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ANALYSIS OF A STOCHASTIC SIS EPIDEMIC MODEL WITH TRANSPORT-RELATED INFECTION*

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Abstract In this paper, we consider a stochastic SIS epidemic model with transport-related infection, which is proposed to investigate the dynamics of disease propagation between two regions. Firstly, we show that the model has a unique global positive solution. Next, the properties of the solution are studied. Especially, by constructing a suitable positive-definite decrescent radially unbounded function and stopping times, we show that the differences between susceptible populations or infected populations in two regions will disappear with probability one. Then we show that the diseases in each region is extinct and the susceptible in each region is stable in the mean. Moreover, we prove that the model has a stationary distribution and the solution has the ergodic property. At last, some numerical simulations are introduced to justify the analytical results.

Keywords Stochastic SIS model, transport-related infection, extinction, stationary distribution.

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1. Introduction

Mathematical modeling can provide an understanding of the underlying mechanisms of disease transmission and the control of their spread. Infectious disease dynamics models can be traced back to the early works by Kermack and McKendrick in 1927 and 1932 (see [9,10]). In the last century, a number of studies appeared on the topic of infectious disease dynamics models (see [1,5,13,18,25,27] and the references therein) and we here do not mention them in detail.

All the above studies ignore the possibility for the individuals to become infective during travel. In [6,23], the authors proposed the following SIS epidemic model to

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understand the effect of transport-related infection on disease spread

$$\begin{cases} \frac{\mathrm{d}S_1}{\mathrm{d}t} = a - \frac{\beta S_1 I_1}{S_1+I_1} - bS_1 + dI_1 - \alpha S_1 + \alpha S_2 - \frac{\gamma\alpha S_2 I_2}{S_2+I_2}, \\ \frac{\mathrm{d}I_1}{\mathrm{d}t} = \frac{\beta S_1 I_1}{S_1+I_1} - (c+d+\alpha)I_1 + \alpha I_2 + \frac{\gamma\alpha S_2 I_2}{S_2+I_2}, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} = a - \frac{\beta S_2 I_2}{S_2+I_2} - bS_2 + dI_2 - \alpha S_2 + \alpha S_1 - \frac{\gamma\alpha S_1 I_1}{S_1+I_1}, \\ \frac{\mathrm{d}I_2}{\mathrm{d}t} = \frac{\beta S_2 I_2}{S_2+I_2} - (c+d+\alpha)I_2 + \alpha I_1 + \frac{\gamma\alpha S_1 I_1}{S_1+I_1}. \end{cases} \tag{1.1}$$

Here S_i and I_i represent the number of susceptible and infected individuals in city i, respectively ($i = 1, 2$). In this model the authors assumed that both cities are identical, i.e. demographic parameters are the same for each city. Here a is the recruitment rate of susceptible individuals per unit time; b represents the natural death rate for susceptible individuals; d stands for the recovery rate of the infected individuals; c ($c > b$) is the mortality rate of the infected individuals, which includes both natural and disease induced mortality; β represents the contact transmission rate within a city; $\frac{\beta S_j I_j}{S_j+I_j}$ stands for the number of new cases of infection per unit time within city j ($j = 1, 2$); α stands for the per capita rate of susceptible and infected individuals of every city i leave to j ($j \neq i, i, j = 1, 2$); $\frac{\gamma\alpha S_j I_j}{S_j+I_j}$ is the incidence of disease transmission with transmission $\gamma\alpha$, when the individuals in city j travel to city i ($j = 1, 2$). All parameters in model (1.1) are assumed to be positive and $\gamma \in [0, 1]$.

From [6], the condition $0 \le \gamma \le 1$ ensures that any solution of model (1.1) is nonnegative if its initial value is nonnegative. Moreover, model (1.1) has a disease free equilibrium $E_0 = (S^0, 0, S^0, 0)$ for all parameter values, and an endemic equilibrium $E^*_\gamma = (S^*_\gamma, I^*_\gamma, S^*_\gamma, I^*_\gamma)$ appears in two cities when $\mathcal{R}_{0\gamma} = \frac{\beta}{c+d} + \frac{\gamma\alpha}{c+d} > 1$, where

$$S^0 = \frac{a}{b}, \quad S^*_\gamma = \frac{a}{b + c(\mathcal{R}_{0\gamma} - 1)}, \quad I^*_\gamma = \frac{a(\mathcal{R}_{0\gamma} - 1)}{b + c(\mathcal{R}_{0\gamma} - 1)}.$$

From [6, 23], for model (1.1), the disease free equilibrium E_0 is locally asymptotically stable provided $\mathcal{R}_{0\gamma} < 1$. When $\mathcal{R}_{0\gamma} > 1$, E_0 is unstable and the endemic equilibrium E^*_γ appears in both cities. Moreover, if $|\beta - \gamma\alpha| < \frac{4(b+2\alpha)(c+d+2\alpha)-d^2}{4(2d+b+4\alpha+c)}$, then the disease free equilibrium point E_0 of model (1.1) is globally asymptotically stable on $X = \big\{(S_1, I_1, S_2, I_2) | S_i \ge 0, I_i \ge 0, i = 1, 2\big\}$ for $\mathcal{R}_{0\gamma} \le 1$ and the endemic equilibrium point E^*_γ is globally asymptotically stable on $X_0 = \big\{(S_1, I_1, S_2, I_2) \in X | I_1 + I_2 > 0\big\}$ for $\mathcal{R}_{0\gamma} > 1$.

However due to the fluctuations in the environment, epidemic models are inevitably affected by environmental noises. In general, such environment fluctuations should be modeled by a colored noise. From [17], if the colored noise is not strongly correlated, then we can approximate the colored noise by a white noise $\dot{w}(t)$, and the approximation works quite well. It turns out that white noise $\dot{w}(t)$ is formally regarded as the derivative of a Brownian motion $w(t)$, i.e. $\dot{w}(t) = \mathrm{d}w(t)/\mathrm{d}t$ (see [20]). In the few past years, stochastic epidemic models with white noise have attracted much attention. In [15, 22, 24], the authors discussed stochastic SIS epidemic models. [24] is mainly concerned with the persistence and extinction for a stochastic SIS epidemic model with nonlinear incidence rate. [15] focused the threshold behavior for a stochastic SIS epidemic model with standard incidence. In [22], the authors considered a stochastic SIS epidemic model with nonlinear saturated incidence rate and double epidemic hypothesis. [7] investigated the permanence and extinction of certain stochastic SIR models perturbed by a complex type of noises. In [3, 11, 12, 28], the authors discussed the dynamics for stochastic SIRS epidemic models. [26] considered the long-time behavior of a stochastic epidemic model with varying population size. [4] discussed a stochastic SIRI epidemic model with relapse and media coverage.

To the best of our knowledge, so far only Liu and Zheng [19] studied the stochastic disease dynamics of an SIS epidemic model on two patches. In [19], the authors only investigated the global existence and positivity of the solutions, and the sufficient conditions for almost surely exponentially stability of the disease-free equilibrium. Motivated by the above discussion, in this paper, we consider a stochastic SIS epidemic model with transport-related infection.

Parameter perturbation induced by white noise is an important and common form to describe the effect of stochasticity. May [21] pointed out that all parameters involved in the population model exhibit random fluctuation as the factors controlling them are not constant. Cai, Kang and Wang [3] pointed out that the death rate is one of the key parameters to disease transmission. In this paper, we assume that the death rates b and c always fluctuate around some average value due to continuous fluctuation in the environment. In this sense $-b \to -b+\sigma_1\dot{w}_1(t)$, $-c \to -c+\sigma_2\dot{w}_2(t)$. Here $\dot{w}_i(t)$ $(i = 1, 2)$ is the white noise. $w_1(t)$ and $w_2(t)$ are mutually independent Brownian motions. σ_1^2 and σ_2^2 are all real constants and are known as the intensity of the noise. Thus, based on model (1.1), we establish the following new stochastic SIS epidemic model with transport-related infection

$$\begin{cases} \mathrm{d}S_1 = \left(a - \frac{\beta S_1 I_1}{S_1+I_1} - bS_1 + dI_1 - \alpha S_1 + \alpha S_2 - \frac{\gamma\alpha S_2 I_2}{S_2+I_2}\right)\mathrm{d}t + \sigma_1 S_1 \mathrm{d}w_1(t), \\ \mathrm{d}I_1 = \left(\frac{\beta S_1 I_1}{S_1+I_1} - (c+d+\alpha)I_1 + \alpha I_2 + \frac{\gamma\alpha S_2 I_2}{S_2+I_2}\right)\mathrm{d}t + \sigma_2 I_1 \mathrm{d}w_2(t), \\ \mathrm{d}S_2 = \left(a - \frac{\beta S_2 I_2}{S_2+I_2} - bS_2 + dI_2 - \alpha S_2 + \alpha S_1 - \frac{\gamma\alpha S_1 I_1}{S_1+I_1}\right)\mathrm{d}t + \sigma_1 S_2 \mathrm{d}w_1(t), \\ \mathrm{d}I_2 = \left(\frac{\beta S_2 I_2}{S_2+I_2} - (c+d+\alpha)I_2 + \alpha I_1 + \frac{\gamma\alpha S_1 I_1}{S_1+I_1}\right)\mathrm{d}t + \sigma_2 I_2 \mathrm{d}w_2(t), \end{cases} \tag{1.2}$$

with initial value $(S_1(0), I_1(0), S_2(0), I_2(0)) = (S_{10}, I_{10}, S_{20}, I_{20}) \in \mathbb{R}_+^4 = \big\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \big| x_i > 0,\ i = 1, 2, 3, 4\big\}$. Here $w = \{w_1(t), w_2(t), t \geq 0\}$ represents the two-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). All meanings of the parameters are exact to or similar as those for model (1.1). Here, all parameters in model (1.1) are assumed to be positive and $\gamma \in [0, 1]$.

The rest of this paper is organized as follows. In the next section, we first prove existence and uniqueness of global positive solution of model (1.2). Then, we investigate the asymptotic property of positive solutions of the model. Especially, we show that the differences between susceptible populations or infected populations in two cities will disappear with probability one. In Section 3, we show that the diseases in each region is extinct and the susceptible in each region is stable in the mean. In Section 4, we prove that model (1.2) has a stationary distribution and the solution has the ergodic property. Numerical simulations under certain parameters are presented to illustrate our main results in Section 5. Finally, a few comments will conclude the paper.

2. Asymptotic properties of the solution

In this section, we first show that model (1.2) has a unique positive global solution with positive initial value. Then, we investigate the asymptotic property of positive solutions of the model. Denote $\mathbb{R}_+ = (0, +\infty)$ and $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\}$. Let $\mathcal{K}$ denote the family of all continuous nondecreasing functions $\mu : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$. For $h > 0$, let $S_h = \{x \in \mathbb{R}^d : |x| < h\}$. For the sake of simplification, we denote

$$X(t) = (S_1(t), I_1(t), S_2(t), I_2(t)), \qquad X_0 = (S_{10}, I_{10}, S_{20}, I_{20}),$$
$$N(t) = S_1(t) + I_1(t) + S_2(t) + I_2(t), \quad \langle u(t) \rangle = \frac{1}{t} \int_0^t u(s) \mathrm{d}s.$$

Definition 2.1 (see [20]). (i) A continuous function $V(x, t)$ defined on $S_h \times [t_0, \infty)$ is said to be positive-definite if $V(0, t) \equiv 0$ and, for some $\mu \in \mathcal{K}$,

$$V(x, t) \geq \mu(|x|) \quad \text{for all } (x, t) \in S_h \times [t_0, \infty).$$

(ii) A non-negative continuous function $V(x, t)$ defined on $S_h \times [t_0, \infty)$ is said to be decrescent if for some $\mu \in \mathcal{K}$,

$$V(x, t) \leq \mu(|x|) \quad \text{for all } (x, t) \in S_h \times [t_0, \infty).$$

(iii) A function $V(x, t)$ defined on $\mathbb{R}^d \times [t_0, \infty)$ is said to be radially unbounded if

$$\lim_{|x| \to \infty} \inf_{t \geq t_0} V(x, t) = \infty.$$

Definition 2.2 (see [14]). Model (1.2) is said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there are positive constants $\varrho = \varrho(\varepsilon)$ and $\chi = \chi(\varepsilon)$, such that for any initial value $X_0 \in \mathbb{R}^4_+$, the solution $X(t)$ of model (1.2) satisfies

$$\liminf_{t \to \infty} \mathbb{P}\{|X(t)| \leq \varrho\} \geq 1 - \varepsilon, \quad \liminf_{t \to \infty} \mathbb{P}\{|X(t)| \geq \chi\} \geq 1 - \varepsilon.$$

Stochastical permanence of model (1.2) means that the total number of the individuals in the model (including the susceptible and the infected) is bounded and permanent. That is, the individuals in the model will not grow wildly or die out.

Theorem 2.1. *For any initial value $(S_{10}, I_{10}, S_{20}, I_{20}) \in \mathbb{R}^4_+$, model (1.2) has a unique global positive solution $(S_1(t), I_1(t), S_2(t), I_2(t))$ defined on $\mathbb{R}_+$. That is, the solution will remain in $\mathbb{R}^4_+$ with probability one.*

Proof. It is easy to show that the coefficients of (1.2) are locally Lipschitz continuous. Thus, for any initial value $(S_{10}, I_{10}, S_{20}, I_{20}) \in \mathbb{R}^4_+$, model (1.2) has a unique maximal local solution $(S_1(t), I_1(t), S_2(t), I_2(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. Let $n_0 > 0$ be sufficiently large such that S_{10}, I_{10}, S_{20} and I_{20} all lie within the interval $(1/n_0, n_0)$. For each integer $n \geq n_0$, define the stopping time

$$\tau_n = \inf \left\{ t \in [0, \tau_e) : \min\{S_1(t), I_1(t), S_2(t), I_2(t)\} \leq \frac{1}{n} \right.$$

$$\text{or } \max\{S_1(t), I_1(t), S_2(t), I_2(t)\} \geq n\Big\},$$

where throughout this paper for the empty set $\emptyset$ we set $\inf \emptyset = \infty$. Clearly, τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to\infty} \tau_n$. It is easy to show that τ_∞ is a stopping time and $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(S_1(t), I_1(t), S_2(t), I_2(t)) \in \mathbb{R}^4_+$ a.s. for all $t \geq 0$.

Now, we show that $\tau_\infty = \infty$ a.s. Assume that the statement does not hold, then there are $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Denote $\Omega_n = \{\omega \in \Omega : \tau_n(\omega) \leq T\}$. Hence,

$$\mathbb{P}(\Omega_n) > \varepsilon, \quad n \geq n_0. \tag{2.1}$$

Define function $V : \mathbb{R}^4_+ \to \mathbb{R}_+$ by

$$\begin{aligned} V(S_1, I_1, S_2, I_2) =&(S_1 - 1 - \ln S_1) + (I_1 - 1 - \ln I_1) + (S_2 - 1 - \ln S_2) \\ &+ (I_2 - 1 - \ln I_2). \end{aligned}$$

By the Itô formula, we have, for any $t \in [0, T]$ and $n \geq n_0$,

$$\begin{aligned} &\mathbb{E}V(S_1(t\wedge\tau_n), I_1(t\wedge\tau_n), S_2(t\wedge\tau_n), S_2(t\wedge\tau_n)) \\ =&V(S_{10}, I_{10}, S_{20}, I_{20}) + \mathbb{E}\int_0^{t\wedge\tau_n} LV(S_1(s), I_1(s), S_2(s), I_2(s))\,\mathrm{d}s, \end{aligned} \tag{2.2}$$

where $LV : \mathbb{R}^4_+ \to \mathbb{R}$ is defined by

$$\begin{aligned} LV =&2a + 2b + 2(c + d + 2\alpha) + \sigma_1^2 + \sigma_2^2 + \frac{\beta I_1}{S_1 + I_1} + \frac{\beta I_2}{S_2 + I_2} - b(S_1 + S_2) \\ &- c(I_1 + I_2) - \frac{a}{S_1} - \frac{a}{S_2} - d\frac{I_1}{S_1} - d\frac{I_2}{S_2} - \alpha\frac{I_1}{I_2} - \alpha\frac{I_2}{I_1} - \frac{\gamma\alpha S_2 I_2}{I_1(S_2 + I_2)} \\ &- \frac{\gamma\alpha S_1 I_1}{I_2(S_1 + I_1)} - \frac{\beta S_1}{S_1 + I_1} - \frac{\beta S_2}{S_2 + I_2} - \frac{1}{S_1}\left(\alpha S_2 - \frac{\gamma\alpha S_2 I_2}{S_2 + I_2}\right) \\ &- \frac{1}{S_2}\left(\alpha S_1 - \frac{\gamma\alpha S_1 I_1}{S_1 + I_1}\right). \end{aligned} \tag{2.3}$$

From $\gamma \in [0, 1]$, it follows that $\alpha S_1 - \frac{\gamma\alpha S_1 I_1}{S_1+I_1} \geq 0$ and $\alpha S_2 - \frac{\gamma\alpha S_2 I_2}{S_2+I_2} \geq 0$. Hence,

$$LV \leq 2a + 2b + 2(c + d + 2\alpha) + 2\beta + \sigma_1^2 + \sigma_2^2 =: K,$$

where K is a positive constant which is independent of S_1, I_1, S_1, I_2 and t. Thus, from (2.2), it follows that

$$\mathbb{E}V(S_1(T\wedge\tau_n), I_1(T\wedge\tau_n), S_2(T\wedge\tau_n), S_2(T\wedge\tau_n)) \leq V(S_{10}, I_{10}, S_{20}, I_{20}) + KT. \tag{2.4}$$

Note that for every $\omega \in \Omega_n$, there is at least one of $S_1(\tau_n, \omega)$, $I_1(\tau_n, \omega)$, $S_2(\tau_n, \omega)$ and $I_2(\tau_n, \omega)$ equalling either $1/n$ or n. Hence

$$V(S_1(\tau_n, \omega), I_1(\tau_n, \omega), S_2(\tau_n, \omega), I_2(\tau_n, \omega)) \geq (n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 - \ln\frac{1}{n}\right). \tag{2.5}$$

It then follows from (2.1), (2.4) and (2.5) that

$$\begin{aligned}V(S_{10}, I_{10}, S_{20}, I_{20}) + KT \geq & \mathbb{E}\Big[I_{\Omega_n}(\omega)V(S_1(\tau_n,\omega), I_1(\tau_n,\omega), S_2(\tau_n,\omega), I_2(\tau_n,\omega))\Big] \\ > & \varepsilon\left[(n-1-\ln n)\wedge\left(\frac{1}{n}-1-\ln\frac{1}{n}\right)\right],\end{aligned}$$

where I_{Ω_n} is the indicator function of Ω_n. Letting $n\to\infty$ leads to the contradiction

$$\infty > V(S_{10}, I_{10}, S_{20}, I_{20}) + KT = \infty.$$

Hence, $\tau_\infty = \infty$ a.s. The proof is complete. □

Theorem 2.2. *For any $X_0 \in \mathbb{R}^4_+$, let $X(t)$ be solution of model (1.2) with initial value X_0. Then*

$$\limsup_{t\to\infty}[S_1(t) + I_1(t) + S_2(t) + I_2(t)] < \infty \quad \text{a.s.}$$

Proof. By model (1.2), we have

$$\mathrm{d}N = \big[2a - b(S_1+S_2) - c(I_1+I_2)\big]\mathrm{d}t + \sigma_1(S_1+S_2)\mathrm{d}w_1(t) + \sigma_2(I_1+I_2)\mathrm{d}w_2(t),$$

which implies

$$\mathrm{d}N = \big[2a - bN - (c-b)(I_1+I_2)\big]\mathrm{d}t + \sigma_1(S_1+S_2)\mathrm{d}w_1(t) + \sigma_2(I_1+I_2)\mathrm{d}w_2(t).$$

Solving this equation, we obtain that

$$\begin{aligned}N(t) = & \frac{2a}{b} + \left(N(0) - \frac{2a}{b}\right)e^{-bt} - (c-b)\int_0^t e^{-b(t-s)}(I_1(s)+I_2(s))\mathrm{d}s \\ & + \sigma_1\int_0^t e^{-b(t-s)}(S_1(s)+S_2(s))\mathrm{d}w_1(s) \\ & + \sigma_2\int_0^t e^{-b(t-s)}(I_1(s)+I_2(s))\mathrm{d}w_2(s) \\ \leq & \frac{2a}{b} + \left(N(0) - \frac{2a}{b}\right)e^{-bt} + M(t) \\ = & N(0) + \frac{2a}{b}(1-e^{-bt}) - N(0)(1-e^{-bt}) + M(t) \quad \text{a.s.},\end{aligned}$$

where

$$M(t) = \sigma_1\int_0^t e^{-b(t-s)}(S_1(s)+S_2(s))\mathrm{d}w_1(s) + \sigma_2\int_0^t e^{-b(t-s)}(I_1(s)+I_2(s))\mathrm{d}w_2(s).$$

Clearly, $M(t)$ is a continuous local martingale with $M(0)=0$. Define

$$Y(t) = Y(0) + A(t) - U(t) + M(t),$$

where $Y(0) = N(0)$, $A(t) = \frac{2a}{b}(1-e^{-bt})$ and $U(t) = N(0)(1-e^{-bt})$. It is clear that $N(t) \leq Y(t)$ a.s. for all $t \geq 0$. Note that $A(t)$ and $U(t)$ are two continuous adapted increasing processes with $A(0) = U(0) = 0$ a.s. By Theorem 1.3.9 in [20], we obtain that $\lim_{t\to\infty} Y(t) < \infty$ a.s. Thus

$$\limsup_{t\to\infty} N(t) = \limsup_{t\to\infty}[S_1(t) + I_1(t) + S_2(t) + I_2(t)] < \infty \quad \text{a.s.}$$

The proof is complete. □

Remark 2.1. From Theorem 2.2, it is easy to see that for any given positive initial value $(S_{10}, I_{10}, S_{20}, I_{20})$, the solution of model (1.2) has the properties that

$$\limsup_{t\to\infty} \frac{S_i(t)}{t} \le 0, \quad \limsup_{t\to\infty} \frac{I_i(t)}{t} \le 0 \ \text{ a.s.}, \quad (i = 1, 2).$$

Theorem 2.3. *For any initial value $X_0 \in \mathbb{R}^4_+$, model (1.2) is stochastically permanent.*

Proof. Define $V_1(X) = N + \frac{1}{N}$, where $X = (S_1, I_1, S_2, I_2)$ and $N = S_1 + I_1 + S_2 + I_2$. It is easy to show

$$\mathrm{d}N = \big[2a - b(S_1 + S_2) - c(I_1 + I_2)\big]\mathrm{d}t + \sigma_1(S_1 + S_2)\mathrm{d}w_1(t) + \sigma_2(I_1 + I_2)\mathrm{d}w_2(t). \tag{2.6}$$

By Itô formula, we have

$$LV_1(X) = \Big(1 - \frac{1}{N^2}\Big)\big[2a - b(S_1 + S_2) - c(I_1 + I_2)\big] + \frac{\sigma_1^2(S_1 + S_2)^2 + \sigma_2^2(I_1 + I_2)^2}{N^3},$$

which, together with $c > b$, yields

$$\begin{aligned} LV_1(X) =& 2a - b(S_1 + S_2) - c(I_1 + I_2) - \frac{2a - b(S_1 + S_2) - c(I_1 + I_2)}{N^2} \\ &+ \frac{\sigma_1^2(S_1 + S_2)^2 + \sigma_2^2(I_1 + I_2)^2}{N^3} \\ \le& 2a - bN - \frac{2a}{N^2} + \frac{c(S_1 + I_1 + S_2 + I_2)}{N^2} + \frac{\sigma^2[(S_1 + S_2)^2 + (I_1 + I_2)^2]}{N^3} \\ \le& -bN - \frac{b}{N} + 2a - \frac{2a}{N^2} + \frac{b + c + \sigma^2}{N} \\ \le& \kappa - bV_1(X(t)), \end{aligned} \tag{2.7}$$

where $\sigma^2 = \sigma_1^2 \vee \sigma_2^2$ and $\kappa = \frac{16a^2 + (b+c+\sigma^2)^2}{8a}$. Applying Itô formula again and from (2.7), it follows that

$$L(e^{bt}V_1(X)) \le be^{bt}V_1(X) + e^{bt}[\kappa - bV_1(X)] = \kappa e^{bt}.$$

Therefore, $\mathbb{E}[e^{bt}V_1(X(t))] \le V_1(X_0) + \mathbb{E}[\int_0^t \kappa e^{bs}\mathrm{d}s] = V_1(X_0) + \frac{\kappa}{b}(e^{bt} - 1)$, which implies

$$\limsup_{t\to\infty} \mathbb{E}[V_1(X(t))] \le \limsup_{t\to\infty} \Big[e^{-bt}V_1(X_0) + \frac{\kappa}{b}(1 - e^{-bt})\Big] = \frac{\kappa}{b}.$$

Thus

$$\limsup_{t\to\infty} \mathbb{E}[N(t)] \le \frac{\kappa}{b}, \qquad \limsup_{t\to\infty} \mathbb{E}\left[\frac{1}{N(t)}\right] \le \frac{\kappa}{b}. \tag{2.8}$$

Note that $N^2 = (S_1 + I_1 + S_2 + I_2)^2 \le 4(S_1^2 + I_1^2 + S_2^2 + I_2^2) = 4|X|^2 \le 4(S_1 + I_1 + S_2 + I_2)^2 = 4N^2$. Then

$$\frac{N(t)}{2} \le |X(t)| \le N(t), \qquad \frac{1}{N(t)} \le \frac{1}{|X(t)|} \le \frac{2}{N(t)}.$$

This, together with (2.8), yields

$$\limsup_{t\to\infty} \mathbb{E}[|X(t)|] \le \frac{\kappa}{b}, \qquad \limsup_{t\to\infty} \mathbb{E}\left[\frac{1}{|X(t)|}\right] \le \frac{2\kappa}{b}. \tag{2.9}$$

For any $\varepsilon \in (0,1)$, let $\varrho = \frac{\kappa}{b\varepsilon}$. Then, by Chebyshev's inequality

$$\mathbb{P}\{|X(t)| > \varrho\} \le \frac{\mathbb{E}[|X(t)|]}{\varrho}.$$

Hence, from (2.9), it follows that

$$\limsup_{t\to\infty} \mathbb{P}\{|X(t)| > \varrho\} \le \frac{\limsup_{t\to\infty} \mathbb{E}[|X(t)|]}{\varrho} = \varepsilon.$$

This implies

$$\liminf_{t\to\infty} \mathbb{P}\{|X(t)| \le \varrho\} \ge 1-\varepsilon.$$

Similarly, let $\chi = \frac{b\varepsilon}{2\kappa}$. Then, by Chebyshev's inequality

$$\mathbb{P}\{|X(t)| < \chi\} = \mathbb{P}\left\{\frac{1}{|X(t)|} > \frac{1}{\chi}\right\} \le \chi\mathbb{E}\left[\frac{1}{|X(t)|}\right].$$

Hence, from (2.9), we have

$$\limsup_{t\to\infty} \mathbb{P}\{|X(t)| < \chi\} \le \limsup_{t\to\infty} \chi\mathbb{E}\left[\frac{1}{|X(t)|}\right] = \varepsilon.$$

This implies

$$\liminf_{t\to\infty} \mathbb{P}\{|X(t)| \ge \chi\} \ge 1-\varepsilon.$$

According to Definition 2.2, model (1.2) is stochastically permanent. The proof is complete. □

Theorem 2.4. *Assume that $p \ge 1$ and $\bar{b} > 0$, where $\bar{b} = b - \frac{p-1}{2}\sigma^2$ and $\sigma^2 = \sigma_1^2 \vee \sigma_2^2$. Let $X(t)$ be the solution of model (1.2) with any given initial value $X_0 \in \mathbb{R}_+^4$. Then, there is a positive constant $M = M(p)$ such that*

$$\mathbb{E}[N^p(t)] \le M, \quad t \ge 0.$$

Proof. Define $V_2(X) = N^p$, where $X = (S_1, I_1, S_2, I_2)$ and $N = S_1 + I_1 + S_2 + I_2$. Applying Itô formula, we obtain

$$\mathrm{d}V_2(X(t)) = LV_2(X(t))\mathrm{d}t + pN^{p-1}(t)\left[\sigma_1(S_1+S_2)(t)\mathrm{d}w_1(t) + \sigma_2(I_1+I_2)(t)\mathrm{d}w_2(t)\right], \tag{2.10}$$

where

$$\begin{aligned} LV_2(X) =& pN^{p-2}\Big[N[2a - b(S_1+S_2) - c(I_1+I_2)] \\ &+ \frac{(p-1)}{2}[\sigma_1^2(S_1+S_2)^2 + \sigma_2^2(I_1+I_2)^2]\Big]. \end{aligned}$$

Using $b < c$ and the condition of theorem, we have

$$\begin{aligned}
&LV_2(X)\\
\le& pN^{p-2}\Big[2aN - b(S_1+S_2)N - b(I_1+I_2)N + \frac{p-1}{2}\sigma^2\big((S_1+S_2)^2 + (I_1+I_2)^2\big)\Big]\\
\le& pN^{p-2}\Big[2aN - \Big(b - \frac{p-1}{2}\sigma^2\Big)N^2\Big]. \qquad (2.11)
\end{aligned}$$

Thus, for $k \in (0, \bar{b}p)$, from the Itô formula, it follows that

$$\mathbb{E}\big[e^{kt}V_2(X(t))\big] = V_2(X_0) + \mathbb{E}\int_0^t L(e^{ks}V_2(X(s)))\mathrm{d}s. \qquad (2.12)$$

Note that $L(e^{kt}V_2(X)) = e^{kt}[kV_2(X) + LV_2(X)]$. This, together with (2.11), yields

$$L(e^{kt}V_2(X(t))) \le pe^{kt}N^{p-2}\Big[-\Big(\bar{b} - \frac{k}{p}\Big)N^2 + 2aN\Big].$$

Let us consider function $f(x) = x^{p-2}\big[-\big(\bar{b} - \frac{k}{p}\big)x^2 + 2ax\big]$ on $(0,\infty)$. It is easy to show that $f(x)$ reaches it's maximum value at $x = 2a(p-1)/(p\bar{b} - k) > 0$ and the maximum value is $f_{\max} = \frac{2a}{p}[\frac{2a(p-1)}{p\bar{b}-k}]^{p-1}$. Thus, there is a positive constant $H = \frac{2a}{p}[\frac{2a(p-1)}{p\bar{b}-k}]^{p-1}$ such that $L(e^{kt}V_2(X(t))) \le pHe^{kt}$ for any $t \ge 0$. This, together with (2.12), yields

$$\mathbb{E}\big[e^{kt}V_2(X(t))\big] \le V_2(X_0) + \mathbb{E}\int_0^t pHe^{ks}\mathrm{d}s \le V_2(X_0) + \frac{pH}{k}e^{kt}.$$

Thus, $\mathbb{E}[N^p(t)] \le V_2(X_0)e^{-kt} + \frac{pH}{k}$, which implies $\limsup_{t\to\infty}\mathbb{E}[N^p(t)] \le \frac{pH}{k}$. Hence, there exists a positive constant $M = M(p)$ such that for all $t \ge 0$,

$$\mathbb{E}[N^p(t)] \le M. \qquad (2.13)$$

Then, by the positivity of the solution, we have $\mathbb{E}[S_i^p(t)] \le M$, $\mathbb{E}[I_i^p(t)] \le M$, $t \ge 0$, $i = 1, 2$. The proof is complete. □

Theorem 2.5. *For any $X_0 \in \mathbb{R}_+^4$, let $X(t)$ be the solution of model (1.2) with initial value X_0. If*

$$\frac{\sigma_1^2}{2} < b + 2\alpha,\ \frac{\sigma_2^2}{2} < c + d + 2\alpha,\ |\beta - \gamma\alpha| < \frac{4(b + 2\alpha - \frac{\sigma_1^2}{2})(c + d + 2\alpha - \frac{\sigma_2^2}{2}) - d^2}{4(b + c + 2d + 4\alpha - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2})}, \qquad (2.14)$$

then

$$\lim_{t\to\infty}\mathbb{E}[S_1(t) - S_2(t)]^2 = 0, \qquad \lim_{t\to\infty}\mathbb{E}[I_1(t) - I_2(t)]^2 = 0. \qquad (2.15)$$

Further,

$$\mathbb{P}\Big\{\lim_{t\to\infty}[S_1(t) - S_2(t)] = 0\Big\} = 1, \quad \mathbb{P}\Big\{\lim_{t\to\infty}[I_1(t) - I_2(t)] = 0\Big\} = 1. \qquad (2.16)$$

Proof. Denote $x = (x_1, x_2) = (S_1 - S_2, I_1 - I_2)$ and $x_0 = (x_{10}, x_{20}) = (S_{10} - S_{20}, I_{10} - I_{20})$. Let us consider the function $V_3(x(t)) = \frac{x_1^2 + x_2^2}{2}$. From Itô formula, it follows that

$$\mathbb{E}V_3(x(t)) = V_3(x_0) + \int_0^t \mathbb{E}LV_3(x(s))\mathrm{d}s. \tag{2.17}$$

Here

$$\begin{aligned} LV_3 =& (S_1 - S_2)\Big[(\beta - \gamma\alpha)\Big(\frac{S_2 I_2}{S_2 + I_2} - \frac{S_1 I_1}{S_1 + I_1}\Big) - (b + 2\alpha)(S_1 - S_2) + d(I_1 - I_2)\Big] \\ &+ (I_1 - I_2)\Big[(\beta - \gamma\alpha)\Big(\frac{S_1 I_1}{S_1 + I_1} - \frac{S_2 I_2}{S_2 + I_2}\Big) - (c + d + 2\alpha)(I_1 - I_2)\Big] \\ &+ \frac{\sigma_1^2}{2}(S_1 - S_2)^2 + \frac{\sigma_2^2}{2}(I_1 - I_2)^2 \\ =& -\Big(b + 2\alpha - \frac{\sigma_1^2}{2}\Big)(S_1 - S_2)^2 - \Big(c + d + 2\alpha - \frac{\sigma_2^2}{2}\Big)(I_1 - I_2)^2 \\ &+ (\beta - \gamma\alpha)\Big(\frac{S_2 I_2}{S_2 + I_2} - \frac{S_1 I_1}{S_1 + I_1}\Big)(S_1 - S_2 - I_1 + I_2) \\ &+ d(S_1 - S_2)(I_1 - I_2). \end{aligned} \tag{2.18}$$

Note that

$$\frac{S_2 I_2}{S_2 + I_2} - \frac{S_1 I_1}{S_1 + I_1} = \frac{S_1 S_2(I_2 - I_1) + I_1 I_2(S_2 - S_1)}{(S_1 + I_1)(S_2 + I_2)} \le |S_1 - S_2| + |I_1 - I_2|.$$

Thus, from (2.18), it follows that

$$\begin{aligned} LV_3 \le& -\Big(b + 2\alpha - \frac{\sigma_1^2}{2} - |\beta - \gamma\alpha|\Big)(S_1 - S_2)^2 \\ &- \Big(c + d + 2\alpha - \frac{\sigma_2^2}{2} - |\beta - \gamma\alpha|\Big)(I_1 - I_2)^2 \\ &+ \Big(d + 2|\beta - \gamma\alpha|\Big)|S_1 - S_2||I_1 - I_2| \\ :=& -A_1(S_1 - S_2)^2 - A_2(I_1 - I_2)^2 + A_3|S_1 - S_2||I_1 - I_2|. \end{aligned}$$

From (2.14), it is easy to show that the following quadratic form

$$-A_1 x^2 - A_2 y^2 + A_3 xy$$

is negative definite. Hence, there exists a positive constant K_1 such that

$$LV_3(x) \le -K_1\big[(S_1 - S_2)^2 + (I_1 - I_2)^2\big] = -2K_1 V_3(x), \tag{2.19}$$

which, together with (2.17), yields $\mathbb{E}V_3(x(t)) \le V_3(x_0) - 2K_1 \int_0^t \mathbb{E}V_3(x(s))\mathrm{d}s$. Since $V_3(x_0) < \infty$, we have

$$\mathbb{E}V_3(x(t)) + 2K_1 \int_0^t \mathbb{E}V_3(x(s))\mathrm{d}s \le V_3(x_0) < \infty,$$

which leads to $\mathbb{E}V_3(x(s)) \in L^1[0, \infty)$. It follows from (2.17) and (2.19) that

$$\frac{\mathrm{d}\mathbb{E}V_3(x(t))}{\mathrm{d}t} = \mathbb{E}LV_3(x(t)) \le -2K_1 \mathbb{E}V_3(x(t)).$$

Thus, we get by the comparison theorem that $\mathbb{E}V_3(x(t)) \le V_3(x_0)e^{-2K_1 t}$, for $t \ge 0$. Further,

$$\lim_{t\to\infty} \mathbb{E}V_3(x(t)) = \lim_{t\to\infty} \frac{\mathbb{E}[S_1(t) - S_2(t)]^2 + \mathbb{E}[I_1(t) - I_2(t)]^2}{2} = 0.$$

Therefore, we can get (2.15).

Now, let us prove (2.16). Note that $\frac{|x|^2}{3} \le V_3(x) \le |x|^2$ and $\lim_{|x|\to\infty} V_3(x) = \lim_{|x|\to\infty} \frac{|x|^2}{2} = \infty$ for any $x \in \mathbb{R}^2$. From Definition 2.1, it follows that $V(x)$ is positive-define, decrescent and radially unbounded. Therefore, for any $\varepsilon \in (0, 1)$, there is an $h > |x_0|$ sufficiently large such that

$$\inf_{|x|\ge h} V_3(x) \ge \frac{4V_3(x_0)}{\varepsilon}. \tag{2.20}$$

Define the stopping time $\tau_h = \inf\{t \ge 0 : |x(t)| \ge h\}$. From (2.19), we can show that for any $t \ge 0$,

$$\begin{aligned}\mathbb{E}V_3(x(\tau_h \wedge t)) =& V_3(x_0) + \mathbb{E}\int_0^{\tau_h\wedge t} LV_3(x(s))\mathrm{d}s \\ \le& V_3(x_0) - K_1\mathbb{E}\int_0^{\tau_h\wedge t} |x(s))|^2\mathrm{d}s \\ \le& V_3(x_0).\end{aligned} \tag{2.21}$$

By (2.20), we have

$$\begin{aligned}\mathbb{E}V_3(x(\tau_h \wedge t)) =& \int_{\{\tau_h\le t\}} V_3(x(\tau_h))\mathrm{d}\mathbb{P} + \int_{\{\tau_h>t\}} V_3(x(t))\mathrm{d}\mathbb{P} \\ \ge& \int_{\{\tau_h\le t\}} V_3(x(\tau_h))\mathrm{d}\mathbb{P} \\ \ge& \frac{4V_3(x_0)}{\varepsilon}\mathbb{P}\{\tau_h \le t\},\end{aligned}$$

which, together with (2.21), yields $\mathbb{P}\{\tau_h \le t\} \le \frac{\varepsilon}{4}$. Letting $t \to \infty$ gives

$$\mathbb{P}\{\tau_h < \infty\} \le \frac{\varepsilon}{4}. \tag{2.22}$$

That is

$$\mathbb{P}\{|x(t)| \le h \text{ for all } t \ge 0\} \ge 1 - \frac{\varepsilon}{4}. \tag{2.23}$$

For any $\chi \in (0, |x_0|)$, choose $0 < \varrho < \chi$ sufficiently small such that

$$\frac{3\varrho^2}{\chi^2} \le \frac{\varepsilon}{4}. \tag{2.24}$$

Define the stopping time $\tau_\varrho = \inf\{t \ge 0 : |x(t)| \le \varrho\}$. It is clear that $|x(t)| \ge \varrho$ on $[0, \tau_\varrho]$. Thus, from (2.19), we can show that for any $t \ge 0$,

$$0 \le \mathbb{E}V_3(x(\tau_\varrho \wedge \tau_h \wedge t)) = V_3(x_0) + \mathbb{E}\int_0^{\tau_\varrho\wedge\tau_h\wedge t} LV_3(x(s))\mathrm{d}s$$

$$\begin{aligned}&\leq V_3(x_0) - K_1\mathbb{E}\int_0^{\tau_\varrho\wedge\tau_h\wedge t}|x(s)|^2\mathrm{d}s\\&\leq V_3(x_0) - K_1\varrho^2\mathbb{E}[\tau_\varrho\wedge\tau_h\wedge t],\end{aligned}$$

which implies $\mathbb{E}[\tau_\varrho\wedge\tau_h\wedge t]\leq\frac{V_3(x_0)}{K_1\varrho^2}$. On the other hand,

$$\begin{aligned}\mathbb{E}\big[\tau_\varrho\wedge\tau_h\wedge t\big]&=\int_{\{(\tau_\varrho\wedge\tau_h)\geq t\}}t\,\mathrm{d}\mathbb{P}+\int_{\{(\tau_\varrho\wedge\tau_h)<t\}}(\tau_\varrho\wedge\tau_h)\,\mathrm{d}\mathbb{P}\\&\geq\int_{\{(\tau_\varrho\wedge\tau_h)\geq t\}}t\,\mathrm{d}\mathbb{P}=t\mathbb{P}\{\tau_\varrho\wedge\tau_h\geq t\}.\end{aligned}$$

Hence,

$$\mathbb{P}\big\{(\tau_\varrho\wedge\tau_h)\geq t\big\}\leq\frac{V_3(x_0)}{K_1\varrho^2 t}.$$

Letting $t\to\infty$ gives $\mathbb{P}\big\{(\tau_\varrho\wedge\tau_h)=\infty\big\}=0$. Therefore, $\mathbb{P}\big\{(\tau_\varrho\wedge\tau_h)<\infty\big\}=1$, which, together with (2.22), yields

$$1=\mathbb{P}\big\{(\tau_\varrho\wedge\tau_h)<\infty\big\}\leq\mathbb{P}\big\{\tau_\varrho<\infty\big\}+\mathbb{P}\big\{\tau_h<\infty\big\}\leq\mathbb{P}\big\{\tau_\varrho<\infty\big\}+\frac{\varepsilon}{4}.$$

Thus,

$$\mathbb{P}\big\{\tau_\varrho<\infty\big\}\geq 1-\frac{\varepsilon}{4}. \tag{2.25}$$

Choose θ sufficiently large such that $\mathbb{P}\big\{\tau_\varrho<\theta\big\}\geq 1-\frac{\varepsilon}{2}$. Obviously,

$$\{\tau_\varrho<(\tau_h\wedge\theta)\}\supset\{\tau_\varrho<\theta\text{ and }\tau_h=\infty\}.$$

Using $\mathbb{P}(AB)\geq\mathbb{P}(A)-\mathbb{P}(B^C)$ for any $A,B\in\mathcal{F}$, we have

$$\begin{aligned}\mathbb{P}\big\{\tau_\varrho<(\tau_h\wedge\theta)\big\}&\geq\mathbb{P}\big(\big\{\tau_\varrho<\theta\big\}\cap\big\{\tau_h=\infty\big\}\big)\\&\geq\mathbb{P}\big\{\tau_\varrho<\theta\big\}-\mathbb{P}\big\{\tau_h<\infty\big\}\\&\geq 1-\frac{3\varepsilon}{4}.\end{aligned} \tag{2.26}$$

Define two stopping times

$$\nu=\begin{cases}\tau_\varrho, & \text{if }\tau_\varrho<(\tau_h\wedge\theta),\\ \infty, & \text{otherwise}\end{cases}$$

and

$$\tau_\chi=\inf\big\{t>\nu:|x(t)|\geq\chi\big\}.$$

It is clear that $\tau_\chi\geq\nu$. Moreover, for any $t\geq\theta$,

$$\mathbb{E}V_3(x(\tau_\chi\wedge t))=\mathbb{E}V_3(x(\nu\wedge t))+\mathbb{E}\int_{\nu\wedge t}^{\tau_\chi\wedge t}LV_3(x(s))\mathrm{d}s,$$

which, together with (2.19), yields $\mathbb{E}V_3(x(\tau_\chi \wedge t)) \leq \mathbb{E}V_3(x(\nu \wedge t))$. It is clear that $V_3(x(\tau_\chi \wedge t)) = V_3(x(\nu \wedge t)) = V_3(x(t))$ on $\big\{\tau_\varrho \geq (\tau_h \wedge \theta)\big\}$. Note that

$$\begin{aligned}\mathbb{E}V_3(x(\tau_\chi \wedge t)) =& \int_{\{\tau_\varrho<(\tau_h\wedge\theta)\}} V_3(x(\tau_\chi \wedge t))\mathrm{d}\mathbb{P} + \int_{\{\tau_\varrho\geq(\tau_h\wedge\theta)\}} V_3(x(\tau_\chi \wedge t))\mathrm{d}\mathbb{P} \\ =&\mathbb{E}\big[I_{\{\tau_\varrho<\tau_h\wedge\theta\}} V_3(x(\tau_\chi \wedge t))\big] + \int_{\{\tau_\varrho\geq\tau_h\wedge\theta\}} V_3(x(t))\mathrm{d}\mathbb{P}. \end{aligned} \tag{2.27}$$

Similarly, for any $t \geq \theta$,

$$\mathbb{E}V_3(x(\nu \wedge t)) =\mathbb{E}\big[I_{\{\tau_\varrho<(\tau_h\wedge\theta)\}} V_3(x(\tau_\varrho))\big] + \int_{\{\tau_\varrho\geq(\tau_h\wedge\theta)\}} V_3(x(t))\mathrm{d}\mathbb{P}. \tag{2.28}$$

From (2.27) and (2.28), for any $t \geq \theta$,

$$\mathbb{E}\big[I_{\{\tau_\varrho<(\tau_h\wedge\theta)\}} V_3(x(\tau_\chi \wedge t))\big] \leq\mathbb{E}\big[I_{\{\tau_\varrho<(\tau_h\wedge\theta)\}} V_3(x(\tau_\varrho))\big]. \tag{2.29}$$

For any $t \geq \theta$, using the fact $\{\tau_\chi \leq t\} \subset \{\tau_\varrho < (\tau_h \wedge \theta)\}$, we have

$$\begin{aligned}\mathbb{E}\big[I_{\{\tau_\varrho<(\tau_h\wedge\theta)\}} V_3(x(\tau_\chi \wedge t))\big] \geq& \int_{\{\tau_\chi\leq t\}} V_3(x(\tau_\chi))\mathrm{d}\mathbb{P} \\ \geq& \int_{\{\tau_\chi\leq t\}} \frac{|x(\tau_\chi)|^2}{3}\mathrm{d}\mathbb{P} \\ \geq& \frac{\chi^2}{3}\mathbb{P}\{\tau_\chi \leq t\} \end{aligned} \tag{2.30}$$

and

$$\mathbb{E}\big[I_{\{\tau_\varrho<(\tau_h\wedge\theta)\}} V_3(x(\tau_\varrho))\big] \leq \int_\Omega V_3(x(\tau_\varrho))\mathrm{d}\mathbb{P} \leq \int_\Omega |x(\tau_\varrho)|^2\mathrm{d}\mathbb{P} \leq \varrho^2. \tag{2.31}$$

From (2.24) and (2.29)–(2.31), it follows that

$$\mathbb{P}\{\tau_\chi \leq t\} \leq \frac{3\varrho^2}{\chi^2} \leq \frac{\varepsilon}{4}.$$

Letting $t \to \infty$ we have $\mathbb{P}\{\tau_\chi < \infty\} \leq \frac{\varepsilon}{4}$. Using (2.26), $\{\nu < \infty \text{ and } \tau_\chi = \infty\} = \{\tau_\varrho < \tau_h \wedge \theta \text{ and } \tau_\chi = \infty\}$ and $\mathbb{P}(B) \geq \mathbb{P}(AB) \geq \mathbb{P}(A) - \mathbb{P}(B^C)$ for any $A, B \in \mathcal{F}$, we have

$$\mathbb{P}\big\{\nu < \infty \text{ and } \tau_\chi = \infty\big\} \geq \mathbb{P}\big\{\tau_\varrho < (\tau_h \wedge \theta)\big\} - \mathbb{P}\big\{\tau_\chi < \infty\big\} \geq 1 - \varepsilon.$$

That is $\mathbb{P}\big\{\limsup_{t\to\infty} |x(t)| \leq \chi\big\} \geq 1 - \varepsilon$. Note that χ is arbitrary. Then

$$\mathbb{P}\Big\{\lim_{t\to\infty} x(t) = 0\Big\} \geq 1 - \varepsilon.$$

Furthermore, from the arbitrariness of ε, it follows that $\mathbb{P}\big\{\lim_{t\to\infty} x(t) = 0\big\} = 1$. That is

$$\mathbb{P}\Big\{\lim_{t\to\infty}[S_1(t) - S_2(t)] = 0,\ \ \lim_{t\to\infty}[I_1(t) - I_2(t)] = 0\Big\} = 1.$$

Thus,

$$\mathbb{P}\Big\{\lim_{t\to\infty}[S_1(t) - S_2(t)] = 0\Big\} = 1, \quad \mathbb{P}\Big\{\lim_{t\to\infty}[I_1(t) - I_2(t)] = 0\Big\} = 1.$$

The proof is complete. □

Remark 2.2. Theorem 2.5 shows that $S_1(t) - S_2(t)$ and $I_1(t) - I_2(t)$ converge to 0 in mean square. Moreover, from Theorem 2.5 and Chebyshev's inequality, $\mathbb{P}\{|S_1(t) - S_2(t)| \geq \varepsilon\} \leq \frac{\mathbb{E}[S_1(t)-S_2(t)]^2}{\varepsilon^2}$ and $\mathbb{P}\{|I_1(t) - I_2(t)| \geq \varepsilon\} \leq \frac{\mathbb{E}[I_1(t)-I_2(t)]^2}{\varepsilon^2}$ for any $\varepsilon > 0$. Thus,

$$\lim_{t\to\infty} \mathbb{P}\{|S_1(t) - S_2(t)| \geq \varepsilon\} \leq \frac{\lim_{t\to\infty} \mathbb{E}[S_1(t) - S_2(t)]^2}{\varepsilon^2} = 0,$$
$$\lim_{t\to\infty} \mathbb{P}\{|I_1(t) - I_2(t)| \geq \varepsilon\} \leq \frac{\lim_{t\to\infty} \mathbb{E}[I_1(t) - I_2(t)]^2}{\varepsilon^2} = 0.$$

That is, $S_1(t) - S_2(t)$ and $I_1(t) - I_2(t)$ converge to 0 stochastically.

Remark 2.3. From Theorem 2.5, it follows that under certain conditions the differences between susceptible populations or infected populations in two cities will disappear with probability 1 as time tends to infinity.

3. Extinction

From [6], if $\mathcal{R}_{0\gamma} < 1$, then disease-free equilibrium $E_0 = (S^0, 0, S^0, 0)$ of the deterministic model (1.1) is locally asymptotically stable. In this section, we provide the conditions of the extinction of disease in model (1.2). First of all, we define

$$\mathcal{R}_{s\gamma} = \frac{\beta + \gamma\alpha}{c + d + \frac{\sigma_2^2}{2}}. \tag{3.1}$$

A similar discussion as Lemma 3.3 in [29], we have the following lemma.

Lemma 3.1. *Let $X(t)$ be the solution of model (1.2) with any given initial value $X_0 \in \mathbb{R}^4_+$. If $p \geq 1$ and $\bar{b} > 0$, then*

$$\lim_{t\to\infty} \frac{S_i(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I_i(t)}{t} = 0 \quad \text{a.s.,} \quad (i = 1, 2).$$

Here $\bar{b} = b - \frac{p-1}{2}\sigma^2$ and $\sigma^2 = \sigma_1^2 \vee \sigma_2^2$.

Lemma 3.2. *Let $X(t)$ be the solution of model (1.2) with any given initial value $X_0 \in \mathbb{R}^4_+$. If $p \geq 1$ and $\bar{b} > 0$, then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S_1(s)\mathrm{d}w_1(s) = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t I_1(s)\mathrm{d}w_2(s) = 0, \quad \text{a.s.}$$
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S_2(s)\mathrm{d}w_1(s) = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t I_2(s)\mathrm{d}w_2(s) = 0, \quad \text{a.s.}$$

Here $\bar{b} = b - \frac{p-1}{2}\sigma^2$ and $\sigma^2 = \sigma_1^2 \vee \sigma_2^2$.

Proof. Denote

$$X_1(t) = \int_0^t S_1(s)\mathrm{d}w_1(s), \quad Y_1(t) = \int_0^t I_1(s)\mathrm{d}w_2(s),$$
$$X_2(t) = \int_0^t S_2(s)\mathrm{d}w_1(s), \quad Y_2(t) = \int_0^t I_2(s)\mathrm{d}w_2(s).$$

By the Burkholder-Davis-Gundy inequality (see Theorem 1.7.3 in [20]) and the Hölder inequality, we obtain that for $p > 0$ and $t \geq 0$,

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t}|X_1(s)|^p\right] \leq& C_p\mathbb{E}\left[\int_0^t S^2(s)\mathrm{d}s\right]^{\frac{p}{2}} \leq C_p\mathbb{E}\left[\left(\int_0^t 1\mathrm{d}s\right)^{1-\frac{2}{p}}\left(\int_0^t (S^2)^{\frac{p}{2}}\mathrm{d}s\right)^{\frac{2}{p}}\right]^{\frac{p}{2}} \\ =& C_p t^{\frac{p}{2}-1}\mathbb{E}\left[\int_0^t S^p(s)\mathrm{d}s\right] \leq C_p t^{\frac{p}{2}-1}\left[\int_0^t \mathbb{E}(S^p(s))\mathrm{d}s\right]. \qquad (3.2)\end{aligned}$$

Here C_p (depending only on p) is a positive constant. Making use of (2.13) and (3.2), for $t \geq 0$

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|X_1(s)|^p\right] \leq C_p t^{\frac{p}{2}-1}\left[\int_0^t \mathbb{E}(S^p(s))\mathrm{d}s\right] \leq C_p M t^{\frac{p}{2}}.$$

Thus, for any positive integer n, we have

$$\mathbb{E}\left[\sup_{n\leq t\leq n+1}|X_1(t)|^p\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq n+1}|X_1(t)|^p\right] \leq C_p M(n+1)^{\frac{p}{2}}.$$

Let $\varepsilon > 0$ be arbitrary. Applying Chebyshev's inequality, we obtain

$$\mathbb{P}\left\{\sup_{n\leq t\leq n+1}|X_1(t)|^p > n^{1+\varepsilon+\frac{p}{2}}\right\} \leq \frac{1}{n^{1+\varepsilon+\frac{p}{2}}}\mathbb{E}\left[\sup_{n\leq t\leq n+1}|X_1(t)|^p\right] \leq \frac{C_p M(n+1)^{\frac{p}{2}}}{n^{1+\varepsilon+\frac{p}{2}}}. \qquad (3.3)$$

Since $\sum_{n=0}^{\infty}\frac{C_pM(n+1)^{\frac{p}{2}}}{n^{1+\varepsilon+\frac{p}{2}}} < \infty$ for $\varepsilon > 0$, the well-known Borel-Cantelli lemma (see Lemma 1.2.1 in [20]) tells us that for almost all $\omega \in \Omega$ there exists a positive integer $n_0 = n_0(\omega)$ such that $\sup_{n\leq t\leq n+1}|X_1(t)|^p \leq n^{1+\varepsilon+\frac{p}{2}}$ for any $n \geq n_0$. That is to say,

$$\frac{\ln|X_1(t)|^p}{\ln t} \leq \frac{\left(1+\varepsilon+\frac{p}{2}\right)\ln n}{\ln n} = 1+\varepsilon+\frac{p}{2}.$$

Hence $\limsup_{t\to\infty}\frac{\ln|X_1(t)|}{\ln t} \leq \frac{1+\varepsilon+\frac{p}{2}}{p}$ a.s. Let $\varepsilon \downarrow 0$, we obtain $\limsup_{t\to\infty}\frac{\ln|X_1(t)|}{\ln t} \leq \frac{1}{p}+\frac{1}{2}$ a.s. Namely, for any small $0 < \xi < \frac{1}{2}-\frac{1}{p}$, there exists a constant $T = T(\omega)$ such that $|X_1(t)| \leq t^{\frac{1}{p}+\frac{1}{2}+\xi}$ for $t \geq T$. Thus,

$$\limsup_{t\to\infty}\frac{|X_1(t)|}{t} \leq \limsup_{t\to\infty}\frac{t^{\frac{1}{p}+\frac{1}{2}+\xi}}{t} = 0.$$

This, together with $\liminf_{t\to\infty}\frac{|X_1(t)|}{t} \geq 0$, yields $\lim_{t\to\infty}\frac{|X_1(t)|}{t} = 0$ a.s. Therefore,

$$\lim_{t\to\infty}\frac{X_1(t)}{t} = \lim_{t\to\infty}\frac{1}{t}\int_0^t S_1(s)\mathrm{d}w_1(s) = 0 \quad \text{a.s.}$$

A similar discussion as in the above, we can get the required assertion. □

Theorem 3.1. *For any $X_0 \in \mathbb{R}^4_+$, let $X(t)$ be solution of model (1.2) with initial value X_0. Then the solution $X(t)$ has the following property:*

$$\limsup_{t\to\infty}\frac{\ln(I_1(t)+I_2(t))}{t} \leq \left(c+d+\frac{\sigma_2^2}{2}\right)(\mathcal{R}_{s\gamma}-1) \quad \text{a.s.}$$

Further, if $\mathcal{R}_{s\gamma} < 1$, then $I_1(t)$ and $I_2(t)$ tend to zero almost surely exponentially.

Proof. Define $V(I_1,I_2)=\ln(I_1+I_2)$. Applying Itô formula, we have

$$\begin{aligned}\ln(I_1(t)+I_2(t)) = &\int_0^t \Big[\frac{1}{I_1(s)+I_2(s)}\Big(\frac{(\beta+\gamma\alpha)S_1(s)I_1(s)}{S_1(s)+I_1(s)}+\frac{(\beta+\gamma\alpha)S_2(s)I_2(s)}{S_2(s)+I_2(s)}\Big)\\ &-(c+d)-\frac{\sigma_2^2}{2}\Big]\mathrm{d}s+\int_0^t\sigma_2\mathrm{d}w_2(s)+\ln(I_{10}+I_{20})\\ \le&\Big[(\beta+\gamma\alpha)-(c+d)-\frac{\sigma_2^2}{2}\Big]t+\sigma_2 w_2(t)+\ln(I_{10}+I_{20}).\end{aligned}$$

Thus

$$\frac{\ln(I_1(t)+I_2(t))}{t}\le\Big[(\beta+\gamma\alpha)-(c+d)-\frac{\sigma_2^2}{2}\Big]+\frac{\sigma_2 w_2(t)}{t}+\frac{\ln(I_{10}+I_{20})}{t}. \quad (3.4)$$

It is clear that $w_2(t)$ is a continuous square-integrable martingale. Thus, from the strong law of large numbers (see Theorem 1.4.2 in [20]), $\lim_{t\to\infty}\frac{\sigma_2 w_2(t)}{t}=0$ a.s. Hence

$$\limsup_{t\to\infty}\frac{\ln(I_1(t)+I_2(t))}{t}\le(\beta+\gamma\alpha)-\Big(c+d+\frac{\sigma_2^2}{2}\Big)=\Big(c+d+\frac{\sigma_2^2}{2}\Big)(\mathcal{R}_{s\gamma}-1)\ \text{ a.s.},$$

which, together with the positivity of the solution, yields

$$\limsup_{t\to\infty}\frac{\ln I_i(t)}{t}\le\limsup_{t\to\infty}\frac{\ln(I_1(t)+I_2(t))}{t}<0\ \text{ a.s.},\quad i=1,2.$$

Thus, $\lim_{t\to\infty}I_1(t)=\lim_{t\to\infty}I_2(t)=0$ a.s. The proof is complete. □

Theorem 3.2. *For any $X_0\in\mathbb{R}_+^4$, let $X(t)$ be solution of model (1.2) with initial value X_0. Assume that for some $p>1$, $\bar b=b-\frac{p-1}{2}\sigma^2\ge0$, where $\sigma^2=\sigma_1^2\vee\sigma_2^2$. If $\mathcal{R}_{s\gamma}<1$, then*

$$\lim_{t\to\infty}\langle S_1(t)\rangle=\lim_{t\to\infty}\langle S_2(s)\rangle=\frac{a}{b}=S^0\ \text{ a.s.}$$

Proof. Integrating from 0 to t on both sides of (2.6) yields

$$\begin{aligned}\frac{N(t)-N(0)}{t}=&2a-b\langle S_1(t)+S_2(t)\rangle-c\langle I_1(t)+I_2(t)\rangle\\ &+\frac{\sigma_1}{t}\int_0^t(S_1(s)+S_2(s))\mathrm{d}w_1(s)\\ &+\frac{\sigma_2}{t}\int_0^t(I_1(s)+I_2(s))\mathrm{d}w_2(s).\end{aligned}$$

Clearly,

$$\langle S_1(t)+S_2(t)\rangle=\frac{2a}{b}-\frac{c}{b}\langle I_1(t)+I_2(t)\rangle+\phi(t), \quad (3.5)$$

where $\phi(t)=\frac{1}{b}[-\frac{N(t)-N(0)}{t}+\frac{\sigma_1}{t}\int_0^t(S_1(s)+S_2(s))\mathrm{d}w_1(s)+\frac{\sigma_2}{t}\int_0^t(I_1(s)+I_2(s))\mathrm{d}w_2(s)]$. This, together with Lemmas 3.1 and 3.2, yields $\lim_{t\to\infty}\phi(t)=0$ a.s. From Theorem 3.1, it follows that $\lim_{t\to\infty}(I_1(t)+I_2(t))=0$ a.s. Applying L'Hospital's rule, it follows that $\lim_{t\to\infty}\langle I_1(t)+I_2(t)\rangle=0$ a.s. Thus,

$$\lim_{t\to\infty}\langle S_1(t)+S_2(t)\rangle=\frac{2a}{b}=2S^0\ \text{ a.s.} \quad (3.6)$$

Moreover, it follows from (1.2) that

$$\begin{aligned}\mathrm{d}[S_1+I_1] = &[a - bS_1 - \alpha(S_1 - S_2) - (c+\alpha)I_1 + \alpha I_2]\,\mathrm{d}t + \sigma_1 S_1 \mathrm{d}w_1(t)\\ &+ \sigma_2 I_1 \mathrm{d}w_2(t).\end{aligned} \tag{3.7}$$

Integrating from 0 to t on both sides of (3.7) yields

$$\begin{aligned}\frac{S_1(t)+I_1(t)-(S_{10}+I_{10})}{t} =& a - b\langle S_1(t)\rangle - \alpha\langle S_1(t)-S_2(t)\rangle - (c+\alpha)\langle I_1(t)\rangle\\ &+ \alpha\langle I_2(t)\rangle + \frac{\sigma_1}{t}\int_0^t S_1(s)\mathrm{d}w_1(s) + \frac{\sigma_2}{t}\int_0^t I_1(s)\mathrm{d}w_2(s).\end{aligned}$$

Obviously,

$$\langle S_1(t)\rangle + \frac{\alpha}{b}\langle S_1(t)-S_2(t)\rangle = \frac{a}{b} - \frac{c+\alpha}{b}\langle I_1(t)\rangle + \frac{\alpha}{b}\langle I_2(t)\rangle + \phi_1(t),$$

where $\phi_1(t) = \frac{1}{b}[-\frac{S_1(t)+I_1(t)-(S_{10}+I_{10})}{t} + \frac{\sigma_1}{t}\int_0^t S_1(s)\mathrm{d}w_1(s) + \frac{\sigma_2}{t}\int_0^t I_1(s)\mathrm{d}w_2(s)]$. This, together with Lemmas 3.1 and 3.2, implies $\lim_{t\to\infty}\phi_1(t) = 0$ a.s. From Theorem 3.1 and L'Hospital's rule, $\lim_{t\to\infty}\langle I_1(t)\rangle = \lim_{t\to\infty}\langle I_2(t)\rangle = 0$ a.s. Thus,

$$\lim_{t\to\infty}\left[\langle S_1(t)\rangle + \frac{\alpha}{b}\langle S_1(t)-S_2(t)\rangle\right] = \frac{a}{b} = S^0 \quad \text{a.s.} \tag{3.8}$$

By a similar way, we have

$$\lim_{t\to\infty}\left[\langle S_2(t)\rangle + \frac{\alpha}{b}\langle S_2(t)-S_1(t)\rangle\right] = \frac{a}{b} = S^0 \quad \text{a.s.} \tag{3.9}$$

From (3.6), (3.8) and (3.9), it follows that

$$\lim_{t\to\infty}\langle S_1(t)\rangle = \lim_{t\to\infty}\langle S_2(t)\rangle = \frac{a}{b} = S^0 \quad \text{a.s.}$$

The proof is complete. □

Remark 3.1. Theorem 3.1 indicates that the extinction of disease in (1.2) occurs if $\mathcal{R}_{s\gamma} = \frac{\beta+\gamma\alpha}{c+d+0.5\sigma_2^2} < 1$. Note that $\mathcal{R}_{s\gamma} < \mathcal{R}_{0\gamma}$. Thus, it is possible that $\mathcal{R}_{s\gamma} < 1 < \mathcal{R}_{0\gamma}$. This is the case when the deterministic model (1.1) has endemic while the stochastic model (1.2) shows that the disease will go extinct with probability one.

4. Stationary distribution and ergodicity

One of the important properties in infectious disease dynamics is the persistence which means the disease will never become extinct. The ergodic stationary distribution reflects the weak stability and persistence of the model to some certain extent. If model (1.2) has an ergodic stationary distribution, we can say that the disease can persist in these two cities. Further, it forms an endemic disease. Thus, in this section, we will show that there is an ergodic stationary distribution for model (1.2).

Let $X(t)$ be a homogeneous Markov process in E_d (denotes d-dimensional Euclidean space), described by the following stochastic differential equation

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t),\ X(0) = X_0. \tag{4.1}$$

The diffusion matrix of $X(t)$ is defined as $J(X) = g(X)g^{\mathbb{T}}(X) = (a_{ij}(X))$.

Definition 4.1 (see [4]). Let $\mathbb{P}(t, X, \cdot)$ be the probability measure induced by $X(t)$ with initial value $X(0) = X_0$. That is, $\mathbb{P}(t, X_0, A) = \mathbb{P}(X(t) \in A | X(0) = X_0)$, for any Borel set $A \in \mathcal{B}(\mathbb{R}^d_+)$. If there exists a probability measure $\mu(\cdot)$ such that $\lim_{t\to\infty} \mathbb{P}(t, X_0, A) = \mu(A)$ for all $X_0 \in \mathbb{R}^d_+$ and $A \in \mathcal{B}(\mathbb{R}^d_+)$, then we say that equation (4.1) has a stationary distribution $\mu(\cdot)$.

Lemma 4.1 (see [4]). *Assume that there exists a bounded domain $D \subset E_d$ with regular boundary Γ and*

(A1) *there is a positive number M such that $\sum_{i,j=1}^{d} a_{ij}(X)\xi_i\xi_j \geq M|\xi|^2$, $X \in D$, $\xi \in \mathbb{R}^d$;*

(A2) *there is a nonnegative C^2-function V such that there exists a positive constant C, such that*

$$LV \leq -C \quad \text{for any } X \in E_d \backslash D.$$

Then the Markov process $X(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$. Moreover, if $f(\cdot)$ is a function integrable with respect to the measure μ, then

$$\mathbb{P}\left\{ \lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))\mathrm{d}t = \int_{E_d} f(x)\mu(\mathrm{d}x) \right\} = 1.$$

Consider the following stochastic SIS epidemic model

$$\begin{cases} \mathrm{d}S = \left[a - \frac{(\beta+\gamma\alpha)SI}{S+I} - bS + dI\right]\mathrm{d}t + \sigma_1 S\mathrm{d}w_1(t), \\ \mathrm{d}I = \left[\frac{(\beta+\gamma\alpha)SI}{S+I} - (c+d)I\right]\mathrm{d}t + \sigma_2 I\mathrm{d}w_2(t), \end{cases} \tag{4.2}$$

with $(S(0), I(0)) = (S_0, I_0) \in \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$. Define parameter

$$\mathcal{R}^s_\gamma = \frac{b(\beta+\gamma\alpha)}{\left(b + \frac{\sigma_1^2}{2}\right)\left(c + d + \frac{\sigma_2^2}{2}\right)}. \tag{4.3}$$

Lemma 4.2. *Let $(S(t), I(t))$ be the solution of model (4.2) with any given initial value $(S_0, I_0) \in \mathbb{R}^2_+$. If $\mathcal{R}^s_\gamma > 1$, then model (4.2) has a stationary distribution $\mu(\cdot)$ and the solution $(S(t), I(t))$ has the ergodic property.*

Proof. By Itô's formula and model (4.2), we have

$$\begin{aligned} L(S+I) &= a - bS - cI = a - b(S+I) - (c-b)I, \\ L(-\ln S) &= -\frac{a}{S} + \frac{(\beta+\gamma\alpha)I}{S+I} - \frac{dI}{S} + b + \frac{\sigma_1^2}{2}, \\ L(-\ln I) &= -\frac{(\beta+\gamma\alpha)S}{S+I} + c + d + \frac{\sigma_2^2}{2}. \end{aligned}$$

Define a function $V_1(S, I) = (S+I) - k_1 \ln S - k_2 \ln I$, where k_1 and k_2 are positive constants to be determined later. Using Itô's formula, we have

$$\begin{aligned} LV_1 = &- b(S+I) - \frac{k_1 a}{S} - \frac{k_2(\beta+\gamma\alpha)S}{S+I} + a + k_1\Big(b + \frac{\sigma_1^2}{2}\Big) + k_2\Big(c + d + \frac{\sigma_2^2}{2}\Big) \\ &- (c-b)I - \frac{k_1 dI}{S} + \frac{k_1(\beta+\gamma\alpha)I}{S+I} \end{aligned}$$

$$\leq -3\big[k_1k_2ab(\beta+\gamma\alpha)\big]^{\frac{1}{3}} + a + k_1\Big(b+\frac{\sigma_1^2}{2}\Big) + k_2\Big(c+d+\frac{\sigma_2^2}{2}\Big) + \frac{k_1(\beta+\gamma\alpha)I}{S+I}.$$

Let $k_1(b+\frac{\sigma_1^2}{2}) = k_2(c+d+\frac{\sigma_2^2}{2}) = a$, then $k_1 = \frac{a}{b+\frac{\sigma_1^2}{2}}$, $k_2 = \frac{a}{c+d+\frac{\sigma_2^2}{2}}$. As a consequence

$$\begin{aligned} LV_1 \leq& -3\left[\left(\frac{a^3b(\beta+\gamma\alpha)}{\big(b+\frac{\sigma_1^2}{2}\big)\big(c+d+\frac{\sigma_2^2}{2}\big)}\right)^{\frac{1}{3}} - a\right] + \frac{k_1(\beta+\gamma\alpha)I}{S+I} \\ =& -3a\Big[(\mathcal{R}_\gamma^s)^{\frac{1}{3}} - 1\Big] + \frac{k_1(\beta+\gamma\alpha)I}{S+I}, \end{aligned}$$

where $\mathcal{R}_\gamma^s$ is defined in (4.3). Further, define

$$\begin{aligned} V_2(S,I) =& MV_1(S,I) - \ln S + (S+I) \\ =& (M+1)(S+I) - (k_1M+1)\ln S - k_2M\ln I, \end{aligned}$$

where the positive constant M satisfies the following condition

$$-M\lambda + a + b + \beta + \gamma\alpha + \frac{\sigma_1^2}{2} \leq -2, \tag{4.4}$$

and $\lambda = 3a[(\mathcal{R}_\gamma^s)^{\frac{1}{3}} - 1] > 0$. It is easy to see that

$$\liminf_{k\to\infty,(S,I)\in\mathbb{R}_+^2\setminus U_k} V_2(S,I) = +\infty,$$

where $U_k = (\frac{1}{k},k)\times(\frac{1}{k},k)$. This, together with the continuity of function $V_2(S,I)$, yields $V_2(S,I)$ has a minimum point $(\bar{S}_0,\bar{I}_0)$ in the interior of $\mathbb{R}_+^2$. Then we define a nonnegative C^2-function V_3: $\mathbb{R}_+^2\to\mathbb{R}$ as follows

$$V_3(S,I) = V_2(S,I) - V_2(\bar{S}_0,\bar{I}_0).$$

Using It$\hat{o}$'s formula, we have

$$\begin{aligned} LV_3 =& MLV_1 + L(-\ln S) + L(S+I) \\ \leq& -M\lambda + \frac{Mk_1(\beta+\gamma\alpha)I}{S+I} - \frac{a}{S} + \frac{(\beta+\gamma\alpha)I}{S+I} + b - \frac{dI}{S} + \frac{\sigma_1^2}{2} + a - bS - cI \\ \leq& -M\lambda + \frac{Mk_1(\beta+\gamma\alpha)I}{S+I} - \frac{a}{S} - b(S+I) - (c-b)I + a + b + \beta + \gamma\alpha + \frac{\sigma_1^2}{2} \\ \leq& -M\lambda + \frac{Mk_1(\beta+\gamma\alpha)I}{S+I} - \frac{a}{S} - b(S+I) + a + b + \beta + \gamma\alpha + \frac{\sigma_1^2}{2}. \end{aligned}$$

Now, define the following bounded closed set

$$D = \left\{(S,I)\in\mathbb{R}_+^2:\ S\geq\varrho,\ I\geq\varrho^2,\ S+I\leq\frac{1}{\varrho}\right\},$$

where $\varrho>0$ is a sufficiently small constant. In the set $\mathbb{R}_+^2\setminus D$, we can choose ϱ sufficiently small such that

$$\varrho \leq \min\left\{\frac{a}{Mk_1(\beta+\gamma\alpha)},\ \frac{b}{Mk_1(\beta+\gamma\alpha)},\ \frac{1}{Mk_1(\beta+\gamma\alpha)}\right\}. \tag{4.5}$$

For convenience, we divide $\mathbb{R}^2_+ \setminus D$ into the following three domains

$$D_1 = \left\{(S,I) \in \mathbb{R}^2_+ :\ 0 < S < \varrho\right\}, \quad D_2 = \left\{(S,I) \in \mathbb{R}^2_+ :\ 0 < I < \varrho^2,\ S \geq \varrho\right\},$$
$$D_3 = \left\{(S,I) \in \mathbb{R}^2_+ :\ S + I > \frac{1}{\varrho}\right\}.$$

Clearly, $\mathbb{R}^2_+ \setminus D = D_1 \cup D_2 \cup D_3$. Now, we show that $LV_3(S,I) \leq -1$ on $\mathbb{R}^2_+ \setminus D$, which is equivalent to proving it on the above three domains.

Case 1. If $(S,I) \in D_1$, then from (4.4) and (4.5), it follows that

$$\begin{aligned} LV_3 \leq& - M\lambda + \frac{Mk_1(\beta+\gamma\alpha)I}{S+I} - \frac{a}{S} - b(S+I) + a + b + \beta + \gamma\alpha + \frac{\sigma_1^2}{2} \\ \leq& -2 + Mk_1(\beta+\gamma\alpha) - \frac{a}{\varrho} \\ \leq& -1. \end{aligned}$$

Case 2. If $(S,I) \in D_2$, then from (4.4) and (4.5), it follows that

$$\begin{aligned} LV_3 \leq& - M\lambda + \frac{Mk_1(\beta+\gamma\alpha)I}{S+I} - \frac{a}{S} - b(S+I) + a + b + \beta + \gamma\alpha + \frac{\sigma_1^2}{2} \\ \leq& -2 + Mk_1(\beta+\gamma\alpha)\varrho \\ \leq& -1. \end{aligned}$$

Case 3. If $(S,I) \in D_3$, then from (4.4) and (4.5), it follows that

$$\begin{aligned} LV_3 \leq& - M\lambda + \frac{Mk_1(\beta+\gamma\alpha)I}{S+I} - \frac{a}{S} - b(S+I) + a + b + \beta + \gamma\alpha + \frac{\sigma_1^2}{2} \\ \leq& -2 + Mk_1(\beta+\gamma\alpha) - \frac{b}{\varrho} \\ \leq& -1. \end{aligned}$$

Obviously, one can see that for a sufficiently small $\varrho > 0$

$$LV_3(S,I) \leq -1 \ \text{ for all } (S,I) \in \mathbb{R}^2_+ \setminus D.$$

Hence (A2) in Lemma 4.1 is satisfied.

The diffusion matrix of model (4.2) is given by $A = \operatorname{diag}(\sigma_1^2 S^2, \sigma_2^2 I^2)$. Then for any $(S,I) \in D$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we have

$$\sum_{i,j=1}^{2} a_{ij}(S,I)\xi_i\xi_j = \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 I^2 \xi_2^2 \geq M|\xi|^2,$$

where $M = \varrho^2\sigma_1^2 \wedge \varrho^4\sigma_2^2$. Thus, condition (A1) of Lemma 4.1 holds. According to Lemma 4.1, model (4.2) has a stationary distribution $\mu(\cdot)$ and the solution $(S(t), I(t))$ has the ergodic property. The proof is complete. □

Theorem 4.1. *For any $X_0 \in \mathbb{R}^4_+$, let $X(t)$ be solution of model (1.2) with initial value X_0. Under the conditions of Theorem 2.5, if $\mathcal{R}^s_\gamma > 1$, then model (1.2) has a stationary distribution and the solution $X(t)$ has the ergodic property.*

Proof. From Theorem 2.5, it follows that

$$\mathbb{P}\Big\{\lim_{t\to\infty}[S_1(t)-S_2(t)]=0\Big\}=1,\quad \mathbb{P}\Big\{\lim_{t\to\infty}[I_1(t)-I_2(t)]=0\Big\}=1.$$

This means that for any small $0<\varepsilon<\frac{a}{\alpha+2\gamma\alpha}$, there exists a positive constant T such that

$$-\alpha\varepsilon\le\alpha\big(S_1(t)-S_2(t)\big)\le\alpha\varepsilon,\quad -\alpha\varepsilon\le\alpha\big(I_1(t)-I_2(t)\big)\le\alpha\varepsilon\quad a.s.\quad \text{for } t\ge T.$$

Note that

$$\left|\frac{\gamma\alpha S_1I_1}{S_1+I_1}-\frac{\gamma\alpha S_2I_2}{S_2+I_2}\right|\le\gamma\alpha|S_1-S_2|+\gamma\alpha|I_1-I_2|.$$

Thus,

$$-2\gamma\alpha\varepsilon\le\frac{\gamma\alpha S_1I_1}{S_1+I_1}-\frac{\gamma\alpha S_2I_2}{S_2+I_2}\le2\gamma\alpha\varepsilon\quad a.s.\quad\text{for } t\ge T.$$

Consider the following two systems

$$\begin{cases}\mathrm{d}Y_1=\Big[(a+\alpha\varepsilon+2\gamma\alpha\varepsilon)-\frac{(\beta+\gamma\alpha)Y_1Z_1}{Y_1+Z_1}-bY_1+dZ_1\Big]\mathrm{d}t+\sigma_1Y_1\mathrm{d}w_1(t),\\ \mathrm{d}Z_1=\Big[\frac{(\beta+\gamma\alpha)Y_1Z_1}{Y_1+Z_1}-(c+d)Z_1+(\alpha\varepsilon+2\gamma\alpha\varepsilon)\Big]\mathrm{d}t+\sigma_2Z_1\mathrm{d}w_2(t),\end{cases}\tag{4.6}$$

with $(Y_1(0),Z_1(0))=(S_{10},I_{10})\in\mathbb{R}^2_+$, and

$$\begin{cases}\mathrm{d}Y_2=\Big[(a-\alpha\varepsilon-2\gamma\alpha\varepsilon)-\frac{(\beta+\gamma\alpha)Y_2Z_2}{Y_2+Z_2}-bY_2+dZ_2\Big]\mathrm{d}t+\sigma_1Y_2\mathrm{d}w_1(t),\\ \mathrm{d}Z_2=\Big[\frac{(\beta+\gamma\alpha)Y_2Z_2}{Y_2+Z_2}-(c+d)Z_2-(\alpha\varepsilon+2\gamma\alpha\varepsilon)\Big]\mathrm{d}t+\sigma_2Z_2\mathrm{d}w_2(t),\end{cases}\tag{4.7}$$

with $(Y_2(0),Z_2(0))=(S_{10},I_{10})\in\mathbb{R}^2_+$. Then it follows from the stochastic comparison theorem that

$$Y_2(t)\le S_1(t)\le Y_1(t),\quad Z_2(t)\le I_1(t)\le Z_1(t)\quad a.s.,\quad\text{for } t\ge T.$$

Similarly, we also have

$$Y_2(t)\le S_2(t)\le Y_1(t),\quad Z_2(t)\le I_2(t)\le Z_1(t)\quad a.s.,\quad\text{for } t\ge T.$$

Let $\varepsilon\to0$, then we have $\lim_{t\to\infty}|Y_1(t)-Y_2(t)|=0$ and $\lim_{t\to\infty}|Z_1(t)-Z_2(t)|=0$ a.s. Then we can conclude that $\lim_{t\to\infty}|Y_1(t)-S_1(t)|=0$, $\lim_{t\to\infty}|Z_1(t)-I_1(t)|=0$, $\lim_{t\to\infty}|Y_1(t)-S_2(t)|=0$ and $\lim_{t\to\infty}|Z_1(t)-I_2(t)|=0$ a.s. Moreover, from the arbitrariness of ε, similar to the proof of Lemma 4.2, we know that if $\mathcal{R}^s_\gamma>1$, then model (4.6) has a stationary distribution and the solution $(Y_1(t),Z_1(t))$ has the ergodic property. Therefore, we can conclude that if $\mathcal{R}^s_\gamma>1$, then model (1.2) has a stationary distribution and the solution $X(t)$ has the ergodic property. □

5. Numerical simulations

In this section, we use the Milstein method (see [8]) to substantiate the main results. To illustrate the theoretical results, we take $a = 0.6$, $b = 0.3$, $c = 0.5$, $d = 0.5$, $\alpha = 0.4$, $\beta = 0.8$, and $\gamma = 0.9$ and $(S_{10}, I_{10}, S_{20}, I_{20}) = (2.4, 0.2, 2, 0.5)$. Denote $Q = \frac{4(b+2\alpha-\frac{\sigma_1^2}{2})(c+d+2\alpha-\frac{\sigma_2^2}{2})-d^2}{4(b+c+2d+4\alpha-\frac{\sigma_1^2}{2}-\frac{\sigma_2^2}{2})}$.

Example 5.1. Let $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.1$. It is clear that $0.05 = \frac{\sigma_1^2}{2} < b+2\alpha = 1.1$ and $0.05 = \frac{\sigma_2^2}{2} < c+d+2\alpha = 1.8$ Moreover, we have $0.44 = |\beta-\gamma\alpha| < Q = 0.53788$. Thus, from Theorem 2.5, it follows that

$$\mathbb{P}\Big\{\lim_{t\to\infty}[S_1(t) - S_2(t)] = 0\Big\} = 1, \quad \mathbb{P}\Big\{\lim_{t\to\infty}[I_1(t) - I_2(t)] = 0\Big\} = 1.$$

The numerical simulations in Fig. 1 support these results clearly.

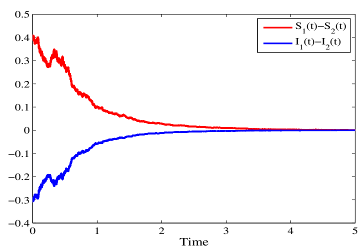
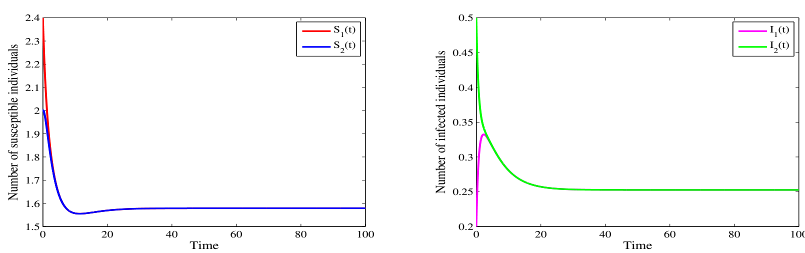


Figure 1. The differences between susceptible populations or infected populations in two cities of model (1.2) with $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.1$.



(a) The susceptible S_1 and S_2. (b) The infected I_1 and I_2.

Figure 2. Numerical simulation of deterministic model (1.1).

Example 5.2. (i) Note that $\mathcal{R}_{0\gamma} = 1.16 > 1$. Thus form [6], model (1.1) admits a unique endemic equilibrium $E_\gamma^* = (1.57895, 0.25263, 1.57895, 0.25263)$ (see Fig. 2).

(ii) For model (1.2), choose $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.4$. Then $\mathcal{R}_{s\gamma} = 0.96 < 1$. If $p = 2$, then $\bar{b} = b - \frac{p-1}{2}\sigma^2 = 0.1 > 0$. Then, from Theorems 3.1 and 3.2, it follows that

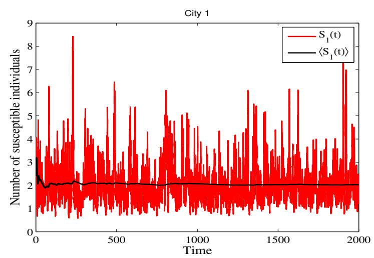
$$\lim_{t\to\infty} I_1(t) = \lim_{t\to\infty} I_2(t) = 0, \quad \lim_{t\to\infty}\langle S_1(t)\rangle = \lim_{t\to\infty}\langle S_2(t)\rangle = \frac{a}{b} = 2 \text{ a.s.}$$

Thus, I_1 and I_2 tend to zero (see Figs. 3(b) and 3(d)) while S_1 and S_2 are persistent in mean (see Figs. 3(a) and 3(c)).

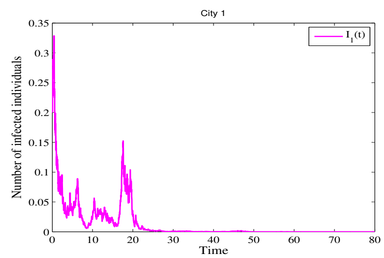
(iii) Set $\sigma_1^2 = 0.02$ and $\sigma_2^2 = 0.03$. By a simple calculation, we have

$$\mathcal{R}_\gamma^s = 1.106 > 1 \quad \text{and} \quad 0.44 = |\beta - \gamma\alpha| < Q = 0.5812.$$

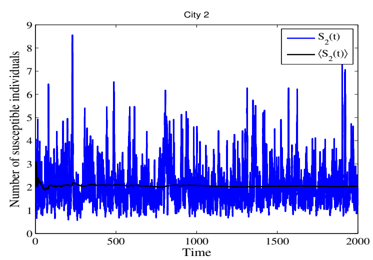
Then, model (1.2) has a stationary distribution. This means that the disease can persist. Fig. 4 gives the solution of (1.2) around the equilibrium E_γ^*.



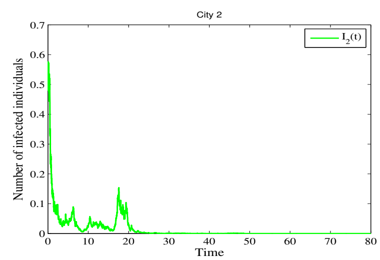
(a) The susceptible individual S_1.



(b) The infected individual I_1.



(c) The susceptible individual S_2.



(d) The infected individual I_2.

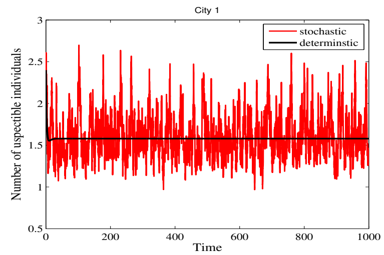
Figure 3. Numerical simulation of model (1.2) with $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.4$.

Example 5.3. To illustrate Lemma 4.2, we take $a = 0.6$, $b = 0.3$, $c = 0.5$, $d = 0.5$, $\alpha = 0.4$, $\beta = 0.8$, $\gamma = 0.9$, $\sigma_1^2 = 0.02$ and $\sigma_2^2 = 0.03$. By a simple calculation, we have $\mathcal{R}_\gamma^s = 1.106 > 1$. Then, from Lemma 4.2, model (4.2) has a stationary distribution and the solution has the ergodic property (see Fig 5).

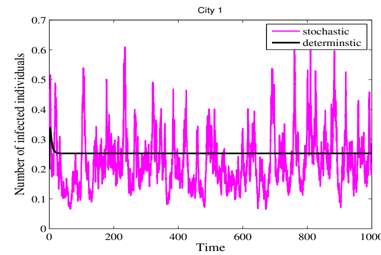
Example 5.4. To illustrate Theorem 4.1, we take $a = 0.6$, $b = 0.3$, $c = 0.5$, $d = 0.5$, $\alpha = 0.4$, $\beta = 0.8$, $\gamma = 0.9$, $\sigma_1^2 = 0.02$ and $\sigma_2^2 = 0.03$. Thus, we have $\mathcal{R}_\gamma^s = 1.106 > 1$ and $0.44 = |\beta - \gamma\alpha| < Q = 0.5812$. Then, from Theorem 4.1, model (1.2) has a stationary distribution and the solution has the ergodic property (see Fig 6).

6. Conclusions and discussions

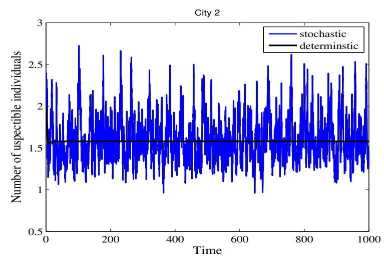
In this work we consider a stochastic SIS epidemic model with transport-related infection. First, we investigate the existence and uniqueness of a global positive solution of the model. Next, we show that under certain conditions the differences between susceptible populations or infected populations in two cities will disappear



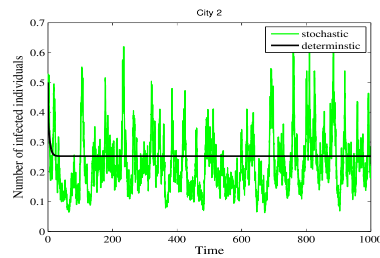
(a) The susceptible individual S_1.



(b) The infected individual I_1.

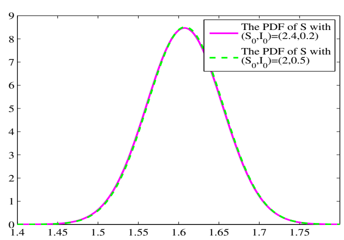


(c) The susceptible individual S_2.

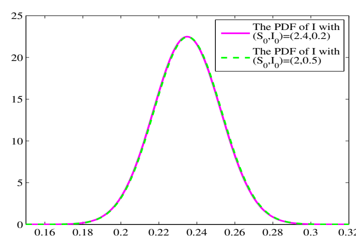


(d) The infected individual I_2.

Figure 4. Numerical simulation of model (1.2) with $\sigma_1^2 = 0.02$ and $\sigma_2^2 = 0.03$.



(a)

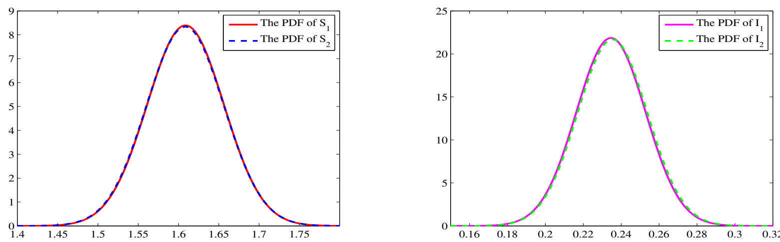


(b)

Figure 5. The density functions of $S(t)$ and $I(t)$ in (4.2) at $t = 30000$ with $(S_0, I_0) = (2.4, 0.2)$ and $(S_0, I_0) = (2, 0.5)$. (a) the density of $S(t)$; (b) the density of $I(t)$.

with probability 1 as time tends to infinity. Then, we show that $I_1(t)$ and $I_2(t)$ in model (1.2) almost surely tend to zero exponentially if $\mathcal{R}_{s\gamma} < 1$. Moreover, we prove that when the difference between the two cities disappears, if $\mathcal{R}_{\gamma}^{s} > 1$, then the model has a stationary distribution and the solution has the ergodic property. Numerical simulations are presented to confirm the theoretical results.

Some interesting problems deserve further consideration. From [16], we know that populations may suffer from sudden environmental fluctuations, such as floods and earthquakes, which cannot be described by Brownian motions. Thus, one can introduce the jumps into (1.2). Moreover, from [2], we know that populations may be perturbed by telegraph noise which is distinguished by factors such as rain falls and nutrition and can be represented by switching among two or more regimes of environment. Thus, one may incorporate the Markovian switching into (1.2). We think that these are challenging problems and leave them to future consideration.



(a) (b)

Figure 6. The density functions of $S_1(t)$, $S_2(t)$, $I_1(t)$ and $I_2(t)$ in model (1.2) at $t = 30000$ with $(S_{10}, I_{10}, S_{20}, I_{20}) = (2.4, 0.2, 2, 0.5)$. (a) the density functions of $S_1(t)$ and $S_2(t)$; (b) the density functions of $I_1(t)$ and $I_2(t)$.

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