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THE EXISTENCE AND STABILITY OF ORDER-1 PERIODIC SOLUTIONS FOR AN IMPULSIVE KOLMOGOROV PREDATOR-PREY MODEL WITH NON-SELECTIVE HARVESTING*

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Abstract In this paper, we focus on a general state-dependent Kolmogorov predator-prey model subject to non-selective harvesting along with delivery. Certain criteria are established for the existence, non-existence and multiplicity of order-1 impulsive periodic solutions to the system. Based on the geometric phase analysis and the method of Poincaré map or successor function with Bendixson domain theory, three typical types of Bendixson domains (i.e., Parallel Domain, Sub-parallel Domain and Semi-ring Domain) are introduced to deal with the discontinuity of the Poincaré map or successor function. We incorporate two discriminants Δ_1 and Δ_2 to link with the existence, non-existence and multiplicity as well as the stability of order-1 periodic solutions. At the same time, we locate the order-1 periodic solutions with the help of three characteristic points and the parameters ratio of delivery over harvesting. The results show that there must exist at least one order-1 periodic solution when the trajectory, that is tangent to the mapping line, can hit the impulsive line. While the trajectory tangent to the mapping line cannot hit the impulsive line, there is not necessary the existence of an order-1 periodic solution, which means the impulsive control may be invalid after finite times stimulation or suppression. In conclusion, we reveal that the delivery can prevent the predator from extinction and stabilize the order-1 periodic solution.

Keywords State dependent impulse, periodic solution, phase analysis, Poincaré map, successor function, Bendixson domain.

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1. Introduction

It is well known that competition between intra-species or inter-species is ubiquitous in nature, causing various phenomena such as co-existence or extinction of certain species. Alternatively, many species such as fish, forest species and wildlife survive in an environment with enemies and the dynamics of their behaviors can be described by the so-called predator-prey system. Microscopically, certain kinds of virus and healthy cells also act as predator and prey species in human body. Mathematical models in this topic have attracted considerable attention from researchers (Butler etc [1]; Smith and Schwartz [11]).

Let $x(t), y(t)$ denote the densities for the prey and predator species at time t.

Consider the following Kolmogorov-type predator-prey system:

$$\begin{cases} \dfrac{dx}{dt} = xF(x,y), \\ \dfrac{dy}{dt} = yG(x,y), \end{cases} \tag{1.1}$$

where $F, G \in C^2(\mathbb{R}^2, \mathbb{R})$, with the assumptions:

Assumption 1.1. (A_1) *There exist two positive numbers* $m < K$ *such that* $F(K,0) = 0$ *and* $G(m,0) = 0$.

Assumption 1.2. (A_2) $F_x < 0$, $F_y < 0$, $G_x > 0$ *and* $G_y < 0$ *for* $x \in [0, K]$.

It is not difficult to find that system (1.1) has a trivial equilibrium $(0,0)$, a boundary equilibrium $(K,0)$ and a unique positive stable equilibrium (x^*, y^*) which will be shown in the next section. Biologically speaking, these three equilibria are corresponding to three rest states: the extinction of two species, the extinction of predator species and the co-existence of two species.

In many cases, one expects to attain more ideal states than the natural ones by implementing a biological control, which depends on the property of the species. To keep a more ideal balance of the system, biological control can be carried out by suppressing certain species, or stimulating others. However, due to the structure of food chain, it is often unrealistic to control a single species only, because the suppression of one species synchronously causes the reduction of other species population, and the protection of one species may correspondingly have negative effects on another one. For example, in the integrated management of pest, spraying pesticides to insects will lead to the damage of plants or the wholesome natural enemy when the pests are killed. Similarly, it is possible to kill healthy cells in human body when one is treated with Antiretroviral Therapy. Such an effect is so-called non-selective harvesting, which originates from fishery industry (Kar etc [7] and Chakraborty etc [4]).

From the perspective of better biological control, the intervention on species, regarded as an impulsive action, can be introduced when the density of a species population reaches a threshold rather than taking measures at fixed moments, which leads us to model the control with a state dependent impulsive system. Assume that the strategy of impulsive control is to suppress the prey in a smaller scale, and to keep the predator in a larger scale with a delivery of predator when a non-selective harvesting occur. This results in the following Komogolov predator-prey model subject to impulsive control:

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= xF(x,y) \\ \frac{dy}{dt} &= yG(x,y) \end{aligned}\right\}, \ x \neq h, \\ \left.\begin{aligned} \Delta x(t) &= -px(t) \\ \Delta y(t) &= -qy(t) + \tau \end{aligned}\right\}, \ x = h, \\ 0 < x(0^+) = x_0^+ < h, \quad y(0^+) = y_0^+ > 0, \end{cases} \tag{1.2}$$

where h and p are positive constants, and q, τ satisfy $q \in [0,1)$ and $\tau \geq 0$. When the density of prey increases to h, one harvests or suppresses the prey at a rate p. Non-selective harvesting will also lead to the reduction of the predator at a rate q. To maintain the population of the predator at a relatively larger scale, people deliver or stimulate the predator at an average amount τ.

Naturally, the property of a semi-continuous impulsive system is different from and hence more complicated than that of the continuous one. The complexity lies in the fact that we have to consider the impulsive strategy, the initial values and the construction of the trajectories. While the state-dependent impulsive differential equations have attracted much attention in recent years (Simeonov and Bainov [10]; Bainov and Simeonov [2]; Chavez etc [3]; Tang etc [13–15]; Zeng etc [21]; Jiang etc [6]; Nie etc [8,9]; Xiao etc [19,20]; Hakl etc [5]; Tang and Fu [12]; Zhang etc [22]; Wang etc [16]; Wang and Xiao [18]; Wang etc [17]). The focus of this paper is on the existence and stability of positive periodic solutions of the realistic model (1.2).

We will propose criteria on the existence, non-existence, multiplicity and the stability of impulsive order-1 periodic solution(s). Notice that there is not only negative effect but also positive one in the impulse function of y. The model is different from most of past references.

The paper is organized as follows. We begin with the phase-plane analysis for system (1.1) in Section 2. Three categories of Bendixson domains are introduced. In section 3, we obtain the main results under cases when the successor function of the tangent point on the mapping line is well defined or not well defined. Furthermore, the stability of the order-1 periodic solution is studied in section 4. Finally, the work ends by conclusion and discussion.

2. Preliminaries

First, we start from system (1.1). For system (1.1), denote any solution $\Big(x(t), y(t)\Big)$ in the phase plane by (x, y).

Under the assumptions (A_1)–(A_2), the solution of system (1.1) is positive for positive initial values and the solution is continuously-dependent on them. There are two implicit functions $y = \varphi_1(x)$ and $y = \varphi_2(x)$ such that $\varphi_1(K) = 0$ and $\varphi_2(m) = 0$, which are defined by the equations $F(x,y) = 0$ and $G(x,y) = 0$, respectively. Denote the isolines $F(x,y) = 0$ and $G(x,y) = 0$ by c_1 and c_2

Let $H(x) = \varphi_1(x) - \varphi_2(x)$. From the assumptions (A_1)–(A_2) and the differentiation for implicity functions, we have

$$\frac{d\varphi_1}{dx} = -\frac{F_x}{F_y} < 0, \quad \frac{d\varphi_2}{dx} = -\frac{G_x}{G_y} > 0, \quad x \in [0, K],$$

which implies the function $\varphi_1(x)$ is decreasing and $\varphi_2(x)$ is increasing, and hence $H(x)$ is continuous and decreasing on $[0, K]$.

Lemma 2.1. *For system* (1.1)*, the equilibria* $(0, 0)$ *and* $(K, 0)$ *are unstable saddles, and the unique positive equilibrium* (x^*, y^*) *is a globally asymptotically stable node or focus, where* (x^*, y^*) *is the positive solution of the equations* $F(x, y) = 0, G(x, y) = 0$*. The region*

$$\Omega = \Big\{(x, y)|x > 0, y > 0, x \leq K, y \leq \sigma\Big\}$$

is positively invariant and there is no positive limit cycle in Ω*, where* $\sigma \triangleq \max\{\varphi_1(0), \varphi_2(K)\} + \varepsilon$ *and* ε *is a positive small number.*

Proof. Firstly, we verify that there is a unique positive equilibrium (x^*, y^*). Since $H(x)$ is a continuous and monotonically decreasing function such that

$$H(0) = \varphi_1(0) - \varphi_2(0) > \varphi_1(K) - \varphi_2(m) = 0$$

and

$$H(K) = \varphi_1(K) - \varphi_2(K) = 0 - \varphi_2(K) < -\varphi_2(m) = 0,$$

by the Intermediate Value Theorem for continuous functions, it follows that there is a unique positive solution $x = x^*$ for equation $H(x) = 0$, which corresponds to the unique positive solution (x^*, y^*) for equations $F(x, y) = 0$ and $G(x, y) = 0$. Hence there is a unique positive equilibrium (x^*, y^*) for system (1.1).

Next, calculating the Jacobian matrix along system (1.1) gives

$$J = \begin{bmatrix} F(x, y) + x\frac{\partial F}{\partial x} & x\frac{\partial F}{\partial y} \\ y\frac{\partial G}{\partial x} & G(x, y) + y\frac{\partial G}{\partial y} \end{bmatrix}.$$

At the equilibrium $E_0(0, 0)$, the Jacobian matrix is given by

$$J_0 = \begin{bmatrix} F(0, 0) & 0 \\ 0 & G(0, 0) \end{bmatrix},$$

which admits two eigenvalues $\lambda_1 = F(0, 0) > 0$ and $\lambda_2 = G(0, 0) < 0$. Hence, E_0 is an unstable saddle.

The Jacobian matrix at the equilibrium $E_1(K, 0)$ is

$$J_1 = \begin{bmatrix} F(K, 0) + K\frac{\partial F}{\partial x}|_{(K,0)} & K\frac{\partial F}{\partial y}|_{(K,0)} \\ 0 & G(K, 0) \end{bmatrix}$$

with two eigenvalues $\lambda_1 = F(K, 0) + K\frac{\partial F}{\partial x}|_{(K,0)} < 0$ and $\lambda_2 = G(K, 0) > 0$, which means the equilibrium E_1 is an unstable saddle.

The Jacobian matrix at $E_2(x^*, y^*)$ is

$$J_2 = \begin{bmatrix} x\frac{\partial F}{\partial x} & x\frac{\partial F}{\partial y} \\ y\frac{\partial G}{\partial x} & y\frac{\partial G}{\partial y} \end{bmatrix}_{(x^*, y^*)}.$$

Since the trace of J_2 is negative, and the determinate is positive, it yields that E_2 is an asymptotically stable node or focus.

In the following, we prove that the domain Ω is positively invariant.

Denote $l_1 : x = K$ and $l_2 : y = \sigma$. Calculating the time derivative of l_1 and l_2 along the trajectories of system (1.1) gives

$$\frac{dl_1}{dt} = \frac{dx}{dt}\Big|_{x=K} = KF(K, y) < 0, \quad y \geq 0$$

and

$$\frac{dl_2}{dt} = \frac{dy}{dt}|_{y=\sigma} = \sigma G(x, \sigma) < \sigma G(x, \varphi_2(K)) < \sigma G(K, \varphi_2(K)) = 0, \quad x \in [0, K],$$

which means that the flow of system (1.1) moves from the right to the left on l_1 and from the upper to the bottom on l_2.

Consequently, the region Ω is positively invariant.

Finally, taking a Dulac function $D = \frac{1}{xy}$, it follows from assumption (A_2) that

$$\frac{\partial(xF(x, y)D)}{\partial x} + \frac{\partial(yG(x, y)D)}{\partial y} = \frac{1}{y}\frac{\partial F}{\partial x} + \frac{1}{x}\frac{\partial G}{\partial y} < 0, \quad (x, y) \in \Omega.$$

Therefore there is no limit cycle or positive periodic solution in Ω, and the unique positive equilibrium is also globally asymptotically stable. □

The first quadrant can be divided into four parts by the curves c_1 and c_2 which intersect at point $E_2(x^*, y^*)$, and four sections of c_1 and c_2 are obtained accordingly.

Define four domains as follows

$$\begin{aligned}
D_1 &= \{(x, y)|F(x, y) > 0, G(x, y) < 0, (x, y) \in \Omega\},\\
D_2 &= \{(x, y)|F(x, y) > 0, G(x, y) > 0, (x, y) \in \Omega\},\\
D_3 &= \{(x, y)|F(x, y) < 0, G(x, y) > 0, (x, y) \in \Omega\},\\
D_4 &= \{(x, y)|F(x, y) < 0, G(x, y) < 0, (x, y) \in \Omega\}.
\end{aligned}$$

Then the signs of the derivative $(\dot{x}, \dot{y})$ are $(+, -), (+, +)$, and $(-, +)$ and $(-, -)$ in the four domains, respectively.

Lemma 2.2. *Under the assumptions* (A_1)–(A_2), *the direction of the flow of system* (1.1) *is counter clockwise.*

Proof. Denote the downward normal vectors of c_1 and c_2 by $\vec{n}_1$ and $\vec{n}_2$, respectively, and the normal vector of trajectories of system (1.1) by $\vec{n}_t$. Then $\vec{n}_1 = (F_x, F_y), \vec{n}_2 = (G_x, G_y)$ and $\vec{n}_t = (yG(x, y), -xF(x, y))$, which implies $\vec{n}_t|_{c_1} = (yG(x, y), 0)$ and $\vec{n}_t|_{c_2} = (0, -xF(x, y))$.

Multiplying $\vec{n}_t|_{c_1}$ by $\vec{n}_1 = (F_x, F_y)$ gives

$$\vec{n}_1 \cdot \vec{n}_t|_{c_1} = F_x[yG(x, y)]|_{c_1} \begin{cases} > 0, \, x < x^*, \\ < 0, \, x > x^*. \end{cases}$$

Similarly, we have

$$\vec{n}_2 \cdot \vec{n}_t|_{c_2} = -G_y[xF(x, y)]|_{c_2} \begin{cases} > 0, \, x < x^*, \\ < 0, \, x > x^*. \end{cases}$$

Thus, the manifolds are downward and rightward on the left sections of c_1 and c_2 respectively, and are upward and leftward on the right parts of c_1 and c_2 respectively, that is, the vector fields is counter clockwise. □

The direction of the manifolds is illustrated in Fig.1. Similar results can be obtained on the line $x = x^*$.

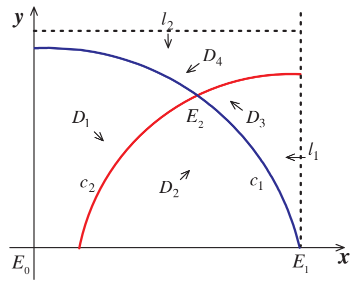


Figure 1. The positive invariant set Ω and four parts of it, and the direction of the manifolds, where $c_1 : F(x, y) = 0$ or $y = \varphi_1(x)$, $c_2 : G(x, y) = 0$ or $y = \varphi_2(x)$ and $l_1 : x = K$ and $l_2 : y = \sigma$.

Let the impulsive function be I and two subsects M and N of $\mathbb{R}^2$ be

$$M = \Big\{(x, y)|y > 0, x = h\Big\}, \quad N = \Big\{(x, y)|y > 0, x = \underline{h}\Big\}.$$

Then $I(M) \subset N$.

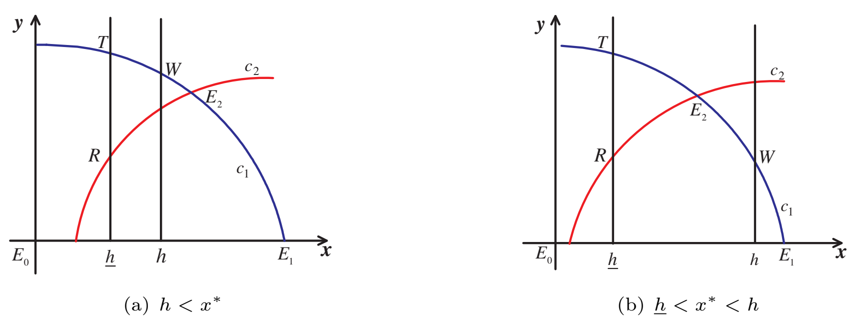
Denote

$$T(\underline{h}, y_T) = N \cap c_1, \quad W(h, y_W) = M \cap c_1, \quad R(\underline{h}, y_R) = N \cap c_2,$$

where

$$\underline{h} = (1 - p)h, \quad y_T = \varphi_1(\underline{h}), \quad y_W = \varphi_1(h), \quad y_R = \varphi_2(\underline{h}).$$

Obviously, there are two trajectories of system (1.1) tangent vertically to the line $x = \underline{h}$ at T and tangent to the line $x = h$ at W, respectively. Also, R is the horizontal tangent point of system (1.1). The location of the three points are illustrated in Fig.2.



(a) $h < x^*$ (b) $\underline{h} < x^* < h$

Figure 2. The location of the three characteristic points T, W, R.

Throughout this paper, we assume that

Assumption 2.1. (A_3) $m < \underline{h} < x^*, h < K$.

Lemma 2.3. *For system* (1.2)*, under the assumptions* (A_1)–(A_3)*, we have*

$$y_T > y_W,\ y_T > y_R. \tag{2.1}$$

Proof. Under the assumptions (A_1)–(A_3), the monotonicity of $\varphi_1(x)$ yields

$$y_T = \varphi_1(\underline{h}) > \varphi_1(h) = y_W.$$

The monotonicity of $H(x)$ and $\underline{h} < x^*$ give

$$H(\underline{h}) = \varphi_1(\underline{h}) - \varphi_2(\underline{h}) > H(x^*) = 0,$$

that is $y_T - y_R > 0$. Hence, $y_T > y_W$ and $y_T > y_R$. □

We define the positive orbit (or solution) starting from $Q\big(x(t), y(t)\big) \in \mathbb{R}_+^2$ by $O^+(Q)$ and the negative orbit arriving at it by $O^-(Q)$.

For any point $Q(\underline{h}, y_Q) \in N$, if the trajectory $O^+(Q)$ hits M at $\bar{Q}(h, y_{\bar{Q}})$ firstly and the impulsive map I maps $\bar{Q}$ to $Q^+(\underline{h}, y_{Q^+})$, we denote the Poincaré maps P, P_N and successor function S as the following

$$\begin{aligned} &P(y_Q) = y_{\bar{Q}}, \quad P_N(y_Q) = I(y_{\bar{Q}}) = (I \circ P)(y_Q) = y_{Q^+}, \\ &S(y_Q) = P_N(y_Q) - y_Q = y_{Q^+} - y_Q. \end{aligned} \tag{2.2}$$

For the sake of convenience, we denote $S(y_Q)$ and $S(Q)$ indiscriminately, so does for the other functions I, P and P_N. Further, we denote $A > B$ if the point A lies above the point B and $A - B = y_A - y_B$.

Thus for a point $Q \in N$, $P(Q)$, $P_N(Q)$ and $S(Q)$ are continuous on Q (provided that they are well defined) due to the continuity of I and the continuous dependence on the initial values of the solutions to system (1.1). If $O^+(Q) \cap M \neq \emptyset$, then we call that $P_N(Q)$ and $S(Q)$ are well defined, or $P_N(y_Q)$ and $S(y_Q)$ are well defined.

Lemma 2.4. *For system* (1.2)*, the Poincaré map* $P_N(y)$ *and successor function* $S(y)$ *are continuous for* $y \in [a, b]$ *provided that they are well defined on* $[a, b]$*.*

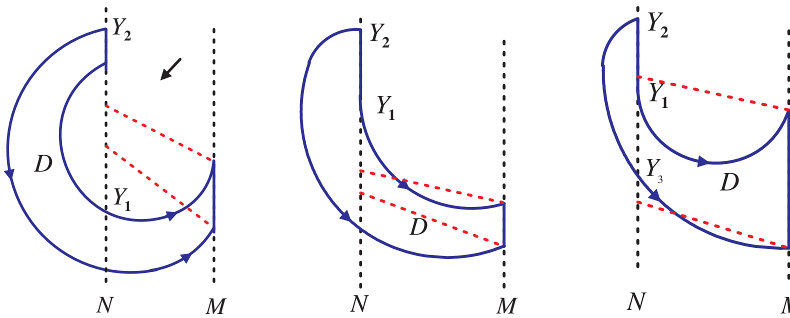
Definition 2.1. A trajectory $O^+(Q)$ along with the line segment Q^+Q is said to be an order-1 periodic circle of system (1.2) if $Q^+ = Q$.

From equation (2.2), if any one of the equations $I(\bar{Q}) = Q$, $P_N(Q) = Q$ and $S(Q) = 0$ holds for a certain point $Q \in N$, then there exists an order-1 periodic solution for equation (1.2).

Remark 2.1. There may be no zero point for the successor function S when point $Y \in (Y_1, Y_2) \subset N$ even if $S(Y_1)S(Y_2) < 0$. And there may exist a periodic solution in a Bendexion domain even if $S(Y_1)S(Y_2) > 0$. Moreover, there may exist multiple order-1 periodic solutions in a domain. Whether the Poincaré map or successor function is well defined depends on the initial values, impulsive strategy and the construction of trajectories (see Fig.3).

To ensure the successor function to be well defined, we consider three categories of Bendexion domains.

Definition 2.2. For system (1.2), suppose that a Bendexion domain D is composed of M, N, L_1 and L_2, and such that



(a) $S(Y_1) \cdot S(Y_2) < 0$ (b) $S(Y_1) \cdot S(Y_2) > 0$ (c) $S(Y_1) \cdot S(Y_2) < 0$ and $S(Y_1) \cdot S(Y_3) < 0$

Figure 3. (a) There is no order-1 periodic solution in D even if $S(Y_1) \cdot S(Y_2) < 0$ holds (b) There is an order-1 periodic solution in D in spite of $S(Y_1) \cdot S(Y_2) > 0$ (c) There may exist two order-1 periodic solutions in D in view of $S(Y_1) \cdot S(Y_2) < 0$ and $S(Y_1) \cdot S(Y_3) < 0$.

(i) there is no singularity (equilibrium) in it;

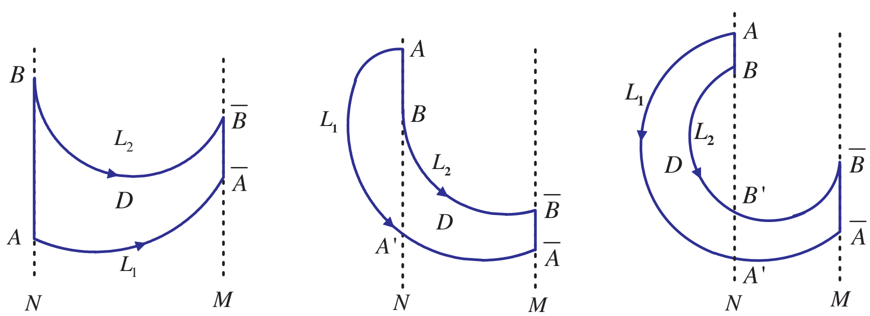
(ii) trajectory L_1 intersects with N, M at A and $\bar{A}$ successively; trajectory L_2 intersects with N, M at B and $\bar{B}$ successively;

(iii) line segments AB and $\bar{A}\bar{B}$ can not be tangent to trajectories of system (1.2) except at the end point.

If $A < B$ gives $\bar{A} < \bar{B}$, then we call the region D a ***Parallel Trajectory Rectangle*** (see Figure 4(a));

If L_2 is tangent to N at B, and $A > B$ gives $\bar{A} < \bar{B}$, then we call the region D a ***Sub-parallel Trajectory Rectangle*** (see Figure 4(b));

If L_1 intersects with N at A and A' successively, and intersects consecutively with M at $\bar{A}$; L_2 intersects with N at B and B' successively, and is consecutively tangent to M at $\bar{B}$; then we call the region D a ***Semi-ring Domain*** provided $A > B$ gives $\bar{A} < \bar{B}$. (see Figure 4(c)).



(a) Parallel domain (b) Sub-parallel domain (c) Semi-ring domain

Figure 4. The illustration for three categories of Bendexion domain. N is a tangent line of the orbit L_2 at B in (b), and M is a tangent line to the orbit L_2 at $\bar{B}$ in (c).

For any $Q \in N$ and $Q > T$, if the trajectory $O^+(Q)$ intersects with N at Q' and hits M at $\bar{Q}$ subsequently, we define $\pi : N \to N, \pi(Q) = Q'$ (for example, in Fig.4

(c),$O^+(A)$ intersects subsequently with N at A', so we define $\pi(A) = A'$). Then the function π is a homeomorphic mapping which is continuous and decreasing when $O^+(Q)$ is in a sub-parallel or semi-ring domain.

According to Lemma 2.4, the Intermediate Value Theorem or Compressing Map Principle for continuous functions leads to the following lemmas.

Lemma 2.5. *Suppose that a parallel domain D is composed of the trajectories $A\bar{A}$ and $B\bar{B}$, and the line segments AB and $\bar{A}\bar{B}$. If $S(A)\cdot S(B) < 0$ or $P_N(AB) \subseteq AB$, then there exists at least one order-1 periodic solution in D.*

Lemma 2.6. *Suppose that a sub-parallel domain D is composed of the trajectories $A\bar{A}$,$B\bar{B}$ and the line segments AB and $\bar{A}\bar{B}$, and $\pi(A) = A'$. If $S(A)S(B) < 0$ or $P_N(AA') \subset AA'$, then there exists at least one order-1 periodic solution in D.*

Proof. Since D is sub-parallel, $P_N(Y)$ is well defined for any $Y \in AB \subset AA'$. Therefore $S(A)\cdot S(B) < 0$ implies the existence of a zero point between A and B.

If $P_N(AA') \subset AA'$, we consider three cases for the function P_N:

Case 1. $P_N(AB) \subseteq AB$; Case 2. $P_N(AB) \subseteq A'B$; Case 3. $P_N(A) \in A'B$ and $P_N(B) \in AB$.

Case 1 implies that P_N is a compressing map. Case 2 means that $(P_N\circ\pi)(A'B) \subseteq A'B$ and hence $(P_N \circ \pi)$ is a compressing map. Thus, there must exist a fixed point in AB and $A'B$, which admits an order-1 periodic solution initiating from AB and $A'B$, respectively.

As for the Case 3, since the function S is well defined and continuous on AB, it follows from $P_N(A) \in A'B$ and $P_N(B) \in AB$ that $S(A) < 0$ and $S(B) > 0$, and hence there must exist a $Y^* \in AB$ such that $S(Y^*) = 0$, which implies the existence of an order-1 periodic solution initiating from AB . □

Remark 2.2. If $P_N(A) < A'$ and $P_N(B) > B$, we have $S(A')S(B) < 0$, which means that there are two order-1 periodic solutions initiating respectively from AB and $A'B$.

Lemma 2.7. *Suppose that a semi-ring domain D is composed of the trajectories $A\bar{A}$, $B\bar{B}$, and the line segments AB and $\bar{A}\bar{B}$ with $\pi(A) = A', \pi(B) = B'$.*

Assume that one of the following condition holds true:

(i) $P_N(AB) \subseteq AB$;

(ii) $P_N(A'B') \subseteq A'B'$;

(iii) $P_N(A'B') \subseteq AB$

(iv) $P_N(AB) \subseteq A'B'$.

Then there exists an order-1 periodic solution which is initiating from AB or $A'B'$. If $P_N(AB) \subseteq BB'$ or $P_N(A'B') \subseteq BB'$, then there is no order-1 periodic solution in D.

Proof. If $P_N(AB) \subseteq AB$, then the continuous map P_N is a compressive map. Thus there exists a fixed point $* \in AB$ such that $P_N(*) = *$, which implies the existence of order-1 periodic solution initiating from AB. Similarly, $P_N(A'B') \subseteq A'B'$ admits an order-1 periodic solution initiates from $A'B'$. Further, $P_N(A'B') \subseteq AB$ implies $(P_N \circ \pi)(AB) \subseteq AB$, and which admits a fixed point between A and B for the compressing map $P_N \circ \pi$. Therefore, there is a periodic solution initiating from AB. Similarly, $P_N(AB) \subseteq A'B'$ implies $(P_M \circ \pi^{-1})(A'B') \subseteq A'B'$, and

$P_N \circ (\pi^{-1})$ is a compressing map, which implies the existence of an order-1 periodic solution initiating from $A'B'$.

If $P_N(AB) \subseteq BB'$ or $P_N(A'B') \subseteq BB'$, then all the trajectories initiating from AB or $A'B'$ will be mapped onto BB', from which the trajectories will not hit M. $\square$

To obtain the existence and non-existence of multiple order-1 periodic solutions, we state the following lemma without proof.

Lemma 2.8. *Suppose that a function g is continuous and concave downward on $[a, b]$. Then the following statements hold true:*

(i) if $g(a) < 0$ and $g(b) < 0$, then there maybe exist a zero point in (a, b) for g;

(ii) if $g(a) > 0$ and $g(b) > 0$, then it is impossible for g to exist a zero point in (a, b).

Suppose that g is continuous and concave upward on $[a, b]$. Then we have the statements as follows:

(i) if $g(a) > 0$ and $g(b) > 0$, then there maybe exist a zero point in (a, b) for g;

(ii) if $g(a) < 0$ and $g(b) < 0$, then it is impossible for g to exist a zero point in (a, b).

3. Main results

3.1. Some Lemmas

Firstly, we give some lemmas which will be applied to obtain the existence results.

Lemma 3.1. *For system (1.2), if $O^+(T) \cap M \neq \emptyset$, then $S(Y)$ is well defined for any $Y \in N$.*

Proof. Since T is a tangent point to the line $x = \underline{h}$ or N, based on the phase analysis for system (1.1), it is obvious that $O^+(T) \cap M \neq \emptyset$ means $O^+(Y) \cap M \neq \emptyset$ whenever $Y < T$. Since the vector fields for system (1.1) is counter clockwise and the equilibrium E_0 is an unstable saddle, all the trajectories initiating above T will be mapped onto the section below T by function π. Thus the fact that $S(T)$ is well defined implies $S(Y)$ is well defined for any $Y \in N$. $\square$

Denote the trajectory $O^+(Y)$ by function $y = y(x, Y)$. We have the following lemma.

Lemma 3.2. *Under the assumptions (A_1)–(A_3), suppose that $S(T)$ is well defined. Then for any $x \in (\underline{h}, h)$, we have*
(i) $y(x, T) < \varphi_1(x)$ and $y_{\bar{T}} < y_W < y_T$;
(ii) $y(x, R) < \varphi_2(x)$ and $y_R < y_{\bar{R}} < y_{\bar{T}} < y_T$.

Proof. Under the assumptions, the geometrical analysis is utilize

(i) It follows from the vector fields analysis in Lemma 2.2 that the trajectories pass through the curve $y = \varphi_1(x)$ downward when $x < x^*$. Therefore, it follows that $y(x, T) < \varphi_1(x)$ for $x < x^*$. We are left to show that $y(x, T) < \varphi_1(x)$ when $x^* < x < h$. Assume to the contrary that there exists a point $x_1 \in (x^*, h)$ such that $y(x_1, T) = \varphi_1(x_1)$. Then the trajectory $O^+(T)$ is tangent to the line $x = x_1$ and goes further leftward, which means that the trajectory $O^+(T)$ cannot hit the line $x = h$. It will contradict the condition that $F(T)$ is well defined. Therefore $y(x,T) < \varphi_1(x)$ when $x^* < x < h$. By the monotonicity of $y = \varphi_1(x)$ and $y(x,T) < \varphi_1(x)$ for $x \in (\underline{h}, h)$, we have that $y_{\bar{T}} = y(h,T) < \varphi_1(h) = y_W < \varphi_1(\underline{h}) = y_T$.

(ii) Since $S(T)$ is well defined, we have $O^+(R) \cap M \neq \emptyset$. According to Lemma 2.1, it gives $\varphi_1(x) > \varphi_2(x)$ for $x < x^*$ and $\varphi_1(x) < \varphi_2(x)$ for $x > x^*$. The vector field analysis shows that the trajectories pass through the curve $y = \varphi_2(x)$ downward when $x < x*$, and hence $y(x,R) < \varphi_2(x) < \varphi_1(x)$ holds for $x \in (\underline{h}, x*]$. It follows from Lemma 2.3 and (2.1) that $y(x,R) < y(x,T)$ in view of the uniqueness of the solutions. Thus, $y(x,R) < \varphi_1(x) < \varphi_2(x)$ when $x \in (x^*, h)$, which means that $y(x,R)$ locates in D_2, where the trajectories are increasing in x. Accordingly, $y_R = y(\underline{h}, R) < y(h,R) = y_{\bar{R}}$. Furthermore, $y_R < y_T$ implies $y_{\bar{R}} < y_{\bar{T}}$. Combining (i), we have $y_R < y_{\bar{R}} < y_{\bar{T}} < y_T$.

□

In the following, for point $Y \in N$, the letter $'Y'$ also represents the ordinate of point Y.

Denote

$$f(x,y) = \frac{xF(x,y)}{yG(x,y)}.$$

Lemma 3.3. *Suppose that $P_N(T)$ is well defined. Then $P_N(Y)$ is increasing in Y for $Y < y_T$ and decreasing for $Y > y_T$, and $S(Y)$ is decreasing for $Y > y_T$.*

Proof. Based on the differentiability of the solutions on the initial values, we have

$$\frac{\partial y(x,Y)}{\partial Y} = \exp\Big(\int_{\underline{h}}^{x} \frac{\partial f(z,y(z,Y))}{\partial y} dz\Big), \quad Y < y_T,$$

which means

$$P'(Y) = \frac{\partial y(x,Y)}{\partial Y}\ |_{x=h} = \exp\Big(\int_{\underline{h}}^{h} \frac{\partial f(z,y(z,Y))}{\partial y} dz\Big), \quad Y < y_T.$$

It follows from $P_N(Y) = I(P(Y))$ that

$$P_N'(Y) = \frac{dI}{dP}\frac{dP}{dY} = (1-q)P'(Y) = (1-q)\exp\Big(\int_{\underline{h}}^{h} \frac{\partial f(z,y(z,Y))}{\partial y} dz\Big), \quad Y < y_T. \tag{3.1}$$

Obviously, $P_N'(Y) > 0$ holds for $Y < y_T$.

When $Y > y_T$, the trajectory $O^+(Y)$ will intersect with N at a point Y' such that $\pi(Y) = Y'$ and $Y' < y_T$, while $\pi(Y)$ is decreasing with Y. Therefore

$$P_N'(Y) = (1-q)\frac{dP}{dY'}\pi'(Y).$$

Since $\frac{dP}{dY'} > 0$ and $\pi'(Y) < 0$, then $P_N'(Y) < 0$, which also implies $S'(Y) = P_N'(Y) - 1 < 0$ for $Y > y_T$. □

Define

$$\Delta_1(Y) = \begin{vmatrix} F & G \\ F_y & G_y \end{vmatrix}_{(x,y(x,Y))}, \quad \Delta_2(Y) = \begin{vmatrix} F & G \\ F_{yy} & G_{yy} \end{vmatrix}_{(x,y(x,Y))},$$

where $\underline{h} \le x \le h, 0 < Y < y_T$.

Lemma 3.4. *Assume that $S(T)$ is well defined and $Y < y_T$. If $\Delta_2(Y) < 0$, then $\Delta_1(Y)$ is decreasing in Y. If $\Delta_2(Y) > 0$, then $\Delta_1(Y)$ is increasing in Y.*

Proof. From the definition of $\Delta_1(Y)$ and $\Delta_2(Y)$, it follows that

$$\frac{\partial \Delta_1(Y)}{\partial Y} = \frac{\partial}{\partial y}(FG_y - GF_y) \cdot \frac{\partial y(x,Y)}{\partial Y} = \Delta_2(Y) \cdot \frac{\partial y(x,Y)}{\partial Y}.$$

Since $\frac{\partial y(x,Y)}{\partial Y} > 0$, the sign of $\frac{\partial \Delta_1(Y)}{\partial Y}$ is the same as that of $\Delta_2(Y)$. Hence, if $\Delta_2(Y) < 0$, then $\Delta_1(Y)$ is decreasing on Y, and $\Delta_2(Y) > 0$ means that $\Delta_1(Y)$ is increasing in Y. □

Lemma 3.5. *Suppose $S(T)$ is well defined and $Y < y_T$. We have:*

(i) for $Y \in [y_R, y_T]$, if $\Delta_2(Y) \le 0$ and $\Delta_1(y_R) < 0$ hold, then $S''(Y) < 0$. Particularly, if $y(x,Y) \subset D_1$, then we have $S'(Y) < 0$;

(ii) if $\Delta_2(Y) \ge 0$ for $Y \in (0, y_R]$ and there exists a $Y^* \in (0, y_R]$ such that $\Delta_1(Y^*) > 0$, then $S''(Y) > 0$ for $Y \in (Y^*, y_R]$. Particularly, if $q = 0$, then we have $S'(Y) > 0$ for $Y \in (Y^*, y_R]$.

Proof. From (3.1), we have

$$P_N''(Y) = (1-q)P'(Y) \int_{\underline{h}}^{h} \frac{\partial^2 f(z, y(z,Y))}{\partial y^2} \cdot \frac{\partial y(z,Y)}{\partial Y} dz. \tag{3.2}$$

Computing the derivative $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial^2 y}$, we have

$$\frac{\partial f}{\partial y} = \frac{xF(G + yG_y) - xyGF_y}{(xF)^2} = \frac{FG + y\Delta_1(Y)}{xF^2} \tag{3.3}$$

and

$$\frac{\partial^2 f}{\partial^2 y} = \frac{2(F - yF_y)\Delta_1(Y) + y\Delta_2(Y)}{xF^3}. \tag{3.4}$$

(i) By Lemma 3.4, the conditions $\Delta_1(y_R) < 0$ and $\Delta_2(Y) \le 0$ for $y_R \le Y \le y_T$ imply $\Delta_1(Y) < 0$ for $y_R \le Y \le y_T$. It follows from (3.4) that $\frac{\partial^2 f}{\partial^2 y} < 0$, which means $P_N''(Y) < 0$. Henceforth, $S''(Y) = P_N''(Y) < 0$ for $y_R \le Y \le y_T$. Particularly, in D_1, we have $F > 0, G < 0$, $G_y \le 0$ and $F_y < 0$. If $y(x,Y) \subset D_1$, then $\Delta_1(Y) < 0$, and hence $\frac{\partial f}{\partial y} < 0$. (3.1) and (3.3) give $P_N'(Y) < 1$. Henceforth, $S'(Y) = P_N'(Y) - 1 < 0$.

(ii) Since $y(x,R) \subset D_2$, we have $y(x,Y) \subset D_2$ for $0 < Y \le y_R$, and hence $F > 0$ and $G > 0$ hold true. If $\Delta_2(Y) > 0$ and $\Delta_1(Y^*) > 0$, then $\Delta_1(Y) > 0$ for $Y \in (Y^*, y_R]$. Therefore $\frac{\partial^2 f}{\partial^2 y} > 0$, which means $P_N''(Y) > 0$, that is $S''(Y) > 0$. Particularly, if $q = 0$, then $P_N'(Y) = P'(Y) > 1$, which means $S'(Y) = P_N'(Y) - 1 > 0$ as $Y^* < Y \le y_R$.

□

Remark 3.1. The conditions $\Delta_1(Y) < 0$ and $\Delta_2(Y) \le 0$ are sufficient for $P_N''(Y) < 0$, which is expressed by the integral of (3.2). It is possible for $P_N''(Y) < 0$ even if $\Delta_1(Y) < 0$ and $\Delta_2(Y) \le 0$ do not hold.

Remark 3.2. Noticing that $F > 0, G > 0$ and $F_y < 0, G_y < 0$ hold in D_2, it is possible for $\Delta_1(Y) \le 0$ or $\Delta_1(Y) \ge 0$ when $y(x, Y) \subset D_2$.

Lemma 3.6. *For $\bar{Y} \in M$, if $\bar{Y} = \frac{\tau}{q}$, then $I(\bar{Y}) = \bar{Y}$; if $\bar{Y} > \frac{\tau}{q}$, then $\frac{\tau}{q} < I(\bar{Y}) < \bar{Y}$; if $\bar{Y} < \frac{\tau}{q}$, then $\bar{Y} < I(\bar{Y}) < \frac{\tau}{q}$.*

Proof. Obviously, $I(\frac{\tau}{q}) = \frac{\tau}{q}$. If $\bar{Y} > \frac{\tau}{q}$, on the one hand, $I(\bar{Y}) > I(\frac{\tau}{q}) = \frac{\tau}{q}$ as I is increasing. On the other hand, $I(\bar{Y}) = (1-q)\bar{Y} + \tau = \bar{Y} + \tau - q\bar{Y} < \bar{Y}$. Similarly, $\bar{Y} < \frac{\tau}{q}$ implies $\bar{Y} < I(\bar{Y}) < \frac{\tau}{q}$. □

We illustrate the above lemma geometrically by Fig.5.

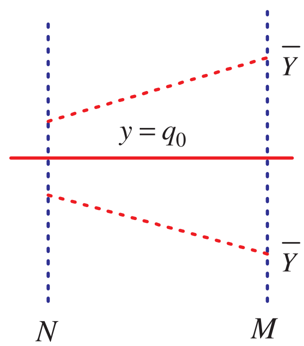


Figure 5. If $\bar{}(Y) = q_0$, then $I(\bar{Y}) = \bar{Y}$; If $\bar{Y} > q_0$, then $q_0 < I(\bar{Y}) < \bar{Y}$; If $\bar{Y} < q_0$, then $\bar{Y} < I(\bar{Y}) < q_0$, where $q_0 = \frac{\tau}{q}$.

Lemma 3.7. *Suppose that $S(T)$ is well defined. Then we have*

(i) if $S(y_T) > 0$, then $\frac{\tau}{q} > y_T$;

(ii) for any $Y \in (0, y_R]$, if $S(Y) < 0$, then $\frac{\tau}{q} < Y$.

Proof. We associate the location of y_T and y_R with the number $\frac{\tau}{q}$, provided that $S(T)$ is well defined.

(i) Assume $\frac{\tau}{q} \le y_T$. Then either $y_{\bar{T}} < \frac{\tau}{q} \le y_T$ or $\frac{\tau}{q} \le y_{\bar{T}} < y_T$ holds. From Lemma 3.6 and Lemma 3.2, it follows that $I(y_{\bar{T}}) < \frac{\tau}{q} \le y_T$ or $< I(y_{\bar{T}}) \le y_{\bar{T}} < y_T$, respectively, which implies $S(y_T) < 0$ and leads to a contradiction;

(ii) Since $Y \in (0, y_R]$, $y(x, Y)$ is increasing with x, and hence $\bar{Y} > Y$. To the contrary if $\frac{\tau}{q} \ge Y$, then either $\frac{\tau}{q} \ge \bar{Y}$ or $Y \le \frac{\tau}{q} < \bar{Y}$ holds. It follows from Lemma 3.6 that $I(\bar{Y}) \ge \bar{Y} > Y$ or $I(\bar{Y}) > \frac{\tau}{q} > Y$, respectively, which implies that $S(Y) > 0$ and this leads to a contradiction to $S(Y) < 0$.

□

Lemma 3.8. *Suppose that $S(T)$ is well defined. The following statements hold true.*

(i) if $S(y_T) > 0$, then $S(y_R) > 0$;

(ii) if $\Delta_1(y_R) < 0$ and $\Delta_2(Y) \le 0$ with $Y \in [y_R, y_T]$, then $S(y_T) > 0$ implies $S(Y) > 0$ for any $Y \in (0, y_T]$.

Proof. Since $S(T)$ is well defined, $S(Y)$ is well defined for $Y \in (0, y_T]$.

(i) From Lemma 3.2, it follows that $y_{\bar{T}} - y_{\bar{R}} < y_T - y_R$. Then we have

$$S(y_T) - S(y_R) = I(y_{\bar{T}}) - y_T - \big(I(y_{\bar{R}}) - y_R\big) = (1-q)(y_{\bar{T}} - y_{\bar{R}}) - (y_T - y_R) < 0.$$

Hence, $S(y_T) > 0$ implies $S(y_R) > 0$.

(ii) Since $\Delta_1(y_R) < 0$ and $\Delta_2(Y) \le 0$ for $y_R \le Y \le y_T$, by Lemma 3.5, the function S is concave on $[y_R, y_T]$. From (i) and Lemma 2.8, $S(y_T) > 0$ implies $S(Y) > 0$ for any $Y \in [y_R, y_T]$.

Furthermore, by Lemma 3.7, $S(y_T) > 0$ implies $\frac{\tau}{q} > y_T > y_R$. If $Y \in (0, Y_R)$, then $Y < \frac{\tau}{q}$. From the item *(ii)* of Lemma 3.7 ,we have $S(Y) > 0$ for any $Y \in (0, Y_R)$.

Therefore, $S(y_T) > 0$ implies $S(Y) > 0$ for any $Y \in (0, y_T]$ under the condition *(ii)*. ☐

Remark 3.3. The condition $S(Y) > 0$ for any $Y \in (0, y_T]$ implies that there is no order-1 periodic solution initiating below T.

Lemma 3.9. *If there exists a small number ϵ^* such that $S(\epsilon^*)$ be well defined, then we have $\lim_{\epsilon\to 0} S(\epsilon) = 0$ for $\tau = 0$, and $\lim_{\epsilon\to 0} S(\epsilon) = \tau$ for $\tau > 0$. If $S(\tau)$ is well defined, then $S(\tau) > 0$.*

Proof. It follows from Lemma 2.2 that $S(\epsilon)$ is well defined when ϵ is a number small enough with $0 < \epsilon < \epsilon^*$. From the definition of the successor function $S(\epsilon) = P_N(\epsilon) - \epsilon = I(P(\epsilon)) - \epsilon$, the former two statements hold true.

We just need to verify that $S(\tau) > 0$ when $S(\tau)$ is well defined. Provided that $S(\tau) \le 0$, we have $P_N(\tau) = (1-q)P(\tau) + \tau \le \tau$, and hence $P(\tau) \le 0$. A contradiction comes up. Thus $S(\tau) > 0$ holds true when $S(\tau)$ is well defined. ☐

3.2. Existence of positive order-1 periodic solution

The existence of an order-1 periodic solution has a lot to do with whether the trajectory $O^+(T)$ hits the impulsive line M, so we consider the existence of periodic solutions under two cases that $S(T)$ is well defined and $S(T)$ is not well defined.

3.2.1. $S(T)$ is well defined

In this subsection, we assume that $S(T)$ is well defined and the assumptions (A_1)–(A_3) hold. From Lemma 3.1, it follows that $S(Y)$ is well defined for any $Y \in N$. In the following, we will not list the case that $S(T) = 0$, which admits an order-1 periodic solution initiating from T, so does for $S(R) = 0$.

Theorem 3.1. *Suppose that $\frac{\tau}{q} \ge y_T$. Then there exists a positive order-1 periodic solution. Specifically, we have the following statements.*

(i) If $S(y_T) > 0$, then there exists an order-1 periodic solution locating in the sub-parallel domain, which initiates above T. Furthermore, the order-1 periodic solution is unique provided that $\Delta_1(y_R) < 0$ and $\Delta_2(Y) \le 0$ for any $Y \in [y_R, y_T]$;

(ii) If $S(y_T) < 0$, then the periodic solution is initiating from the point between R and T, which locates in a parallel domain.

Proof. When $\frac{\tau}{q} \geq y_T$, we consider two cases that $S(y_T) > 0$ and $S(y_T) < 0$.

(i) Obviously, $S(y_T) > 0$ implies $T^+ > T$,and hence $\pi(T^+) < \pi(T) = T$. Thus $P_N(T^+) < P_N(T) = T^+$, that is $S(y_{T^+}) < 0$. Since $S(y_T) \cdot S(y_{T^+}) < 0$ holds in the sub-parallel domain composed of $N, M, O^+(T), O^+(T^+)$, following Lemma 2.6, there is an order-1 periodic solution initiating above T. Further, provided that $\Delta_1(y_R) < 0$ and $\Delta_2(Y) \leq 0$ hold for $Y \in [y_R, y_T]$, it follows from Lemma 3.8 that $S(Y) > 0$ for $Y \in (0, y_T]$, and hence there exists no order-1 periodic solution initiating below T. Moreover, $S(Y)$ is decreasing when $Y > y_T$ in view of Lemma 3.3. Thus, there is a unique order-1 periodic solution when $S(y_T) > 0$.

(ii) If $S(y_T) < 0$, on account of Lemma 3.7, we have $S(y_R) > 0$ because of $\frac{\tau}{q} \geq y_T > y_R$. Henceforth, we have $S(y_T) \cdot S(y_R) < 0$. Based on Lemma 2.5, there exists an order-1 periodic solution in the parallel domain composed of $N, M, O^+(T)$ and $O^+(R)$.

□

Theorem 3.2. *Suppose $y_R < \frac{\tau}{q} < y_T$. Then there is an order-1 periodic solution initiating from the point between R and T.*

Proof. We claim that $S(y_T) < 0$ and $S(y_R) > 0$. Otherwise, provided that $S(y_T) > 0$, then $\frac{\tau}{q} > y_T$ in view of Lemma 3.7, which leads to a contradiction, and hence $S(y_T) < 0$ holds true. Similarly, we have $S(y_R) > 0$. Consequently, there is an order-1 periodic solution initiating between R and T, which is in the parallel domain composed of $N, M, O^+(T)$ and $O^+(R)$. □

Denote $Q_\tau = (\underline{h}, \tau)$ and $Q'_\tau = (\underline{h}, \frac{\tau}{q})$.

Theorem 3.3. *Suppose $0 < \frac{\tau}{q} \leq y_R$. Then there exists a positive order-1 periodic solution initiating below T. Specifically,*

(i) If $S(y_R) > 0$, then the periodic solution initiates between R and T;

(ii) If $S(y_R) < 0$, then the order-1 periodic solution initiates between Q'_τ and R. In addition, if $\Delta_1(y_R) < 0$ and $\Delta_2(Y) \leq 0$ for $Y \in [y_R, y_T]$, then there may exist another order-1 periodic solution initiating between R and T. If $\Delta_1(\tau) > 0$ and $\Delta_2(Y) \geq 0$ for $Y \in [\tau, \frac{\tau}{q}]$, there may exist another order-1 periodic solution initiating between Q_τ and Q'_τ

Proof. Since $\frac{\tau}{q} \leq y_R$, then $\frac{\tau}{q} < y_T$, which means $S(y_T) < 0$ owing to Lemma 3.7. At the same time, $y(x, \frac{\tau}{q})$ is increasing in x on account of $\frac{\tau}{q} \leq y_R$, which means $S(\frac{\tau}{q}) > 0$.

(i) If $S(y_R) > 0$, there exists an order-1 periodic solution initiating from the point between R and T because of $S(y_T) \cdot S(y_R) < 0$;

(ii) If $S(y_R) < 0$, then there is an order-1 periodic solution locating between Q'_τ and R because of $S(y_R) \cdot S(\frac{\tau}{q}) < 0$. In addition, if $\Delta_1(y_R) < 0$ and $\Delta_2(Y) \leq 0$ for $Y \in [y_R, y_T]$, then it follows from Lemma 3.5 that $S(Y)$ is concave on the interval (y_R, y_T). While $S(y_R) < 0$ and $S(y_T) < 0$ hold true, and hence by Lemma 2.8, there may exist a zero point on the interval (y_R, y_T) for function S, which corresponds to an order-1 periodic solution.

Similarly, if $\Delta_1(\tau) > 0$ and $\Delta_2(Y) \geq 0$ for $Y \in [\tau, \frac{\tau}{q}]$, then $S(Y)$ is convex on the interval $(\tau, \frac{\tau}{q})$. Since $S(\tau) > 0$ and $S(\frac{\tau}{q}) > 0$ both hold, there may exist a zero point in $(\tau, \frac{\tau}{q})$, and hence an order-1 periodic solution initiating between Q_τ and Q'_τ. □

Particularly, when $\tau = 0$, we have $S(y_T) < 0$, and $\lim_{\varepsilon \to 0} F(\varepsilon) = 0$ is always satisfied. If $q = 0$, then $S(Y) > 0$ holds for $Y \in (0, y_R)$. Therefore, we have the following corollaries.

Corollary 3.1. *Suppose $\tau = 0$. If $S(y_R) > 0$, then there exists an order-1 periodic solution initiating from the point between R and T. If $S(y_R) < 0$, then there exists a limit circle, i.e a semi-trivial periodic solution. Moreover, if $S(y_R) < 0$, and $\Delta_1(y_R) < 0$ and $\Delta_2(Y) \leq 0$ for $Y \in [y_R, y_T]$, then there may exist another order-1 periodic solution initiating between R and T.*

Corollary 3.2. *Suppose $q = 0$. If $S(y_T) > 0$, then there exists an order-1 periodic solution initiating above T. If $S(y_T) < 0$, then there exists an order-1 periodic solution initiating between R and T.*

3.2.2. $S(T)$ is not well defined

In this subsection, we consider the case that $S(T)$ is not well defined. As $E_2(x^*, y^*)$ is asymptotically stable, any trajectory initiating from N will hit M if $h < x^*$. If $S(T)$ is not well defined, then $h > x^*$.

Suppose that $O^-(W)$ intersects with N at least two points. We denote

$$W_- = \max\{W_k | O^+(W_k) \cap M = W, W_k < T\}$$

and $W^- = \min\{W_k | O^+(W_k) \cap M = W, W_k > T\}$. Obviously, $\pi(W^-) = W_-$, and $y_{W^-} > y_T$ and $y_{W_-} < y_T$.

If $I(W) = W^+ > W^-$, then the domain, composed of segments W^-W^+, $W\overline{W^+}$, and the trajectories W^-W and $W^+\overline{W^+}$, is a semi-ring one. The location of such points is illustrated in Fig.6.

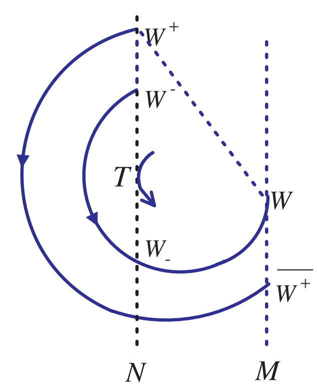


Figure 6. The illustration for the location of W, W^+, W^-, and $\overline{W^+}$ when $O^-(W)$ intersects with N at least two points.

Based on Lemma 2.7, we have the following theorems. We will not list the cases that $I(y_W) = y_{W^-}$ and $I(y_W) = y_{W_-}$ because one can take granted that the order-1 periodic solutions exist.

Theorem 3.4. *Suppose $\tau > y_{W^-}$. Then there is a periodic solution initiating above W^-, which is locating in the semi-ring domain composed of the segments W^-W^+, $W\overline{W^+}$, and the trajectories W^-W and $W^+\overline{W^+}$.*

Proof. It follows from $\tau > y_{W^-}$ that $P_N(\epsilon) > y_{W^-}$ for any $\epsilon \in (0, y_{W_-})$, which means $P_N(W^-W^+) \subset W^-W^+$. From Lemma 2.7, there is a periodic solution in the semi-ring domain. □

Theorem 3.5. *Suppose $y_{W_-} < \tau < y_{W^-}$. If $I(y_W) < y_{W^-}$, then there is no order-1 periodic solution in Ω.*

Proof. Since $y_{W_-} < \tau$, then $P_N(\epsilon) > y_{W_-}$ for any $\epsilon \in (0, y_{W_-})$. It follows from $I(y_W) < y_{W^-}$ that $P_N(\epsilon) < y_{W^-}$. Furthermore, W is a tangent point and $\underline{h} < x^* < h$. All the trajectories initiating from W_-W^- will not hit M any longer. According to Lemma 2.7, there is no periodic solution locating in the semi-ring domain or in Ω. □

Remark 3.4. The fact that there is no impulsive periodic solution implies the system is out of impulsive control after finite-time stimulation or suppression.

Theorem 3.6. *Suppose $0 < \tau \leq y_{W_-}$ and $I(y_W) < y_{W_-}$. Then there is at least one positive order-1 periodic solution initiating between Q_τ and W_-. Specifically, if $y_R \geq y_{W_-}$, then the periodic solution initiates between Q'_τ and W_-; if $y_R < y_{W_-}$, then the periodic solution initiates between R and W_- or between Q'_τ and W_-.*

Proof. If $0 < \tau < y_{W_-}$, then $S(\tau)$ is well defined. It follows from Lemma 3.9 that $S(\tau) > 0$. The condition $I(y_W) < y_{W_-}$ implies $S(y_{W_-}) < 0$, and hence there exists an order-1 periodic solution initiating between Q_τ and W_-.

More specifically, we consider the location of order-1 periodic solution in the following two cases: Case 1: $y_R > y_{W_-}$; Case 2: $y_R < y_{W_-}$.

In the Case 1, $S(y_R)$ is not well defined. Since $y_{W_-} < y_R$ and $S(y_{W_-}) < 0$ hold, by Lemma 3.7, we have $\frac{\tau}{q} < y_{W_-} < y_R$. Henceforth, $P(\frac{\tau}{q}) > \frac{\tau}{q}$ holds. Following Lemma 3.6, we have $S(\frac{\tau}{q}) > 0$. Consequently, $S(y_{W_-}) \cdot S(\frac{\tau}{q}) < 0$ holds, and there exists an order-1 periodic solution initiating between Q'_τ and W_-.

In the Case 2, $S(y_R)$ is well defined. If $y_R < \frac{\tau}{q}$, then $S(y_R) > 0$, and hence there is an order-1 periodic solution initiating between R and W_- . If $y_R > \frac{\tau}{q}$, then $S(\frac{\tau}{q}) > 0$, which means that there is an order-1 periodic solution initiating between Q'_τ and W_-. □

Theorem 3.7. *Suppose $0 < \tau \leq y_{W_-}$ and $I(y_W) > y_{W_-}$. Then there maybe exist a positive order-1 periodic solution initiating between Q_τ and W_-, provided that $\Delta_1(\tau) > 0$ and $\Delta_2(Y) \geq 0$ for $Y \in [\tau, y_{W_-}]$.*

Proof. $I(y_W) > y_{W_-}$ gives $S(y_{W_-}) > 0$. Henceforth $S(\tau) > 0$ and $S(y_{W_-}) > 0$ both hold. Based on Lemma 3.4, we have $\Delta_1(Y) > 0$ for $Y \in [\tau, y_{W_-}]$. Since $\Delta_1(\tau) > 0$ and $\Delta_2(Y) \geq 0$ for $Y \in [\tau, y_{W_-}]$, according to the second part of Lemma 3.5, S is concave downward on interval $[\tau, y_{W_-}]$. From Lemma 2.8, there may exist a zero point in (τ, y_{W_-}), which is corresponding to an order-1 periodic solution. □

Particulary, if $\tau = 0$, then it is impossible to have a semi-ring type periodic solution because of $y_W < y_T < y_{W^-}$. We just state the existence of periodic solution or limit circle for the case $\tau = 0$ and $q = 0$.

Corollary 3.3. *Assume that $\tau = 0$. We have the following statements.*

(i) If $y_R > y_{W_-}$ and $I(y_W) < y_{W_-}$, then there is a limit circle or a semi-trivial periodic solution;

(ii) If $y_R < y_{W_-}$,and $I(y_W) < y_{W_-}$ and $S(y_R) > 0$, then there exists a periodic solution initiating between R and W_- ;

(iii) If $y_R < y_{W_-}$, and $I(y_W) < y_{W_-}$ and $S(R) < 0$, then there is a semi-trivial periodic solution. In addition, there may exist an order-1 periodic solution initiating between W_- and R, provided that $\Delta_1(y_R) < 0$ and $\Delta_2(Y) \leq 0$ for $Y \in [y_R, y_{W_-}]$.

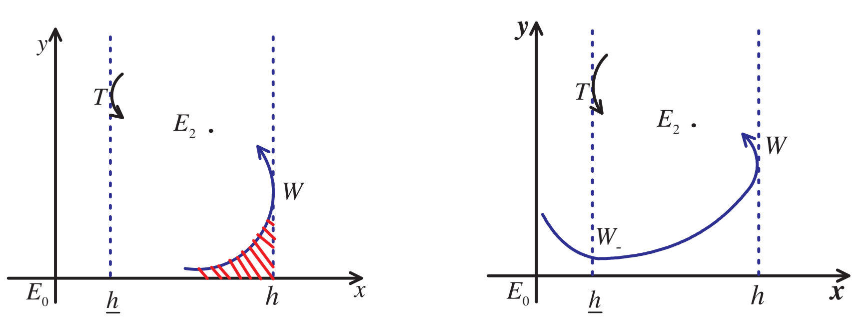
Corollary 3.4. *Assume that $q = 0$. We have:*

(i) if $\tau > y_{W^-}$, then there is a periodic solution locating in the semi-ring domain composed of the segments W^-W^+ and $W\overline{W^+}$ along with the trajectories W^-W and $W^+\overline{W^+}$;

(ii) if $y_R < y_{W_-}$ and $y_W + \tau < y_{W_-}$, then there is an order-1 periodic solution initiating between R and W_-;

(iii) if $y_R > y_{W_-}$ and $0 < \tau < y_{W_-}$, there maybe exist a positive order-1 periodic solution initiating between Q_τ and W_-, provided that $\Delta_1(\tau) > 0$ and $\Delta_2(Y) \geq 0$ for $Y \in [\tau, y_{W_-}]$.

Remark 3.5. Suppose that $O^-(W) \cap N = \emptyset$. Then impulsive impact just has the effect on the trajectories initiating from the shadowed domain in Figure 7(a), which will be out of the impulsive control. Suppose that $O^-(W)$ intersects with N at a unique point W_- (see Figure 7(b)). Then the system will be out of the impulsive control for $\tau > y_{W_-}$, and the results are the same as the Theorem 3.6 and Theorem 3.7 for $0 < \tau < y_{W_-}$.



(a) $O^-(W) \cap N = \emptyset$ (b) $O^-(W) \cap N = W_-$(unique)

Figure 7. Two possible cases of that the trajectory $O^-(W)$ intersects with N.

3.3. Stability of the order-1 periodic solutions

Now, we will consider the stability of the order-1 periodic solution for system (1.2) based on the Analogue of Poincaré Criterion.

Lemma 3.10 (Analogue of Poincaré Criterion, [2, 6, 10]). *The ω-periodic solution $x = \xi(t), y = \eta(t)$ of system*

$$\begin{cases} \dfrac{dx}{dt} = P(x,y), \dfrac{dy}{dt} = Q(x,y), \quad if \quad \phi(x,y) \neq 0, \\ \Delta x = I_1(x,y), \quad \Delta y = I_2(x,y), \quad if \quad \phi(x,y) = 0, \end{cases} \tag{3.5}$$

is orbitally asymptotically stable, where P, Q are continuous differentiable functions and ϕ is a sufficiently smooth function with $\nabla\phi \neq 0$, if the Floquet multiplier μ satisfies $\mid \mu \mid < 1$, where

$$\mu = \prod_{j=1}^{n} \kappa_j \exp\Big\{ \int_0^{\omega} \Big[\frac{\partial P\big(\xi(t),\eta(t)\big)}{\partial x} + \frac{\partial Q\big(\xi(t),\eta(t)\big)}{\partial y}\Big] dt \Big\} \tag{3.6}$$

with

$$\kappa_j = \frac{\big(\frac{\partial I_2}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial I_2}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}\big)P_+ + \big(\frac{\partial I_1}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial I_1}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\big)Q_+}{\frac{\partial \phi}{\partial x}P + \frac{\partial \phi}{\partial y}Q} \tag{3.7}$$

and $P, Q, \frac{\partial I_1}{\partial x}, \frac{\partial I_1}{\partial y}, \frac{\partial I_2}{\partial x}, \frac{\partial I_2}{\partial y}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$ are calculated at the point $\big(\xi(\tau_j), \eta(\tau_j)\big)$,$P_+ = P\big(\xi(\tau_j^+), \eta(\tau_j^+)\big)$, $Q_+ = Q\big(\xi(\tau_j^+), \eta(\tau_j^+)\big)$, and τ_j is the time of the jth jump.

Theorem 3.8. *Let $\big(X(t), Y(t)\big)$ be the order-1 periodic solution of system* (1.2) *with period ω. If*

$$|\mu| = \frac{(1-q)Y(\omega)}{(1-q)Y(\omega)+\tau} \cdot \exp\Big\{ \int_0^{\omega} Y[G_y - \frac{F_y}{F}G]\big|_{(X(t),Y(t))} dt \Big\} < 1, \tag{3.8}$$

then $\big(X(t), Y(t)\big)$ is orbitally asymptotically stable.

Proof. Suppose (X, Y) intersects the sections M and N respectively at points $\big(h, Y(\omega)\big)$ and $\big(\underline{h}, Y(\omega^+)\big)$, where $Y(\omega^+) = (1-q)Y(\omega) + \tau$.

We rewrite the system (1.2) as the form of (3.5). Then we have

$$\begin{aligned} &P(x,y) = xF(x,y), \quad Q(x,y) = yG(x,y), \\ &I_1(x,y) = -px, \quad I_2(x,y) = -qy + \tau, \quad \phi(x,y) = x - h \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial P}{\partial x} = F(x,y) + xF_x, \quad \frac{\partial Q}{\partial y} = G(x,y) + yG_y, \quad \frac{\partial I_1}{\partial x} = -p, \quad \frac{\partial I_2}{\partial y} = -q, \\ &\frac{\partial \phi}{\partial x} = 1, \quad \frac{\partial I_1}{\partial y} = \frac{\partial I_2}{\partial x} = \frac{\partial \phi}{\partial y} = 0. \end{aligned}$$

It follows that

$$\begin{aligned} \kappa_1 &= \frac{\big(\frac{\partial I_2}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial I_2}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}\big)P_+ + \big(\frac{\partial I_1}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial I_1}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial v}\big)Q_+}{\frac{\partial \phi}{\partial x}P + \frac{\partial \phi}{\partial y}Q} \\ &= \frac{(1-q)P^+}{P}. \end{aligned} \tag{3.9}$$

Moreover,

$$
\begin{aligned}
\int_0^\omega \frac{\partial P(X,Y)}{\partial X}dt &= \int_0^\omega [F(X,Y) + XF_x(X,Y)]dt \\
&= \int_{\underline{h}}^{h} \frac{1}{X}dX + \int_0^\omega XF_x(X,Y)dt \\
&= \ln\frac{h}{\underline{h}} + \int_0^\omega XF_x(X,Y)dt
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
\int_0^\omega \frac{\partial Q(X,Y)}{\partial Y}dt &= \int_0^\omega \Big(G(X,Y) + YG_y\Big)dt \\
&= \int_{Y(\omega^+)}^{Y(\omega)} \frac{1}{Y}dY + \int_0^\omega YG_y(X,Y)dt \\
&= \ln\frac{Y(\omega)}{Y(\omega^+)} + \int_0^\omega YG_y(X,Y)dt.
\end{aligned}
\tag{3.11}
$$

Further, the last items of (3.10) becomes

$$
\begin{aligned}
\int_0^\omega XF_x(X,Y)dt &= \int_0^\omega \frac{X[F_x(X,Y) + F_y(X,Y)\frac{dY}{dX}]}{XF(X,Y)}\dot{X}(t)dt \\
&\quad - \int_0^\omega \frac{F_y(X,Y)}{F(X,Y)}\dot{Y}(t)dt \\
&= \int_{\underline{h}}^{h} \frac{dF(X,Y(X))}{F(X,Y(X))} - \int_0^\omega \frac{F_y(X,Y)}{F(X,Y)}YG(X,Y)dt \\
&= \ln\frac{F(h,Y(\omega))}{F(\underline{h},Y(\omega^+)} - \int_0^\omega \frac{F_y(X,Y)}{F(X,Y)}YG(X,Y)dt.
\end{aligned}
\tag{3.12}
$$

Hence, it follows from (3.10), (3.11) and (3.12) that

$$
\begin{aligned}
&\exp\Big\{\int_0^\omega \Big[\frac{\partial P\big(X,Y\big)}{\partial X} + \frac{\partial Q\big(X,Y\big)}{\partial Y}\Big]dt\Big\} \\
&= \exp\Big\{\ln\frac{h}{\underline{h}} + \ln\frac{F(h,Y(\omega))}{F(\underline{h},Y(\omega^+)} + \ln\frac{Y(\omega)}{Y(\omega^+)} \\
&\quad + \int_0^\omega YG_y(X,Y)dt - \int_0^\omega \frac{F_y(X,Y)}{F(X,Y)}YG(X,Y)dt\Big\} \\
&= \frac{h}{\underline{h}}\cdot\frac{F(h,Y(\omega))}{F(\underline{h},Y(\omega^+)}\cdot\frac{Y(\omega)}{Y(\omega^+)} \\
&\quad \times\exp\Big\{\int_0^\omega Y[G_y(X,Y) - \frac{F_y(X,Y)}{F(X,Y)}G(X,Y)]dt\Big\}.
\end{aligned}
\tag{3.13}
$$

Therefore, according to (3.6)–(3.7), (3.8)–(3.9) and (3.13), we have

$$
|\mu| = \frac{(1-q)Y(\omega)}{(1-q)Y(\omega)+\tau}\cdot\exp\Big\{\int_0^\omega Y[G_y - \frac{F_y}{F}G]\big|_{(X(t),Y(t))}dt\Big\} < 1,
$$

which implies the order-1 periodic solution $(X(t),Y(t))$ is orbitally asymptotically stable. □

Theorem 3.9. *If $\Delta_1(Y(\omega^+)) < 0$ and $F(X(t), Y(t)) > 0$ for $t \in [0, \omega]$, then the order-1 periodic solution $\big(X(t), Y(t)\big)$ is orbitally asymptotically stable.*

Proof. Since

$$\Delta_1(Y(\omega^+)) = \begin{vmatrix} F & G \\ F_y & G_y \end{vmatrix}_{(X(t),Y(t))},$$

$\Delta_1(Y(\omega^+)) < 0$ and $F > 0$ imply $[G_y - \frac{GF_y}{F}]_{(X,Y)} < 0$. From (3.8), it follows that $|\mu| < 1$. □

In view of the fact that $F > 0, G_y < 0$ and $G < 0, F_y < 0$ in Ω_1, we have the following corollary.

Corollary 3.5. *If $h < x^*$ and the order-1 periodic solution locates in Ω_1, then it must be orbitally asymptotically stable.*

4. Conclusion and discussion

The dynamic property of a state-dependent impulsive system is rich and it depends on the impulsive strategy, the initial values and the intrinsic dynamics of the system without impulse. For the purpose of biologic control, it is quite challenging to investigate a general model subject to not only positive impulsive effect but also negative one, which exhibits in a Kolmogorov predator-prey model with non-selective harvesting along with delivery. Such an impulsive strategy enhanced the complexity of the analysis.

In this paper, we focused on theoretical analysis on the existence and location of order-1 periodic solutions of the system as well as their stability. To deal with the discontinuity of the Poincaré map and the successor function, we introduced three types of Bendixson domain and established new existence results theoretically. We revealed that the Intermediate Value Theorem may be not applicable when the domain is semi-ring, because the Poincaré map may be not well defined and hence the property of continuity does not hold (see Remark 2.1).

From technique point of view, to locate the order-1 periodic solution, we combined the geometrical phase analysis with the method of successor function or Bendixson Domain Theory. We derived two discriminates Δ_1 and Δ_2 to define the concavity or convexity and the monotonicity of the Poincaré map P_N and the successor function S. With the help of the definitions of three characteristic points T, R and W and the ratio of τ/q, we obtained the detailed existing locations along with the conditions, respectively in the case that $S(T)$ is well defined and the case that $S(T)$ is not well defined, which involves the particular subcases that $\tau = 0$ and/or $q = 0$. Furthermore, we incorporated the Floquet multiplier μ with Δ_1 when the stability of order-1 periodic solution is investigated.

From biological control point of view, we were aiming at the protect of the predator in a relatively larger scale . We summarize our main results in brief as follows:

We can take a large enough delivery τ such that $S(y_T) > 0$ (when $S(T)$ is well defined), or $I(y_W) > y_{W^-}$ (when $S(T)$ is not well defined) to ensure an order-1 periodic solution initiating above T or W^-, which locates in a sub-parallel domain or a semi-ring domain, respectively. However, it may not be economic.

In fact, since the harvesting is non-selective, we need to select the delivery τ responsing to the harvesting q.

If q is small enough such that $(1-q)y_{\bar{R}} \geq y_R$ (or $W_- < (1-q)y_W < W^-$), then there is an order-1 periodic solution initiating between R and T (or the impulsive control will be invalid).

If q is large enough such that $(1-q)y_{\bar{R}} < y_R$ or $(1-q)y_W < y_{W_-}$, we have to take a deliver to protect the predator against extinction. Given a small delivery τ, the predator will be retrieved because of $S(\varepsilon) \to \tau > 0$. With good luck, if the initial densities of the predator and the prey are high enough to locate in Ω_1, the balance is possible to be kept without delivery.

The conditions $\Delta_1(Y) < 0$ and $\Delta_2(Y) \leq 0$ are sufficient for $P''_N(Y) < 0$, since $P''_N(Y)$ is expressed by an integral as (3.2). So does for $P'_N(Y) < 1$. That is, it is possible for $P''_N(Y) < 0$ even if $\Delta_1(Y) < 0$ and $\Delta_2(Y) \leq 0$ does not hold.

Finally, the strategy of impulsive control depends on the trajectories structure and the purpose of control. For example, if $O^-(W) \cap N = \emptyset$, the impulsive control will only be valid for the initial values illustrated in Figure 8(a) and the harvesting can not be sustainable. However, an intrinsic balance can be kept. If $O^-(W)$ intersects N at a unique point, then we need adjust τ relatively small so that the harvesting is sustainable, and choose τ relatively large so that the intrinsic balance can be realized. Similarly, in equation (3.8), provided $exp(*) = 1$. If $\tau = 0$, then $\mu = 1$, which means the periodic solution may be unstable. If $\tau > 0$, then $\mu < 1$, which means the periodic solution is orbitally asymptotically stable. Henceforth, the delivery τ can stabilize the order-1 periodic solution.

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References

[1] G. Butler, S. Hsu and P. Waltman, *Coexistence ofcompeting predators in achemastat*. Journal of Mathematical Biology., 1983, 17(2), 133–151.

[2] D. Bainov and P. Simeonov, *Impulsive differential equations: periodic solutions and applications*, Longman Scientific & Technical, New York, 1993.

[3] J. Chavez, D. Jungmann and S. Siegmund, *A comparative study of integrated pest management strategies based on impulsive control*, Journal of Biological Dynamics, 2018, 12(1), 318–341.

[4] K. Chakraborty, S. Das and T. K. Kar, *On non-selective harvesting of a multi-species fishery incorporating partial closure for the populations*, Applied Mathematics and Computation, 2013, 221(C), 581–597.

[5] R. Hakl, M. Pinto and V. Tkachenko, *Almost periodic evolution systems with impulse action at state-dependent moments*, Journal of Mathematical Analysis and Application, 2017, 446(1), 1030–1045.

[6] G. Jiang, Q. Lu and L. Qian, *Complex dynamics of a Holling type II prey-predator system with state feedback control*, Chaos, Solitions and Fractals, 2007, 31(2), 448–461.

[7] T. Kar and S. Chattopadhyay, *A focus on long-run sustainability of a havested prey predator system in the presence of alternative prey*, Competes Rendus Biologies, 2010, 333(11–12), 841–849.

[8] L. Nie, Z. Teng and B. Guo, *A state dependent pulse control strategy for a SIRS epidemic system*, Bulletin of Mathematical Biology, 2013, 75(10), 1697–1715.

[9] L. Nie, J. Shen and C. Yang, *Dynamic behavior analysis of SIVS epidemic models with state-dependent pulse vaccination*, Nonlinear analysis- Hybrid System, 2018, 27, 258–270.

[10] P. Simeonov and D. Bainov, *Orbital stability of periodic solutions of autonomous systems with impulse effect*, International Journal of Systems Science, 1989, 19(12), 2561–2585.

[11] R. Smith and E. Schwartz, *Predicting the potential impact of a cytotoxic T-lymphocyte HIV vaccine: How often should you vaccinate and how strong should the vaccine be?*, Mathematical Biosciences, 2008, 212(2), 180–187.

[12] X. Tang and X. Fu, *On period-k solution for a population system with state-dependent impulsive effect*, Journal of Applied Analysis and Computation, 2017, 7(2), 439–454.

[13] S. Tang and R. Cheke, *State-dependent impulsive models of integrated pest management (IPM) strategies and their dynamic consequences*, Mathematical Biology, 2015, 50(3), 257–292.

[14] S. Tang, W. Pang and R. Cheke, *Global dynamics of a state-dependent feedback control system*, Advances in Difference Equations, 2015. DOI: 10.1186/s13662–015–0661–x.

[15] S. Tang, X. Tan and J. Yang, *Periodic solution bifurcation and spiking dynamics of impacting predator-prey dynamical model*, International Journal of Bifurcation and Chaos, 2018, 28(12), 1850147.

[16] J. Wang, H. Chen, Y. Li and X. Zhang, *The geometrical analysis of a predator-prey model with multi-state dependent impulses*, Journal of Applied Analysis and Computation, 2018, 8(2), 427–442.

[17] H. Wang, B. Dai and Q. Xiao, *Existence of order-1 periodic solutions for a viral infection model with state-dependent impulsive control*, Advances in Difference Equations, 2019. DOI: 10.1186/s13662–019–1967–x.

[18] A. Wang, Y. Xiao and R. Smith, *Using non-smooth models to determine thresholds for microbial pest management*, Journal of Mathematical Biology, 2019, 78(5), 1389–1424.

[19] Q. Xiao, B. Dai, B. Xu and L. Bao, *Homoclinic bifurcation for a general state-dependent Kolmogorov type predator-prey model with harvesting*, Nonlinear Analysis: Real World Applications, 2015, 26, 263–273.

[20] Q. Xiao and B. Dai, *Dynamics of an impulsive predator-prey logistic population model with state-dependent*, Applied Mathematics and Computation, 2015, 259, 220–230.

[21] G. Zeng, L. Chen and L. Sun, *Existence of periodic solution of order one of planar impulsive autonomous system*, Journal of Computational and Applied Mathematics, 2006, 186(2), 466–481.

[22] Q. Zhang, B. Tang and S. Tang, *Vaccination threshold size and backward bifurcation of SIR model with state-dependent pulse control*, Journal of Theoretical Biology, 2018, 455, 75–85.