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EXISTENCE AND UPPER SEMI-CONTINUITY OF RANDOM ATTRACTORS FOR NON-AUTONOMOUS STOCHASTIC PLATE EQUATIONS WITH MULTIPLICATIVE NOISE ON $\mathbb{R}^{N}$*

Qiaozhen Ma[1], Xiaobin Yao[2] and Tingting Liu[1,†]

Abstract Based on the abstract theory of random attractors of non-autonomous non-compact dynamical systems, we investigate existence and the upper semi-continuity of random attractors for the non-autonomous stochastic plate equations with multiplicative noise defined on the entire space $\mathbb{R}^n$. We extend and improve the results of [42] not only from the additive white noise to the multiplicative white noise, but also from the time-independent of forcing term $g(x)$ to the time-dependent forcing term $g(x,t)$.

Keywords Random attractor, stochastic plate equation, upper semi-continuity.

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1. Introduction

We consider the following non-autonomous stochastic plate equations with multiplicative noise:

$$u_{tt} + \alpha u_t + \Delta^2 u_t + \Delta^2 u + \lambda u + f(x,u) = g(x,t) + \varepsilon u \circ \frac{dw}{dt},\ x \in \mathbb{R}^n,\ t > \tau, \tag{1.1}$$

and the initial value conditions are raised as follows:

$$u(x,\tau) = u_0(x), \quad u_t(x,\tau) = u_1(x),\ x \in \mathbb{R}^n, \tag{1.2}$$

where $\tau \in \mathbb{R}$, α, λ and ε are positive constants, $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, and w is a two-sided real-valued Wiener process on a probability space. The problem (1.1) is understood in the sense of Stratonovich integration.

The study of plate equations have been paid extensive attention to by some of the researches due to their importance in both the physical and engineering areas such as vibration and elasticity theory of solid mechanics; besides, the long-time dynamics of solutions associated with this problem has also located to an important position

†The corresponding author. Email address: lttnwnu91@126.com (T. Liu)
[1]College of Mathematics and Statistics, Northwest Normal University, AnningDong Road, Lanzhou 730070, China
[2]School of Mathematics and Statistics, Qinghai Nationalities University, Xining 810007, China
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and become more and more outstanding in the field of the infinite-dimensional dynamical systems .

As we know, attractor is a proper concept describing the long-time dynamics of the dynamical systems, and there are many classical literatures and monographs not only for the deterministic but also for the stochastic dynamical systems over the last two decades years, see for instance([1, 2, 6–12, 14–19, 22, 23, 26, 27, 30–32, 36, 41]) and references therein. In order to scrutinize the asymptotic behavior of solution for the stochastic partial differential equations driven by noise, H. Crauel & F. Flandoi([11, 12]), F. Flandoi & B. Schmalfuss([15]) and B. Schmalfuss([27]) et al. introduced a concept of pullback attractors respectively, and established some abstract results proving existence of such attractors([1, 12, 15, 22]). However, the compactness of pullback absorbing set was necessary for obtaining the existence of random attractors if we exploit above mentioned methods, so it could not be used to deal with the stochastic PDEs on unbounded domains. In order to implement such defects, P. W. Bates, H. Lisei & K. Lu presented the concept of asymptotic compactness, and applied this technique into the lattic dynamical systems([5]) and the reaction-diffusion equations on unbounded domain([4]), respectively. B. Wang in [32] further developed the concept of asymptotic compactness, and obtained existence of a unique pullback attractor for the stochastic reaction-diffusion equations with additive noise on $\mathbb{R}^n$. As far as the corresponding other works on stochastic PDEs, we refer to ([13, 14, 30, 33–38, 41]) and references therein.

Only for our problem (1.1)–(1.2), under the deterministic case (i.e., $\varepsilon = 0$), existence of global attractors has been studied by several authors, see for instance [3, 19–21, 39, 40, 43] and reference therein. As far as the stochastic case driven by additive noise, when the forcing term g is independent of time, that is, $g(x,\ t) \equiv g(x)$, existence of random pullback attractors on bounded domain was obtained in [24, 28, 29]. Recently, X. Yao, Q. Ma & T. Liu investigated existence and upper semi-continuity of random attractors for stochastic plate equations with rotational inertia and Kelvin-Voigt dissipative term on an unbounded domain(see [42] for details). To the best of our knowledge, it is not yet considered by any predecessors for the stochastic plate equations with multiplicative noise, so we focus on this problem on unbounded domain in the present paper. It is well known that the multiplicative noise makes the problem more complex and interesting even to the environment of bounded domain. Motivated by the theory and applications of B. Wang in [32, 36, 38], and also based on the works of the stochastic plate equations with rotational inertia and Kelvin-Voight dissipative term on unbounded domains by Yao, Ma & Liu, we are concerned with the existence and upper semi-continuity of random attractors for problem (1.1)–(1.2).

Notice that (1.1) is a non-autonomous stochastic differential equation with the time-dependent external forcing term g, like in [32], we need to introduce two parametric spaces so that describe its dynamics: one is responsible for the deterministic non-autonomous perturbations, and another for the stochastic perturbations. On the other hand, since Sobolev embeddings are not compact on unbounded domain, we can not get the desired asymptotic compactness directly via the regularity of solutions. In order to move these obstacles, we take advantage of the uniform estimates on the tails of solutions outside a bounded ball in $\mathbb{R}^n$ and along with the splitting technique([33]), as well as the compactness methods(that is so called "C-Condition" or "flattening Condition") introduced in [17, 18].

The remainder of this paper is as follows. In the next Section, we recall a sufficient and necessary conditions proving existence of random attractors for cocycle or non-autonomous random dynamical systems. In Section 3, we define a continuous cocycle for (1.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ under the condition that insure the well-posedness of solutions. Then we derive all uniform estimates of solutions in Section 4, and prove the existence of random attractors in Sections 5, Finally, we further show the upper semi-continuity of random attractors in the last Section.

Throughout the paper, we use $||\cdot||$ and $(\cdot,\cdot)$ to denote the norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. The norms of $L^p(\mathbb{R}^n)$ and a Banach space X are generally written as $||\cdot||_p$ and $||\cdot||_X$. The letters c and c_i $(i = 1, 2, \ldots)$ are generic positive constants which may change their values from line to line or even in the same line and do not depend on ε.

2. Preliminaries

In this section, we recall some definitions and known results regarding pullback attractors of non-autonomous random dynamical systems from ([1, 10, 11, 32]) which they are useful to our problem.

In the sequel, we use $(\Omega, \mathcal{F}, \mathcal{P})$ and (X, d) to denote a probability space and a complete separable metric space, respectively. If A and B are two nonempty subsets of X, then we use $d(A, B)$ to denote their Hausdorff semi-distance.

Definition 2.1. Let $\theta : \mathbb{R} \times \Omega \to \Omega$ be a $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable mapping. We say $(\Omega, \mathcal{F}, \mathcal{P}, \theta)$ is a parametric dynamical system if $\theta(0, \cdot)$ is the identity on Ω, $\theta(s + t, \cdot) = \theta(t, \cdot) \circ \theta(s, \cdot)$ for all $t, s \in \mathbb{R}$, and $\mathcal{P}\theta(t, \cdot) = \mathcal{P}$ for all $t \in \mathbb{R}$.

Definition 2.2. Let $K : \mathbb{R}\times\Omega \to 2^X$ be a set-valued mapping with closed nonempty images. We say K is measurable with respect to $\mathcal{F}$ in Ω if the mapping $\omega \in \Omega \to d(x, K(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$.

Definition 2.3. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (1)–(4) are satisfied:

(1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(2) $\Phi(\tau, \tau, \omega, \cdot)$ is the identity on X;

(3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;

(4) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Hereafter, we assume Φ is a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$, and $\mathcal{D}$ is the collection of some families of nonempty bounded subsets of X parameterized by $\tau \in \mathbb{R}$ and $\omega \in \Omega$:

$$\mathcal{D} = \{D = \{D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\}\}.$$

Definition 2.4. (i) A mapping $\Psi : \mathbb{R} \times \mathbb{R} \times \Omega \to X$ is called a complete orbit (solution) of Φ if for every $t \in \mathbb{R}^+, s, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t, \tau + s, \theta_s\omega, \Psi(s, \tau, \omega)) = \Psi(t + s, \tau, \omega).$$

If, in addition, there exists $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ such that $\Psi(t, \tau, \omega)$ belongs to $D(\tau + t, \theta_t\omega)$ for every $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$, then Ψ is called a $\mathcal{D}$-complete orbit (solution) of Φ.

(ii) A mapping $\xi : \mathbb{R} \times \Omega \to X$ is called a complete quasi-solution of Φ if for every $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t, \tau, \omega, \xi(\tau, \omega)) = \xi(\tau + t, \theta_t \omega).$$

If, in addition, there exists $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ such that $\xi(\tau, \omega)$ belongs to $D(\tau, \omega)$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then ξ is called a $\mathcal{D}$-complete quasi-solution of Φ.

Definition 2.5. A collection $\mathcal{D}$ of some families of nonempty subsets of X is said to be inclusion-closed if for each $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the family

$$\{B(\tau, \omega) : \ B(\tau, \omega) \text{ is a nonempty subset of } D(\tau, \omega), \ \forall\, \tau \in \mathbb{R}, \ \forall\, \omega \in \Omega\}$$

also belongs to $\mathcal{D}$.

Definition 2.6. Let $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of nonempty subsets of X. For every $\tau \in \mathbb{R}, \omega \in \Omega$, let

$$\Omega(B, \tau, \omega) = \bigcap_{r \ge 0} \overline{\bigcup_{t \ge r} \Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))}.$$

Then the family $\{\Omega(B, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called the Ω-limit set of B and is denoted by $\Omega(B)$.

Definition 2.7. Let $\mathcal{D}$ be a collection of some families of nonempty bounded subsets of X and $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then K is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $B \in \mathcal{D}$, there exists $T = T(B, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \ \text{ for all } t \ge T.$$

If, in addition, $K(\tau, \omega)$ is closed in X and is measurable in ω with respect to $\mathcal{F}$, then K is called a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 2.8. Let $\mathcal{D}$ be a collection of some families of nonempty bounded subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \ \text{ has a convergent subsequence in } X$$

whenever $t_n \to \infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.9. Let $\mathcal{D}$ be a collection of some families of nonempty bounded subsets of X and $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (1)-(3) are fulfilled: that is, for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(1) $\mathcal{A}(\tau, \omega)$ is compact in X and is measurable in ω with respect to $\mathcal{F}$.

(2) $\mathcal{A}$ is invariant, that is,

$$\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t \omega).$$

(3) For every $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t \to \infty} d(\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0.$$

Proposition 2.1. Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty bounded subsets of X, and Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$. If Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$, then Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ which is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned}\mathcal{A}(\tau,\omega) =&\Omega(K,\tau,\omega) = \bigcup_{B\in\mathcal{D}} \Omega(B,\tau,\omega)\\ =&\{\Psi(0,\tau,\omega) : \Psi \text{ is a } \mathcal{D}\text{-complete solution of } \Phi \text{ under Definition 2.4(i)}\}\\ =&\{\xi(\tau,\omega) : \xi \text{ is a } \mathcal{D}\text{-complete quasi-solution of } \Phi \text{ under Definition 2.4(ii)}\}.\end{aligned}$$

3. Cocycles associated with stochastic plate equation

In this section, we outline some basic settings about (1.1)–(1.2) and show that it generates a continuous cocycle in $\mathcal{H} = \mathcal{H}(\mathbb{R}^n) = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

We define a new norm $\|\cdot\|_{\mathcal{H}}$ by

$$\|Y\|_{\mathcal{H}} = (\|v\|^2 + (\delta^2 + \lambda - \delta\alpha)\|u\|^2 + (1-\delta)\|\Delta u\|^2)^{\frac{1}{2}}, \tag{3.1}$$

for $Y = (u, v)^\top \in \mathcal{H}$, where $\top$ stands for the transposition.

Let $z = u_t + \delta u$, where δ is a small positive constant whose value will be determined later. Substituting $u_t = z - \delta u$ into (1.1) we find

$$\frac{du}{dt} + \delta u = z, \tag{3.2}$$

$$\frac{dz}{dt}+(\alpha-\delta)z+\Delta^2 z+(\delta^2+\lambda-\delta\alpha)u+(1-\delta)\Delta^2 u+f(x,u)=g(x,t)+\varepsilon u\circ\frac{dw}{dt}, \tag{3.3}$$

with the initial value conditions

$$u(x,\tau) = u_0(x), \qquad z(x,\tau) = z_0(x), \tag{3.4}$$

where $z_0(x) = u_1(x) + \delta u_0(x),\ x \in \mathbb{R}^n$.

Let $F(x,u) = \int_0^u f(x,s)ds$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$. The function $f \in C^2(\mathbb{R}^n \times \mathbb{R},\ \mathbb{R})$ will be assumed to satisfy the following conditions:

(F1) $|f(x,u)| \le c_1|u|^\gamma + \phi_1(x)$,

(F2) $f(x,u)u - c_2F(x,u) \ge \phi_2(x)$,

(F3) $F(x,u) \ge c_3|u|^{\gamma+1} - \phi_3(x)$,

(F4) $|f_u(x,u)| \le c_4|u|^{\gamma-1} + \phi_4(x)$,

where $c_i,\ i = 1,\ 2,\ 3,\ 4$, are positive constants, $1 \le \gamma \le \frac{n+4}{n-4}(n \ge 5)$, $\phi_1 \in L^2(\mathbb{R}^n), \phi_2 \in L^1(\mathbb{R}^n), \phi_3 \in L^1(\mathbb{R}^n)$ and $\phi_4 \in H^2(\mathbb{R}^n)$. Note that (F1) and (F2) imply

$$F(x,u) \le c(|u|^2 + |u|^{\gamma+1} + \phi_1^2 + \phi_2). \tag{3.5}$$

We also need the following condition on g like in [32]: there exists a positive constant σ such that

$$\int_{-\infty}^0 e^{\frac{1}{2\gamma+2}\sigma s}\|g(s+\tau,\cdot)\|^2 ds < \infty,\ \forall\ \tau \in \mathbb{R}, \tag{3.6}$$

where γ is a given number by (F1), which implies that

$$\lim_{k\to\infty}\int_{-\infty}^{0}\int_{|x|\geq k} e^{\frac{1}{2\gamma+2}\sigma s}|g(s+\tau,x)|^2 dxds = 0,\ \forall\ \tau\in\mathbb{R}, \tag{3.7}$$

where $|\cdot|$ denotes the absolute value of real number in $\mathbb{R}$. Since $\gamma \geq 1$, by (3.6) it is easy to see that for every $\tau\in\mathbb{R}$,

$$\int_{-\infty}^{0} e^{\frac{1}{4}\sigma s}\|g(s+\tau,\cdot)\|^2 ds < \infty,\ \forall\ \tau\in\mathbb{R}. \tag{3.8}$$

For our purpose, let $(\Omega,\mathcal{F},\mathcal{P})$ be the standard probability space, where $\Omega = \{\omega\in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega,\mathcal{F})$. There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega,\mathcal{F},\mathcal{P})$ which is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega\in\Omega,\ t\in\mathbb{R}.$$

Then $(\Omega,\mathcal{F},\mathcal{P},\{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system.

Now we convert the problem (3.2)–(3.4) into a deterministic system with a random parameter. To this end, we consider the Ornstein-Uhlenbeck equation $dy(\theta_t\omega) + \varepsilon y(\theta_t\omega)dt = dw(t)$, and Ornstein-Uhlenbeck process

$$y(\omega) = -\varepsilon\int_{-\infty}^{0} e^{\varepsilon\tau}\omega(\tau)d\tau. \tag{3.9}$$

From [1, 14, 22], it is known that the random variable $|y(\omega)|$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega}\subset\Omega$ of full $\mathcal{P}$ measure such that $y(\theta_t\omega)$ is continuous in t for every $\omega\in\widetilde{\Omega}$.

Let $v(t,\tau,\omega) = z(t,\tau,\omega) - \varepsilon y(\theta_t\omega)u(t,\tau,\omega)$, we obtain the equivalent system of (3.2)–(3.4),

$$\frac{du}{dt} + \delta u - v = \varepsilon y(\theta_t\omega)u, \tag{3.10}$$

$$\begin{aligned}&\frac{dv}{dt} + (\alpha-\delta)v + \Delta^2 v + (\delta^2+\lambda-\delta\alpha)u + (1-\delta)\Delta^2 u + \varepsilon y(\theta_t\omega)\Delta^2 u + f(x,u)\\ =&g(x,t) - \varepsilon y(\theta_t\omega)v - \varepsilon(\varepsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)u,\end{aligned} \tag{3.11}$$

with the initial value conditions

$$u(x,\tau,\tau) = u_0(x), \qquad v(x,\tau,\tau) = v_0(x), \tag{3.12}$$

where $v_0(x) = z_0(x) - \varepsilon y(\theta_\tau\omega)u_0(x),\ x\in\mathbb{R}^n$.

In line with the standard discussion like in [29, 32, 42], we show that the dynamics of solutions for (1.1)–(1.2) is the same as that of cocycle Φ associated with (3.10)–(3.12) in $\mathcal{H}$, so from now on, we will only consider the dynamics of solutions for (3.10)–(3.12) for $\omega\in\widetilde{\Omega}$ and write $\widetilde{\Omega}$ as Ω for brevity.

The well-posedness of (3.10)–(3.12) in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ can be established by standard methods as in [5, 20, 25, 29, 36], more precisely, if (F1)–(F4) are fulfilled, then we obtain the following Lemma.

Lemma 3.1. *Put* $\varphi(t+s,\tau,\theta_{-s}\omega,\varphi_0) = (u(t+s,\tau,\theta_{-s}\omega,u_0), v(t+s,\tau,\theta_{-s}\omega,v_0))^\top$, *where* $\varphi_0 = (u_0,v_0)^\top$, *and let* (F1)–(F4) *hold. Then for every* $\omega \in \Omega$, $\tau \in \mathbb{R}$ *and* $\varphi_0 \in \mathcal{H}(\mathbb{R}^n)$, *problem* (3.10)–(3.12) *has a unique* $(\mathcal{F},\ \mathcal{B}(H^2(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$-*measurable solution* $\varphi(\cdot,\tau,\omega,\varphi_0) \in C([\tau,\infty),\mathcal{H}(\mathbb{R}^n))$ *with* $\varphi(\tau,\tau,\omega,\varphi_0) = \varphi_0$, $\varphi(t,\tau,\omega,\varphi_0) \in \mathcal{H}(\mathbb{R}^n)$ *being continuous in* φ_0 *with respect to the usual norm of* $\mathcal{H}(\mathbb{R}^n)$ *for each* $t > \tau$. *Moreover, for every* $(t,\tau,\omega,(u_0,v_0)) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{H}(\mathbb{R}^n)$, *the mapping*

$$\Phi(t,\tau,\omega,(u_0,v_0)) = (u(t+\tau,\tau,\theta_{-\tau}\omega,u_0), v(t+\tau,\tau,\theta_{-\tau}\omega,v_0)) \tag{3.13}$$

generates a continuous cocycle for (3.2)–(3.4) *from* $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{H}(\mathbb{R}^n)$ *to* $\mathcal{H}(\mathbb{R}^n)$ *over* $\mathbb{R}$ *and* $(\Omega,\mathcal{F},\mathcal{P},\{\theta_t\}_{t\in\mathbb{R}})$, *where* v *is defined by*

$$v(t+\tau,\tau,\theta_{-\tau}\omega,z_0) = z(t+\tau,\tau,\theta_{-\tau}\omega,v_0) - \varepsilon y(\theta_t\omega)u(t+\tau,\tau,\theta_{-\tau}\omega,u_0) \tag{3.14}$$

with $v_0 = z_0 - \varepsilon y(\omega)u_0$. *By* (3.13)–(3.14), *for each* $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$, *we have*

$$\begin{aligned}\Phi(t,\tau-t,\theta_{-t}\omega,(u_0,v_0)) &= (u(\tau,\tau-t,\theta_{-\tau}\omega,u_0), v(\tau,\tau-t,\theta_{-\tau}\omega,v_0))\\ &=(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0), z(\tau,\tau-t,\theta_{-\tau}\omega,z_0) - \varepsilon y(\omega)u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)).\end{aligned} \tag{3.15}$$

Let δ be small enough such that

$$\delta^2 + \lambda - \delta\alpha > 0, \quad 1 - \delta > 0,$$

and define σ appearing in (3.6) by

$$\sigma = \min\{\frac{\alpha-\delta}{8}, \frac{\delta}{4}, \frac{c_2\delta}{8}\}, \tag{3.16}$$

where c_2 is the positive constant in (F2).

Given a bounded nonempty subset B of $\mathcal{H}$, we write $\|B\| = \sup_{\phi\in B}\|\phi\|_{\mathcal{H}}$. Let $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $\mathcal{H}$ such that for every $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\lim_{s\to\infty} e^{-\sigma s}\|D(\tau - s, \theta_{-s}\omega)\|^{\gamma+1} = 0. \tag{3.17}$$

Let $\mathcal{D}$ be the collection of all such families, that is,

$$\mathcal{D} = \{D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} :\ D \text{ satisfies } (3.17)\}. \tag{3.18}$$

By (3.18), we see $\mathcal{D}$ is inclusion-closed.

4. Uniform estimates of solutions

In this subsection, we derive uniform estimates on the solutions of problem (3.10)–(3.12) defined on $\mathbb{R}^n$ when $t \to \infty$. These estimates are necessary for proving the existence of bounded absorbing sets and the asymptotic compactness of the random dynamical system associated with the equations. In particular, we will show that the tails of the solutions for large space variables are uniformly small when time is sufficiently large. We first obtain the estimates in $\mathcal{H}$.

Lemma 4.1. *Assume that (F1)–(F4) and* (3.6) *hold. Let* $\tau \in \mathbb{R}, \omega \in \Omega$, *and* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. *Then there exists* $T = T(\tau,\omega,D) > 0$ *such that for all* $t \geq T$, *the solution* $(u,v)^T$ *of problem* (3.10)–(3.12) *satisfies*

$$\begin{aligned}
&\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}\\
&+\int_{\tau-t}^{\tau} e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}ds\\
&+\int_{\tau-t}^{\tau} e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2 ds\\
\leq& R(\tau,\omega),
\end{aligned} \tag{4.1}$$

where $\varphi_0 = (u_0,v_0)^\top \in D(\tau-t,\theta_{-t}\omega)$ *and* $R(\tau,\omega)$ *is given by*

$$R(\tau,\omega) = M + M\int_{-\infty}^0 e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(s+\tau,\cdot)\|^2+\varepsilon|y(\theta_s\omega)|^2)ds, \tag{4.2}$$

where M *and* c *are positive constants depending on* λ,σ,α *and* δ, *but independent of* τ,ω,D *and* ε. *In addition, the random variable* R *has the property:*

$$\lim_{t\to\infty} e^{-\frac{1}{\gamma+1}\sigma t}R(\tau-t,\theta_{-t}\omega) = 0.$$

Proof. Taking the inner product of (3.11) with v in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|^2 + (\alpha-\delta)(v,v) + (\lambda+\delta^2-\delta\alpha)(u,v) + (1-\delta)(\Delta^2 u,v)\\
&+(\Delta^2 v,v) + \varepsilon y(\theta_t\omega)(\Delta^2 u,v) + (f(x,u),v)\\
=&(g(x,t),v) - \varepsilon y(\theta_t\omega)(v,v) - \varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(u,v).
\end{aligned} \tag{4.3}$$

By (3.10), we have

$$v = \frac{du}{dt} + \delta u - \varepsilon y(\theta_t\omega)u. \tag{4.4}$$

Then substituting the above v into the third, fourth and last terms on the left-hand side of (4.3), there holds

$$(u,v) = (u,\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u) = \frac{1}{2}\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 - \varepsilon y(\theta_t\omega)\|u\|^2, \tag{4.5}$$

$$(\Delta^2 u,v) = (\Delta^2 u,\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u) = \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \delta\|\Delta u\|^2 - \varepsilon y(\theta_t\omega)\|\Delta u\|^2, \tag{4.6}$$

$$\begin{aligned}
(f(x,u),v) &= (f(x,u),\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u)\\
&= \frac{d}{dt}\int_{\mathbb{R}^n}F(x,u)dx + \delta(f(x,u),u) - \varepsilon y(\theta_t\omega)(f(x,u),u).
\end{aligned} \tag{4.7}$$

It follows from (4.3)–(4.7) that

$$\begin{aligned}
&\frac{d}{dt}(\|v\|^2 + (\delta^2+\lambda-\delta\alpha)\|u\|^2 + (1-\delta)\|\Delta u\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx)\\
&+2(\alpha-\delta)\|v\|^2 + 2\delta(\delta^2+\lambda-\delta\alpha)\|u\|^2 + 2\delta(1-\delta)\|\Delta u\|^2 + 2\delta(f(x,u),u) + 2\|\Delta v\|^2\\
=&2(g,v) - 2\varepsilon y(\theta_t\omega)\|v\|^2 - 2\varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(u,v) - 2\varepsilon y(\theta_t\omega)(\Delta^2 u,v)
\end{aligned}$$

$$+ 2\varepsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\|u\|^2 + 2\varepsilon(1-\delta)y(\theta_t\omega)\|\Delta u\|^2 + 2\varepsilon y(\theta_t\omega)(f(x,u),u). \tag{4.8}$$

For the last term on the right-hand side of (4.8), by (F1) and (F3), we have

$$\begin{aligned}
&2\varepsilon y(\theta_t\omega)(f(x,u),u)\\
\le\ & 2\varepsilon c_1|y(\theta_t\omega)|\int_{\mathbb{R}^n}|u|^{\gamma+1}dx + \varepsilon|y(\theta_t\omega)|\|\phi_1\|^2 + \varepsilon|y(\theta_t\omega)|\|u\|^2\\
\le\ & 2\varepsilon c_1 c_3^{-1}|y(\theta_t\omega)|\int_{\mathbb{R}^n}(F(x,u)+\phi_3(x))dx + \varepsilon|y(\theta_t\omega)|\|\phi_1\|^2 + \varepsilon|y(\theta_t\omega)|\|u\|^2\\
\le\ & \varepsilon c|y(\theta_t\omega)|\int_{\mathbb{R}^n}F(x,u)dx + \varepsilon c|y(\theta_t\omega)| + \varepsilon|y(\theta_t\omega)|\|u\|^2,
\end{aligned} \tag{4.9}$$

where c depends on c_1, c_3^{-1}, $\|\phi_3\|_{L^1(\mathbb{R}^n)}$ and $\|\phi_1\|_{L^2(\mathbb{R}^n)}$. In line with Young's inequality, we find

$$\begin{aligned}
&2(g,v) - 2\varepsilon y(\theta_t\omega)\|v\|^2 - 2\varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(u,v) - 2\varepsilon y(\theta_t\omega)(\Delta^2 u,v)\\
&+ 2\varepsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\|u\|^2 + 2\varepsilon(1-\delta)y(\theta_t\omega)\|\Delta u\|^2\\
\le&(\alpha-\delta)\|v\|^2 + c\|g\|^2 + \varepsilon c|y(\theta_t\omega)|(1+|y(\theta_t\omega)|)(\|u\|^2+\|v\|^2+\|\Delta u\|^2) + \|\Delta v\|^2.
\end{aligned} \tag{4.10}$$

Then by (F2), (4.8)–(4.10) we get

$$\begin{aligned}
&\frac{d}{dt}(\|v\|^2 + (\delta^2+\lambda-\delta\alpha)\|u\|^2 + (1-\delta)\|\Delta u\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx)\\
&+(\alpha-\delta)\|v\|^2+2\delta(\delta^2+\lambda-\delta\alpha)\|u\|^2+2\delta(1-\delta)\|\Delta u\|^2+2\delta c_2\int_{\mathbb{R}^n}F(x,u)dx+\|\Delta v\|^2\\
\le& c+c\|g\|^2+\varepsilon c|y(\theta_t\omega)|\int_{\mathbb{R}^n}F(x,u)dx+\varepsilon c(1+|y(\theta_t\omega)|^2)(1+\|u\|^2+\|v\|^2+\|\Delta u\|^2).
\end{aligned} \tag{4.11}$$

According to (F3) we know $F(x,u)+\phi_3(x)\ge 0$ for all $x\in\mathbb{R}^n$ and $u\in\mathbb{R}$. This along with (3.16) implies

$$\delta c_2\int_{\mathbb{R}^n}(F(x,u)+\phi_3(x))dx \ge 4\sigma\int_{\mathbb{R}^n}(F(x,u)+\phi_3(x))dx,$$

that is,

$$\delta c_2\int_{\mathbb{R}^n}F(x,u)dx \ge 4\sigma\int_{\mathbb{R}^n}F(x,u)dx + (4\sigma-\delta c_2)\int_{\mathbb{R}^n}\phi_3(x)dx. \tag{4.12}$$

Thus, combining with (4.11) and (4.12), it leads to

$$\begin{aligned}
&\frac{d}{dt}(\|v\|^2 + (\delta^2+\lambda-\delta\alpha)\|u\|^2 + (1-\delta)\|\Delta u\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx)\\
&+ 4\sigma(\|v\|^2 + (\delta^2+\lambda-\delta\alpha)\|u\|^2 + (1-\delta)\|\Delta u\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx) + \|\Delta v\|^2\\
\le& c + c\|g\|^2 + \varepsilon c|y(\theta_t\omega)|\int_{\mathbb{R}^n}F(x,u)dx
\end{aligned}$$

$$+\varepsilon c(1+|y(\theta_t\omega)|^2)(1+\|u\|^2+\|v\|^2+\|\Delta u\|^2)+(\delta c_2-4\sigma)\int_{\mathbb{R}^n}\phi_3(x)dx$$
$$\leq \varepsilon c(1+|y(\theta_t\omega)|^2)(\|v\|^2+(\delta^2+\lambda-\delta\alpha)\|u\|^2+(1-\delta)\|\Delta u\|^2+2\int_{\mathbb{R}^n}F(x,u)dx)$$
$$+c(1+\|g\|^2+\varepsilon|y(\theta_t\omega)|^2). \tag{4.13}$$

Recalling the norm $\|\cdot\|_{\mathcal{H}}$ in (3.1), we obtain from (4.13) that

$$\frac{d}{dt}(\|\varphi\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u)dx)+4\sigma(\|\varphi\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u)dx)+\|\Delta v\|^2$$
$$\leq \varepsilon c(1+|y(\theta_t\omega)|^2)(\|\varphi\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u)dx)+c(1+\|g\|^2+\varepsilon|y(\theta_t\omega)|^2). \tag{4.14}$$

Multiplying (4.14) by $e^{\int_0^t(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}$ and then integrating over $(\tau-t,\tau)$ with $t\geq 0$, we get

$$\|\varphi(\tau,\tau-t,\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u(\tau,\tau-t,\omega,u_0))dx$$
$$+2\sigma\int_{\tau-t}^{\tau}e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(\|\varphi(s,\tau-t,\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}$$
$$+2\int_{\mathbb{R}^n}F(x,u(s,\tau-t,\omega,\varphi_0))dx)ds$$
$$+\int_{\tau-t}^{\tau}e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\|\Delta v(s,\tau-t,\omega,v_0)\|^2ds$$
$$\leq e^{\int_\tau^{\tau-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx)$$
$$+c\int_{\tau-t}^{\tau}e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(s,\cdot)\|^2+\varepsilon|y(\theta_s\omega)|^2)ds. \tag{4.15}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above inequality we claim that for every $t\in\mathbb{R}^+,\ \tau\in\mathbb{R},$ and $\omega\in\Omega$,

$$\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))dx$$
$$+2\sigma\int_{\tau-t}^{\tau}e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}(\|\varphi(s,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}$$
$$+2\int_{\mathbb{R}^n}F(x,u(s,\tau-t,\theta_{-\tau}\omega,\varphi_0))dx)ds$$
$$+\int_{\tau-t}^{\tau}e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2ds$$
$$\leq e^{\int_\tau^{\tau-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx)$$
$$+c\int_{\tau-t}^{\tau}e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}(1+\|g(s,\cdot)\|^2+\varepsilon|y(\theta_{s-\tau}\omega)|^2)ds$$
$$\leq e^{\int_0^{-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx)$$
$$+c\int_{-t}^{0}e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(s+\tau,\cdot)\|^2+\varepsilon|y(\theta_s\omega)|^2)ds. \tag{4.16}$$

Since $|y(\theta_t\omega)|$ is stationary and ergodic, we get from (3.9) and the ergodic theorem (see [14] for details) that

$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}|y(\theta_r\omega)|^2dr=E(|y(\omega)|^2)=\frac{1}{2\alpha}. \tag{4.17}$$

By (4.17), there exists $T_1(\omega)>0$ such that for all $t\geq T_1(\omega)$,

$$\int_{-t}^{0}|y(\theta_r\omega)|^2dr<\frac{1}{\alpha}t. \tag{4.18}$$

Let

$$\varepsilon\leq\min\{1,\ \frac{\alpha\sigma}{c(1+\alpha)},\ \frac{(2\gamma+1)\sigma}{c(1+\gamma)},\ \frac{\alpha\sigma}{2c(1+\gamma)}\}, \tag{4.19}$$

where c is the positive number in (4.16). By virtue of (4.18)–(4.19), we have

$$e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\leq e^{\sigma s},\ \text{for all } s\leq -T_1. \tag{4.20}$$

$|y(\theta_t\omega)|$ is tempered, so integer with (3.7) and (4.20) we conclude that the following integral

$$R_1(\tau,\omega)=c\int_{-\infty}^{0}e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(s+\tau,\cdot)\|^2+\varepsilon|y(\theta_s\omega)|^2)ds \tag{4.21}$$

is convergent. Due to $D\in\mathcal{D}$, and $(u_0,v_0)^\top\in D(\tau-t,\theta_{-t}\omega)$, we infer from (3.5), (4.20) and (4.21) that, for all $t\geq T_1$,

$$\begin{aligned}&e^{\int_0^{-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx)\\ \leq&ce^{-\sigma t}(1+\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n)}+\|u_0\|^2_{H^2}+\|u_0\|^{\gamma+1}_{H^2})\\ \leq&ce^{-\sigma t}(1+\|D(\tau-t,\theta_{-t}\omega)\|^2+\|D(\tau-t,\theta_{-t}\omega)\|^{\gamma+1})\to 0,\quad \text{as } t\to\infty.\end{aligned} \tag{4.22}$$

Therefore, it follows from (4.16), (4.21) and (4.22) that there exists $T_2=T_2(\tau,\omega,D)\geq T_1$ such that for all $t\geq T_2$,

$$\begin{aligned}&\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))dx\\ &+2\sigma\int_{\tau-t}^{\tau}e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}(\|\varphi(s,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}\\ &+2\int_{\mathbb{R}^n}F(x,u(s,\tau-t,\theta_{-\tau}\omega,\varphi_0))dx)ds\\ &+\int_{\tau-t}^{\tau}e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2ds\\ \leq&1+R_1(\tau,\omega),\end{aligned} \tag{4.23}$$

which along with (F3) implies that

$$\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&+\int_{\tau-t}^{\tau} e^{\int_\tau^s (2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\|\varphi(s,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}ds\\
&+\int_{\tau-t}^{\tau} e^{\int_\tau^s (2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2 ds\\
\le& c(1+R_1(\tau,\omega)). \qquad (4.24)
\end{aligned}
$$

Next, we prove $R_1(\tau,\omega)$ has the following property: for every $\tau\in\mathbb{R},\ \omega\in\Omega$,

$$
\lim_{t\to\infty} e^{-\frac{1}{\gamma+1}\sigma t} R_1(\tau-t,\theta_{-t}\omega)=0. \qquad (4.25)
$$

Collecting all (4.18)–(4.21) we get, for every $t\ge T_1,\ \tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$
\begin{aligned}
&R_1(\tau-t,\theta_{-t}\omega)\\
=&c\int_{-\infty}^{0} e^{\int_0^s (2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-t}\omega)|^2)dr}(1+\|g(s+\tau-t,\cdot)\|^2+\varepsilon|y(\theta_{s-t}\omega)|^2)ds\\
=&c\int_{-\infty}^{-t} e^{\int_{-t}^s (2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r}\omega)|^2)dr}(1+\|g(s+\tau,\cdot)\|^2+\varepsilon|y(\theta_{s}\omega)|^2)ds\\
\le&c\int_{-\infty}^{-t} e^{(2\sigma-\varepsilon c)(s+t)+\varepsilon c\int_s^0 |y(\theta_r\omega)|^2dr}(1+\|g(s+\tau,\cdot)\|^2+\varepsilon|y(\theta_{s}\omega)|^2)ds\\
\le&c\int_{-\infty}^{-t} e^{(2\sigma-\varepsilon c)(s+t)-\frac{\varepsilon c}{\alpha}s}(1+\|g(s+\tau,\cdot)\|^2+\varepsilon|y(\theta_{s}\omega)|^2)ds\\
\le&c\int_{-\infty}^{-t} e^{\frac{1}{\gamma+1}\sigma(s+t)-\frac{\varepsilon c}{\alpha}s}(1+\|g(s+\tau,\cdot)\|^2+\varepsilon|y(\theta_{s}\omega)|^2)ds\\
\le&ce^{\frac{1}{\gamma+1}\sigma t}\int_{-\infty}^{-t} e^{\frac{1}{\gamma+1}\sigma s-\frac{\varepsilon c}{\alpha}s}(1+\|g(s+\tau,\cdot)\|^2+\varepsilon|y(\theta_{s}\omega)|^2)ds\\
\le&ce^{\frac{1}{\gamma+1}\sigma t}\int_{-\infty}^{-t} e^{\frac{1}{2\gamma+2}\sigma s}(1+\|g(s+\tau,\cdot)\|^2+\varepsilon|y(\theta_{s}\omega)|^2)ds, \qquad (4.26)
\end{aligned}
$$

which along with (3.6) implies (4.25). □

The following estimates are used to prove pullback asymptotic compactness of solutions.

Lemma 4.2. *Assume that (F1)–(F4) and* (3.6) *hold. Let $\tau\in\mathbb{R},\ \omega\in\Omega$, and $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$. Then there exists $T=T(\tau,\omega,D)>0$ such that for all $t\ge T$ and $s\in[-t,0]$, the solution (u,v) of problem* (3.10)–(3.12) *satisfies*

$$
\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}\le M+e^{\int_s^0 (2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}R_2(\tau,\omega),
$$

where $\varphi_0=(u_0,v_0)^\top\in D(\tau-t,\theta_{-t}\omega)$, M and c are positive constants independent of τ,ω,D and ε, and $R_2(\tau,\omega)$ is given by (4.29).

Proof. Multiplying (4.14) by $e^{\int_0^t (2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}$ and then integrating over $(\tau-t,\tau+s)$ with $t\ge0$ and $s\in[-t,0]$, we deduce

$$
\begin{aligned}
&\|\varphi(\tau+s,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_0))dx\\
\le& e^{\int_{\tau+s}^{\tau-t} (2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx)
\end{aligned}
$$

$$
\begin{aligned}
&+c\int_{\tau-t}^{\tau+s} e^{\int_{\tau+s}^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}(1+\|g(\xi,\cdot)\|^2+|y(\theta_{\xi-\tau}\omega)|^2)d\xi\\
\leq& e^{\int_s^{-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx)\\
&+c\int_{-t}^{s} e^{\int_s^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(\xi+\tau,\cdot)\|^2+|y(\theta_\xi\omega)|^2)d\xi. \qquad (4.27)
\end{aligned}
$$

On one hand, for all $t\geq T_1$, $s\in[-t,0]$, we have the following estimates for the last term on the right-hand side of (4.27):

$$
\begin{aligned}
&c\int_{-t}^{s} e^{\int_s^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(\xi+\tau,\cdot)\|^2+|y(\theta_\xi\omega)|^2)d\xi\\
=&c\int_{-t}^{-T_1} e^{\int_s^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(\xi+\tau,\cdot)\|^2+|y(\theta_\xi\omega)|^2)d\xi\\
&+c\int_{-T_1}^{s} e^{\int_s^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(\xi+\tau,\cdot)\|^2+|y(\theta_\xi\omega)|^2)d\xi\\
\leq& ce^{\int_s^{0}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{-t}^{-T_1} e^{\int_0^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(\xi+\tau,\cdot)\|^2)d\xi\\
&+ce^{\int_s^{0}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{-t}^{-T_1} e^{\int_0^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}|y(\theta_\xi\omega)|^2d\xi\\
&+ce^{\int_s^{0}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{-T_1}^{0} e^{\int_0^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(\xi+\tau,\cdot)\|^2)d\xi\\
&+ce^{\int_s^{0}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{-T_1}^{0} e^{\int_0^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}|y(\theta_\xi\omega)|^2d\xi\\
\leq& ce^{\int_s^{0}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{-t}^{-T_1} e^{\sigma\xi}(1+\|g(\xi+\tau,\cdot)\|^2+|y(\theta_\xi\omega)|^2)d\xi\\
&+ce^{\int_s^{0}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{-T_1}^{0} e^{\int_0^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(\xi+\tau,\cdot)\|^2)d\xi\\
&+ce^{\int_s^{0}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{-T_1}^{0} e^{\int_0^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}|y(\theta_\xi\omega)|^2d\xi\\
\leq& e^{\int_s^{0}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}R_2(\tau,\omega), \qquad (4.28)
\end{aligned}
$$

where $R_2(\tau,\omega)$ is given by

$$
\begin{aligned}
R_2(\tau,\omega)=&c\int_{-\infty}^{0} e^{\frac{1}{4}\sigma\xi}(1+\|g(\xi+\tau,\cdot)\|^2+|y(\theta_\xi\omega)|^2)d\xi\\
&+c\int_{-T_1}^{0} e^{\int_0^{\xi}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|g(\xi+\tau,\cdot)\|^2+|y(\theta_\xi\omega)|^2)d\xi. \qquad (4.29)
\end{aligned}
$$

On the other hand, there exists $T_2=T_2(\tau,\omega,D)\geq T_1$ such for all $t\geq T_2$

$$
\begin{aligned}
&e^{\int_s^{-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx)\\
\leq& ce^{\int_s^{0}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}e^{\int_0^{-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\|D(\tau-t,\theta_{-t}\omega)\|^{\gamma+1}
\end{aligned}
$$

$$\leq e^{\int_s^0 (2\sigma - \varepsilon c - \varepsilon c|y(\theta_r \omega)|^2) dr} R_2(\tau, \omega). \tag{4.30}$$

It follows from (4.28)-(4.30) that, for all $t \geq T_2,\ s \in [-t, 0]$

$$\begin{aligned}&\|\varphi(\tau+s, \tau-t, \theta_{-\tau}\omega, \varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n)} + 2\int_{\mathbb{R}^n} F(x, u(\tau+s, \tau-t, \theta_{-\tau}\omega, u_0))dx \\ \leq& 2e^{\int_s^0 (2\sigma - \varepsilon c - \varepsilon c|y(\theta_r \omega)|^2) dr} R_2(\tau, \omega),\end{aligned} \tag{4.31}$$

which along with (F3) complete the proof. □

Next, we establish uniform smallness of solutions for large space and time variables. These estimates are important for proving asymptotic compactness of solutions on the unbounded domain $\mathbb{R}^n$. For simplicity, we denote $Q_k = \{x \in \mathbb{R}^n : |x| \leq k\}$ and $\mathbb{R}^n \setminus Q_k$ the complement of Q_k in the sequel.

Lemma 4.3. *Assume that (F1)–(F4) and* (3.6) *hold. Let any* $\eta > 0, \tau \in \mathbb{R}, \omega \in \Omega$ *and* $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. *Then there exists* $T = T(\tau, \omega, D, \eta) > 0$ *and* $K = K(\tau, \omega, \eta) \geq 1$ *such that for all* $t \geq T,\ k \geq K$, *the solution* (u, v) *of problem* (3.10)-(3.12) *satisfies*

$$\|\varphi(\tau, \tau-t, \theta_{-\tau}\omega, \varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n \setminus Q_k)} \leq \eta, \tag{4.32}$$

where $\varphi_0 = (u_0, v_0)^\top \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Choose a smooth function ρ, such that $0 \leq \rho \leq 1$ for any $s \in \mathbb{R}$, and

$$\rho(s) = \begin{cases} 0, & \text{if } \ 0 \leq |s| \leq 1, \\ 1, & \text{if } \ |s| \geq 2, \end{cases} \tag{4.33}$$

and there exist positive constants $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ such that $|\rho'(s)| \leq \mu_1,\ |\rho''(s)| \leq \mu_2,\ |\rho'''(s)| \leq \mu_3,\ |\rho''''(s)| \leq \mu_4$ for any $s \in \mathbb{R}$.

Taking the inner product of (3.11) with $\rho(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx + (\alpha - \delta)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx \\ &+ (\lambda + \delta^2 - \delta\alpha)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) uv dx + (1-\delta)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) v\Delta^2 u dx \\ &+ \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) v\Delta^2 v dx + \varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) v\Delta^2 u dx + \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) f(x,u) v dx \\ =& \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) g v dx - \varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx - \varepsilon(\varepsilon y(\theta_t\omega) - 2\delta) y(\theta_t\omega)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) uv dx.\end{aligned} \tag{4.34}$$

Substituting $v = \frac{du}{dt} + \delta u - \varepsilon y(\theta_t(\omega)u$ into the third, fourth and seventh terms on the left-hand side of (4.34), using Young inequality and the Sobolev interpolation inequality

$$\|\nabla v\| \leq \varsigma\|v\| + C_\varsigma\|\Delta v\|, \quad \forall\ \varsigma > 0, \tag{4.35}$$

we conclude that

$$\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) uv dx$$

$$=\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})u(\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u)dx$$
$$=\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(\frac{1}{2}\frac{d}{dt}u^2+\delta u^2-\varepsilon y(\theta_t\omega)u^2)dx$$
$$=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx-\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx, \quad (4.36)$$

and

$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})v(\Delta^2u)dx$$
$$=\int_{\mathbb{R}^n}(\Delta^2u)\rho(\frac{|x|^2}{k^2})(\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u)dx$$
$$=\int_{\mathbb{R}^n}(\Delta u)\Delta(\rho(\frac{|x|^2}{k^2})(\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u))dx$$
$$=\int_{\mathbb{R}^n}(\Delta u)((\frac{2}{k^2}\rho'(\frac{|x|^2}{k^2})+\frac{4x^2}{k^4}\rho''(\frac{|x|^2}{k^2}))(\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u)$$
$$+2\cdot\frac{2|x|}{k^2}\rho'(\frac{|x|^2}{k^2})\nabla(\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u)+\rho(\frac{|x|^2}{k^2})\Delta(\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u))dx$$
$$\geq-\int_{k<x<\sqrt{2}k}(\frac{2\mu_1}{k^2}+\frac{4\mu_2x^2}{k^4})|(\Delta u)v|dx-\int_{k<x<\sqrt{2}k}\frac{4\mu_1x}{k^2}|(\Delta u)(\nabla v)|dx$$
$$+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx-\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$\geq-\int_{\mathbb{R}^n}(\frac{2\mu_1+8\mu_2}{k^2})|(\Delta u)v|dx-\int_{\mathbb{R}^n}\frac{4\sqrt{2}\mu_1}{k}|(\Delta u)(\nabla v)|dx+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx-\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$\geq-\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)-\frac{4\sqrt{2}\mu_1}{k}\|\Delta u\|\|\nabla v\|+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx-\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$\geq-\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)-\frac{4\sqrt{2}\mu_1}{k}\|\Delta u\|(\varsigma\|v\|+C_\varsigma\|\Delta v\|)+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx-\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$\geq-\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)-\frac{2\sqrt{2}\mu_1}{k}(\|\Delta u\|^2+2\varsigma^2\|v\|^2+2C_\varsigma^2\|\Delta v\|^2)$$
$$+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx-\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx, \quad (4.37)$$

and

$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)vdx$$

$$=\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)(\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u)dx$$
$$=\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx-\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx. \tag{4.38}$$

Exploiting (4.34) and (4.36)-(4.38) we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v|^2+(\delta^2+\lambda-\delta\alpha)|u|^2+(1-\delta)|\Delta u|^2+2F(x,u))dx\\&+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(2(\alpha-\delta)|v|^2+2\delta(\delta^2+\lambda-\delta\alpha)|u|^2+2\delta(1-\delta)|\Delta u|^2+2\delta f(x,u)u))dx\\ \le&2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})gvdx-2\varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx\\&-2\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+2\varepsilon(1-\delta)y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx\\&+2\varepsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx+2\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx\\&-2\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})v\Delta^2udx-2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})v\Delta^2vdx\\&+2(1-\delta)\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)+2(1-\delta)\frac{2\sqrt{2}\mu_1}{k}(\|\Delta u\|^2+2\varsigma^2\|v\|^2+2C_\varsigma^2\|\Delta v\|^2).\end{aligned} \tag{4.39}$$

We now estimate the nonlinear terms of (4.39). First,using (F2) we have

$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx\ge c_2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\phi_2(x)dx. \tag{4.40}$$

According to (F1) and (F3), it follows that

$$\begin{aligned}&2\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx\\ \le&\varepsilon cy(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx+\varepsilon cy(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx\\&+\varepsilon cy(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\phi_1|^2+|\phi_3|)dx.\end{aligned} \tag{4.41}$$

For the seventh and eighth terms on the right-hand side of (4.39), using Young inequality and (4.35) we conclude that

$$\begin{aligned}&-2\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})v\Delta^2udx\\=&-2\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}(\Delta u)\Delta\big(\rho(\frac{|x|^2}{k^2})v\big)dx\\=&-2\varepsilon y(\theta_t\omega)\int_{\mathbb{R}^n}(\Delta u)((\frac{2}{k^2}\rho'(\frac{|x|^2}{k^2})+\frac{4x^2}{k^4}\rho''(\frac{|x|^2}{k^2}))v\end{aligned}$$

$$
\begin{aligned}
&+ 2 \cdot \frac{2|x|}{k^2}\rho'(\frac{|x|^2}{k^2})\nabla v + \rho(\frac{|x|^2}{k^2})\Delta v)dx \\
\leq& 2|\varepsilon||y(\theta_t\omega)|(\int_{k<x<\sqrt{2}k}(\frac{2\mu_1}{k^2} + \frac{4\mu_2 x^2}{k^4})|(\Delta u)v|dx \\
&+ \int_{k<x<\sqrt{2}k}\frac{4\mu_1 x}{k^2}|(\Delta u)(\nabla v)|dx + \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u||\Delta v|dx) \\
\leq& 2|\varepsilon||y(\theta_t\omega)|\int_{\mathbb{R}^n}(\frac{2\mu_1 + 8\mu_2}{k^2})|(\Delta u)v|dx + 2|\varepsilon||y(\theta_t\omega)|\int_{\mathbb{R}^n}\frac{4\sqrt{2}\mu_1}{k}|(\Delta u)(\nabla v)|dx \\
&+ 2|\varepsilon||y(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u||\Delta v|dx \\
\leq& \frac{2\mu_1 + 8\mu_2}{k^2}|\varepsilon||y(\theta_t\omega)|(\|\Delta u\|^2 + \|v\|^2) + \frac{8\sqrt{2}\mu_1}{k}|\varepsilon||y(\theta_t\omega)|\|\Delta u\|\|\nabla v\| \\
&+ |\varepsilon||y(\theta_t\omega)|^2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx + \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta v|^2dx \\
\leq& \frac{2\mu_1 + 8\mu_2}{k^2}|\varepsilon||y(\theta_t\omega)|(\|\Delta u\|^2 + \|v\|^2) + \frac{8\sqrt{2}\mu_1}{k}|\varepsilon||y(\theta_t\omega)|\|\Delta u\|(\varsigma\|v\| + C_\varsigma\|\Delta v\|) \\
&+ |\varepsilon||y(\theta_t\omega)|^2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx + \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta v|^2dx \\
\leq& \frac{2\mu_1 + 8\mu_2}{k^2}|\varepsilon||y(\theta_t\omega)|(\|\Delta u\|^2 + \|v\|^2) + \frac{4\sqrt{2}\mu_1}{k}|\varepsilon||y(\theta_t\omega)|(\|\Delta u\|^2 + 2\varsigma^2\|v\|^2 \\
&+ 2C_\varsigma^2\|\Delta v\|^2) + |\varepsilon||y(\theta_t\omega)|^2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx + \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta v|^2dx, \qquad (4.42)
\end{aligned}
$$

and

$$
\begin{aligned}
&- \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})v\Delta^2 v dx \\
=& - \int_{\mathbb{R}^n}(\Delta v)\Delta(\rho(\frac{|x|^2}{k^2})v))dx \\
=& - \int_{\mathbb{R}^n}(\Delta v)((\frac{2}{k^2}\rho'(\frac{|x|^2}{k^2}) + \frac{4x^2}{k^4}\rho''(\frac{|x|^2}{k^2}))v + 2 \cdot \frac{2|x|}{k^2}\rho'(\frac{|x|^2}{k^2})\nabla v + \rho(\frac{|x|^2}{k^2})\Delta v)dx \\
\leq& \int_{k<x<\sqrt{2}k}(\frac{2\mu_1}{k^2} + \frac{4\mu_2 x^2}{k^4})|(\Delta v)v|dx + \int_{k<x<\sqrt{2}k}\frac{4\mu_1 x}{k^2}|(\Delta v)(\nabla v)|dx \\
&- \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta v|^2dx \\
\leq& \int_{\mathbb{R}^n}(\frac{2\mu_1 + 8\mu_2}{k^2})|(\Delta v)v|dx + \int_{\mathbb{R}^n}\frac{4\sqrt{2}\mu_1}{k}|(\Delta v)(\nabla v)|dx - \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta v|^2dx \\
\leq& \frac{\mu_1 + 4\mu_2}{k^2}(\|\Delta v\|^2 + \|v\|^2) + \frac{4\sqrt{2}\mu_1}{k}\|\Delta v\|\|\nabla v\| - \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta v|^2dx \\
\leq& \frac{\mu_1 + 4\mu_2}{k^2}(\|\Delta v\|^2 + \|v\|^2) + \frac{4\sqrt{2}\mu_1}{k}\|\Delta v\|(\varsigma\|v\| + C_\varsigma\|\Delta v\|) - \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta v|^2dx \\
\leq& \frac{\mu_1 + 4\mu_2}{k^2}(\|\Delta v\|^2 + \|v\|^2) + \frac{2\sqrt{2}\mu_1}{k}(\|\Delta v\|^2 + 2\varsigma^2\|v\|^2 + 2C_\varsigma^2\|\Delta v\|^2)
\end{aligned}
$$

$$- \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta v|^2 dx. \tag{4.43}$$

Controlling other terms of (4.39) by Young's inequality, and together with (4.39)–(4.43) we claim

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\big(|v|^2 + (\delta^2+\lambda-\delta\alpha)|u|^2 + (1-\delta)|\Delta u|^2 + 2F(x,u)\big)dx \\
&+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\big((\alpha-\delta)|v|^2+2\delta(\delta^2+\lambda-\delta\alpha)|u|^2+2\delta(1-\delta)|\Delta u|^2+2\delta c_2 F(x,u)\big)dx \\
\le& c|\varepsilon||y(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx + c|\varepsilon||y(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\phi_1|^2+|\phi_3|)dx \\
&+ c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\phi_2|+|g|^2)dx + c|\varepsilon|\big(1+|y(\theta_t\omega)|^2\big)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|u|^2+|\Delta u|^2+|v|^2)dx \\
&+\frac{(2\mu_1+8\mu_2)}{k^2}\Big(\big(1-\delta+|\varepsilon||y(\theta_t\omega)|\big)\|\Delta u\|^2+\big(2-\delta+|\varepsilon||y(\theta_t\omega)|\big)\|v\|^2+\|\Delta v\|^2\Big) \\
&+\frac{4\sqrt{2}\mu_1}{k}\Big(\big(1-\delta+|\varepsilon||y(\theta_t\omega)|\big)\|\Delta u\|^2 + 2\varsigma^2\big(2-\delta+|\varepsilon||y(\theta_t\omega)|\big)\|v\|^2 \\
&+\big(2C_\varsigma^2(2-\delta+|\varepsilon||y(\theta_t\omega)|)+1\big)\|\Delta v\|^2\Big),
\end{aligned}$$

which along with (4.12) implies

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\big(|v|^2+(\delta^2+\lambda-\delta\alpha)|u|^2+(1-\delta)|\Delta u|^2+2F(x,u)\big)dx+\big(2\sigma-c|\varepsilon| \\
&-c|\varepsilon||y(\theta_t\omega)|^2\big)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\big(|v|^2+(\delta^2+\lambda-\delta\alpha)|u|^2+(1-\delta)|\Delta u|^2+2F(x,u)\big)dx \\
\le& c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\phi_2|+|\phi_3|+|g|^2)dx + c|\varepsilon|\big(1+|y(\theta_t\omega)|^2\big)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\phi_1|^2+|\phi_3|)dx \\
&+\frac{(2\mu_1+8\mu_2)}{k^2}\Big(\big(1-\delta+|\varepsilon||y(\theta_t\omega)|\big)\|\Delta u\|^2+\big(2-\delta+|\varepsilon||y(\theta_t\omega)|\big)\|v\|^2+\|\Delta v\|^2\Big) \\
&+\frac{4\sqrt{2}\mu_1}{k}\Big(\big(1-\delta+|\varepsilon||y(\theta_t\omega)|\big)\|\Delta u\|^2 + 2\varsigma^2\big(2-\delta+|\varepsilon||y(\theta_t\omega)|\big)\|v\|^2 \\
&+\big(2C_\varsigma^2(2-\delta+|\varepsilon||y(\theta_t\omega)|)+1\big)\|\Delta v\|^2\Big).
\end{aligned} \tag{4.44}$$

Since $\phi_1 \in L^2(\mathbb{R}^n)$ and $\phi_2,\ \phi_3 \in L^1(\mathbb{R}^n)$, given $\eta > 0$, there exists $K_1 = K_1(\eta) \ge 1$ such that for all $k \ge K_1$,

$$\begin{aligned}
&c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\phi_1|^2+|\phi_2|+|\phi_3|)dx \\
=&c\int_{|x|\ge k}\rho(\frac{|x|^2}{k^2})(|\phi_1|^2+|\phi_2|+|\phi_3|)dx \\
\le& c\int_{|x|\ge k}(|\phi_1|^2+|\phi_2|+|\phi_3|)dx \le \eta.
\end{aligned} \tag{4.45}$$

By (4.44) and (4.45), there exists $K_2 = K_2(\eta) \geq K_1$ such that for all $k \geq K_2$,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v|^2 + (\delta^2 + \lambda - \delta\alpha)|u|^2 + (1-\delta)|\Delta u|^2 + 2F(x,u))dx \\
&+(2\sigma - \varepsilon c - \varepsilon c|y(\theta_t\omega)|^2)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v|^2 + (\delta^2+\lambda-\delta\alpha)|u|^2 + (1-\delta)|\Delta u|^2 + 2F(x,u))dx \\
\leq &\varepsilon\eta(1 + |y(\theta_t\omega)|^2) + c\int_{|x|\geq k} |g(t,x)|^2 dx + \eta\ (1 + \|\Delta u\|^2 + \|\Delta v\|^2 + \|v\|^2). \quad (4.46)
\end{aligned}$$

Let

$$X(t) = |v|^2 + (\delta^2 + \lambda - \delta\alpha)|u|^2 + (1-\delta)|\Delta u|^2.$$

We obtain from (4.46) that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(X(t) + 2F(x,u))dx \\
&+ (2\sigma - \varepsilon c - \varepsilon c|y(\theta_t\omega)|^2)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(X(t) + 2F(x,u))dx \\
\leq &\varepsilon\eta(1 + |y(\theta_t\omega)|^2) + c\int_{|x|\geq k} |g(t,x)|^2 dx + \eta\ (1 + \|\Delta u\|^2 + \|\Delta v\|^2 + \|v\|^2). \quad (4.47)
\end{aligned}$$

Integrating (4.47) over $(\tau - t, \tau)$ for $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, we obtain for all $k \geq K_2$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(X(\tau, \tau-t, \omega, X_0) + 2F(x, u(\tau, \tau-t, \omega, u_0)))dx \\
\leq &e^{\int_\tau^{\tau-t}(2\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(X_0 + 2F(x, u_0(x)))dx \\
&+ \varepsilon\eta\int_{\tau-t}^{\tau} e^{\int_\tau^s(2\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr}(1 + |y(\theta_s\omega)|^2)ds \\
&+ c\int_{\tau-t}^{\tau} e^{\int_\tau^s(2\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr}\int_{|x|\geq k} |g(s,x)|^2 dxds \\
&+ \eta\int_{\tau-t}^{\tau} e^{\int_\tau^s(2\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr}(1 + \|\Delta u(s, \tau-t, \omega, u_0)\|^2 \\
&+ \|\Delta v(s, \tau-t, \omega, v_0)\|^2 + \|v(s, \tau-t, \omega, v_0)\|^2)ds. \quad (4.48)
\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.48) we deduce, for every $t \in \mathbb{R}^+,\ \tau \in \mathbb{R}, \omega \in \Omega$ and $k \geq K_2$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(X(\tau, \tau-t, \theta_{-\tau}\omega, X_0) + 2F(x, u(\tau, \tau-t, \theta_{-\tau}\omega, u_0)))dx \\
\leq &e^{\int_\tau^{\tau-t}(2\sigma - \varepsilon c - \varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(X_0 + 2F(x, u_0(x)))dx \\
&+ \varepsilon\eta\int_{\tau-t}^{\tau} e^{\int_\tau^s(2\sigma - \varepsilon c - \varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}(1 + |y(\theta_{s-\tau}\omega)|^2)ds \\
&+ c\int_{\tau-t}^{\tau} e^{\int_\tau^s(2\sigma - \varepsilon c - \varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\int_{|x|\geq k} |g(s,x)|^2 dxds \\
&+ \eta\int_{\tau-t}^{\tau} e^{\int_\tau^s(2\sigma - \varepsilon c - \varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}(1 + \|\Delta u(s, \tau-t, \theta_{-\tau}\omega, u_0)\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds\\
\le& ce^{\int_0^{-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n\setminus Q_k)}+\|u_0\|^2_{H^2}+\|u_0\|^{\gamma+1}_{H^2})\\
&+\varepsilon\eta\int_{-t}^0 e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+|y(\theta_s\omega)|^2)ds\\
&+c\int_{-t}^0 e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{|x|\ge k}|g(s+\tau,x)|^2dxds\\
&+\eta\int_{-t}^0 e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}ds\\
&+\eta\int_{\tau-t}^{\tau} e^{\int_\tau^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds.
\end{aligned}\tag{4.49}
$$

Taking advantage of (4.20), for $(u_0,v_0)^\top\in D(\tau-t,\theta_{-\tau}\omega)$, we have

$$
\begin{aligned}
&\limsup_{t\to\infty} ce^{\int_0^{-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n\setminus Q_k)}+\|u_0\|^2_{H^2}+\|u_0\|^{\gamma+1}_{H^2})\\
\le&\limsup_{t\to\infty} ce^{-\sigma t}(1+\|D(\tau-t,\theta_{-t}\omega)\|^{\gamma+1})\\
\le&0,
\end{aligned}
$$

which shows that there exists a positive $T_3=T_3(\tau,\omega,D,\eta)$ such that for all $t\ge T_3$,

$$
ce^{\int_0^{-t}(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(\|\varphi_0\|^2_{\mathcal{H}(\mathbb{R}^n\setminus Q_k)}+\|u_0\|^2_{H^2}+\|u_0\|^{\gamma+1}_{H^2})\le\eta. \tag{4.50}
$$

For the third term on the right-hand side of (4.49), by (4.20) we get

$$
\begin{aligned}
&\int_{-\infty}^0 e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{|x|\ge k}|g(s+\tau,x)|^2dxds\\
=&\int_{-\infty}^{-T_1} e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{|x|\ge k}|g(s+\tau,x)|^2dxds\\
&+\int_{-T_1}^0 e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{|x|\ge k}|g(s+\tau,x)|^2dxds\\
\le&\int_{-\infty}^{-T_1} e^{\sigma s}\int_{|x|\ge k}|g(s+\tau,x)|^2dxds+e^{c_5}\int_{-T_1}^0 e^{\sigma s}\int_{|x|\ge k}|g(s+\tau,x)|^2dxds,
\end{aligned}\tag{4.51}
$$

where $c_5=(\sigma+\varepsilon c+\varepsilon c\max\limits_{-T_1\le r\le 0}y^2(\theta_r\omega))T_1$. Using (4.51) we get

$$
\begin{aligned}
&\int_{-\infty}^0 e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{|x|\ge k}|g(s+\tau,x)|^2dxds\\
\le& e^{c_5}\int_{-\infty}^0 e^{\frac{1}{4}\sigma s}\int_{|x|\ge k}|g(s+\tau,x)|^2dxds,
\end{aligned}
$$

which along with (3.8) implies that there exists $K_3=K_3(\tau,\eta)\ge K_2$ such for all $k\ge K_3$

$$
c\int_{-\infty}^0 e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\int_{|x|\ge k}|g(s+\tau,x)|^2dxds\le\eta. \tag{4.52}
$$

On the other hand, due to (4.20), we find the following integral is convergent:

$$R_3(\tau,\omega)=\int_{-\infty}^{0} e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+|y(\theta_s\omega)|^2)ds. \tag{4.53}$$

Collecting all (4.49)-(4.50), (4.52)-(4.53) and Lemma 4.1, we achieve, for all $t\geq T_3, k\geq K_3$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(X(\tau,\tau-t,\theta_{-\tau}\omega,X_0)+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)))dx\\ \leq&3\eta(1+R(\tau,\omega)+R_3(\tau,\omega)),\end{aligned} \tag{4.54}$$

where $R(\tau,\omega)$ and $R_3(\tau,\omega)$ are given by (4.2) and (4.53), respectively. It follows from (F3) and (4.54) that there exists $K_4=K_4(\tau,\eta)\geq K_3$ such that for all $k\geq K_4, t\geq T_3$,

$$\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{\mathcal{H}(\mathbb{R}^n\setminus Q_k)}\leq 4\eta(1+R(\tau,\omega)+R_3(\tau,\omega)).$$

Then we complete the proof. □

We now derive uniform estimates of solutions in bounded domains. These estimates will be used to establish pullback asymptotic compactness. Let $\widehat{\rho}=1-\rho$ with ρ given by (4.33). Fix $k\geq 1$, and set

$$\begin{cases}\widehat{u}(t,\tau,\omega,\widehat{u_0})=\widehat{\rho}(\frac{|x|^2}{k^2})u(t,\tau,\omega,u_0),\\ \widehat{v}(t,\tau,\omega,\widehat{v}_0)=\widehat{\rho}(\frac{|x|^2}{k^2})v(t,\tau,\omega,v_0).\end{cases} \tag{4.55}$$

By (3.10)–(3.12) we find that $\widehat{u}$ and $\widehat{v}$ satisfy the following system in $Q_{2k}=\{x\in\mathbb{R}^n:|x|\leq 2k\}$:

$$\frac{d\widehat{u}}{dt}+\delta\widehat{u}-\widehat{v}=\varepsilon y(\theta_t\omega)\widehat{u}, \tag{4.56}$$

$$\begin{aligned}&\frac{d\widehat{v}}{dt}+(\alpha-\delta)\widehat{v}+\Delta^2\widehat{v}+(\delta^2+\lambda-\delta\alpha)\widehat{u}+(1-\delta)\Delta^2\widehat{u}+\varepsilon y(\theta_t\omega)\Delta^2\widehat{u}+\widehat{\rho}f(x,u)\\ =&\widehat{\rho}g-\varepsilon y(\theta_t\omega)\widehat{v}-\varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)\widehat{u}+4\Delta\nabla\widehat{\rho}\nabla v+6\Delta\widehat{\rho}\Delta v+4\nabla\widehat{\rho}\Delta\nabla v+v\Delta^2\widehat{\rho}\\ &+4(1-\delta)\Delta\nabla\widehat{\rho}\nabla u+6(1-\delta)\Delta\widehat{\rho}\Delta u+4(1-\delta)\nabla\widehat{\rho}\Delta\nabla u+(1-\delta)u\Delta^2\widehat{\rho}\\ &+4\varepsilon y(\theta_t\omega)\Delta\nabla\widehat{\rho}\nabla u+6\varepsilon y(\theta_t\omega)\Delta\widehat{\rho}\Delta u+4\varepsilon y(\theta_t\omega)\nabla\widehat{\rho}\Delta\nabla u+\varepsilon y(\theta_t\omega)u\Delta^2\widehat{\rho},\end{aligned} \tag{4.57}$$

with boundary conditions

$$\widehat{u}=\widehat{v}=0\quad\text{for}\quad |x|=2k. \tag{4.58}$$

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L^2(Q_{2k})$ such that $\Delta^2e_n=\lambda_ne_n$ with boundary condition (4.58) in Q_{2k}. Given n, let $X_n=\text{span}\{e_1,\cdots,e_n\}$ and $P_n:L^2(Q_{2k})\to X_n$ be the projection operator.

Lemma 4.4. *Assume that (F1)–(F4) and* (3.6) *hold. Let any* $\eta>0,\tau\in\mathbb{R},\omega\in\Omega$ *and* $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$. *Then there exists* $T=T(\tau,\omega,D,\eta)>0$, $K=K(\tau,\omega,\eta)\geq 1$ *and* $N=N(\tau,\omega,\eta)\geq 1$ *such that for all* $t\geq T,\ k\geq K$ *and* $n\geq N$, *the solution* $(\widehat{u},\widehat{v})$ *of problem* (4.56)–(4.58) *satisfies*

$$\|(I-P_n)\widehat{\varphi}(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{\varphi}_0)\|^2_{\mathcal{H}(Q_{2k})}\leq\eta,$$

where $\widehat{\varphi}(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{\varphi}_0) = (\widehat{u}(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{u}_0),\widehat{v}(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{v}_0))$, $\widehat{\varphi}_0 = \widehat{\rho}(\frac{|x|^2}{k^2})\varphi_0$, $\varphi_0=(u_0,v_0)^\top\in D(\tau-t,\theta_{-t}\omega)$.

Proof. Let $\widehat{u}_{n,1}=P_n\widehat{u}$, $\widehat{u}_{n,2}=(I-P_n)\widehat{u}$, $\widehat{v}_{n,1}=P_n\widehat{v}$, $\widehat{v}_{n,2}=(I-P_n)\widehat{v}$. Applying $I-P_n$ to (4.56), we obtain

$$\widehat{v}_{n,2}=\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-\varepsilon y(\theta_t\omega)\widehat{u}_{n,2}. \tag{4.59}$$

Then applying $I-P_n$ to (4.57) and taking the inner product of the resulting equation with $\widehat{v}_{n,2}$ in $L^2(Q_{2k})$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{v}_{n,2}\|^2+(\alpha-\delta)\|\widehat{v}_{n,2}\|^2+\|\Delta\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)(\widehat{u}_{n,2},\widehat{v}_{n,2})\\
&\quad+(1-\delta)(\Delta^2\widehat{u}_{n,2},\widehat{v}_{n,2})+\varepsilon y(\theta_t\omega)(\Delta^2\widehat{u}_{n,2},\widehat{v}_{n,2})+(\widehat{\rho}f(x,u),\widehat{v}_{n,2})\\
=&(\widehat{\rho}g+4\Delta\nabla\widehat{\rho}\nabla v+6\Delta\widehat{\rho}\Delta v+4\nabla\widehat{\rho}\Delta\nabla v+v\Delta^2\widehat{\rho}+4(1-\delta)\Delta\nabla\widehat{\rho}\nabla u\\
&\quad+6(1-\delta)\Delta\widehat{\rho}\Delta u+4(1-\delta)\nabla\widehat{\rho}\Delta\nabla u+(1-\delta)u\Delta^2\widehat{\rho}+4\varepsilon y(\theta_t\omega)\Delta\nabla\widehat{\rho}\nabla u\\
&\quad+6\varepsilon y(\theta_t\omega)\Delta\widehat{\rho}\Delta u+4\varepsilon y(\theta_t\omega)\nabla\widehat{\rho}\Delta\nabla u+\varepsilon y(\theta_t\omega)u\Delta^2\widehat{\rho},\widehat{v}_{n,2})\\
&\quad-\varepsilon y(\theta_t\omega)\|\widehat{v}_{n,2}\|^2-\varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(\widehat{u}_{n,2},\widehat{v}_{n,2}).
\end{aligned}\tag{4.60}$$

Using (4.59) we get from (4.59)–(4.60) that

$$\begin{aligned}
&\frac{d}{dt}(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\
&\quad+2(\alpha-\delta)\|\widehat{v}_{n,2}\|^2+2\delta(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+2\delta(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2\\
=&2(4\Delta\nabla\widehat{\rho}\nabla v+6\Delta\widehat{\rho}\Delta v+4\nabla\widehat{\rho}\Delta\nabla v+v\Delta^2\widehat{\rho}+4(1-\delta)\Delta\nabla\widehat{\rho}\nabla u\\
&\quad+6(1-\delta)\Delta\widehat{\rho}\Delta u+4(1-\delta)\nabla\widehat{\rho}\Delta\nabla u+(1-\delta)u\Delta^2\widehat{\rho}+4\varepsilon y(\theta_t\omega)\Delta\nabla\widehat{\rho}\nabla u\\
&\quad+6\varepsilon y(\theta_t\omega)\Delta\widehat{\rho}\Delta u+4\varepsilon y(\theta_t\omega)\nabla\widehat{\rho}\Delta\nabla u+\varepsilon y(\theta_t\omega)u\Delta^2\widehat{\rho},\widehat{v}_{n,2})\\
&\quad+2(\widehat{\rho}g,\widehat{v}_{n,2})-2\varepsilon y(\theta_t\omega)(\Delta^2\widehat{u}_{n,2},\widehat{v}_{n,2})-2\varepsilon y(\theta_t\omega)\|\widehat{v}_{n,2}\|^2\\
&\quad-2\varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(\widehat{u}_{n,2},\widehat{v}_{n,2})+2\varepsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\|\widehat{u}_{n,2}\|^2\\
&\quad+2\varepsilon(1-\delta)y(\theta_t\omega)\|\Delta\widehat{u}_{n,2}\|^2-2\|\Delta\widehat{v}_{n,2}\|^2-2(\widehat{\rho}f(x,u),\widehat{v}_{n,2}).
\end{aligned}\tag{4.61}$$

Now we estimate each term one by one for the right-hand side of (4.61).

$$\begin{aligned}
&(4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\nabla v+6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta v+4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta\nabla v+v\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}),\widehat{v}_{n,2})\\
=&(4\nabla v\cdot(\frac{12|x|}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{8|x|^3}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2}))+6\Delta v\cdot(\frac{2}{k^2}\widehat{\rho}\,'(\frac{|x|^2}{k^2})\\
&+\frac{4x^2}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2}))+\frac{8|x|}{k^2}\Delta\nabla v\cdot\widehat{\rho}\,'(\frac{|x|^2}{k^2})+v(\frac{12}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{48x^2}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2})\\
&+\frac{16x^4}{k^8}\widehat{\rho}\,''''(\frac{|x|^2}{k^2})),\widehat{v}_{n,2})\\
\le&\frac{16\sqrt{2}(3\mu_2+4\mu_3)}{k^3}\lambda_{n+1}^{-\frac{1}{4}}\|\Delta v\|\cdot\|\widehat{v}_{n,2}\|+\frac{12(\mu_1+4\mu_2)}{k^2}\|\Delta v\|\cdot\|\widehat{v}_{n,2}\|\\
&+\frac{8\sqrt{2}\mu_1}{k}\lambda_{n+1}^{-\frac{1}{4}}\|\Delta v\|\cdot\|\Delta\widehat{v}_{n,2}\|+\frac{4(3\mu_2+24\mu_3+16\mu_4)}{k^4}\|v\|\cdot\|\widehat{v}_{n,2}\|\\
\le&\lambda_{n+1}^{-\frac{1}{2}}(\frac{18(48\mu_2+64\mu_3)^2}{(\alpha-\delta)k^6}+\frac{192\mu_1^2}{k^2})\|\Delta v\|^2+\frac{1}{6}\|\Delta\widehat{v}_{n,2}\|^2+\frac{9}{(\alpha-\delta)}
\end{aligned}$$

$$\times (\frac{(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|v\|^2+\frac{(12\mu_1+48\mu_2)^2}{k^4}\|\Delta v\|^2)+\frac{(\alpha-\delta)}{12}\|\widehat{v}_{n,2}\|^2, \tag{4.62}$$

and

$$\begin{aligned}
&(1-\delta)(4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\nabla u+6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta u+4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta\nabla u+u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}),\widehat{v}_{n,2})\\
=&(1-\delta)(4\nabla u\cdot(\frac{12|x|}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{8|x|^3}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2}))+6\Delta u\cdot(\frac{2}{k^2}\widehat{\rho}\,'(\frac{|x|^2}{r^2})\\
&+\frac{4x^2}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2}))+\frac{8|x|}{k^2}\Delta\nabla u\cdot\widehat{\rho}\,'(\frac{|x|^2}{k^2})+u(\frac{12}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{48x^2}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2})\\
&+\frac{16x^4}{k^8}\widehat{\rho}\,''''(\frac{|x|^2}{k^2})),\widehat{v}_{n,2})\\
\le&\frac{16\sqrt{2}(1-\delta)(3\mu_2+4\mu_3)}{k^3}\lambda_{n+1}^{-\frac14}\|\Delta u\|\cdot\|\widehat{v}_{n,2}\|+\frac{12(1-\delta)(\mu_1+4\mu_2)}{k^2}\|\Delta u\|\cdot\|\widehat{v}_{n,2}\|\\
&+\frac{8\sqrt{2}(1-\delta)\mu_1}{k}\lambda_{n+1}^{-\frac14}\|\Delta u\|\cdot\|\Delta\widehat{v}_{n,2}\|+\frac{4(1-\delta)(3\mu_2+24\mu_3+16\mu_4)}{k^4}\|u\|\cdot\|\widehat{v}_{n,2}\|\\
\le&(1-\delta)^2\lambda_{n+1}^{-\frac12}(\frac{18(48\mu_2+64\mu_3)^2}{(\alpha-\delta)k^6}+\frac{192\mu_1^2}{k^2})\|\Delta u\|^2+\frac16\|\Delta\widehat{v}_{n,2}\|^2+\frac{9(1-\delta)^2}{(\alpha-\delta)}\\
&\times(\frac{(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2+\frac{(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2)+\frac{(\alpha-\delta)}{12}\|\widehat{v}_{n,2}\|^2,
\end{aligned} \tag{4.63}$$

and

$$\begin{aligned}
&\varepsilon y(\theta_t\omega)(4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\nabla u+6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta u+4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta\nabla u+u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}),\widehat{v}_{n,2})\\
=&\varepsilon y(\theta_t\omega)(4\nabla u\cdot(\frac{12|x|}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{8|x|^3}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2}))+6\Delta u\cdot(\frac{2}{k^2}\widehat{\rho}\,'(\frac{|x|^2}{k^2})\\
&+\frac{4x^2}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2}))+\frac{8|x|}{k^2}\Delta\nabla u\cdot\widehat{\rho}\,'(\frac{|x|^2}{k^2})+u(\frac{12}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{48x^2}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2})\\
&+\frac{16x^4}{k^8}\widehat{\rho}\,''''(\frac{|x|^2}{k^2})),\widehat{v}_{n,2})\\
\le&\frac{16\sqrt{2}|\varepsilon y(\theta_t\omega)|(3\mu_2+4\mu_3)}{k^3}\lambda_{n+1}^{-\frac14}\|\Delta u\|\cdot\|\widehat{v}_{n,2}\|+\frac{12|\varepsilon y(\theta_t\omega)|(\mu_1+4\mu_2)}{k^2}\|\Delta u\|\cdot\|\widehat{v}_{n,2}\|\\
&+\frac{8\sqrt{2}|\varepsilon y(\theta_t\omega)|\mu_1}{k}\lambda_{n+1}^{-\frac14}\|\Delta u\|\cdot\|\Delta\widehat{v}_{n,2}\|+\frac{4|\varepsilon y(\theta_t\omega)|(3\mu_2+24\mu_3+16\mu_4)}{k^4}\|u\|\cdot\|\widehat{v}_{n,2}\|\\
\le&(\varepsilon y(\theta_t\omega))^2\lambda_{n+1}^{-\frac12}(\frac{18(48\mu_2+64\mu_3)^2}{(\alpha-\delta)k^6}+\frac{192\mu_1^2}{k^2})\|\Delta u\|^2+\frac16\|\Delta\widehat{v}_{n,2}\|^2+\frac{9(\varepsilon y(\theta_t\omega))^2}{(\alpha-\delta)}\\
&\times(\frac{(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2+\frac{(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2)+\frac{(\alpha-\delta)}{12}\|\widehat{v}_{n,2}\|^2.
\end{aligned} \tag{4.64}$$

By Young's inequality again, we have

$$\begin{aligned}
&-2\varepsilon y(\theta_t\omega)(\Delta^2\widehat{u}_{n,2},\widehat{v}_{n,2})-2\varepsilon y(\theta_t\omega)\|\widehat{v}_{n,2}\|^2-2\varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(\widehat{u}_{n,2},\widehat{v}_{n,2})\\
&+2\varepsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\|\widehat{u}_{n,2}\|^2+2\varepsilon(1-\delta)y(\theta_t\omega)\|\Delta\widehat{u}_{n,2}\|^2\\
\le&\varepsilon c(1+|y(\theta_t\omega)|^2)(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)+\frac12\|\Delta\widehat{v}_{n,2}\|^2;
\end{aligned} \tag{4.65}$$

$$2(\widehat{\rho}g,\widehat{v}_{n,2})=2((I-P_n)\widehat{\rho}g,\widehat{v}_{n,2})\leq\frac{1}{2}(\alpha-\delta)\|\widehat{v}_{n,2}\|^2+c_6\|(I-P_n)(\widehat{\rho}g)\|^2. \quad (4.66)$$

It follows from (3.16) and (4.61)–(4.66) that

$$\begin{aligned}
&\frac{d}{dt}(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\
&+7\sigma(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\
&+\sigma\|\widehat{v}_{n,2}\|^2+\sigma(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2+\sigma(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2\\
\leq&\varepsilon c(1+|y(\theta_t\omega)|^2)(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\
&+2\lambda_{n+1}^{-\frac{1}{2}}(\frac{18(48\mu_2+64\mu_3)^2}{(\alpha-\delta)k^6}+\frac{192\mu_1^2}{k^2})(\|\Delta v\|^2+(1-\delta)^2\|\Delta u\|^2\\
&+(\varepsilon y(\theta_t\omega))^2\|\Delta u\|^2)+\frac{18}{\alpha-\delta}((1-\delta)^2+(\varepsilon y(\theta_t\omega))^2)(\frac{(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2\\
&+\frac{(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2)+\frac{18}{\alpha-\delta}(\frac{(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|v\|^2-\frac{1}{2}\|\Delta\widehat{v}_{n,2}\|^2\\
&+\frac{(12\mu_1+48\mu_2)^2}{k^4}\|\Delta v\|^2)+c_6\|(I-P_n)(\widehat{\rho}g)\|^2-2(\widehat{\rho}f(x,u),\widehat{v}_{n,2}). \quad (4.67)
\end{aligned}$$

Next we estimate the nonlinear terms in (4.67). Let $\theta=\frac{n(\gamma-1)}{4(\gamma+1)}$. thanks to $1\leq\gamma\leq\frac{n+4}{n-4}$, we find that $0\leq\theta\leq1$. Then by (F1) and interpolation inequalities, we get

$$\begin{aligned}
&|-2(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{v}_{n,2})|\\
\leq&c_7\int_{\mathbb{R}^5}\widehat{\rho}(\frac{|x|^2}{k^2})|u|^\gamma|\widehat{v}_{n,2}|dx+\int_{\mathbb{R}^5}\widehat{\rho}(\frac{|x|^2}{k^2})|\phi_1(x)||\widehat{v}_{n,2}|dx\\
\leq&c_8\|u\|_{\gamma+1}^\gamma\|\widehat{v}_{n,2}\|_{\gamma+1}+\|\phi_1\|\|\widehat{v}_{n,2}\|\\
\leq&c_8\|u\|_{\gamma+1}^\gamma\|\Delta\widehat{v}_{n,2}\|^\theta\|\widehat{v}_{n,2}\|^{1-\theta}+\lambda_{n+1}^{-\frac{1}{2}}\|\phi_1\|\|\Delta\widehat{v}_{n,2}\|\\
\leq&c_9\lambda_{n+1}^{\frac{\theta-1}{2}}\|u\|_{H^2}^\gamma\|\Delta\widehat{v}_{n,2}\|+\lambda_{n+1}^{-\frac{1}{2}}\|\phi_1\|\|\Delta\widehat{v}_{n,2}\|\\
\leq&\lambda_{n+1}^{-\frac{1}{2}}\|\Delta\widehat{v}_{n,2}\|(c_9\lambda_{n+1}^{\frac{\theta}{2}}\|u\|_{H^2}^k+\|\phi_1\|)\\
\leq&\frac{1}{2}\|\Delta\widehat{v}_{n,2}\|^2+\frac{1}{2}\lambda_{n+1}^{-1}(c_9\lambda_{n+1}^{\frac{\theta}{2}}\|u\|_{H^2}^k+\|\phi_1\|)^2. \quad (4.68)
\end{aligned}$$

Combining with (4.67)–(4.68), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\
&+7\sigma(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\
\leq&\varepsilon c(1+|y(\theta_t\omega)|^2)(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\
&+2\lambda_{n+1}^{-\frac{1}{2}}(\frac{18(48\mu_2+64\mu_3)^2}{(\alpha-\delta)k^6}+\frac{192\mu_1^2}{k^2})(\|\Delta v\|^2+(1-\delta)^2\|\Delta u\|^2\\
&+(\varepsilon y(\theta_t\omega))^2\|\Delta u\|^2)+\frac{18}{\alpha-\delta}((1-\delta)^2+(\varepsilon y(\theta_t\omega))^2)(\frac{(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2\\
&+\frac{(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2)+\frac{18}{\alpha-\delta}(\frac{(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|v\|^2
\end{aligned}$$

$$+\frac{(12\mu_1+48\mu_2)^2}{k^4}\|\Delta v\|^2)+c_6\|(I-P_n)(\widehat{\rho}g)\|^2+\frac{1}{2}\lambda_{n+1}^{-1}(c_9\lambda_{n+1}^{\frac{\theta}{2}}\|u\|_{H^2}^k+\|\phi_1\|)^2.$$

Note that $1\le\gamma\le\frac{n+4}{n-4}(n\ge5)$ and $\lambda_n\to\infty$. Therefore, given $\eta>0$, there exist $N_1=N_1(\eta)\ge1$ and $K_1=K_1(\eta)\ge1$ such for all $n\ge N_1$ and $k\ge K_1$,

$$\begin{aligned}&\frac{d}{dt}(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\&+(7\sigma-\varepsilon c-\varepsilon c|y(\theta_t\omega)|^2)(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\\le&\eta(1+\|u\|_{H^2(\mathbb{R}^n)}^{2\gamma})+c_6\|(I-P_n)(\widehat{\rho}g)\|^2.\end{aligned}\tag{4.69}$$

Recalling the norm $\|\cdot\|_{\mathcal{H}}$ in (3.1), from (4.69) we conclude that

$$\begin{aligned}&\frac{d}{dt}(\|\widehat{\varphi}_{n,2}\|_{\mathcal{H}(Q_{2k})}^2)+(7\sigma-\varepsilon c-\varepsilon c|y(\theta_t\omega)|^2)(\|\widehat{\varphi}_{n,2}\|_{\mathcal{H}(Q_{2k})}^2)\\\le&\eta(1+\|u\|_{H^2(\mathbb{R}^n)}^{2\gamma})+c_6\|(I-P_n)(\widehat{\rho}g)\|^2.\end{aligned}\tag{4.70}$$

Integrating (4.70) over $(\tau-t,\tau)$ with $t\ge0$, we get for all $n\ge N_1$ and $k\ge K_1$,

$$\begin{aligned}&\|\widehat{\varphi}_{n,2}(\tau,\tau-t,\omega,\widehat{\varphi}_{n,2,0})\|_{\mathcal{H}(Q_{2k})}^2\\\le&c_{10}e^{\int_\tau^{\tau-t}(7\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\|\widehat{\varphi}_{n,2,0}\|_{\mathcal{H}(Q_{2k})}^2\\&+\eta\int_{\tau-t}^{\tau}e^{\int_\tau^s(7\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}(1+\|u(s,\tau-t,\omega,u_0)\|_{H^2(\mathbb{R}^n)}^{2\gamma})ds\\&+c_6\int_{\tau-t}^{\tau}e^{\int_\tau^s(7\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\|(I-P_n)(\widehat{\rho}g)(s)\|^2ds.\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above inequality we obtain, for every $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, and $\omega\in\Omega$, $n\ge N_1$ and $k\ge K_1$,

$$\begin{aligned}&\|\widehat{\varphi}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{\varphi}_{n,2,0})\|_{\mathcal{H}(Q_{2k})}^2\\\le&c_{10}e^{\int_\tau^{\tau-t}(7\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\|\widehat{\varphi}_{n,2,0}\|_{\mathcal{H}(Q_{2k})}^2\\&+\eta\int_{\tau-t}^{\tau}e^{\int_\tau^s(7\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}ds\\&+\eta\int_{\tau-t}^{\tau}e^{\int_\tau^s(7\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|_{H^2(\mathbb{R}^n)}^{2\gamma}ds\\&+c_6\int_{\tau-t}^{\tau}e^{\int_\tau^s(7\sigma-\varepsilon c-\varepsilon c|y(\theta_{r-\tau}\omega)|^2)dr}\|(I-P_n)(\widehat{\rho}g)(s)\|^2ds\\\le&c_{10}e^{\int_0^{-t}(7\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\|\widehat{\varphi}_{n,2,0}\|_{\mathcal{H}(Q_{2k})}^2\\&+\eta\int_{-t}^{0}e^{\int_0^s(7\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}ds\\&+\eta\int_{-t}^{0}e^{\int_0^s(7\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|_{H^2(\mathbb{R}^n)}^{2\gamma}ds\\&+c_6\int_{-t}^{0}e^{\int_0^s(7\sigma-\varepsilon c-\varepsilon c|y(\theta_r\omega)|^2)dr}\|(I-P_n)(\widehat{\rho}g)(s+\tau)\|^2ds.\end{aligned}\tag{4.71}$$

We estimate each term on the right-hand side of (4.71). For the first term, there exists $T_1 = T_1(\tau, \omega, D, \eta) > 0$ such that for all $t \geq T_1$,

$$c_{10} e^{\int_0^{-t}(7\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr} \|\widehat{\varphi}_{n,2,0}\|^2_{\mathcal{H}(Q_{2k})} \leq \eta. \tag{4.72}$$

For the second term on the right-hand side of (4.71), by Lemma 4.2 we have

$$\begin{aligned}
&\eta \int_{-t}^{0} e^{\int_0^s(7\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr} \|u(s+\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|^{2\gamma}_{H^2(\mathbb{R}^n)} ds \\
\leq &\eta c_{11} \int_{-t}^{0} e^{\int_0^s(7\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr} ds \\
&+ \eta R^\gamma(\tau, \omega) \int_{-t}^{0} e^{\int_0^s((7-2\gamma)\sigma + (\gamma-1)\varepsilon c + (\gamma-1)\varepsilon c|y(\theta_r\omega)|^2)dr} ds \\
\leq &\eta c_{11} \int_{-t}^{0} e^{\int_0^s(7\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr} ds + \eta c_{12} R^\gamma(\tau, \omega),
\end{aligned} \tag{4.73}$$

where $R(\tau, \omega)$ is given by Lemma 4.2.

For the last term of (4.71). Exploiting (3.6) and (4.20), we can easily get the following integral is convergent,

$$\int_{-\infty}^{0} e^{\int_0^s(7\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr} \|(\widehat{\rho} g)(s+\tau, \cdot)\|^2 ds < \infty,$$

and hence by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{-\infty}^{0} e^{\int_0^s(7\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr} \|(I - P_n)(\widehat{\rho} g)(s+\tau)\|^2 ds = 0.$$

This shows that there exists $N_2 = N_2(\tau, \omega, \eta) \geq N_1$ such for all $n \geq N_2$,

$$\int_{-\infty}^{0} e^{\int_0^s(7\sigma - \varepsilon c - \varepsilon c|y(\theta_r\omega)|^2)dr} \|(I - P_n)(\widehat{\rho} g)(s+\tau)\|^2 ds \leq \eta. \tag{4.74}$$

It follows from (4.71)–(4.74) that, for every $\tau \in \mathbb{R}, \omega \in \Omega, t \geq T_1, n \geq N_2$ and $k \geq K_1$,

$$\|\widehat{\varphi}_{n,2}(\tau, \tau - t, \theta_{-\tau}\omega, \widehat{\varphi}_{n,2,0})\|^2_{\mathcal{H}(Q_{2k})} \leq \eta c_{13}(1 + R_4(\tau, \omega)),$$

where $R_4(\tau, \omega)$ is a positive constant depending only on τ and ω. □

5. Random attractors

In this section, we shall prove the existence of a $\mathcal{D}$-pullback attractor for the random system (3.10)–(3.12) by using Proposition 2.1. First we apply the lemmas shown in Section 4 to prove pullback asymptotic compactness of solutions of (3.10)–(3.12) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Lemma 5.1. *Assume that (F1)–(F4) and* (3.6) *hold. Then for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the solution sequence of* (3.10)–(3.12)*, $\{(u(\tau, \tau - t_m, \theta_{-\tau}\omega, u_{0,m}), v(\tau, \tau - t_m, \theta_{-\tau}\omega, v_{0,m})\}_{m=1}^{\infty}$, has a convergent subsequence in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ whenever $t_m \to \infty$ and $(u_{0,m}, v_{0,m}) \in D(\tau - t_m, \theta_{-t_m}\omega)$ with $D \in \mathcal{D}$.*

Proof. According to Lemma 4.1 and the assumption $t_m \to \infty$, we see that, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $m_1 = m_1(\tau, \omega, D) > 0$ such for all $m \geq m_1$,

$$\|\varphi(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m})\|^2_{\mathcal{H}(\mathbb{R}^n)} \leq R_1(\tau, \omega). \tag{5.1}$$

In line with Lemma 4.3 , we find that for every $\eta > 0$, there exist $k_0 = k_0(\tau, \omega, \eta) \geq 1$ and $m_2 = m_2(\tau, \omega, D, \eta) \geq m_1$ and such that for all $m \geq m_2$,

$$\|\varphi(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m})\|^2_{\mathcal{H}(\mathbb{R}^n \setminus Q_{k_0})} \leq \eta. \tag{5.2}$$

Let $\widehat{u}$ and $\widehat{v}$ be the functions defined by (4.55). Then by Lemma 4.4, there are $k_1 = k_1(\tau, \omega, \eta) \geq k_0$, $m_3 = m_3(\tau, \omega, D, \eta) \geq m_2$ and $n_1 = n_1(\tau, \omega, \eta) \geq 1$ such for all $m \geq m_3$,

$$\|(I - P_{n_1})\widehat{\varphi}(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m})\|^2_{\mathcal{H}(Q_{2k_1})} \leq \eta. \tag{5.3}$$

On the other hand, due to (5.1) we obtain for all $m \geq m_3$,

$$\|\widehat{\varphi}(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m})\|^2_{\mathcal{H}(\mathbb{R}^n)} \leq c_{14} R_1(\tau, \omega),$$

which along with (5.3) implies the precompactness of $\{(\widehat{u}(\tau, \tau - t_m, \theta_{-\tau}\omega), \widehat{v}(\tau, \tau - t_m, \theta_{-\tau}\omega)\}$ in $H^2(Q_{2k_1}) \times L^2(Q_{2k_1})$ based on the abstract result introduced in [34]. Therefore, the sequence $\{(u(\tau, \tau - t_m, \theta_{-\tau}\omega, u_{0,m}), v(\tau, \tau - t_m, \theta_{-\tau}\omega, v_{0,m})\}$ is precompact in $H^2(Q_{k_1}) \times L^2(Q_{k_1})$ due to (4.55) and the fact that $\widehat{\rho}(\frac{|x|^2}{k^2}) = 1$ for $|x| \leq 1$. So together with (5.2) we get the precompactness of the sequence in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. □

Theorem 5.1. *Assume that (F1)–(F4) and* (3.6) *hold. Then the cocycle* Φ *generated by the stochastic plate equation problem* (3.10)–(3.12) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A} \in \mathcal{D}$ *in* $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *whose structure is characterized by Proposition 2.1 .*

Proof. Note that Φ is $\mathcal{D}$-pullback asymptotically compact in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ by Lemma 5.1 , and has a closed measurable $\mathcal{D}$-pullback absorbing set by Lemma 4.1 . Therefore, the existence and uniqueness of $\mathcal{D}$-pullback attractor of Φ follows from Proposition 2.1 immediately. □

6. Upper semicontinuity of pullback attractors

In this section, we will consider the upper semicontinuity of pullback attractors for the stochastic plate equation (3.10)–(3.12) on $\mathbb{R}^n$. As the critical exponent of $f(x, u)$ is $\frac{n+4}{n-4}$, we can't derive the upper semicontinuity of pullback attractors, so we must supplement the following additional condition (this condition has nothing to do with the proof of pullback attractors):

$$|f'_u(x, u)| \leq l, \quad \forall\, x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \tag{6.1}$$

where the constant $l > 0$. First, we present a criteria concerning the upper semicontinuity of non-autonomous random attractors with respect to a parameter in [38].

Theorem 6.1. *Let $(X, \|\cdot\|_X)$ be a separable Banach space, Φ_ϵ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Suppose that*
(i) Φ_ϵ has a closed measurable random absorbing set $K_\varepsilon = \{K_\varepsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(X)$ and a unique random attractor $\mathcal{A}_\epsilon = \{\mathcal{A}_\varepsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(X)$.
(ii) There exists a map $\varsigma : \mathbb{R} \to \mathbb{R}$ such that for each $\tau \in \mathbb{R}, \omega \in \Omega, K_0(\tau) = \{u \in X : \|u\|_X \le \varsigma(\tau)\}$ and

$$\limsup_{\varepsilon\to 0} \|K_\varepsilon(\tau,\omega)\|_X = \limsup_{\varepsilon\to 0} \limsup_{x\in K_\varepsilon(\tau,\omega)} \|x\|_X \le \varsigma(\tau). \tag{6.2}$$

(iii) There exists $\varepsilon_0 > 0$, such that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\bigcup_{|\varepsilon|\le\epsilon_0} \mathcal{A}_\varepsilon(\tau,\omega) \text{ is precompact in } X.$$

(iv) For $t > 0, \tau \in \mathbb{R}, \omega \in \Omega, \varepsilon_n \to 0$ when $n \to \infty$, and $x_n, x_0 \in X$ with $x_n \to x_0$ when $n \to \infty$, it holds

$$\lim_{n\to\infty} \Phi_{\varepsilon_n}(t,\tau,\omega)x_n = \Phi_0(t,\tau)x_0. \tag{6.3}$$

Then for $\tau \in \mathbb{R}, \omega \in \Omega$,

$$d_H(\mathcal{A}_\varepsilon(\tau,\omega), \mathcal{A}_0(\tau)) = \sup_{u\in\mathcal{A}_\varepsilon(\tau,\omega)} \inf_{v\in\mathcal{A}_0(\tau)} \|u - v\| \to 0, \quad \text{as } \varepsilon \to 0. \tag{6.4}$$

Next, we will use Theorem 6.1 to prove the upper semicontinuity of random attractors $\mathcal{A}_\varepsilon(\tau,\omega)$ when $\varepsilon \to 0$. To indicate the dependence of solutions on ε, we will write the solutions of problem (3.10)–(3.12) as $(u^{(\varepsilon)},\ v^{(\varepsilon)})$, that is, $\varphi^\varepsilon = (u^{(\varepsilon)}, v^{(\varepsilon)})^T$ satisfies

$$\begin{cases} \dfrac{du^{(\varepsilon)}}{dt} + \delta u^{(\varepsilon)} - v^{(\varepsilon)} = \varepsilon y(\theta_t\omega)u^{(\varepsilon)}, \\ \dfrac{dv^{(\varepsilon)}}{dt} + (\alpha-\delta)v^{(\varepsilon)} + \Delta^2 v^{(\varepsilon)} + (\delta^2 + \lambda - \delta\alpha)u^{(\varepsilon)} + (1-\delta)\Delta^2 u^{(\varepsilon)} \\ + \varepsilon y(\theta_t\omega)\Delta^2 u^{(\varepsilon)} + f(x,u^{(\varepsilon)}) = g - \varepsilon y(\theta_t\omega)v^{(\varepsilon)} - \varepsilon(\varepsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)u^{(\varepsilon)}, \\ u^{(\varepsilon)}(x,\tau,\tau) = u_0^{(\varepsilon)}(x), \qquad v^{(\varepsilon)}(x,\tau,\tau) = v_0^{(\varepsilon)}(x). \end{cases} \tag{6.5}$$

When $\varepsilon = 0$, the random dynamical system (6.5) reduces to a deterministic dynamical system:

$$\begin{cases} \dfrac{du^{(0)}}{dt} + \delta u^{(0)} - v^{(0)} = 0, \\ \dfrac{dv^{(0)}}{dt} + (\alpha-\delta)v^{(0)} + \Delta^2 v^{(0)} + (\delta^2 + \lambda - \delta\alpha)u^{(0)} + (1-\delta)\Delta^2 u^{(0)} + f(x,u^{(0)}) = g, \\ u^{(0)}(x,\tau,\tau) = u_0^{(0)}(x), \qquad v^{(0)}(x,\tau,\tau) = v_0^{(0)}(x) = u_1^{(0)}(x) + \delta u_0^{(0)}(x). \end{cases} \tag{6.6}$$

Accordingly, by virtue of the above similar discussion step by step and together with Lemma 5.1, the deterministic non-autonomous system Φ_0 generated by (6.6) is readily verified to admit a unique $\mathcal{D}_0(\mathcal{H}(\mathbb{R}^n))$-pullback attractor $\mathcal{A}_0(\tau)$ if $g(x,\cdot) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, and $\alpha,\ \lambda$ are positive constants.

Theorem 6.2. *Assume that* (F1)-(F4) *and* (3.6) *hold. Then the cocycle* Φ_ε *generated by* (3.10)–(3.12) *has a unique* $\mathcal{D}$*-pullback attractor* $\{\mathcal{A}_\varepsilon(\tau,\omega)\}_{\omega\in\Omega}$ *in* $\mathcal{H}(\mathbb{R}^n)$*. Moreover, the family of random attractors* $\{\mathcal{A}_\varepsilon\}_{\varepsilon>0}$ *is upper semicontinuous.*

Proof. (i) From Lemma 4.1 and Theorem 5.1, we know that Φ_ϵ has a closed measurable random absorbing set $E_\varepsilon = \{E_\varepsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\big(\mathcal{H}(\mathbb{R}^n)\big)$, where $E_\varepsilon(\tau,\omega) = \big\{\varphi^{(\varepsilon)} \in \mathcal{H}(\mathbb{R}^n) : \|\varphi^{(\varepsilon)}\|^2_{\mathcal{H}(\mathbb{R}^n)} \le R(\varepsilon,\tau,\omega)\big\}$, and a unique random attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(\mathcal{H}(\mathbb{R}^n))$, for each $\tau \in \mathbb{R}, \omega \in \Omega$, $\mathcal{A}_\varepsilon(\tau,\omega) \subseteq E_\varepsilon(\tau,\omega)$.

(ii) Given $\varepsilon \le 1$, by (4.2), we have

$$R(\varepsilon,\tau,\omega) \le R(1,\tau,\omega) < \infty,$$

and

$$\limsup_{\varepsilon\to 0} R(\varepsilon,\tau,\omega) \le R(1,\tau,\omega).$$

So, for every $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\limsup_{\varepsilon\to 0} \|E_\varepsilon(\tau,\omega)\| = \limsup_{\varepsilon\to 0} \sup_{x\in E_\varepsilon(\tau,\omega)} \|x\|_{\mathcal{H}(\mathbb{R}^n)} \le R^{\frac{1}{2}}(1,\tau,\omega). \tag{6.7}$$

Let $E_1(\tau,\omega) = \{\varphi^{(\varepsilon)} \in \mathcal{H}(\mathbb{R}^n) : \|\varphi^{(\varepsilon)}\|^2_{\mathcal{H}(\mathbb{R}^n)} \le R(1,\tau,\omega)\}$, then

$$\bigcup_{\varepsilon\le 1} \mathcal{A}_\varepsilon(\tau,\omega) \subseteq \bigcup_{\varepsilon\le 1} E_\varepsilon(\tau,\omega) \subseteq E_1(\tau,\omega). \tag{6.8}$$

(iii) Given $\varepsilon \le 1$. Let us prove the precompactness of $\bigcup\limits_{\varepsilon\le 1} \mathcal{A}_\varepsilon(\tau,\omega)$ for every $\tau \in \mathbb{R}, \omega \in \Omega$. For one thing, by (6.8), Lemma 4.3 and the invariance of $\mathcal{A}_\varepsilon(\tau,\omega)$, for every $\eta > 0, \varepsilon > 0, \tau \in \mathbb{R}, \omega \in \Omega$, there exist $T = T(\tau,\omega,E_1,\varepsilon,\eta) > 0, K = K(\tau,\omega,\varepsilon,\eta) \ge 1$, such that for all $t \ge T,\ k \ge K$, the solution $\varphi^{(\varepsilon)}$ of (6.5) satisfies

$$\sup_{\varphi^{(\varepsilon)} \in \bigcup\limits_{\varepsilon\le 1} \mathcal{A}_\varepsilon(\tau,\omega)} \big\|\varphi^{(\varepsilon)}\big(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0^{(\varepsilon)}\big)\big\|^2_{\mathcal{H}(\mathbb{R}^n\setminus Q_k)} \le \eta.$$

For another thing, by (6.8)we find that the set $\bigcup\limits_{\varepsilon\le 1} \mathcal{A}_\varepsilon(\tau,\omega)$ is precompact in $\mathcal{H}(Q_k)$ and hence $\bigcup\limits_{\varepsilon\le 1} \mathcal{A}_\varepsilon(\tau,\omega)$ is precompact in $\mathcal{H}(\mathbb{R}^n)$.

(iv) Let $\varphi^{(0)} = (u^{(0)}, v^{(0)})$ be a solution of (6.6) with initial data $\varphi_0^{(0)} = (u_0^{(0)}, v_0^{(0)})$, and $U = u^{(\varepsilon)} - u^{(0)}$, $V = v^{(\varepsilon)} - v^{(0)}$. It follows from (6.5) and (6.6) that

$$\begin{cases} \dfrac{dU}{dt} + \delta U - V = \varepsilon y(\theta_t\omega)U + \varepsilon y(\theta_t\omega)u^{(0)}, \\ \dfrac{dV}{dt} + (\alpha-\delta)V + \Delta^2 V + (\delta^2+\lambda-\delta\alpha)U + (1-\delta)\Delta^2 U + f(x,u^{(\varepsilon)}) - f(x,u^{(0)}) \\ = -\varepsilon y(\theta_t\omega)\Delta^2 U - \varepsilon y(\theta_t\omega)\Delta^2 u^{(0)} - \varepsilon y(\theta_t\omega)V - \varepsilon y(\theta_t\omega)v^{(0)} \\ \quad - \varepsilon(\varepsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)U - \varepsilon(\varepsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)u^{(0)}. \end{cases} \tag{6.9}$$

Taking the inner product of the second equation of (6.9) with V in $L^2(\mathbb{R}^n)$, and then using the first equation of (6.9) to simplify the resulting equality, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|V\|^2 + (\delta^2+\lambda-\delta\alpha)\|U\|^2 + (1-\delta)\|\Delta U\|^2)$$

$$
\begin{aligned}
&+(\alpha-\delta)\|V\|^2+\delta(\delta^2+\lambda-\delta\alpha)\|U\|^2+\delta(1-\delta)\|\Delta U\|^2+\Delta^2 V\\
=&(f(x,u^{(0)})-f(x,u^{(\varepsilon)}),V)+\varepsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\|U\|^2\\
&+\varepsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)(U,u^{(0)})+\varepsilon(1-\delta)y(\theta_t\omega)\|\Delta U\|^2\\
&+\varepsilon(1-\delta)y(\theta_t\omega)(\Delta U,\Delta u^{(0)})-(\varepsilon y(\theta_t\omega)\Delta^2 U+\varepsilon y(\theta_t\omega)\Delta^2 u^{(0)},V)\\
&-\varepsilon y(\theta_t\omega)\|V\|^2-\varepsilon y(\theta_t\omega)(V,v^{(0)})-\varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(V,u^{(0)})\\
&-\varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(V,U).
\end{aligned}
\tag{6.10}
$$

Due to (6.1), it leads to

$$
|(f(x,u^{(0)})-f(x,u^{(\varepsilon)}),V)|\le l\|U\|\|V\|\le c(\|V\|^2+(\delta^2+\lambda-\delta\alpha)\|U\|^2). \tag{6.11}
$$

Thanks to Young's inequality, we find the remaining terms on the right hand side of (6.10) are controlled by $\varepsilon c(1+|y(\theta_t\omega)|^2)(\|U\|^2_{H^2(\mathbb{R}^n)}+\|V\|^2+\|u^{(0)}\|^2_{H^2(\mathbb{R}^n)}+\|v^{(0)}\|^2)$ for all $\varepsilon\le 1$, which along with (6.10)–(6.11) implies

$$
\begin{aligned}
&\frac{d}{dt}(\|V\|^2+(\delta^2+\lambda-\delta\alpha)\|U\|^2+(1-\delta)\|\Delta U\|^2)\\
\le& c(\|V\|^2+(\delta^2+\lambda-\delta\alpha)\|U\|^2+(1-\delta)\|\Delta U\|^2)\\
&+\varepsilon c(1+|y(\theta_t\omega)|^2)(\|U\|^2_{H^2(\mathbb{R}^n)}+\|V\|^2+\|u^{(0)}\|^2_{H^2(\mathbb{R}^n)}+\|v^{(0)}\|^2).
\end{aligned}
\tag{6.12}
$$

Applying Lemma 4.1 , there exists a constant $c_0=c_0(\tau,\omega,R,T)>0$ such that for all $t\ge T$,

$$
\|u^{(0)}\|^2_{H^2(\mathbb{R}^n)}+\|v^{(0)}\|^2\le c_0. \tag{6.13}
$$

Together with (6.12) and (6.13) we get

$$
\begin{aligned}
&\frac{d}{dt}(\|V\|^2+(\delta^2+\lambda-\delta\alpha)\|U\|^2+(1-\delta)\|\Delta U\|^2)\\
\le& c(\|V\|^2+(\delta^2+\lambda-\delta\alpha)\|U\|^2+(1-\delta)\|\Delta U\|^2)+\varepsilon c(1+|y(\theta_t\omega)|^2).
\end{aligned}
\tag{6.14}
$$

Therefore, applying the Gronwall inequality to (6.14) over (τ,t), we have

$$
\begin{aligned}
&\|u^{(\varepsilon)}(t,\tau,\omega,u_0^{(\varepsilon)})-u^{(0)}(t,\tau,u_0^{(0)})\|^2_{H^2(\mathbb{R}^n)}+\|v^{(\varepsilon)}(t,\tau,\omega,v_0^{(\varepsilon)})-v^{(0)}(t,\tau,v_0^{(0)})\|^2_{L^2(\mathbb{R}^n)}\\
\le& ce^{c(t-\tau)}(\|u_0^{(\varepsilon)}-u_0^{(0)}\|^2_{H^2(\mathbb{R}^n)}+\|v_0^{(\varepsilon)}-v_0^{(0)}\|^2_{L^2(\mathbb{R}^n)})\\
&+\varepsilon c\int_\tau^t e^{c(t-s)}(1+|y(\theta_s\omega)|^2)ds,
\end{aligned}
\tag{6.15}
$$

which along with (i),(ii), (iii) and Theorem 6.1 complete the proof. □

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