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r}} \le \|u\|_{L^\rho_T\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r}} \text{ if } \rho \le r,$$
$$\|u\|_{L^\rho_T\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r}} \le \|u\|_{\widetilde{L}^\rho_T\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r}} \text{ if } r \le \rho.$$

To obtain the local well-posedness of grMHD equations in the space with variable, we need to introduce the following spaces.

Definition 2.2 (Homogeneous Fourier-Besov spaces with variable exponents)**.** let $p(\cdot), q(\cdot) \in C^{log}\mathbb{R}^n \cap \mathcal{P}_0(\mathbb{R}^n)$ and $s(\cdot) \in C^{log}(\mathbb{R}^n)$, the variable exponent Fourier-Besov space $\mathcal{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j\widehat{f}\}^\infty_{-\infty}\|_{\ell^{q(\cdot)}L^{p(\cdot)}} < \infty.$$

Similarly, we denote by $\widetilde{L}^\rho(0,T,\mathcal{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})$ the set of all tempered distribution u satisfying

$$\|u\|_{\widetilde{L}^\rho(0,T,\mathcal{F}\dot{\mathscr{B}}^{s(\cdot)}_{p(\cdot),r})} := (\sum_{k\in\mathbb{Z}} \|2^{js(\cdot)}\varphi_j\widehat{u}\|^r_{L^\rho_T L^{p(\cdot)}})^{\frac{1}{r}} < \infty.$$

Proposition 2.1. The following inclusions holds for the variable exponent function spaces.

(1) Hölder inequality [12]: $(\int_{\mathbb{R}^n} |f(x)g(x|^\eta dx)^{\frac{1}{\eta}} \le C\|f\|_{L^{\eta_1(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{\eta_1(\cdot)}(\mathbb{R}^n)}$, where $\frac{1}{\eta} = \frac{1}{\eta_1} + \frac{1}{\eta_2}$, $1 \le \eta \le \infty$, $f \in L^{\eta_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{\eta_1(\cdot)}(\mathbb{R}^n)$. Moreover, let $h(x) \in L^{p_1}$ and $\frac{1}{p(\cdot)} = \frac{1}{p_1} + \frac{1}{p_2(\cdot)}$, from the definition of $L^{p(\cdot)}$, we also have

$$\begin{aligned}\|fh\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= inf\{\lambda : \int |\frac{f(x)h(x)}{\lambda}|^{p(x)}dx < 1\}\\ &\le inf\{\lambda : \|f\|_{L^{p_2(\cdot)}}\|\frac{h}{\lambda}\|_{L^{p_1}}\}\\ &\le \|f\|_{L^{p_2(\cdot)}\mathbb{R}^n}\|h\|_{L^{p_1}\mathbb{R}^n}.\end{aligned}$$

(2) Sobolev inequality [2]: Let $p_0, p_1, q \in \mathcal{P}_0$ and $s_0, s_1 \in L^\infty \cap C^{log}(\mathbb{R}^n)$ with $s_0 > s_1$. If $\frac{1}{q}$ and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$$

are locally log- Hölder continuous, then

$$\dot{\mathscr{B}}^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

(3) [2] Let $p_0, p_1, q_0, q_1 \in \mathcal{P}_0$ and $s_0, s_1 \in L^\infty \cap C^{log}(\mathbb{R}^n)$ with $s_0 > s_1$. If $\frac{1}{q_0}$, $\frac{1}{q_1}$and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} + \varepsilon(x)$$

are locally log- Hölder continuous and $essinf_{x\in\mathbb{R}^n}\varepsilon(x) > 0$, then

$$\dot{\mathscr{B}}^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow \dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}.$$

(4) Molification inequality [12]: For $p(\cdot) \in C^{log}(\mathbb{R}^n)$ and $\psi \in L^1(\mathbb{R}^n)$, assume that $\Psi(x) = \sup_{y \notin B(0,|x|)} |\psi(y)|$ is integrable. Then

$$\|f * \psi_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\Psi\|_{L^1(\mathbb{R}^n)},$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, where $\psi_\varepsilon = \frac{1}{\varepsilon^n}\psi(\frac{1}{\varepsilon})$ and C depends only on n.

Proposition 2.2 ([32]). *Let $s > 0$, $1 \leq \eta, \lambda, q, p, r, \lambda_1, \lambda_2 \leq \infty$, and $\frac{1}{\eta} = \frac{1}{r} + \frac{1}{p}$, $\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$. Then we have*

$$\|uv\|_{\widetilde{L}^\lambda_t \dot{\mathscr{B}}^s_{\eta,q}} \lesssim \|u\|_{\widetilde{L}^{\lambda_1}_t \dot{\mathscr{B}}^s_{p,q}}\|v\|_{L^{\lambda_2}_t L^r} + \|v\|_{\widetilde{L}^{\lambda_1}_t \dot{\mathscr{B}}^s_{P,q}}\|u\|_{L^{\lambda_2}_t L^r}.$$

3. Well-posedness

In this section, firstly we need to introduce the generalized Stokes-Coriolis semi group S, which is closely related to grMHD equations. For $B = 0$, the grMHD equations deduce to the fractional Navier-Stokes equations with Coriolis force. In fact we have to consider the following linear generalized problem:

$$\begin{cases} u_t + (-\Delta)^\alpha u + \Omega e_3 \times u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ divu = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \end{cases} \tag{LNSC}$$

The solution of equation (LNSC) can be given by the generalized Stokes-Coriolis semi group $S_{\Omega,\alpha}$, which has the following explicit representation [20, 36]:

$$\begin{aligned} S_{\Omega,\alpha}(t)u &= \mathcal{F}^{-1}[\cos(\Omega\frac{\xi_3}{|\xi|}t)e^{-|\xi|^{2\alpha}t}I + \sin(\Omega\frac{\xi_3}{|\xi|}t)e^{-|\xi|^{2\alpha}t}R(\xi)] * u \\ &= \mathcal{F}^{-1}[\cos(\Omega\frac{\xi_3}{|\xi|}t)I + \sin(\Omega\frac{\xi_3}{|\xi|}t)R(\xi)] * (e^{-(-\Delta)^\alpha t}u), \end{aligned}$$

where divergence free vector field $u \in \mathcal{S}(\mathbb{R}^3)$, I is the unit matrix in $M_{3\times3}(\mathbb{R})$ and $R(\xi)$ is skew-symmetric matrix defined by

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}, \xi \in \mathbb{R}^3 \setminus \{0\}.$$

So, we can write a semigroup:

$$\mathcal{A}_{\Omega,\alpha}(t) = \begin{pmatrix} S_{\Omega,\alpha}(t) & 0 \\ 0 & H_\alpha(t) \end{pmatrix}$$

where $H_\alpha(t) := e^{-(-\Delta)^\alpha t} = \mathcal{F}^{-1}(e^{-|\xi|^{2\alpha}t})$.

Theorem 3.1. Let $p(\cdot) \in C^{log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $\frac{1}{2} < \alpha \leq 1$, $2 \leq p(\cdot) \leq \frac{6}{5-4\alpha}$, $1 \leq \gamma < \infty$, $1 \leq q < \frac{3}{2\alpha-1}$ and there exist a sufficiently small $\varepsilon > 0$, such that

$$\|u_0\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}} + \|B_0\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}} < \varepsilon.$$

Then the grMHD equations has a unique small global solution u in the class

$$u, B \in \widetilde{L}^{\infty}(\mathbb{R}_+, \mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}) \cap \widetilde{L}^{\gamma}(\mathbb{R}_+, \mathcal{F}\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q}) \cap \widetilde{L}^{\infty}(\mathbb{R}_+, \mathcal{F}\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q}).$$

Moreover, let $p_1(\cdot) \in C^{log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $s_1(\cdot) = \frac{2\alpha}{\gamma} - \frac{3}{p_1(\cdot)} + 4 - 2\alpha$ and $s_1(\cdot) \in C^{log}(\mathbb{R}^n)$, if there exists $c > 0$ such that $2 \leq p_1(\cdot) \leq c \leq p(\cdot)$, then in addition we obtain that $u, B \in \widetilde{L}^{\gamma}(\mathbb{R}_+, \mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q})$.

Remark 3.1. It is pointed out that the variable exponent Fourier-Besov space $\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}$ is important as it gives the scaling invariant function space. In fact, if $u(t,x)$ is the solution of grMHD equations, then

$$u_\lambda(t,x) = \lambda^{2\alpha-1} u(\lambda^{2\alpha} t, \lambda x)$$

is also a solution of the same equation and

$$\|u(0,x)\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}} \sim \|u_\lambda(0,x)\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}}.$$

Remark 3.2. Theroem 3.1 extends the result of [32].

Proof of Theorem 3.1. Let $\delta_0 > 0, \delta > 0$ will be choosen later, consider

$$\begin{aligned}\mathcal{D}=\{u, B : &\|u\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q})} + \|B\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q})} \leq \delta_0,\\ &\|u\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} + \|B\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \leq \delta\},\end{aligned}$$

which is equipped with the metric

$$\begin{aligned}d(\begin{pmatrix} u \\ B \end{pmatrix} - \begin{pmatrix} v \\ \beta \end{pmatrix}) =&\|u-v\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q})\cap\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\\ &+\|B-\beta\|_{\widetilde{L}^{\infty}(\mathbb{R}_+\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q})\cap\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}.\end{aligned}$$

It is easy to see that $(\mathcal{D}, d)$ is a complete metric space. Next we consider the following mapping

$$g : \begin{pmatrix} u \\ B \end{pmatrix} \to \begin{pmatrix} u_0 \\ B_0 \end{pmatrix} - \int_0^t \mathcal{A}_{\Omega,\alpha}(t-\tau)\mathbb{P} \begin{pmatrix} (u.\nabla)u - (B.\nabla)B \\ (u.\nabla)B - (B.\nabla)u \end{pmatrix} d\tau.$$

We shall prove there exist $\delta_0, \delta > 0$, such that $g : (\mathcal{D}, d) \to (\mathcal{D}, d)$ is a strict contraction mapping.

In order to solve grMHD equations, we need to consider the following integral equation.

$$\begin{pmatrix} u \\ B \end{pmatrix} = \mathcal{A}_{\Omega,\alpha}(t) \begin{pmatrix} u_0 \\ B_0 \end{pmatrix} - \int_0^t \mathcal{A}_{\Omega,\alpha}(t-\tau)\mathbb{P} \begin{pmatrix} (u.\nabla)u - (B.\nabla)B \\ (u.\nabla)B - (B.\nabla)u \end{pmatrix} d\tau$$

where $\mathbb{P} = I - \nabla(-\Delta)^{-1}div$ is the Leray-Hopf projection.

Step 1. Estimate of $\mathcal{A}_{\Omega,\alpha}(t)\begin{pmatrix} u_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} S_{\Omega,\alpha}(t)u_0 \\ H_\alpha(t)B_0 \end{pmatrix}$

Since $\mathbb{P}$ and $S_{\Omega,\alpha}$ are bounded Fourier multipliers, we simply estimate by an absolute constant. From proposition 2.1,

$$\begin{aligned}
&\|S_{\Omega,\alpha}(t)u_0\|_{\widetilde{L}^\gamma(\mathbb{R}_+,\dot{\mathscr{B}}_{2,q}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha})} \\
&\lesssim \|\|2^{j(\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha)}\varphi_j e^{-t|\cdot|^{2\alpha}}\widehat{u_0}\|_{L^\gamma(R_+,L^2)}\|_{\ell^q} \\
&\lesssim \|\textstyle\sum_{\ell=0,\pm1}\|2^{j(4-2\alpha-\frac{3}{p(\cdot)})}\varphi_j\widehat{u_0}\|_{L^{p(\cdot)}}\|2^{j(\frac{2\alpha}{\gamma}-\frac{3}{2}+\frac{3}{p(\cdot)})}\varphi_{j+\ell}e^{-t|\cdot|^{2\alpha}}\|_{L^\gamma(R_+,L^{\frac{2p(\cdot)}{p(\cdot)-2}})}\|_{\ell^q} \\
&\lesssim \|\|2^{j(4-2\alpha-\frac{3}{p(\cdot)})}\varphi_j\widehat{u_0}\|_{L^{p(\cdot)}}\|_{\ell^q} = \|u_0\|_{\mathcal{F}\dot{\mathscr{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}},
\end{aligned}$$

where we used the following estimates

$$\begin{aligned}
(1),\quad &\|2^{j(\frac{2\alpha}{\gamma}-\frac{3}{2}+\frac{3}{p(\cdot)})}\varphi_{j+\ell}e^{-t|\cdot|^{2\alpha}}\|_{L^\gamma(\mathbb{R}_+,L^{\frac{2p(\cdot)}{p(\cdot)-2}})} \\
&\lesssim \|2^{j\frac{2\alpha}{\gamma}}2^{j(-\frac{3}{2}+\frac{3}{p(\cdot)})}\varphi_{j+\ell}e^{-t2^{2\alpha(j+\ell)}}\|_{L^\gamma(\mathbb{R}_+,L^{\frac{2p(\cdot)}{p(\cdot)-2}})} \\
&\lesssim \|2^{j\frac{2\alpha}{\gamma}}e^{-t2^{2\alpha(j+\ell)}}\|_{L^\gamma(\mathbb{R}_+)}\|2^{j(-\frac{3}{2}+\frac{3}{p(\cdot)})}\varphi_{j+\ell}\|_{L^{\frac{2p(\cdot)}{p(\cdot)-2}}} \\
&\lesssim \|2^{j(-\frac{3}{2}+\frac{3}{p(\cdot)})}\varphi_{j+\ell}\|_{L^{\frac{2p(\cdot)}{p(\cdot)-2}}}, \\
(2),\quad &\|2^{j(-\frac{3}{2}+\frac{3}{p(\cdot)})}\varphi_{j+\ell}\|_{L^{\frac{2p(\cdot)}{p(\cdot)-2}}} \\
&= inf\{\lambda>0:\textstyle\int|\frac{2^{j(-\frac{3}{2}+\frac{3}{p(\cdot)})}\varphi_{j+\ell}}{\lambda}|^{\frac{2p(x)}{p(x)-2}}dx<1\} \\
&\lesssim inf\{\lambda>0:\textstyle\int|\frac{\varphi_{j+\ell}}{\lambda}|^{\frac{2p(x)}{p(x)-2}}2^{-3j}dx<1\} \\
&\lesssim inf\{\lambda>0:\textstyle\int|\frac{\varphi_\ell}{\lambda}|^{\frac{2p(2^jx)}{p(2^jx)-2}}dx<1\} \lesssim C.
\end{aligned}$$

Also, for $p_1(\cdot) \le c \le p(\cdot)$, we have

$$\begin{aligned}
&\|S_{\Omega,\alpha}(t)u_0\|_{\widetilde{L}^\gamma(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}_{p_1(\cdot),q}^{s_1(\cdot)})} \\
&\lesssim \|\|2^{js_1(\cdot)}\varphi_j e^{-t|\cdot|^{2\alpha}}\widehat{u_0}\|_{L^\gamma(\mathbb{R}_+,L^{p_1(\cdot)})}\|_{\ell^q} \\
&\lesssim \|\textstyle\sum_{\ell=0,\pm1}\|2^{j(4-2\alpha-\frac{3}{c})}\varphi_j\widehat{u_0}\|_{L^c}\|2^{j(\frac{2\alpha}{\gamma}-\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+\ell}e^{-t2^{2\alpha(j+l)}}\|_{L^\gamma(\mathbb{R}_+,L^{\frac{cp(\cdot)}{c-p_1(\cdot)}})}\|_{\ell^q} \\
&\lesssim \|\textstyle\sum_{\ell=0,\pm1}\|2^{j(4-2\alpha-\frac{3}{p(\cdot)})}\varphi_j\widehat{u_0}\|_{L^{p(\cdot)}}\|_{\ell^q} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathscr{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}},
\end{aligned}$$

where we used the following estimate

$$\begin{aligned}
&\|2^{j(\frac{2\alpha}{\gamma}-\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+\ell}e^{-t2^{2\alpha(j+l)}}\|_{L^\gamma(\mathbb{R}_+,L^{\frac{cp(\cdot)}{c-p_1(\cdot)}})} \\
&= \|2^{j\frac{2\alpha}{\gamma}}e^{-t2^{2\alpha(j+\ell)}}\|_{L^\gamma(\mathbb{R}_+)} inf\{\lambda>0:\int|\frac{2^{j(\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{k+\ell}}{\lambda}|^{\frac{cp_1(x)}{c-p_1(x)}}dx<1\} \\
&\lesssim inf\{\lambda>0:\int|\frac{\varphi_{j+\ell}}{\lambda}|^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}}2^{-3j}dx<1\} \lesssim C.
\end{aligned}$$

Similarly we can obtain

$$\|H_\alpha(t)B_0\|_{\widetilde{L}^\gamma(\mathbb{R}_+,\dot{\mathscr{B}}_{2,q}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha})} \lesssim \|B_0\|_{\mathcal{F}\dot{\mathscr{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}}$$

and

$$\|H_\alpha(t)B_0\|_{\widetilde{L}^\gamma(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}_{p_1(\cdot),q}^{s_1(\cdot)})} \lesssim \|B_0\|_{\mathcal{F}\dot{\mathscr{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}}.$$

It is easy to see that the estimates for $S_{\Omega,\alpha}(t)u_0$ also hold for $\gamma=\infty$ and $p_1(\cdot)=p(\cdot)$, i.e.,

$$\|S_{\Omega,\alpha}(t)u_0\|_{\widetilde{L}^\infty(R_+,\dot{\mathscr{B}}_{2,q}^{\frac{5}{2}-2\alpha})} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathscr{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}}$$

and

$$\|S_{\Omega,\alpha}(t)u_0\|_{\widetilde{L}^\infty(R_+,\dot{\mathscr{B}}_{2,q}^{4-2\alpha-\frac{3}{p(\cdot)}})} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathscr{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}}.$$

Also

$$\|H_\alpha(t)B_0\|_{\widetilde{L}^\infty(R_+,\dot{\mathscr{B}}_{2,q}^{\frac{5}{2}-2\alpha})} \lesssim \|B_0\|_{\mathcal{F}\dot{\mathscr{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}}$$

and

$$\|H_\alpha(t)B_0\|_{\widetilde{L}^\infty(R_+,\dot{\mathscr{B}}_{2,q}^{4-2\alpha-\frac{3}{p(\cdot)}})} \lesssim \|B_0\|_{F\dot{B}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}}.$$

Step 2. Estimate of $\int_0^t \mathcal{A}_{\Omega,\alpha}(t-\tau)\mathbb{P}\begin{pmatrix}(u.\nabla)u-(B.\nabla)B\\(u.\nabla)B-(B.\nabla)u\end{pmatrix}d\tau$

let $\bar{p}_\alpha(\cdot)=6-(5-4\alpha)p_1(\cdot)$, $\frac{1}{2}<\alpha\le 1$ and $1\le q<\frac{3}{2\alpha-1}$, from proposition 2.1 and 2.2, we have

$$\begin{aligned}
&\|\int_0^t S_{\Omega,\alpha}(t-\tau)\mathbb{P}[(u.\nabla)u]d\tau\|_{\widetilde{L}^\gamma(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}_{p_1(\cdot),q}^{s_1(\cdot)})}\\
\lesssim&\|\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|.|^{2\alpha}}[\widehat{(u\cdot\nabla)u}]d\tau\|_{L^\gamma(\mathbb{R}_+,L^{p_1(\cdot)})}\|_{\ell^q}\\
\lesssim&\|\|\int_0^t \|2^{j(s_1(\cdot)+1)}\varphi_j e^{-(t-\tau)|.|^{2\alpha}}\|_{L^{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}}\|\Delta_j(u\otimes u)\|_{M^{h(\cdot)}_{\frac{6}{4\alpha+1}}}d\tau\|_{L^\gamma(\mathbb{R}_+)}\|_{\ell^q}\\
\lesssim&\|\|\int_0^t 2^{j(\frac{2\alpha}{\gamma}+\frac{5}{2})}\|2^{-3j\frac{\bar{p}_\alpha}{6p_1(\cdot)}}\varphi_j e^{-(t-\tau)|.|^{2\alpha}}\|_{L^{\frac{6p_1(\cdot)}{\bar{p}_\alpha(\cdot)}}}\|\Delta_j(u\otimes u)\|_{M^{h(\cdot)}_{\frac{6}{4\alpha+1}}}d\tau\|_{L^\gamma(\mathbb{R}_+)}\|_{\ell^q}\\
\lesssim&\|\|\int_0^t 2^{j(\frac{2\alpha}{\gamma}+\frac{5}{2})}e^{-(t-\tau)2^{2\alpha j}}\|2^{-3j\frac{\bar{p}_\alpha}{6p_1(\cdot)}}\varphi_j\|_{L^{\frac{6p_1(\cdot)}{\bar{p}_\alpha(\cdot)}}}\|\Delta_j(u\otimes u)\|_{L^{\frac{6}{4\alpha+1}}}d\tau\|_{L^\gamma(\mathbb{R}_+)}\|_{\ell^q}\\
\lesssim&\|\|\int_0^t 2^{j(\frac{2\alpha}{\gamma}+\frac{5}{2})}e^{-(t-\tau)2^{2\alpha j}}\|\Delta_j(u\otimes u)\|_{L^{\frac{6}{4\alpha+1}}}d\tau\|_{L^\gamma(\mathbb{R}_+}\|_{\ell^q}\\
\lesssim&\|\|2^{j(\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha)}\|\Delta_j(u\otimes u)\|_{L^{\frac{6}{4\alpha+1}}}\|_{L^\gamma(\mathbb{R}_+)}\|2^{2\alpha j}e^{-t2^{2\alpha j}}\|_{L^1(\mathbb{R}_+)}\|_{\ell^q}\\
\lesssim&\|u\|_{\widetilde{L}^\gamma(\mathbb{R}_+,\dot{\mathscr{B}}_{2,q}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha})}\|u\|_{L^\infty(\mathbb{R}_+,L^{\frac{3}{2\alpha-1}})}\\
\lesssim&\|u\|_{\widetilde{L}^\gamma(\mathbb{R}_+,\dot{\mathscr{B}}_{2,q}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha})}\|u\|_{\widetilde{L}^\infty(\mathbb{R}_+,\dot{\mathscr{B}}_{2,q}^{\frac{5}{2}-2\alpha})}.
\end{aligned}$$

Similarly

$$\|\int_0^t S_{\Omega,\alpha}(t-\tau)\mathbb{P}[(B.\nabla)B]d\tau\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})} \lesssim \|B\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})}\|B\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})},$$
$$\|\int_0^t H_{\alpha}(t-\tau)\mathbb{P}[(u.\nabla)B]d\tau\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})} \lesssim \|u\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})}\|B\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})},$$
$$\|\int_0^t H_{\alpha}(t-\tau)\mathbb{P}[(B.\nabla)u]d\tau\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q})} \lesssim \|B\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})}\|u\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}.$$

Also, we have

$$\begin{aligned}
&\|\int_0^t S_{\Omega,\alpha}(t-\tau)\mathbb{P}[(u.\nabla)u]d\tau\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\\
=&\|\int_0^t S_{\Omega,\alpha}(t-\tau)\mathbb{P}[(u.\nabla)u]d\tau\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\\
\lesssim&\|u\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\|u\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\|\int_0^t S_{\Omega,\alpha}(t-\tau)\mathbb{P}[(B.\nabla)B]d\tau\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\\
\lesssim&\|B\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\|B\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})},\\
&\|\int_0^t H_{\alpha}(t-\tau)\mathbb{P}[(u.\nabla)B]d\tau\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\\
\lesssim&\|u\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{B}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\|B\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})},\\
&\|\int_0^t H_{\alpha}(t-\tau)\mathbb{P}[(B.\nabla)u]d\tau\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\\
\lesssim&\|B\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\|u\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}.
\end{aligned}$$

Step 3. Let $Y := \widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\gamma}(\mathbb{R}_+,\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q})$. Then we have

$$\begin{aligned}
\|gu\|_Y+\|gB\|_Y \lesssim&\|u_0\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}}+\|B_0\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}}\\
&+\|u\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\|u\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\\
&+\|B\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\|B\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\\
&+\|B\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\|u\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\\
&+\|u\|_{\widetilde{L}^{\gamma}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}\|B\|_{\widetilde{L}^{\infty}(\mathbb{R}_+,\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}.
\end{aligned}$$

Denote

$$\delta=\delta_0=2C(\|u\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}}+\|u\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}})<2C_{\varepsilon},$$

if ε is small enough, then we have

$$\|gu\|_Y + \|gB\|_Y \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Similarly, we have

$$d(g\begin{pmatrix} u \\ B \end{pmatrix}, g\begin{pmatrix} v \\ \beta \end{pmatrix}) \leq d(\begin{pmatrix} u \\ B \end{pmatrix}, \begin{pmatrix} v \\ \beta \end{pmatrix}).$$

In view of Banach's contraction mapping principle, there exist a unique $u, B \in \mathcal{D}$ satisfying

$$\begin{pmatrix} u \\ B \end{pmatrix} = \mathcal{A}_{\Omega,\alpha}(t)\begin{pmatrix} u_0 \\ B_0 \end{pmatrix} - \int_0^t \mathcal{A}_{\Omega,\alpha}(t-\tau)\mathbb{P}\begin{pmatrix} (u.\nabla)u - (B.\nabla)B \\ (u.\nabla)B - (B.\nabla)u \end{pmatrix} d\tau.$$

On the other hand, let

$$\begin{aligned} Z :=&\widetilde{L}^\gamma(\mathbb{R}_+, \mathcal{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q}) \cap \widetilde{L}^\gamma(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q}) \\ &\cap \widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q}) \cap \widetilde{L}^\infty(\mathbb{R}_+, \mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}), \end{aligned}$$

then we have

$$\begin{aligned} \|gu\|_Z + \|gB\|_Z \lesssim& \|u_0\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}} + \|B_0\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}} \\ &+ \|u\|_{\widetilde{L}^\gamma(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q}) \cap \widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \|u\|_{\widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \\ &+ \|B\|_{\widetilde{L}^\gamma(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q}) \cap \widetilde{L}^\infty(\mathbb{R}_+, \dot{B}^{\frac{5}{2}-2\alpha}_{2,q})} \|B\|_{\widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \\ &+ \|B\|_{\widetilde{L}^\gamma(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q}) \cap \widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \|u\|_{\widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \\ &+ \|u\|_{\widetilde{L}^\gamma(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q}) \cap \widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \|B\|_{\widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})}. \end{aligned}$$

Put

$$\begin{aligned} \mathcal{D} = \{u, B :& \|u\|_{\widetilde{L}^\gamma(\mathbb{R}_+, \mathcal{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q}) \cap \widetilde{L}^\gamma(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q}) \cap \widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \\ &+ \|B\|_{\widetilde{L}^\gamma(\mathbb{R}_+, \mathcal{F}\dot{\mathscr{B}}^{s_1(\cdot)}_{p_1(\cdot),q}) \cap \widetilde{L}^\gamma(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,q}) \cap \widetilde{L}^\infty(\mathbb{R}_+, \dot{\mathscr{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \leq \delta, \\ &\|u\|_{\widetilde{L}^\infty(\mathbb{R}_+, F\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q})} + \|B\|_{\widetilde{L}^\infty(\mathbb{R}_+, FB\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q})} \leq \delta_0\}. \end{aligned}$$

In a similar way to the case of space Y, it can be obtain that for $\|u_0\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}} + \|B_0\|_{\mathcal{F}\dot{\mathscr{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}} < \varepsilon$ with small enough ε, grMHD equations has a unique global solution in $\mathcal{D}$. □

Acknowledgements

The authors would like to thank the anonymous referee for their careful reading of the paper and valuable suggestions. The research was supported by Zhejiang Normal University Postdoctoral Research fund under grant No. ZC304020909 and NSF of China (No. 12071437, 10271437).

References

[1] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Archive for Rational Mechanics and Analysis, 2002, 164(3), 213–259.

[2] A. Almeida and P. Hästö, *Besov spaces with variable smoothness and integrability*, Journal of Functional Analysis, 2010, 258(5), 1628–1655.

[3] A. Babin, A. Mahalov and B. Nicolaenko, *Regularity and integrability of 3d euler and Navier-Stokes equations for rotating fluids*, Asymptotic Analysis, 1997, 15(2), 103–150.

[4] A. Babin, A. Mahalov and B. Nicolaenko, *3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity*, Indiana University Mathematics Journal, 2001, 1–35.

[5] A. Biswas and D. Swanson, *Gevrey regularity of solutions to the 3-D Navier-Stokes equations with weighted ℓ_p initial data*, Indiana University mathematics journal, 2007, 1157–1188.

[6] J. Bourgain and N. Pavlović, *Ill-posedness of the Navier-Stokes equations in a critical space in 3D*, Journal of Functional Analysis, 2008, 255(9), 2233–2247.

[7] J. Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical geophysics: An introduction to rotating fluids and the Navier-Stokes equations*, 32, Oxford University Press on Demand, 2006.

[8] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM journal on Applied Mathematics, 2006, 66(4), 1383–1406.

[9] D. Cruz-Uribe, L. Diening and P. Hästö, *The maximal operator on weighted variable Lebesgue spaces*, Fractional Calculus and Applied Analysis, 2011, 14(3), 361–374.

[10] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces: Foundations and harmonic analysis*, Springer Science & Business Media, 2013.

[11] C. Deng and X. Yao, *Well-posedness and ill-posedness for the 3D generalized Navier-Stokes equations in f- α, r*, Dynamical Systems, 2014, 34(2), 437–459.

[12] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Springer, 2011.

[13] H. Dong, D. Li et al., *Optimal local smoothing and analyticity rate estimates for the generalized Navier-Stokes equations*, Communications in Mathematical Sciences, 2009, 7(1), 67–80.

[14] G. Duvaut and J. L. Lions, *Inéquations en thermoélasticité et magnétohydrodynamique*, Archive for Rational Mechanics and Analysis, 1972, 46(4), 241–279.

[15] A. El Baraka and M. Toumlilin, *Global well-posedness and decay results for 3d generalized magneto-hydrodynamic equations in critical Fourier-Besov-Morrey spaces*, Electronic Journal of Differential Equations, 2017, 2017(65), 1–20.

[16] A. El Baraka and M. Toumlilin, *Global well-posedness for fractional Navier-Stokes equations in critical Fourier-Besov-Morrey spaces*, Moroccan Journal of Pure and Applied Analysis, 2017, 3(1), 1–13.

[17] X. Fan, *Global c1, α regularity for variable exponent elliptic equations in divergence form*, Journal of Differential Equations, 2007, 235(2), 397–417.

[18] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem. i*, Archive for rational mechanics and analysis, 1964, 16(4), 269–315.

[19] M. Hieber and Y. Shibata, *The Fujita-Kato approach to the Navier-Stokes equations in the rotational framework*, Mathematische Zeitschrift, 2010, 265(2), 481–491.

[20] M. Hieber and Y. Shibata, *The Fujita-Kato approach to the Navier-Stokes equations in the rotational framework*, Mathematische Zeitschrift, 2010, 265(2), 481–491.

[21] T. Iwabuchi and R. Takada, *Global solutions for the Navier-Stokes equations in the rotational framework*, Mathematische Annalen, 2013, 357(2), 727–741.

[22] T. Iwabuchi and R. Takada, *Global well-posedness and ill-posedness for the Navier-Stokes equations with the Coriolis force in function spaces of Besov type*, Journal of Functional Analysis, 2014, 267(5), 1321–1337.

[23] T. Kato, *Strongl p-solutions of the Navier-Stokes equation inr m, with applications to weak solutions*, Mathematische Zeitschrift, 1984, 187(4), 471–480.

[24] T. Kato and H. Fujita, *On the nonstationary Navier-Stokes system*, Rendiconti del Seminario Matematico della Università di Padova, 1962, 32, 243–260.

[25] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Advances in Mathematics, 2001, 157(1), 22–35.

[26] P. Konieczny and T. Yoneda, *On dispersive effect of the coriolis force for the stationary Navier-Stokes equations*, Journal of Differential Equations, 2011, 250(10), 3859–3873.

[27] Z. Lei, *On axially symmetric incompressible magnetohydrodynamics in three dimensions*, Journal of Differential Equations, 2015, 259(7), 3202–3215.

[28] Z. Lei and F. H. Lin, *Global mild solutions of Navier-Stokes equations*, Communications on Pure and Applied Mathematics, 2011, 64(9), 1297–1304.

[29] P. Li and Z. Zhai, *Well-posedness and regularity of generalized Navier-Stokes equations in some critical Q-spaces*, Journal of functional analysis, 2010, 259(10), 2457–2519.

[30] J. L. Lions, *Quelques méthodes de résolution des problemes aux limites non linéaires*, 1969.

[31] H. Nakano, *Topology and linear topological spaces*, 3, Maruzen Company, 1951.

[32] S. Ru and M. Z. Abidin, *Global well-posedness of the incompressible fractional Navier-Stokes equations in Fourier-Besov spaces with variable exponents*, Computers & Mathematics with Applications, 2019, 77(4), 1082–1090.

[33] M. Ruzicka, *Electrorheological fluids: modeling and mathematical theory*, Springer Science & Business Media, 2000.

[34] L. W.A.J, *Banach function spaces*, Ph.D. thesis, Delft Technical University, 1955.

[35] W. Wang and G. Wu, *Global mild solution of the generalized Navier-Stokes equations with the Coriolis force*, Applied Mathematics Letters, 2018, 76, 181–186.

[36] W. Wang and G. Wu, *Global mild solution of the generalized Navier-Stokes equations with the Coriolis force*, Applied Mathematics Letters, 2018, 76, 181–186.

[37] Y. Wang and K. Wang, *Global well-posedness of the three dimensional magnetohydrodynamics equations*, Nonlinear Analysis: Real World Applications, 2014, 17, 245–251.

[38] J. Wu, *Generalized MHD equations*, Journal of Differential Equations, 2003, 195(2), 284–312.

[39] J. Wu, *Lower bounds for an integral involving fractional laplacians and the generalized Navier-Stokes equations in Besov spaces*, Communications in mathematical physics, 2006, 263(3), 803–831.

[40] J. Wu, *Regularity criteria for the generalized MHD equations*, Communications in Partial Differential Equations, 2008, 33(2), 285–306.

[41] J. Xiao, *Homothetic variant of fractional Sobolev space with application to Navier-Stokes system*, Dynamics of Partial Differential Equations, 2007, 4(3), 227–245.

[42] Z. Zhai, *Well-posedness for fractional Navier-Stokes equations in critical spaces close to* $\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)$, arXiv preprint arXiv:0906.5140, 2009.