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UNCONDITIONALLY OPTIMAL CONVERGENCE ANALYSIS OF SECOND-ORDER BDF SCHEME FOR LANDAU-LIFSHITZ EQUATION*

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Abstract The Landau-Lifshitz equation is used to describe the evolution of spin fields in continuum ferromagnets and is a highly nonlinear parabolic problem with the constraint of unit length in the point-wise sense. This paper focuses on the unconditionally optimal error estimates of a linearized second-order BDF scheme for the numerical approximations of the solution to the Landau-Lifshitz equation. Since the point-wise constraint can be deduced from the partial differential equation, we do not take into account it in designing the numerical scheme. A rigorous error analysis is done and we derive the unconditionally optimal $\mathbf{L}^2$ error estimate by using the error splitting technique. Numerical result is shown to check the theoretical analysis.

Keywords Landau-Lifshitz equation, BDF2 scheme, finite element method, optimal error estimates.

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1. Introduction

The Landau-Lifshitz equation is used to describe the evolution of magnetization in continuum ferromagnets and plays a fundamental role in the understanding of non-equilibrium magnetism [19]. The unknown magnetization field $\mathbf{m}$ satisfies the following nonlinear partial differential equation with an exchange fields:

$$\mathbf{m}_t = \gamma \mathbf{m} \times \Delta \mathbf{m} - \lambda \gamma \mathbf{m} \times (\mathbf{m} \times \Delta \mathbf{m}), \quad \text{in } (0, T] \times \Omega \tag{1.1}$$

for some $T > 0$, where $\lambda > 0$ represents the damping constant and $\gamma > 0$ denotes the exchange constant. The domain $\Omega \subset \mathbf{R}^d, d = 2, 3$ is assumed to be a bounded and convex domain with a smooth boundary $\partial\Omega$.

For the well-posedness of the solution to (1.1), the suitable initial and boundary conditions are needed. In this paper, the initial and boundary conditions are taken as

$$\begin{cases} \nabla \mathbf{m} \cdot \mathbf{n} = 0, & \text{on } (0, T] \times \partial\Omega, \\ \mathbf{m}(0) = \mathbf{m}_0, & \text{in } \Omega, \end{cases} \tag{1.2}$$

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where $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$.

It is clear that if we multiply (1.1) by $\mathbf{m}$, then $\frac{d}{dt}|\mathbf{m}(t)|^2 \equiv 0$, which results in $|\mathbf{m}(t)| \equiv |\mathbf{m}_0|$ during the evolution process. Generally, the initial value $\mathbf{m}_0$ is required to satisfy $|\mathbf{m}_0| = 1$. Then we can see that the solution $\mathbf{m}$ to (1.1) always satisfies $|\mathbf{m}(t)| = 1$ in the point-wise sense for any $t > 0$.

Except for (1.1), there have other equivalent forms of the Landau-Lifshitz equation. For example, from $|\mathbf{m}| = 1$ and the following vector formula:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \qquad \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3,$$

an equivalent form of (1.1) is

$$\mathbf{m}_t - \lambda\gamma\Delta\mathbf{m} - \mathbf{m} \times \Delta\mathbf{m} = \lambda\gamma|\nabla\mathbf{m}|^2\mathbf{m}. \tag{1.3}$$

The equation (1.3) is a highly nonlinear parabolic problem with the point-wise constraint $|\mathbf{m}| = 1$ in general domains. We need to note that the point-wise constraint $|\mathbf{m}| = 1$ can be deduced from (1.3) with $|\mathbf{m}_0| = 1$ if the solution $\mathbf{m}$ is a classical solution, which will be given in next section. In addition, since the exchange constant γ is not critical in the designing of numerical schemes, we set $\gamma = 1$ for simplicity. Instead of (1.1), in this paper, we will consider the equivalent Landau-Lifshitz equation (1.3).

Since the Landau-Lifshitz equation (1.3) is a nonlinear problem, we can not find the exact solution $\mathbf{m}$. Then how to solve the numerical approximation solution becomes more and more important by studying the efficiently numerical algorithms. In views of the characteristics of (1.3), a key issue in designing the numerical algorithms is that we have to take into account the point-wise constraint $|\mathbf{m}| = 1$. Usually, there have two strategies to deal with the point-wise constraint: one is the preserving unit length, and another is the approximating unit length.

For the numerical schemes of the preserving unit length, a natural way is to project the numerical solutions onto the unit sphere. This projection method was firstly studied by E and Wang in [13] for the Landau-Lifshitz equation by using the finite difference method. Based upon this method, many works were reported. For example, a Gauss-Seidel projection scheme was suggested by Wang et al. in [25], where the stability of numerical scheme was shown numerically. A mimetic finite difference method was suggested in [18], where the stability of numerical algorithm was proved and no convergence analysis were done. Recently, a second-order BDF finite difference scheme was studied for the Landau-Lifshitz equation (1.1) by Chen et al. in [11], where the optimal second-order convergence rate were derived under the condition $\tau \leq Ch$. Another projection method is the orthogonal projection method proposed by Alouges and Jaisson in [1]. However, at each time step, one has to build a new finite element space which is orthogonal point-wisely to the finite element solution at the previous time step. Bartels et al. in [8] studied the Alouges and Jaisson's scheme in [1] and proved the convergence of the finite element solution under the strong time step condition $\tau = o(h^{1+d/2})$. Although there have some other orthogonal projection algorithms were suggested in [2, 4–7], no convergence rate was obtained. Another strategy to preserve the unit length is using the nonlinear fully implicit scheme [9]. However, a nonlinear system should be solved by using some iterative method at each time step, which is very expensive, especially for the three-dimensional problem.

For the numerical schemes of the approximating unit length, there have two different methods: one is that the point-wise constraint is relaxed by introducing some penalty functions, and another is that the point-wise constraint is not taken into account in designing the numerical schemes since $|\mathbf{m}| = 1$ can be deduced from (1.1) or (1.3). For the penalty methods, Pistella and Valente in [23] used the Ginzburg-Landau penalty function to relax the point-wise constraint and suggested an explicit finite difference scheme to solve the penalty approximating problem numerically. The authors proved the convergence of the numerical solution under the time step condition $\tau \leq C(\varepsilon)h^2$. Based upon different penalty functions, some different penalty schemes for (1.1) or (1.3) were studied by Prohl in [24]. Generally, the penalty methods need a very small time step as h and ε tend to zero, which will result in extremely time-consuming in the practical computations. For the second method, many fully discrete finite element schemes were suggested without taking account of the point-wise constraint, such as the first-order Euler semi-implicit scheme [12, 14] and the second-order Crank-Nicolson semi-implicit scheme [3]. In particular, the unconditionally optimal temporal-spatial error estimates were proved in [3, 14] by using the error splitting technique proposed in [20–22].

In this paper, we will propose and study a second-order BDF finite element scheme for the numerical approximation of the solution to the Landau-Lifshitz equation (1.3) with (1.2). As like in [3,12,14], the point-wise constraint is not taken into account in designing the proposed scheme. Thus, this scheme belongs to the method of the approximating unit length. In addition, by using the linearized extrapolated technique, the proposed BDF scheme is a semi-implicit scheme, which means that we only solve a linear system at each time step. Based upon the error splitting technique, we proved that the BDF finite element scheme is of the unconditionally optimal second-order convergence rate $\mathcal{O}(\tau^2 + h^2)$ in $\mathbf{L}^2-$norm when the piecewise linear element is used to approximate the magnetization field.

The rest of the paper is organized as follows. The proposed BDF scheme and the main result in this paper are presented in Section 2. To prove the optimal error estimates in main result, the temporal and spatial error analysis are given in Section 3. Numerical results are provided to confirm the theoretical analysis in Section 4.

2. BDF scheme and main result

Firstly, we introduce some classical notations. For $k \in \mathbb{N}^+$ and $1 \leq p \leq +\infty$, we use $W^{k,p}(\Omega)$ to denote the classical Sobolev space. In particular, when $p = 2$, $W^{k,2}(\Omega)$ is the Hilbert space $H^k(\Omega)$. When $k = 0$, $W^{0,p}(\Omega)$ is the Lebesgue space $L^p(\Omega)$. The norms in $W^{k,p}(\Omega)$, $H^k(\Omega)$ and $L^p(\Omega)$ are denoted by $\|\cdot\|_{W^{k,p}}$, $\|\cdot\|_{H^k}$ and $\|\cdot\|_{L^p}$, respectively, and defined by a classical way. The boldface spaces $\mathbf{H}^k(\Omega), \mathbf{W}^{k,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ are used to denote the vector Sobolev spaces $H^k(\Omega)^d, W^{k,p}(\Omega)^d$ and $L^p(\Omega)^d$, respectively. In addition, we use $(\cdot,\cdot)$ to denotes the $\mathbf{L}^2(\Omega)$ inner product.

As described in the above section, the point-wise constraint $|\mathbf{m}| = 1$ can be deduced from the Landau-Lifshitz equation (1.3) with $|\mathbf{m}_0| = 1$ if the solution $\mathbf{m}$ is a classical solution such that we can ignore this constraint condition in designing the numerical scheme. Now, we will prove this fact. By setting the exchange constant $\gamma = 1$ in (1.3), we have

$$\mathbf{m}_t - \lambda\Delta\mathbf{m} - \mathbf{m} \times \Delta\mathbf{m} = \lambda|\nabla\mathbf{m}|^2\mathbf{m}. \tag{2.1}$$

Taking the inner product with (2.1) by $2\mathbf{m}$ and setting $z = |\mathbf{m}|^2$ leads to

$$z_t - \lambda\Delta z = 2\lambda|\nabla\mathbf{m}|^2(z-1), \tag{2.2}$$

where we use

$$z_t = 2(\mathbf{m}_t \cdot \mathbf{m}), \quad \Delta z = 2(\Delta\mathbf{m} \cdot \mathbf{m}) + 2|\nabla\mathbf{m}|^2, \quad (\mathbf{m} \times \Delta\mathbf{m}) \cdot \mathbf{m} = 0.$$

Denote $w = z - 1$. Then (2.2) reduces to

$$w_t - \lambda\Delta w = 2\lambda|\nabla\mathbf{m}|^2 w. \tag{2.3}$$

Testing (2.3) by $2w$ and using Gronwall inequality, we obtain

$$\|w(t)\|_{L^2}^2 + 2\lambda\int_0^t \|\nabla w(s)\|_{L^2}^2 ds \le \|w(0)\|_{L^2}^2 \exp\left(4\lambda\int_0^t \|\nabla\mathbf{m}(s)\|_{L^2}^2 ds\right) = 0$$

for any $t > 0$. In the above inequality, we use $w(0) = |\mathbf{m}_0|^2 - 1 = 0$. Thus, $w \equiv 0$, which implies $|\mathbf{m}| \equiv 1$ in the point-wise sense. Based upon the above observations and the linearized extrapolated technique, we will suggest a second-order BDF scheme such that the numerical solution has the approximating unit length.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with time step $\tau = T/N$ and $t_n = n\tau$ with $0 \le n \le N$. For any sequence of functions $\{f^n\}_{n=0}^N$, we denote

$$D_\tau f^n = \frac{3f^n - 4f^{n-1} + f^{n-2}}{2\tau}, \qquad \widehat{f}^n = 2f^{n-1} - f^{n-2}, \quad n \ge 2,$$

where $\widehat{f}^n$ is the second-order extrapolated formula. For the discrete time-derivative operator D_τ, there has the following telescope formula:

$$(D_\tau f^n, f^n) = \frac{1}{4\tau}\left(\|f^n\|_{L^2}^2 - \|f^{n-1}\|_{L^2}^2 + \|\widehat{f}^n\|_{L^2}^2 - \|\widehat{f}^{n-1}\|_{L^2}^2 + \|f^n - 2f^{n-1} + f^{n-2}\|_{L^2}^2\right)$$

for $n \ge 2$.

Next, we propose the following second-order BDF time discretization scheme for the Landau-Lifshitz equation (2.1) with (1.2):

Step I: For $\mathbf{M}^0 = \mathbf{m}_0$, we find $\mathbf{M}^1$ by

$$\frac{\mathbf{M}^1 - \mathbf{M}^0}{\tau} - \lambda\Delta\overline{\mathbf{M}}^1 - \widehat{\mathbf{M}}^0 \times \Delta\overline{\mathbf{M}}^1 - \nabla\widehat{\mathbf{M}}^0 \times \nabla\overline{\mathbf{M}}^1 = \lambda|\nabla\widehat{\mathbf{M}}^0|^2\widehat{\mathbf{M}}^0 \tag{2.4}$$

with the boundary condition $\nabla\mathbf{M}^1 \cdot \mathbf{n}|_{\partial\Omega} = 0$, where $\overline{\mathbf{M}}^1 = \frac{\mathbf{M}^1 + \mathbf{M}^0}{2}$ and $\widehat{\mathbf{M}}^0$ satisfies

$$\frac{\widehat{\mathbf{M}}^0 - \mathbf{m}_0}{\tau/2} - \lambda\Delta\widehat{\mathbf{M}}^0 - \mathbf{m}_0 \times \Delta\widehat{\mathbf{M}}^0 - \nabla\mathbf{m}_0 \times \nabla\widehat{\mathbf{M}}^0 = \lambda|\nabla\mathbf{m}_0|^2\mathbf{m}_0$$

with the boundary condition $\nabla\widehat{\mathbf{M}}^0 \cdot \mathbf{n}|_{\partial\Omega} = 0$.

Step II: For $2 \le n \le N$, we find $\mathbf{M}^n$ by

$$D_\tau\mathbf{M}^n - \lambda\Delta\mathbf{M}^n - \widehat{\mathbf{M}}^{n-1} \times \Delta\mathbf{M}^n - \nabla\widehat{\mathbf{M}}^{n-1} \times \nabla\mathbf{M}^n = \lambda|\nabla\widehat{\mathbf{M}}^{n-1}|^2\widehat{\mathbf{M}}^{n-1} \tag{2.5}$$

with the boundary condition $\nabla\mathbf{M}^n \cdot \mathbf{n}|_{\partial\Omega} = 0$.

Next, we give the finite element fully discrete scheme corresponding to the above semi-discrete scheme (2.4)–(2.5). Let $\mathcal{T}_h$ be a quasi-uniform triangular or tetrahedral partition of Ω into triangles or tetrahedrons of diameters bounded by h with

$0 < h < 1$. We use the piecewise linear polynomial to approximate the magnetization field and denote the finite element space by $\mathbf{V}_h$.

Then the finite element fully discrete scheme is written as follows:

Step I: For given $\mathbf{M}_h^0 = \rho_h \mathbf{m}_0 \in \mathbf{V}_h$, we find $\mathbf{M}_h^1 \in \mathbf{V}_h$ such that for all $\mathbf{v}_h \in \mathbf{V}_h$,

$$\begin{aligned}&\frac{1}{\tau}(\mathbf{M}_h^1 - \mathbf{M}_h^0, \mathbf{v}_h) + \lambda(\nabla \overline{\mathbf{M}}_h^1, \nabla \mathbf{v}_h) + (\widehat{\mathbf{M}}_h^0 \times \nabla \overline{\mathbf{M}}_h^1, \nabla \mathbf{v}_h)\\ =&\lambda(|\nabla \widehat{\mathbf{M}}_h^0|^2 \widehat{\mathbf{M}}_h^0, \mathbf{v}_h),\end{aligned} \tag{2.6}$$

where ρ_h is the Ritz projection from $\mathbf{H}^1(\Omega)$ onto $\mathbf{V}_h$, and $\widehat{\mathbf{M}}_h^0 \in \mathbf{V}_h$ is solved by

$$\frac{2}{\tau}(\widehat{\mathbf{M}}_h^0 - \mathbf{M}_h^0, \mathbf{v}_h) + \lambda(\nabla \widehat{\mathbf{M}}_h^0, \nabla \mathbf{v}_h) + (\mathbf{M}_h^0 \times \nabla \widehat{\mathbf{M}}_h^0, \nabla \mathbf{v}_h) = \lambda(|\nabla \mathbf{M}_h^0|^2 \mathbf{M}_h^0, \mathbf{v}_h).$$

Step II: For $2 \le n \le N$, we find $\mathbf{M}_h^n \in \mathbf{V}_h$ such that for all $\mathbf{v}_h \in \mathbf{V}_h$,

$$\begin{aligned}&(D_\tau \mathbf{M}_h^n, \mathbf{v}_h) + \lambda(\nabla \mathbf{M}_h^n, \nabla \mathbf{v}_h) + (\widehat{\mathbf{M}}_h^{n-1} \times \nabla \mathbf{M}_h^n, \nabla \mathbf{v}_h)\\ =&\lambda(|\nabla \widehat{\mathbf{M}}_h^{n-1}|^2 \widehat{\mathbf{M}}_h^{n-1}, \mathbf{v}_h).\end{aligned} \tag{2.7}$$

Remark 2.1. Taking $\mathbf{v}_h = \mathbf{M}_h^1$ and $\mathbf{v}_h = \mathbf{M}_h^n$ in (2.6) and (2.7), respectively, the existence and uniqueness of $\mathbf{M}_h^1$ and $\mathbf{M}_h^n$ are from the Lax-Milgram theorem due to $(\mathbf{u} \times \nabla \mathbf{v}, \nabla \mathbf{v}) = 0$.

Remark 2.2. To start up the second-order BDF scheme, here, we use the Crank-Nicolson scheme to compute $\mathbf{M}_h^1$. The temporal and spatial error analysis for $\mathbf{M}_h^1$ were studied in [3].

To prove the unconditionally optimal error estimates for second-order BDF scheme (2.6)–(2.7), we assume that the Landau-Lifshitz equation (1.2) and (1.3) has a unique local strong solution $\mathbf{m}$ which satisfies the following regularity:

$$\begin{aligned}&\|\mathbf{m}\|_{L^\infty(0,T;W^{2,4}(\Omega))} + \|\mathbf{m}_t\|_{L^\infty(0,T;H^2(\Omega))}\\ &+ \|\mathbf{m}_{tt}\|_{L^2(0,T;H^1(\Omega))} + \|\mathbf{m}_{ttt}\|_{L^2(0,T;L^2(\Omega))} \le C.\end{aligned} \tag{2.8}$$

Remark 2.3. Regularity assumption is essential for the error analysis of numerical methods. It is noted that the solution of the Landau-Lifshitz equation may blow up at finite time. Thus, we assume that the exact solution exists locally in time such that the analysis presented in this paper is not applicable for the problem near the blow-up.

We present the main result in the following theorem. The proof will be given in next section.

Theorem 2.1. *Suppose that the solution $\boldsymbol{m}$ to the Landau-Lifshitz equation satisfies the regularity assumption (2.8). Then there exists sufficiently small constants $h_0 > 0$ and $\tau_0 > 0$ such that when $h < h_0$ and $\tau < \tau_0$, the following optimal temporal-spatial error estimate holds:*

$$\max_{1 \le n \le N} \left(\|\boldsymbol{m}^n - \boldsymbol{M}_h^n\|_{L^2} + \|1 - |\boldsymbol{M}_h^n|^2\|_{L^2} \right) \le C(\tau^2 + h^2), \tag{2.9}$$

where $\boldsymbol{m}^n = \boldsymbol{m}(t_n)$ and $C > 0$ is some constant independent of h and τ.

Before beginning to prove Theorem 2.1, we recall the inverse inequalities [10] and the discrete Gronwall's inequality [17] which will be frequently used in error analysis.

Lemma 2.1. *There exists some $C > 0$ independent of h such that*

$$\|\boldsymbol{v}_h\|_{L^\infty} \le Ch^{-d/2}\|\boldsymbol{v}_h\|_{L^2}, \quad \|\boldsymbol{v}_h\|_{L^3} \le Ch^{-d/6}\|\boldsymbol{v}_h\|_{L^2}, \quad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.10}$$

Lemma 2.2. *Let a_k, b_k, c_k and γ_k be the nonnegative numbers such that*

$$a_n + \tau\sum_{k=0}^{n} b_k \le \tau\sum_{k=0}^{n}\gamma_k a_k + \tau\sum_{k=0}^{n} c_k + B, \quad \text{for } n \ge 1. \tag{2.11}$$

Suppose $\tau\gamma_k < 1$ and set $\sigma_k = (1-\tau\gamma_k)^{-1}$. Then there holds:

$$a_n + \tau\sum_{k=0}^{n} b_k \le \exp(\tau\sum_{k=0}^{n}\gamma_k\sigma_k)(\tau\sum_{k=0}^{n} c_k + B), \quad \text{for } n \ge 1. \tag{2.12}$$

Remark 2.4. If the first sum on the right-hand side of (2.11) extends only up to $n-1$, then the estimate (2.12) still holds for all $k \ge 1$ with $\sigma_k = 1$.

3. Error analysis

The proof of Theorem 2.1 is based upon the following error splitting:

$$\begin{aligned}\|\mathbf{m}^n - \mathbf{M}_h^n\|_{L^2} \le& \|\mathbf{m}^n - \mathbf{M}^n\|_{L^2} + \|\Pi_h^n\mathbf{M}^n - \mathbf{M}_h^n\|_{L^2} + \|\mathbf{M}^n - \Pi_h^n\mathbf{M}^n\|_{L^2} \\ :=& \|\mathbf{e}^n\|_{L^2} + \|\mathbf{e}_h^n\|_{L^2} + \|\mathbf{E}^n\|_{L^2}, \quad \forall\ 1 \le n \le N,\end{aligned} \tag{3.1}$$

where $\mathbf{e}^n$, $\mathbf{e}_h^h$ and $\mathbf{E}^n$ are the temporal error, the spatial error and the projection error, respectively. $\Pi_h^n : \mathbf{V} \to \mathbf{V}_h$ is a projection operator defined in subsection 3.2.

3.1. Temporal error analysis

In this subsection, we will prove that the BDF time discretization scheme (2.4)-(2.5) is of the second-order convergence accuracy $\mathcal{O}(\tau^2)$. Note that $\mathbf{M}^1$ is solved by a second-order Crank-Nicolson scheme studied in [3], where the following temporal error estimate for $\mathbf{e}^1$ is derived:

$$\|\mathbf{e}^1\|_{L^2}^2 + \tau\|\mathbf{e}^1\|_{H^1}^2 + \tau^2\|\mathbf{e}^1\|_{H^2}^2 \le C\tau^5. \tag{3.2}$$

Moreover, there exists some $C > 0$ such that

$$\|\mathbf{M}^1\|_{W^{2,4}} \le C. \tag{3.3}$$

The main result in this subsection is the following optimal temporal convergence accuracy.

Theorem 3.1. *Suppose that the solution $\boldsymbol{m}$ to the Landau-Lifshitz equation satisfies the regularity assumption (2.8). Then there exist $\tau_1 > 0$ such that when $\tau < \tau_1$, there holds*

$$\max_{1\le m\le N}\left(\|\boldsymbol{e}^m\|_{H^1}^2 + \tau\sum_{k=1}^{m}\|\boldsymbol{e}^k\|_{H^2}^2\right) \le C_1\tau^4, \tag{3.4}$$

where $C_1 > 0$ is independent of τ.

Proof. We will prove this theorem by using the method of mathematical induction. In view of (3.2), the error estimate (3.4) is valid for $m = 1$. Now, we assume that (3.4) is valid for $m = n-1$ with $2 \le n \le N$. Then

$$\|\widehat{\mathbf{e}}^{n-1}\|_{H^1}^2 + \tau \sum_{k=1}^{n-1} \|\widehat{\mathbf{e}}^k\|_{H^2}^2 \le CC_1\tau^4, \tag{3.5}$$

for some $C > 0$ independent of C_1. In addition, we have

$$\|\widehat{\mathbf{M}}^{n-1}\|_{H^2} \le \|\widehat{\mathbf{m}}^{n-1}\|_{H^2} + \|\widehat{\mathbf{e}}^{n-1}\|_{H^2} \le C + C\sqrt{C_1\tau^3} \le C \tag{3.6}$$

for $C_1\tau_1^3 \le 1$. From the Sobolev imbedding $\mathbf{H}^2(\Omega) \hookrightarrow \mathbf{W}^{1,6}(\Omega)$, it holds that

$$\|\widehat{\mathbf{M}}^{n-1}\|_{W^{1,6}} \le C. \tag{3.7}$$

To close the mathematical induction, we need to prove that (3.4) is valid for $m = n$. It is easy to check that the exact solution $\mathbf{m}^n$ satisfies

$$D_\tau \mathbf{m}^n - \lambda\Delta\mathbf{m}^n - \widehat{\mathbf{m}}^{n-1} \times \Delta\mathbf{m}^n - \nabla\widehat{\mathbf{m}}^{n-1} \times \nabla\mathbf{m}^n = \mathbf{R}^n + \lambda|\nabla\widehat{\mathbf{m}}^{n-1}|^2\widehat{\mathbf{m}}^{n-1}, \tag{3.8}$$

where

$$\begin{aligned}\mathbf{R}^n = & D_\tau\mathbf{m}^n - \mathbf{m}_t^n - (\widehat{\mathbf{m}}^{n-1} - \mathbf{m}^n) \times \Delta\mathbf{m}^n - \nabla(\widehat{\mathbf{m}}^{n-1} - \mathbf{m}^n) \times \nabla\mathbf{m}^n \\ & + \lambda|\nabla\mathbf{m}^n|^2\mathbf{m}^n - \lambda|\nabla\widehat{\mathbf{m}}^{n-1}|^2\widehat{\mathbf{m}}^{n-1}.\end{aligned}$$

By using the regularity assumption (2.8) and Taylor formula, we have

$$\tau \sum_{n=2}^{N} \|\mathbf{R}^n\|_{L^2}^2 \le C\tau^4. \tag{3.9}$$

Subtracting (2.5) from (3.8) leads to

$$\begin{aligned}D_\tau\mathbf{e}^n - \lambda\Delta\mathbf{e}^n = & \mathbf{R}^n + (\widehat{\mathbf{m}}^{n-1} \times \Delta\mathbf{m}^n - \widehat{\mathbf{M}}^{n-1} \times \Delta\mathbf{M}^n) \\ & + (\nabla\widehat{\mathbf{m}}^{n-1} \times \nabla\mathbf{m}^n - \nabla\widehat{\mathbf{M}}^{n-1} \times \nabla\mathbf{M}^n) \\ & + \lambda(|\nabla\widehat{\mathbf{m}}^{n-1}|^2\widehat{\mathbf{m}}^{n-1} - |\nabla\widehat{\mathbf{M}}^{n-1}|^2\widehat{\mathbf{M}}^{n-1}).\end{aligned} \tag{3.10}$$

Testing (3.10) by $4\tau\mathbf{e}^n$ and using the telescope formula, we get

$$\begin{aligned}& \|\mathbf{e}^n\|_{L^2}^2 - \|\mathbf{e}^{n-1}\|_{L^2}^2 + \|\widehat{\mathbf{e}}^n\|_{L^2}^2 - \|\widehat{\mathbf{e}}^{n-1}\|_{L^2}^2 + 4\lambda\tau\|\nabla\mathbf{e}^n\|_{L^2}^2 \\ = & 4\tau(\mathbf{R}^n, \mathbf{e}^n) - 4\tau(\widehat{\mathbf{e}}^{n-1} \times \nabla\mathbf{m}^n, \nabla\mathbf{e}^n) \\ & + 4\lambda\tau(|\nabla\widehat{\mathbf{m}}^{n-1}|^2\widehat{\mathbf{m}}^{n-1} - |\nabla\widehat{\mathbf{M}}^{n-1}|^2\widehat{\mathbf{M}}^{n-1}, \mathbf{e}^n) \\ \le & \tau\|\mathbf{e}^n\|_{L^2}^2 + \lambda\tau\|\nabla\mathbf{e}^n\|_{L^2}^2 + C\tau(\|\mathbf{R}^n\|_{L^2}^2 + \|\widehat{\mathbf{e}}^{n-1}\|_{L^2}^2) \\ & + 4\lambda\tau(|\nabla\widehat{\mathbf{m}}^{n-1}|^2\widehat{\mathbf{m}}^{n-1} - |\nabla\widehat{\mathbf{M}}^{n-1}|^2\widehat{\mathbf{M}}^{n-1}, \mathbf{e}^n)\end{aligned} \tag{3.11}$$

where we use the Sobolev imbedding theorem $\mathbf{W}^{2,4}(\Omega) \hookrightarrow \mathbf{W}^{1,\infty}(\Omega)$ for $d = 2, 3$, and

$$(\widehat{\mathbf{m}}^{n-1} \times \Delta\mathbf{m}^n - \widehat{\mathbf{M}}^{n-1} \times \Delta\mathbf{M}^n, \mathbf{e}^n) + (\nabla\widehat{\mathbf{m}}^{n-1} \times \nabla\mathbf{m}^n - \nabla\widehat{\mathbf{M}}^{n-1} \times \nabla\mathbf{M}^n, \mathbf{e}^n)$$

$= -(\widehat{\mathbf{e}}^{n-1} \times \nabla \mathbf{m}^n, \nabla \mathbf{e}^n).$

Since

$$\begin{aligned} &|\nabla \widehat{\mathbf{m}}^{n-1}|^2 \widehat{\mathbf{m}}^{n-1} - |\nabla \widehat{\mathbf{M}}^{n-1}|^2 \widehat{\mathbf{M}}^{n-1} \\ =&(\nabla \widehat{\mathbf{e}}^{n-1} \cdot (\nabla \widehat{\mathbf{m}}^{n-1} + \nabla \widehat{\mathbf{M}}^{n-1}))\nabla \widehat{\mathbf{m}}^{n-1} + |\nabla \widehat{\mathbf{M}}^{n-1}|^2 \widehat{\mathbf{e}}^{n-1}, \end{aligned} \tag{3.12}$$

the last term in the right-hand side of (3.11) can be bounded by

$$\begin{aligned} &4\lambda\tau(|\nabla \widehat{\mathbf{m}}^{n-1}|^2 \widehat{\mathbf{m}}^{n-1} - |\nabla \widehat{\mathbf{M}}^{n-1}|^2 \widehat{\mathbf{M}}^{n-1}, \mathbf{e}^n) \\ \le& C\tau\Big(\|\nabla \widehat{\mathbf{m}}^{n-1}\|_{L^\infty} \|\nabla(\widehat{\mathbf{m}}^{n-1} + \widehat{\mathbf{M}}^{n-1})\|_{L^3} \|\nabla \widehat{\mathbf{e}}^{n-1}\|_{L^2} + \|\nabla \widehat{\mathbf{M}}^{n-1}\|_{L^6}^2 \|\widehat{\mathbf{e}}^{n-1}\|_{L^2}\Big)\|\mathbf{e}^n\|_{L^6} \\ \le& \lambda\tau\|\mathbf{e}^n\|_{H^1}^2 + C\tau\|\widehat{\mathbf{e}}^{n-1}\|_{H^1}^2, \end{aligned}$$

where $C > 0$ is independent of C_1. Substituting the above estimates into (3.11) leads to

$$\begin{aligned} &\|\mathbf{e}^n\|_{L^2}^2 - \|\mathbf{e}^{n-1}\|_{L^2}^2 + \|\widehat{\mathbf{e}}^n\|_{L^2}^2 - \|\widehat{\mathbf{e}}^{n-1}\|_{L^2}^2 + \lambda\tau\|\nabla \mathbf{e}^n\|_{L^2}^2 \\ \le& C\tau(\|\mathbf{R}^n\|_{L^2}^2 + \|\mathbf{e}^n\|_{L^2}^2 + \|\widehat{\mathbf{e}}^{n-1}\|_{H^1}^2). \end{aligned} \tag{3.13}$$

Testing (3.10) by $-4\tau\Delta\mathbf{e}^n$ gives

$$\begin{aligned} &\|\nabla \mathbf{e}^n\|_{L^2}^2 - \|\nabla \mathbf{e}^{n-1}\|_{L^2}^2 + \|\nabla \widehat{\mathbf{e}}^n\|_{L^2}^2 - \|\nabla \widehat{\mathbf{e}}^{n-1}\|_{L^2}^2 + 4\lambda\tau\|\Delta \mathbf{e}^n\|_{L^2}^2 \\ =& 4\tau(\mathbf{R}^n, \Delta\mathbf{e}^n) + 4\tau(\widehat{\mathbf{m}}^{n-1} \times \Delta\mathbf{m}^n - \widehat{\mathbf{M}}^{n-1} \times \Delta\mathbf{M}^n, \Delta\mathbf{e}^n) \\ &+ 4\tau(\nabla\widehat{\mathbf{m}}^{n-1} \times \nabla\mathbf{m}^n - \nabla\widehat{\mathbf{M}}^{n-1} \times \nabla\mathbf{M}^n, \Delta\mathbf{e}^n) \\ &+ 4\lambda\tau(|\nabla \widehat{\mathbf{m}}^{n-1}|^2 \widehat{\mathbf{m}}^{n-1} - |\nabla \widehat{\mathbf{M}}^{n-1}|^2 \widehat{\mathbf{M}}^{n-1}, \Delta\mathbf{e}^n) := \sum_{i=1}^{4} I_i. \end{aligned} \tag{3.14}$$

Next, we estimate I_1 to I_4 as follows. It is easy to see that

$$\begin{aligned} I_1 &\le \frac{\lambda\tau}{2}\|\Delta\mathbf{e}^n\|_{L^2}^2 + C\tau\|\mathbf{R}^n\|_{L^2}^2, \\ I_2 &\le \frac{\lambda\tau}{2}\|\Delta\mathbf{e}^n\|_{L^2}^2 + C\tau\|\widehat{\mathbf{e}}^{n-1}\|_{H^1}^2. \end{aligned}$$

For I_3, we have

$$\begin{aligned} I_3 &= 4\tau(\nabla\widehat{\mathbf{e}}^{n-1} \times \nabla\mathbf{m}^n - \nabla\widehat{\mathbf{e}}^{n-1} \times \nabla\mathbf{e}^n + \nabla\widehat{\mathbf{m}}^{n-1} \times \nabla\mathbf{e}^n, \Delta\mathbf{e}^n) \\ &\le C\tau(\|\nabla\widehat{\mathbf{e}}^{n-1}\|_{L^2} + \|\nabla\mathbf{e}^n\|_{L^2})\|\Delta\mathbf{e}^n\|_{L^2} + C\tau\|\nabla\widehat{\mathbf{e}}^{n-1}\|_{L^6}\|\nabla\mathbf{e}^n\|_{L^3}\|\Delta\mathbf{e}^n\|_{L^2} \\ &\le \frac{\lambda\tau}{2}\|\Delta\mathbf{e}^n\|_{L^2}^2 + C\tau(\|\nabla\mathbf{e}^n\|_{L^2}^2 + \|\nabla\widehat{\mathbf{e}}^{n-1}\|_{L^2}^2 + \|\widehat{\mathbf{e}}^{n-1}\|_{H^2}^4\|\nabla\mathbf{e}^n\|_{L^2}^2) \\ &\le \frac{\lambda\tau}{2}\|\Delta\mathbf{e}^n\|_{L^2}^2 + C\tau(\|\nabla\mathbf{e}^n\|_{L^2}^2 + \|\nabla\widehat{\mathbf{e}}^{n-1}\|_{L^2}^2) \end{aligned}$$

for $C_1^2\tau_1^6 \le 1$. From (3.12), we estimate I_4 by

$$\begin{aligned} I_4 &\le C\tau(\|\nabla\widehat{\mathbf{e}}^{n-1}\|_{L^2} + \|\nabla\widehat{\mathbf{e}}^{n-1}\|_{L^4}^2 + \|\widehat{\mathbf{e}}^{n-1}\|_{L^6})\|\Delta\mathbf{e}^n\|_{L^2} \\ &\le \frac{\lambda\tau}{2}\|\Delta\mathbf{e}^n\|_{L^2}^2 + C\tau(\|\widehat{\mathbf{e}}^{n-1}\|_{H^1}^2 + \|\widehat{\mathbf{e}}^{n-1}\|_{H^1}\|\widehat{\mathbf{e}}^{n-1}\|_{H^2}^3) \end{aligned}$$

$$\leq \frac{\lambda\tau}{2}\|\Delta \mathbf{e}^n\|_{L^2}^2 + \frac{\lambda\tau}{2}\|\widehat{\mathbf{e}}^{n-1}\|_{H^2}^2 + C\tau\|\widehat{\mathbf{e}}^{n-1}\|_{H^1}^2$$

for $CC_1\tau_1^{7/2} \leq \lambda/2$. Combining these estimates into (3.14) yields

$$\begin{aligned}&\|\nabla \mathbf{e}^n\|_{L^2}^2 - \|\nabla \mathbf{e}^{n-1}\|_{L^2}^2 + \|\nabla \widehat{\mathbf{e}}^n\|_{L^2}^2 - \|\nabla \widehat{\mathbf{e}}^{n-1}\|_{L^2}^2 + \lambda\tau\|\Delta \mathbf{e}^n\|_{L^2}^2 \\ \leq &\frac{\lambda\tau}{2}\|\widehat{\mathbf{e}}^{n-1}\|_{H^2}^2 + C\tau(\|\mathbf{R}^n\|_{L^2}^2 + \|\mathbf{e}^n\|_{H^1}^2 + \|\widehat{\mathbf{e}}^{n-1}\|_{H^1}^2).\end{aligned} \tag{3.15}$$

Taking the sum of (3.13) and (3.15) leads to

$$\begin{aligned}&\|\mathbf{e}^n\|_{H^1}^2 - \|\mathbf{e}^{n-1}\|_{H^1}^2 + \|\widehat{\mathbf{e}}^n\|_{H^1}^2 - \|\widehat{\mathbf{e}}^{n-1}\|_{H^1}^2 + \lambda\tau\|\mathbf{e}^n\|_{H^2}^2 \\ \leq &\frac{\lambda\tau}{2}\|\widehat{\mathbf{e}}^{n-1}\|_{H^2}^2 + C\tau(\|\mathbf{R}^n\|_{L^2}^2 + \|\mathbf{e}^n\|_{H^1}^2 + \|\widehat{\mathbf{e}}^{n-1}\|_{H^1}^2).\end{aligned}$$

Summing up the above inequality and using the discrete Gronwall inequality, we get

$$\|\mathbf{e}^n\|_{H^1}^2 + \tau\sum_{k=1}^n \|\mathbf{e}^k\|_{H^2}^2 \leq C\exp(CT)\tau^4 := C_1\tau^4. \tag{3.16}$$

Thus, we prove that (3.4) is valid for $m = n$ and close the mathematical induction. The proof of Theorem 3.1 is completed. □

It follows from (3.4) that

$$\|D_\tau \mathbf{M}^n\|_{H^2} \leq \|D_\tau \mathbf{m}^n\|_{H^2} + \|D_\tau \mathbf{e}^n\|_{H^2} \leq C, \tag{3.17}$$

$$\|\mathbf{M}^n\|_{H^2} \leq \|\mathbf{m}^n\|_{H^2} + \|\mathbf{e}^n\|_{H^2} \leq C \tag{3.18}$$

for $2 \leq n \leq N$. From the regularity of elliptic problem, we have

$$\begin{aligned}\|\mathbf{M}^n\|_{W^{2,3}} &\leq C\|D_\tau \mathbf{M}^n\|_{L^3} + C\|\nabla \widehat{\mathbf{M}}^{n-1} \times \nabla \mathbf{M}^n\|_{L^3} + C\||\nabla \widehat{\mathbf{M}}^{n-1}|^2 \widehat{\mathbf{M}}^{n-1}\|_{L^3} \\ &\leq C\|D_\tau \mathbf{M}^n\|_{H^2} + C\|\nabla \widehat{\mathbf{M}}^{n-1}\|_{L^6}\|\nabla \mathbf{M}^n\|_{L^6} + C\|\nabla \widehat{\mathbf{M}}^{n-1}\|_{L^6}^2\|\widehat{\mathbf{M}}^{n-1}\|_{L^\infty} \\ &\leq C,\end{aligned}$$

which implies $\mathbf{M}^n \in \mathbf{W}^{1,p}$ for any $1 \leq p < +\infty$. Furthermore, from the regularity of elliptic problem, again, we have

$$\begin{aligned}\|\mathbf{M}^n\|_{W^{2,4}} \leq &C\|D_\tau \mathbf{M}^n\|_{L^4} + C\|\nabla \widehat{\mathbf{M}}^{n-1} \times \nabla \mathbf{M}^n\|_{L^4} + C\||\nabla \widehat{\mathbf{M}}^{n-1}|^2 \widehat{\mathbf{M}}^{n-1}\|_{L^4} \\ \leq &C\|D_\tau \mathbf{M}^n\|_{H^2} + C\|\nabla \widehat{\mathbf{M}}^{n-1}\|_{L^6}\|\nabla \mathbf{M}^n\|_{L^{12}} + C\|\nabla \widehat{\mathbf{M}}^{n-1}\|_{L^8}^2\|\widehat{\mathbf{M}}^{n-1}\|_{L^\infty} \\ \leq &C.\end{aligned} \tag{3.19}$$

3.2. Spatial error analysis

In this subsection, we will prove the optimal spatial convergence accuracy. For $2 \leq n \leq N$, we define the projection operator Π_h^n from $\mathbf{H}^1(\Omega)$ onto $\mathbf{V}_h$ by

$$\lambda(\nabla(\Pi_h^n \mathbf{u} - \mathbf{u}), \nabla \mathbf{v}_h) + \lambda(\Pi_h^n \mathbf{u} - \mathbf{u}, \mathbf{v}_h) + (\widehat{\mathbf{M}}^{n-1} \times \nabla(\Pi_h^n \mathbf{u} - \mathbf{u}), \nabla \mathbf{v}_h) = 0$$

for all $\mathbf{v}_h \in \mathbf{V}_h$. By the classical finite element theory [10], we have

$$\|\mathbf{M}^n - \Pi_h^n \mathbf{M}^n\|_{L^2} + h\|\mathbf{M}^n - \Pi_h^n \mathbf{M}^n\|_{W^{1,4}} \leq Ch^2\|\mathbf{M}^n\|_{W^{2,4}}, \tag{3.20}$$

$$\|\Pi_h^n \mathbf{M}^n\|_{W^{1,p}} \le C\|\mathbf{M}^n\|_{W^{2,4}} \tag{3.21}$$

for $2 \le n \le N$ and $1 \le p \le \infty$.

Remark 3.1. The definition of new elliptic projection operator Π_h^n is borrowed from that in [14]. Although there has no proof for the approximation and stability properties (3.20)-(3.21), but they still hold. The reason is that the bilinear form

$$\lambda(\nabla\Pi_h^n\mathbf{u}, \nabla\mathbf{v}_h) + \lambda(\Pi_h^n\mathbf{u}, \mathbf{v}_h) + (\widehat{\mathbf{M}}^{n-1} \times \nabla\Pi_h^n\mathbf{u}, \nabla\mathbf{v}_h)$$

satisfies

$$\lambda(\nabla\Pi_h^n\mathbf{u}, \nabla\Pi_h^n\mathbf{u}) + \lambda(\Pi_h^n\mathbf{u}, \Pi_h^n\mathbf{u}) + (\widehat{\mathbf{M}}^{n-1} \times \nabla\Pi_h^n\mathbf{u}, \nabla\Pi_h^n\mathbf{u}) = \lambda\|\Pi_h^n\mathbf{u}\|_{H^1}^2.$$

Then $\Pi_h^n\mathbf{u}$ is well defined and the classical elliptic projection approximation theory is still suitable for this new projection.

Recall the error estimate $\|\mathbf{e}_h^1\|_{L^2}$ established in [3]:

$$\|\mathbf{e}_h^1\|_{L^2} \le Ch^2. \tag{3.22}$$

The main result in this subsection is the following optimal $\mathbf{L}^2$ spatial convergence accuracy.

Theorem 3.2. *Suppose that the solution $\boldsymbol{m}$ to the Landau-Lifshitz equation satisfies the regularity assumption (2.8). Then there exist some $h_1 > 0$ such that when $h < h_1$, there holds*

$$\max_{1\le m\le N} \|\boldsymbol{e}_h^m\|_{L^2} \le C_2 h^2, \tag{3.23}$$

where $C_2 > 0$ is independent of h and τ.

Proof. In terms of (3.22), the error estimate (3.23) holds for $m = 1$. Now, we suppose that (3.23) is valid for $m = n-1$ with $2 \le n \le N$. Under this assumption, from the inverse inequality, one has

$$\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} \le CC_2h^2 \quad \text{and} \quad \|\mathbf{e}_h^{n-1}\|_{H^1} \le CC_2h, \tag{3.24}$$

where $C > 0$ is independent of h, τ and C_2. Furthermore, we have

$$\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^\infty} \le Ch^{-d/2}\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} \le CC_2h^{1/2} \le C \tag{3.25}$$

for $C_2h_1^{1/2} \le 1$, which results in

$$\|\mathbf{M}_h^n\|_{L^\infty} \le \|\Pi_h^n\mathbf{M}^n\|_{L^\infty} + \|\widehat{\mathbf{e}}_h^{n-1}\|_{L^\infty} \le C. \tag{3.26}$$

To close the mathematical induction, we need to prove that (3.23) is valid for $m = n$.

Testing (2.5) by $\mathbf{v}_h$ and subtracting the resulting equation from (2.7) and setting $\mathbf{v}_h = 4\tau\mathbf{e}_h^n$ leads to

$$\begin{aligned}
&\|\mathbf{e}_h^n\|_{L^2}^2 - \|\mathbf{e}_h^{n-1}\|_{L^2}^2 + \|\widehat{\mathbf{e}}_h^n\|_{L^2}^2 - \|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 + 4\lambda\tau\|\nabla\mathbf{e}_h^n\|_{L^2}^2 \\
=&4\tau(D_\tau\mathbf{E}^n, \mathbf{e}_h^n) - 4\lambda\tau(\mathbf{E}^n, \mathbf{e}_h^n) + 4\tau\left((\widehat{\mathbf{e}}_h^{n-1} - \widehat{\mathbf{E}}^{n-1}) \times \nabla\Pi_h^n\mathbf{M}^n, \nabla\mathbf{e}_h^n\right) \\
&+ 4\lambda\tau\left(|\nabla\widehat{\mathbf{M}}^{n-1}|^2\widehat{\mathbf{M}}^{n-1} - |\nabla\widehat{\mathbf{M}}_h^{n-1}|^2\widehat{\mathbf{M}}_h^{n-1}, \mathbf{e}_h^n\right) := \sum_{i=1}^4 J_i.
\end{aligned} \tag{3.27}$$

Next, we estimate J_1 to J_4 term by term. It is easy to prove that J_1, J_2 and J_3 satisfy

$$\begin{aligned} J_1 &\le C\tau\|\mathbf{e}_h^n\|_{L^2}^2 + C\tau\|D_\tau\mathbf{E}^n\|_{L^2}^2 \le C\tau\|\mathbf{e}_h^n\|_{L^2}^2 + C\tau h^4, \\ J_2 &\le C\tau\|\mathbf{e}_h^n\|_{L^2}^2 + C\tau\|\mathbf{E}^n\|_{L^2}^2 \le C\tau\|\mathbf{e}_h^n\|_{L^2}^2 + C\tau h^4, \\ J_3 &\le \lambda\tau\|\nabla\mathbf{e}_h^n\|_{L^2}^2 + C\tau(\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 + \|\widehat{\mathbf{E}}^{n-1}\|_{L^2}^2)\|\nabla\Pi_h^n\mathbf{M}^n\|_{L^\infty}^2 \\ &\le \lambda\tau\|\nabla\mathbf{e}_h^n\|_{L^2}^2 + C\tau(\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 + h^4) \end{aligned}$$

where we use (3.17)-(3.18) and (3.20)-(3.21). An alternative of J_5 is

$$\begin{aligned} J_5 =& 4\lambda\tau\left(|\nabla\widehat{\mathbf{M}}^{n-1}|^2(\widehat{\mathbf{e}}_h^{n-1} - \widehat{\mathbf{E}}^{n-1}), \mathbf{e}_h^n\right) - 4\lambda\tau\left(|\nabla(\widehat{\mathbf{e}}_h^{n-1} - \widehat{\mathbf{E}}^{n-1})|^2\widehat{\mathbf{M}}_h^{n-1}, \mathbf{e}_h^n\right) \\ &+ 8\lambda\tau\left((\nabla\widehat{\mathbf{M}}^{n-1}\cdot\nabla(\widehat{\mathbf{e}}_h^{n-1} - \widehat{\mathbf{E}}^{n-1}))\widehat{\mathbf{e}}_h^{n-1}, \mathbf{e}_h^n\right) \\ &- 8\lambda\tau\left((\nabla\widehat{\mathbf{M}}^{n-1}\cdot\nabla(\widehat{\mathbf{e}}_h^{n-1} - \widehat{\mathbf{E}}^{n-1}))\Pi_h^n\widehat{\mathbf{M}}^{n-1}, \mathbf{e}_h^n\right) \\ :=& K_1 + K_2 + K_3 + K_4. \end{aligned}$$

To bound J_5, we need to estimate K_1 to K_4 term by term. It is easy to find that K_1 satiefies

$$\begin{aligned} K_1 &\le C\tau\|\nabla\widehat{\mathbf{M}}^{n-1}\|_{L^\infty}^2(\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} + \|\widehat{\mathbf{E}}^{n-1}\|_{L^2})\|\mathbf{e}_h^n\|_{L^2} \\ &\le C\tau\|\mathbf{e}_h^n\|_{L^2} + C\tau(\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 + h^4), \end{aligned}$$

where we use (3.18) and (3.21). It follows from (3.24) and (3.26) that

$$\begin{aligned} K_2 &\le C\tau\|\widehat{\mathbf{M}}_h^{n-1}\|_{L^\infty}(\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} + \|\nabla\widehat{\mathbf{E}}^{n-1}\|_{L^2})(\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^3} + \|\nabla\widehat{\mathbf{E}}^{n-1}\|_{L^3})\|\mathbf{e}_h^n\|_{L^6} \\ &\le C\tau(\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} + h)(h^{-d/6}\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} + h)\|\mathbf{e}_h^n\|_{H^1} \\ &\le C\tau h^2\|\mathbf{e}_h^n\|_{H^1} + C\tau(C_2h^{1-d/6} + h + h^{1-d/6})\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}\|\mathbf{e}_h^n\|_{H^1} \\ &\le \frac{\lambda\tau}{2}\|\mathbf{e}_h^n\|_{H^1}^2 + C\tau(C_2^2h^{2-d/3} + h^2 + h^{2-d/3})\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 \\ &\le \frac{\lambda\tau}{2}\|\mathbf{e}_h^n\|_{H^1}^2 + \frac{\lambda\tau}{2}\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} \end{aligned}$$

for $C(C_2^2h_1^{2-d/3} + h_1^2 + h_1^{2-d/3}) \le \lambda/2$. For K_3, we have

$$\begin{aligned} K_3 &\le C\tau\|\nabla\widehat{\mathbf{M}}^{n-1}\|_{L^\infty}(\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^3} + \|\nabla\widehat{\mathbf{E}}^{n-1}\|_{L^3})\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}\|\mathbf{e}_h^n\|_{L^6} \\ &\le C\tau(C_2h^{1-d/6} + h)\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}\|\mathbf{e}_h^n\|_{H^1} \le \frac{\lambda\tau}{2}\|\mathbf{e}_h^n\|_{H^1}^2 + C\tau\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 \end{aligned}$$

for $C(C_2^2h_1^{2-d/3} + h_1^2) \le 1$. Finally, by using the integration by parts, we can estimate K_4 by

$$\begin{aligned} K_4 \le& C\tau\|\nabla^2\widehat{\mathbf{M}}^{n-1}\|_{L^3}(\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} + \|\widehat{\mathbf{E}}^{n-1}\|_{L^2})\|\Pi_h^n\widehat{\mathbf{M}}^{n-1}\|_{L^\infty}\|\mathbf{e}_h^n\|_{L^6} \\ &+ C\tau\|\nabla\widehat{\mathbf{M}}^{n-1}\|_{L^\infty}(\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} + \|\widehat{\mathbf{E}}^{n-1}\|_{L^2})\|\nabla\Pi_h^n\widehat{\mathbf{M}}^{n-1}\|_{L^\infty}\|\mathbf{e}_h^n\|_{L^2} \\ &+ C\tau\|\nabla\widehat{\mathbf{M}}^{n-1}\|_{L^\infty}(\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} + \|\widehat{\mathbf{E}}^{n-1}\|_{L^2})\|\Pi_h^n\widehat{\mathbf{M}}^{n-1}\|_{L^\infty}\|\nabla\mathbf{e}_h^n\|_{L^2} \end{aligned}$$

$$\leq \frac{\lambda\tau}{2}\|\mathbf{e}_h^n\|_{H^1}^2 + C\tau\|\mathbf{e}_h^n\|_{L^2} + C\tau(\|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 + h^4).$$

Thus, J_5 is bounded by

$$J_5 \leq \frac{3\lambda\tau}{2}\|\mathbf{e}_h^n\|_{H^1}^2 + \frac{\lambda\tau}{2}\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} + C\tau(\|\mathbf{e}_h^n\|_{L^2} + \|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 + h^4).$$

Taking into account these estimates for J_1 to J_5, from (3.27), we get

$$\begin{aligned}&\|\mathbf{e}_h^n\|_{L^2}^2 - \|\mathbf{e}_h^{n-1}\|_{L^2}^2 + \|\widehat{\mathbf{e}}_h^n\|_{L^2}^2 - \|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 + \frac{3\lambda\tau}{2}\|\mathbf{e}_h^n\|_{H^1}^2\\ \leq &\frac{\lambda\tau}{2}\|\nabla\widehat{\mathbf{e}}_h^{n-1}\|_{L^2} + C\tau(\|\mathbf{e}_h^n\|_{L^2} + \|\widehat{\mathbf{e}}_h^{n-1}\|_{L^2}^2 + h^4).\end{aligned}$$

By using (3.22) and the discrete Gronwall inequality, we get

$$\|\mathbf{e}_h^n\|_{L^2} \leq C\exp(CT)h^2 := C_2h^2$$

for some $C_2 > 0$. Thus, we close the mathematical induction and complete the proof of Theorem 3.2. □

3.3. Proof of Theorem 2.1

It is clear that the optimal $\mathbf{L}^2$ error estimate in (2.9) is from the error splitting (3.1):

$$\|\mathbf{m}^n - \mathbf{M}_h^n\|_{L^2} \leq \|\mathbf{e}^n\|_{L^2} + \|\mathbf{E}^n\|_{L^2} + \|\mathbf{e}_h^n\|_{L^2} \leq C(\tau^2 + h^2),$$

where we use (3.4), (3.20) and (3.23). Although the proposed BDF finite element fully scheme (2.6)–(2.7) can not preserve the point-wise constraint $|\mathbf{M}_h^n| = 1$, the convergence rate between 1 and $|\mathbf{M}_h^n|^2$ in $\mathbf{L}^2$-norm can be obtained. In fact, by using (3.26), we have

$$\begin{aligned}\|1 - |\mathbf{M}_h^n|^2\|_{L^2} &= \||\mathbf{m}^n|^2 - |\mathbf{M}_h^n|^2\|_{L^2}\\ &= \|\mathbf{m}^n - \mathbf{M}_h^n\|_{L^2}(\|\mathbf{m}^n\|_{L^\infty} + \|\mathbf{M}_h^n\|_{L^\infty})\\ &\leq C\|\mathbf{m}^n - \mathbf{M}_h^n\|_{L^2}\\ &\leq C(\tau^2 + h^2).\end{aligned}$$

Thus, we complete the proof of Theorem 2.1. □

4. Numerical Results

In this section, we will give the numerical results to verify the optimal $\mathbf{L}^2$ error estimates derived in Theorem 2.1. All programs are implemented by the finite element software FreeFem++ [16]. We consider the Landau-Lifshitz equation in the unit circle $\Omega = \{(x, y) : x^2 + y^2 < 1\}$. The initial value $\mathbf{m}_0$ is taken as

$$\mathbf{m}_0 = (\sin(x)\cos(y), \cos(x)\cos(y), \sin(y)).$$

Gilbert damping constant is set as $\lambda = 1$. We take a uniform triangular partition with M nodes on $\partial\Omega$. We solve the Landau-Lifshitz equation (2.1) and (1.2) by using the linearized second-order BDF scheme (2.6)–(2.7). The final time of computation is $T = 1$.

Since there has no exact solution to (1.1)–(1.2), the reference solution is taken as the numerical solution corresponding to $M = 300$ and $\tau = 1/(20M)$. We take different $M = 25, 50, 100$ and 150 and the time step $\tau = 1/(20M)$. In this case, the optimal error estimates are rewritten as

$$\|\mathbf{m}^n - \mathbf{M}_h^n\|_{L^2} + \|1 - |\mathbf{M}_h^n|^2\|_{L^2} + h\|\mathbf{m}^n - \mathbf{M}_h^n\|_{H^1} \leq Ch^2, \tag{4.1}$$

where the optimal $\mathbf{H}^1$ error estimate is from (3.20), (3.23) and the inverse inequality. The numerical errors are displayed in Tables 1, from which we can see that $\mathbf{L}^2$ convergence rates of $\|\mathbf{m}^n - \mathbf{M}_h^n\|_{L^2}$ and $\|1 - |\mathbf{M}_h^n|^2\|_{L^2}$ are in good agreement with our theoretical analysis in Theorem 2.1. Moreover, $\mathbf{H}^1$ convergence rate has reached the optimal first-order convergence order $\mathcal{O}(h)$.

Table 1. Numerical errors and their spatial convergence rates for different M

M	$\|\|\mathbf{m}(\cdot,1) - \mathbf{M}_h^J\|\|_{L^2}$	$\|\|\mathbf{m}(\cdot,1) - \mathbf{M}_h^J\|\|_{H^1}$	$\|\|1 - \|\mathbf{M}_h^J\|^2\|\|_{L^2}$
25	2.51763E-02	2.77701E-02	4.96490E-02
50	6.15460E-03	1.39321E-02	1.23555E-02
100	1.38291E-03	6.82318E-03	2.96039E-03
150	6.02944E-04	4.37703E-03	1.32224E-03
rate	2.08	1.04	2.02

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