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ANALYSIS FOR 3D DEGENERATE CAHN-LARCHÉ MODEL WITH PERIODIC BOUNDARY CONDITIONS

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Abstract Our aim in this article is to study the existence of weak solutions to the degenerate Cahn-Larché model. Under appropriate assumptions on the degenerate mobility and chemical free energy density, we prove the existence of weak solutions to the approximate problem with positive mobility by applying the method of continuation of local solutions, then we use the solutions of approximate problem to approach the solutions of degenerate problem. Furthermore, we perform the numerical simulations to investigate the microstructure evolution during the spinodal decomposition by utilizing this model.

Keywords Degenerate Cahn-Larché model, weak solutions, phase-field simulations.

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1. Introduction

Spinodal decomposition is a kind of solid phase transition, it refers to the supersaturated solid solution decomposes into two phases with the same structure but different components at a certain temperature. In the process of decomposition, the solute atoms diffuse from the low concentration area to the high concentration area, which is called uphill diffusion, then the solute atoms in the rich area will be further enriched, and the solute atoms in the poor area will be gradually depleted, moreover, there is no clear boundary between the two regions and the composition at the interface is continuous. Since the product of spinodal decomposition is the rich and poor regions of solute atoms, it has certain influence on the strength and magnetism of materials.

Spinodal decomposition is a phase transition dominated by concentration diffusion, the evolution of each phase in the phase transition process can be represented by a conserved order parameter S, we usually use the Cahn-Hilliard nonlinear diffusion equation to describe the evolution of S. Cahn [4] presented a theory to describe the phase separation process of spinodal decomposition in binary alloys, and used this theory to simulate the microstructure evolution during phase transformation of alloys in 1965.

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A number of studies in recent years have shown that the elastic strain energy comes from the difference of lattice parameters between parent and product phases is the resistance of phase transition, which affects the shape and the volume fraction of product phase. Therefore, the Cahn-Hilliard equation with elastic effect (which is also called Cahn-Larché model [16]) has caused concern of many researchers (see, e.g, [3,10–14,17]), Garcke [11] studied the Cahn-Larché model without considering the volume force ($b = 0$)

$$\begin{aligned} &\mathrm{div}\sigma = b, \\ &\sigma = D(\varepsilon(\nabla u) - \bar{\varepsilon}S), \\ &S_t = \mathrm{div}(m\nabla\frac{\delta F}{\delta S}), \end{aligned}$$

in which, mobility m is a constant matrix, $\frac{\delta F}{\delta S}$ is a variational derivative of the total free energy $F(\varepsilon(\nabla u), S, \nabla S)$ with respect to order parameter S, the total free energy of system is

$$F(\varepsilon(\nabla u), S, \nabla S) = \int_\Omega \hat{\psi}(S) + \frac{\nu}{2}|\nabla S|^2 + f_{el}(S, \varepsilon(\nabla u)) + f_{ex}(\varepsilon(\nabla u))dx,$$

the chemical free energy density $\hat{\psi}(S)$ was selected as a logarithmic double well function, the elastic strain energy density $f_{el}(S, \varepsilon(\nabla u)) = \frac{1}{2}(D(\varepsilon(\nabla u) - \bar{\varepsilon}S)) \cdot (\varepsilon(\nabla u) - \bar{\varepsilon}S)$, then the existence of weak solution to that model was obtained. After that, Garcke and Kuak [12] studied the total free energy of the model does not conclude the external load energy $\int_\Omega f_{ex}(\varepsilon(\nabla u))dx$, the above model converges to the sharp interface model when the interface thickness tends to 0. The mobility is a function dependent on order parameter S, which is more accord with the physical phenomenon, [5,8,19] studied the existence of weak solutions to the above model without considering the elastic strain energy and the external load energy. To our knowledge, when the mobility dependents on the order parameter S and degenerates, the existence of weak solutions to the Cahn-Larché model has not been studied. In this paper we consider periodic boundary conditions and study the Cahn-Larché model with nonnegative mobility.

Let $\Omega = (0, L)^d$, $L \in (0, \infty)$, be a cube in $\mathbb{R}^d$. We write $\Gamma_j = \partial\Omega \cap \{x_j = 0\}$ and $\Gamma_{j+d} = \partial\Omega \cap \{x_j = L\}$, where $j = 1, \cdots, d$. The different phases are characterized by the order parameter $S(t, x) \in \mathbb{R}$, the other unknowns are the displacement $u(t, x) \in \mathbb{R}^d$ of the material point x at time t and the Cauchy stress tensor $\sigma(t, x) \in \mathcal{S}^d$, where $\mathcal{S}^d$ denotes the set of symmetric $d \times d$-matrices. Then we study the following initial-boundary value problem for $d = 3$

$$-\mathrm{div}\sigma = b, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$\sigma = D(\varepsilon(\nabla u) - \bar{\varepsilon}S), \quad x \in \Omega,\ t > 0, \tag{1.2}$$

$$S_t = \mathrm{div}(m(S)\nabla(\hat{\psi}'(S) - \bar{\varepsilon} \cdot \sigma - \nu\Delta S)), \quad x \in \Omega,\ t > 0, \tag{1.3}$$

periodic boundary and initial conditions are

$$u|_{\Gamma_j} = u|_{\Gamma_{j+d}}, \quad \nabla u|_{\Gamma_j} = \nabla u|_{\Gamma_{j+d}}, \tag{1.4}$$

$$\nabla S|_{\Gamma_j} = \nabla S|_{\Gamma_{j+d}}, \quad \nabla(\psi_S - \nu\Delta S)|_{\Gamma_j} = \nabla(\psi_S - \nu\Delta S)|_{\Gamma_{j+d}}, \tag{1.5}$$

$$m(S)\nabla(\hat{\psi}'(S) - \bar{\varepsilon} \cdot \sigma - \nu\Delta S)|_{\Gamma_j} = m(S)\nabla(\hat{\psi}'(S) - \bar{\varepsilon} \cdot \sigma - \nu\Delta S)|_{\Gamma_{j+d}}, \tag{1.6}$$

$$S(0,x) = S_0(x), \quad x \in \Omega, \tag{1.7}$$

here, $m(S) \geq 0$ represents the coefficient of atomic mobility; $D : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping, the elastic modulus tensor of material; ∇u denotes the 3×3-matrix of the first order spatial derivatives of u, the deformation gradient; $\varepsilon(\nabla u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$ is the strain tensor; $\bar{\varepsilon} \in \mathcal{S}^3$ is a given matrix, the misfit strain; $\nu > 0$ represents the interfacial energy coefficient; $b : [0,\infty) \times \Omega \to \mathbb{R}^3$ is the volume force and $S_0 : \Omega \to \mathbb{R}$ is a given initial function.

The total free energy of system is

$$F(\varepsilon, S, \nabla S) = \int_\Omega \psi^*(\varepsilon, S, \nabla S) dx, \tag{1.8}$$

where

$$\psi^*(\varepsilon, S, \nabla S) = \psi(\varepsilon, S) + \frac{\nu}{2}|\nabla S|^2, \tag{1.9}$$

in the free energy

$$\psi(\varepsilon, S) = \frac{1}{2}(D(\varepsilon - \bar{\varepsilon}S)) \cdot (\varepsilon - \bar{\varepsilon}S) + \hat{\psi}(S), \tag{1.10}$$

the scalar product of two matrices is expressed as $A \cdot B = \sum a_{ij} b_{ij}$, $\hat{\psi}(S)$ is the chemical free energy density.

When the mobility depends on the order parameter S and degenerates, it increases the difficulty of studying the model. For $m(S)$ and $\hat{\psi}(S)$, we make the following assumptions

(1) $m(S) \in C^1(\mathbb{R}; [0,\infty))$, and for any $S_1, S_2 \in \mathbb{R}$, there exist constants $M_0, M_1 > 0$ such that

$$0 \leq m(S) \leq M_0, \quad |m(S_1) - m(S_2)| \leq M_1 |S_1 - S_2|. \tag{1.11}$$

(2) $\hat{\psi}(S) \in C^3(\mathbb{R}; \mathbb{R})$, there is a constant $C > 0$, such that for all $S \in \mathbb{R}$

$$C(S^{2r+2} - 1) \leq \hat{\psi}(S) \leq C(S^{2r+2} + 1), \tag{1.12}$$

$$|\hat{\psi}'(S)| \leq C(|S|^{2r+1} + 1), \tag{1.13}$$

$$|\hat{\psi}''(S)| \leq C(|S|^{2r} + 1), \tag{1.14}$$

$$|\hat{\psi}'''(S)| \leq C(|S|^{2r-1} + 1). \tag{1.15}$$

We write $Q_{T_e} := (0, T_e) \times \Omega$, T_e is a positive constant, $H^m_{\mathrm{per}}(\Omega)$ ($W^{m,p}_{\mathrm{per}}(\Omega)$) is the Sobolev spaces with periodic boundary conditions, X' denotes the dual space of X and $\|\cdot\|$ denotes the usual L^2-norm. Then the main result of this article is

Theorem 1.1. *We define*

$$G_S = \{(t,x) \in Q_{T_e};\ m(S(t,x)) > 0\}. \tag{1.16}$$

For all $S_0 \in H^2_{\mathrm{per}}(\Omega)$, $b \in L^\infty(0, T_e; L^2(\Omega))$ and $b_t \in L^2(Q_{T_e})$. A function (u, σ, S) with

$$u \in L^\infty(0, T_e; H^2_{\mathrm{per}}(\Omega)), \quad \sigma \in L^\infty(0, T_e; H^1_{\mathrm{per}}(\Omega)), \tag{1.17}$$

$$S \in L^\infty(0, T_e; H^1_{\mathrm{per}}(\Omega)) \cap L^2(0, T_e; H^2_{\mathrm{per}}(\Omega)), \tag{1.18}$$

$$\nabla \Delta S \in L^2(G_S), \quad m(S)^{\frac{1}{2}} \nabla \Delta S \in L^2(G_S) \tag{1.19}$$

is a weak solution to initial-boundary value problem (1.1)-(1.7)*, if* (1.1)*,* (1.2) *and* (1.4) *are satisfied weakly, and if for all* $\varphi \in C_0^\infty(-\infty, T_e; C_{\text{per}}^\infty(\Omega))$

$$\begin{aligned}&(S, \varphi_t)_{Q_{T_e}} + (S_0, \varphi(0))_\Omega - (m(S)\nabla(\hat{\psi}'(S) - \sigma \cdot \bar{\varepsilon}), \nabla\varphi)_{Q_{T_e}} \\ &+ \nu(m(S)\nabla\Delta S, \nabla\varphi)_{G_S} = 0. \end{aligned} \tag{1.20}$$

The method to prove Theorem 1.1 is: replacing the degenerate parabolic equation by the following non-degenerate equation

$$S_t = \operatorname{div}(m_\kappa(S)\nabla(\hat{\psi}'(S) - \bar{\varepsilon} \cdot \sigma - \nu\Delta S)), \tag{1.21}$$

here

$$m_\kappa(S) = m(S) + \kappa, \quad \kappa \in (0, 1]. \tag{1.22}$$

By making uniform a-priori estimates independent of κ for the solutions $(u^\kappa, \sigma^\kappa, S^\kappa)$ of approximate problem(1.1), (1.2) and (1.21), we study the convergence of $(u^\kappa, \sigma^\kappa, S^\kappa)$ as $\kappa \to 0$ to get the solutions of degenerate problem (1.1)–(1.7). However, although (1.21) is a non-degenerate parabolic equation, we don't know whether the solutions of (1.1), (1.2) and (1.21) exist or not, we replace $m_\kappa(S)$ in (1.21) by $m_\kappa(\widetilde{\hat{S}})$, $\widetilde{\hat{S}}$ represents the smoothing of a given function $\hat{S}$, then (1.21) becomes a semilinear parabolic equation with smooth coefficient

$$S_t = \operatorname{div}(m_\kappa(\widetilde{\hat{S}})\nabla(\hat{\psi}'(S) - \bar{\varepsilon} \cdot \sigma - \nu\Delta S)), \tag{1.23}$$

in this case we can use the classical theory of the fourth order parabolic equation to obtain Hölder continuous solutions, and we derive suitable uniform a-priori estimates, then apply the method of continuation of local solutions to get the existence of weak solutions to approximate problem (1.1), (1.2) and (1.21).

It is easy to get from (1.11) and (1.22) that

$$\kappa \le m_\kappa(S) \le m(S) + 1 \le M_0 + 1. \tag{1.24}$$

2. Existence of weak solutions to approximate problem

For a given $\kappa > 0$, we study the following quasilinear, uniformly parabolic initial-boundary value problem

$$-\operatorname{div}\sigma = b, \ x \in \Omega, \ t > 0, \tag{2.1}$$

$$\sigma = D(\varepsilon(\nabla u) - \bar{\varepsilon}S), \ x \in \Omega, \ t > 0, \tag{2.2}$$

$$S_t = \operatorname{div}(m_\kappa(S)\nabla(\hat{\psi}'(S) - \bar{\varepsilon} \cdot \sigma - \nu\Delta S)), \ x \in \Omega, \ t > 0, \tag{2.3}$$

periodic boundary and initial conditions are

$$u|_{\Gamma_j} = u|_{\Gamma_{j+d}}, \quad \nabla u|_{\Gamma_j} = \nabla u|_{\Gamma_{j+d}}, \tag{2.4}$$

$$\nabla S|_{\Gamma_j} = \nabla S|_{\Gamma_{j+d}}, \quad \nabla(\psi_S - \nu\Delta S)|_{\Gamma_j} = \nabla(\psi_S - \nu\Delta S)|_{\Gamma_{j+d}}, \tag{2.5}$$

$$m_\kappa(S)\nabla(\hat{\psi}'(S) - \bar{\varepsilon} \cdot \sigma - \nu\Delta S)|_{\Gamma_j} = m_\kappa(S)\nabla(\hat{\psi}'(S) - \bar{\varepsilon} \cdot \sigma - \nu\Delta S)|_{\Gamma_{j+d}}, \tag{2.6}$$

$$S(0, x) = S_0(x), \quad x \in \Omega. \tag{2.7}$$

We are going to prove the existence of weak solutions to approximate problem (2.1)–(2.7).

Theorem 2.1. *For any $S_0 \in H^1_{\text{per}}(\Omega)$, $b \in L^\infty(0,T_e;L^2(\Omega))$ and $b_t \in L^2(Q_{T_e})$. A function (u,σ,S) with*

$$u \in L^\infty(0,T_e;H^2_{\text{per}}(\Omega)), \quad \sigma \in L^\infty(0,T_e;H^1_{\text{per}}(\Omega)), \tag{2.8}$$

$$S \in L^\infty(0,T_e;H^1_{\text{per}}(\Omega)) \cap L^2(0,T_e;H^3_{\text{per}}(\Omega)) \tag{2.9}$$

is a weak solution to approximate problem (2.1)–(2.7)*, if* (2.1)*,* (2.2) *and* (2.4) *are satisfied weakly, and if for all $\varphi \in C_0^\infty(-\infty,T_e;C^\infty_{\text{per}}(\Omega))$*

$$(S,\varphi_t)_{Q_{T_e}} + (S_0,\varphi(0))_\Omega - (m_\kappa(S)\nabla(\hat\psi'(S) - \bar\varepsilon\cdot\sigma - \nu\Delta S),\nabla\varphi)_{Q_{T_e}} = 0. \tag{2.10}$$

In addition, for almost all $t \in [0,T_e]$, we have the following energy inequality

$$\begin{aligned}&\|\nabla S(t)\|^2 + \nu\int_{Q_{T_e}} m_\kappa(S)|\nabla\Delta S|^2 d(\tau,x)\\ \le&\|\nabla S_0\|^2 + \int_{Q_{T_e}} m_\kappa(S)\nabla\hat\psi'(S)\cdot\nabla\Delta S d(\tau,x) + \int_{Q_{T_e}} m_\kappa(S)\nabla(\bar\varepsilon\cdot\sigma)\cdot\nabla\Delta S d(\tau,x).\end{aligned} \tag{2.11}$$

The way to prove the Theorem 2.1 is: applying the method of continuation of local solutions. We take a change of the S in (2.2) by $\hat S$ and replace the $m_\kappa(S)$ in (2.3) by $m_\kappa(\widetilde{\hat S})$, where $\widetilde{\hat S}$ denotes the smoothing of a given function $\hat S$, we thus obtain the following semilinear problem

$$-\operatorname{div}\sigma = b,\ x\in\Omega,\ t>0, \tag{2.12}$$

$$\sigma = D(\varepsilon(\nabla u) - \bar\varepsilon\hat S),\ x\in\Omega,\ t>0, \tag{2.13}$$

$$S_t = \operatorname{div}(m_\kappa(\widetilde{\hat S})\nabla(\hat\psi'(S) - \bar\varepsilon\cdot\sigma - \nu\Delta S)),\ x\in\Omega,\ t>0, \tag{2.14}$$

periodic boundary and initial conditions are

$$u|_{\Gamma_j} = u|_{\Gamma_{j+d}}, \quad \nabla u|_{\Gamma_j} = \nabla u|_{\Gamma_{j+d}}, \tag{2.15}$$

$$\nabla S|_{\Gamma_j} = \nabla S|_{\Gamma_{j+d}}, \quad \nabla(\psi_S - \nu\Delta S)|_{\Gamma_j} = \nabla(\psi_S - \nu\Delta S)|_{\Gamma_{j+d}}, \tag{2.16}$$

$$m_\kappa(\widetilde{\hat S})\nabla(\hat\psi'(S) - \bar\varepsilon\cdot\sigma - \nu\Delta S)|_{\Gamma_j} = m_\kappa(\widetilde{\hat S})\nabla(\hat\psi'(S) - \bar\varepsilon\cdot\sigma - \nu\Delta S)|_{\Gamma_{j+d}}, \tag{2.17}$$

$$S(0,x) = S_0(x), \quad x\in\Omega, \tag{2.18}$$

where, $\hat S \in L^2(0,T_e;H^1_{\text{per}}(\Omega))$ is a given function, and

$$\widetilde{\hat S}(t,x) = (\chi_\eta * \hat S)(t,x) = \int_{Q_{T_e}} \chi_\eta(t-\tau,x-y)\hat S(\tau,y)d(\tau,y), \tag{2.19}$$

with the standard mollifier $\chi_\eta \in C_0^\infty(\{x\in\mathbb{R}^2|\ |x|\le\eta\})$.

Lemma 2.1. *Let $0<\alpha<1$, to every $\hat S \in L^2(0,T_e;H^1_{\text{per}}(\Omega))$, $b \in C^{\frac{\alpha}{4},\alpha}_{\text{per}}(Q_{T_e})$ and $S_0 \in C^{4+\alpha}_{\text{per}}(\Omega)$, the initial-boundary value problem* (2.12)–(2.18) *exists a unique solution (u,σ,S), and the solution belongs to the following spaces*

$$L^\infty(0,T_e;C^{2+\alpha}_{\text{per}}(\Omega)) \times L^\infty(0,T_e;C^{1+\alpha}_{\text{per}}(\Omega)) \times C^{1+\frac{\alpha}{4},4+\alpha}_{\text{per}}(Q_{T_e}),$$

and $\Delta S_t \in L^2(Q_{T_e})$.

Proof. Since (2.14) is a semilinear, strictly parabolic equation with Hölder continuous coefficient, we can prove the lemma by applying the theorem on page 616 in [15]. □

2.1. Local solutions

In what follows, C_{t_0} denotes a constant depending on t_0 but independent of n, t_0 is a sufficiently small positive constant; $C_{t_0,\kappa}$ is a constant depending on t_0 and κ but independent of n; C_κ is a constant dependent on κ; C denotes a constant independent of n and κ.

Using the elliptic regularity theory to the linear elasticity system (2.12), (2.13) and (2.15), we have

$$\|u\|_{H^2_{\rm per}(\Omega)} + \|\sigma\|_{H^1_{\rm per}(\Omega)} \le C(\|S\|_{H^1_{\rm per}(\Omega)} + \|b\|_{L^2(\Omega)}). \tag{2.20}$$

For a sufficiently small constant t_0, we assume that for a given function $\hat{S}$ satisfies $\|\hat{S}\|_{L^\infty(0,T_e;H^1_{\rm per}(\Omega))} \le K$, then

$$\|\hat{\sigma}\|_{H^1_{\rm per}(\Omega)} \le C(\|\hat{S}\|_{H^1_{\rm per}(\Omega)} + \|b\|_{L^2(\Omega)}) \le C(K+1). \tag{2.21}$$

Lemma 2.2. *Assume that t_0 is a sufficiently small constant, then for a given function $\hat{S} \in L^\infty(0,T_e;H^1_{\rm per}(\Omega))$, there hold*

$$\|S\|_{L^\infty(0,t_0;H^1_{\rm per}(\Omega))} \le C_{t_0}, \tag{2.22}$$

$$\|u\|_{L^\infty(0,t_0;H^2_{\rm per}(\Omega))} + \|\sigma\|_{L^\infty(0,t_0;H^1_{\rm per}(\Omega))} \le C_{t_0}, \tag{2.23}$$

$$\int_{Q_{t_0}} m_\kappa(\widetilde{\hat{S}})|\nabla(\hat{\psi}'(S) - \nu\Delta S)|^2 d(\tau,x) \le C_{t_0}. \tag{2.24}$$

Proof. We consider the following energy density

$$\psi_1^*(S,\nabla S) = \frac{\nu}{2}|\nabla S|^2 + \hat{\psi}(S),$$

differentiating it with respect to t and substituting (2.14) to the resulting equation, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \psi_1^*(S,\nabla S)dx &= \int_\Omega \nu\nabla S\cdot\nabla S_t + \hat{\psi}'(S)S_t dx = \int_\Omega (\hat{\psi}'(S) - \nu\Delta S)S_t dx \\
&= \int_\Omega (\hat{\psi}'(S) - \nu\Delta S)\mathrm{div}(m_\kappa(\widetilde{\hat{S}})\nabla(\hat{\psi}'(S) - \bar{\varepsilon}\cdot\sigma - \nu\Delta S))dx \\
&= -\int_\Omega \nabla(\hat{\psi}'(S) - \nu\Delta S)\cdot m_\kappa(\widetilde{\hat{S}})\nabla(\hat{\psi}'(S) - \bar{\varepsilon}\cdot\sigma - \nu\Delta S))dx \\
&= -\int_\Omega m_\kappa(\widetilde{\hat{S}})|\nabla(\hat{\psi}'(S) - \nu\Delta S)|^2 \\
&\quad + \int_\Omega \nabla(\hat{\psi}'(S) - \nu\Delta S)\cdot m_\kappa(\widetilde{\hat{S}})\nabla(\bar{\varepsilon}\cdot\sigma)dx,
\end{aligned} \tag{2.25}$$

integrating the above equality with respect to t from 0 to t_0

$$\begin{aligned}
&\int_\Omega \psi_1^*(S,\nabla S)(t,x)dx + \int_{Q_{t_0}} m_\kappa(\widetilde{\hat{S}})|\nabla(\hat{\psi}'(S) - \nu\Delta S)|^2 d(\tau,x) \\
&= \int_\Omega \psi_1^*(S,\nabla S)(0,x) + \int_{Q_{t_0}} \nabla(\hat{\psi}'(S) - \nu\Delta S)\cdot m_\kappa(\widetilde{\hat{S}})\nabla(\bar{\varepsilon}\cdot\sigma)d(\tau,x).
\end{aligned} \tag{2.26}$$

Invoking that $S_0 \in H^1_{\text{per}}s(\Omega)$, then from (1.12) we deduce that

$$\begin{aligned}|\int_\Omega \psi_1^*(S,\nabla S)(0,x)| &= |\int_\Omega \frac{\nu}{2}|\nabla S_0|^2 + \hat{\psi}(S_0)dx| \\ &\leq \int_\Omega \frac{\nu}{2}|\nabla S_0|^2 + C(|S_0|^{2r+2}+1)dx \\ &\leq C(\|S_0\|^2_{H^1_{\text{per}}(\Omega))} + \|S_0\|^{2r+2}_{H^1_{\text{per}}(\Omega))} + 1) \\ &\leq C, \end{aligned} \tag{2.27}$$

and

$$\begin{aligned}&|\int_{Q_{t_0}} \nabla(\hat{\psi}'(S) - \nu\Delta S)\cdot m_\kappa(\widetilde{\hat{S}})\nabla(\bar{\varepsilon}\cdot\sigma)d(\tau,x)| \\ =&|\int_{Q_{t_0}} m_\kappa(\widetilde{\hat{S}})^{\frac{1}{2}}\nabla(\hat{\psi}'(S) - \nu\Delta S)\cdot m_\kappa(\widetilde{\hat{S}})^{\frac{1}{2}}\nabla(\bar{\varepsilon}\cdot\sigma)d(\tau,x)| \\ \leq&\frac{1}{2}\int_{Q_{t_0}} m_\kappa(\widetilde{\hat{S}})|\nabla(\hat{\psi}'(S) - \nu\Delta S)|^2 d(\tau,x) + \frac{1}{2}\int_{Q_{t_0}} m_\kappa(\widetilde{\hat{S}})|\nabla(\bar{\varepsilon}\cdot\sigma)|^2 d(\tau,x),\end{aligned} \tag{2.28}$$

it follows from (2.26)-(2.28) and (1.24) that

$$\begin{aligned}&\int_\Omega \psi_1^*(S,\nabla S)(t,x)dx + \frac{1}{2}\int_{Q_{t_0}} m_\kappa(\widetilde{\hat{S}})|\nabla(\hat{\psi}'(S) - \nu\Delta S)|^2 d(\tau,x) \\ \leq&\frac{1}{2}\int_{Q_{t_0}} m_\kappa(\widetilde{\hat{S}})|\nabla(\bar{\varepsilon}\cdot\sigma)|^2 d(\tau,x) \\ \leq&\frac{C(M_0+1)}{2}(K+1)^2 t_0 \leq C_{t_0}.\end{aligned} \tag{2.29}$$

We thus deduce from (1.12) that

$$\int_\Omega (\frac{\nu}{2}|\nabla S|^2 + A_0 S^{2r+2})dx + \int_{Q_{t_0}} m_\kappa(\widetilde{\hat{S}})|\nabla(\hat{\psi}'(S) - \nu\Delta S)|^2 d(\tau,x) \leq C_{t_0}, \tag{2.30}$$

then we get (2.24). Combining $r \geq \frac{1}{2}$ with (2.30) we obtain that $S \in L^\infty(0,t_0;H^1_{\text{per}}(\Omega))$. Because of $b \in L^\infty(0,T_e;L^2(\Omega))$, then we use (2.20) to arrive at (2.23). □

Lemma 2.3. *For a sufficiently small constant t_0, there hold*

$$\int_{Q_{t_0}} |\nabla\Delta S|^2 d(\tau,x) \leq C_{t_0,\kappa}, \tag{2.31}$$

$$\|S\|_{L^2(0,t_0;H^3_{\text{per}}(\Omega))} \leq C_{t_0,\kappa}. \tag{2.32}$$

Proof. From the estimates (1.24) and (2.24) we get

$$\int_{Q_{t_0}} |\nabla(\hat{\psi}'(S) - \nu\Delta S)|^2 d(\tau,x) \leq C_{t_0,\kappa}, \tag{2.33}$$

we infer from the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ that

$$\nu^2\int_{Q_{t_0}} |\nabla\Delta S|^2 d(\tau,x) \leq 2\int_{Q_{t_0}} |\nabla(\hat{\psi}'(S) - \nu\Delta S)|^2 d(\tau,x) + 2\int_{Q_{t_0}} |\nabla\hat{\psi}'(S)|^2 d(\tau,x)$$

$$\leq 2C_{t_0,\kappa} + 2\int_{Q_{t_0}} |\nabla\hat{\psi}'(S)|^2 d(\tau,x), \tag{2.34}$$

utilizing (1.14), (2.22) and Hölder's inequality, we find

$$\begin{aligned}
\int_{Q_{t_0}} |\nabla\hat{\psi}'(S)|^2 d(\tau,x) &= \int_{Q_{t_0}} |\hat{\psi}''(S)|^2 |\nabla S|^2 d(\tau,x)\\
&\leq C\int_{Q_{t_0}} (|S|^{2r}+1)^2 |\nabla S|^2 d(\tau,x)\\
&\leq C\int_{Q_{t_0}} |S|^{4r}|\nabla S|^2 d(\tau,x) + C\int_{Q_{t_0}} |\nabla S|^2 d(\tau,x)\\
&\leq C\int_{Q_{t_0}} |S|^{4r}|\nabla S|^2 d(\tau,x) + C_{t_0}\\
&\leq C\int_0^{t_0} \|S\|^{4r}_{L^6(\Omega)} \|\nabla S\|^2_{L^{\frac{6}{3-2r}}(\Omega)} d\tau + C_{t_0}\\
&\leq C\int_0^{t_0} \|\nabla S\|^{4r} \|\nabla S\|^2_{L^{\frac{6}{3-2r}}(\Omega)} d\tau + C_{t_0},
\end{aligned} \tag{2.35}$$

here we used the Sobolev inequality $\|S\|_{L^6(\Omega)} \leq C\|\nabla S\|$. By the Gagliardo-Nirenberg inequality, we deduce that

$$\|\nabla S\|^2_{L^{\frac{6}{3-2r}}(\Omega)} \leq C\|\nabla S\|^{\frac{2-r}{2}} \|\nabla\Delta S\|^{\frac{r}{2}} + C\|\nabla S\|^2. \tag{2.36}$$

Substituting (2.36) into (2.35), and using the Young inequality, we have

$$\begin{aligned}
\int_{Q_{t_0}} |\nabla\hat{\psi}'(S)|^2 d(\tau,x) &\leq C\int_0^{t_0} \|\nabla S\|^{4r}\|\nabla S\|^{\frac{2-r}{2}} \|\nabla\Delta S\|^{\frac{r}{2}} d\tau + C\int_0^{t_0} \|\nabla S\|^{4r}\|\nabla S\|^2 d\tau\\
&\leq C\int_0^{t_0} \|\nabla S\|^{\frac{7r+2}{2}} \|\nabla\Delta S\|^{\frac{r}{2}} d\tau + C\int_0^{t_0} \|\nabla S\|^{4r+2} d\tau\\
&\leq \frac{\nu^2}{2}\int_{Q_{t_0}} |\nabla\Delta S|^2 d(\tau,x) + C_\nu \int_0^{t_0} \|\nabla S\|^{\frac{14r+4}{4-r}} d\tau\\
&\quad + C\int_0^{t_0} \|\nabla S\|^{4r+2} d\tau.
\end{aligned} \tag{2.37}$$

Then, we derive from (2.34)-(2.37) that

$$\int_{Q_{t_0}} |\nabla\Delta S|^2 d(\tau,x) \leq C_{t_0,\kappa} + C_\nu \int_0^{t_0} \|\nabla S\|^{\frac{14r+4}{4-r}} d\tau + C\int_0^{t_0} \|\nabla S\|^{4r+2} d\tau \leq C_{t_0,\kappa}, \tag{2.38}$$

therefore, (2.31) is proved.

In order to prove (2.32), we apply the Gagliardo-Nirenberg inequality again

$$\|\Delta S\| \leq C\|\nabla S\|^{\frac{1}{2}} \|\nabla\Delta S\|^{\frac{1}{2}} + C\|\nabla S\|,$$

integrating the above inequality with respect to t from 0 to t_0, using (2.22), (2.31) and Hölder's inequality, we obtain

$$\int_0^{t_0} \|\Delta S\|^2 d\tau \leq C\int_0^{t_0} \|\nabla S\|\|\nabla\Delta S\| d\tau + C\int_0^{t_0} \|\nabla S\|^2 d\tau$$

$$\leq (\int_0^{t_0} \|\nabla S\|^2 d\tau)^{\frac{1}{2}} (\int_0^{t_0} \|\nabla \Delta S\|^2 d\tau)^{\frac{1}{2}} + C\int_0^{t_0} \|\nabla S\|^2 d\tau$$
$$\leq C_{t_0,\kappa}, \tag{2.39}$$

thus

$$\Delta S \in L^2(Q_{t_0}). \tag{2.40}$$

Let

$$\Delta S = f,$$

Δ is a isomorphic mapping: $H^1_{\text{per}}(\Omega)$ is mapped onto $(H^1_{\text{per}}(\Omega))'$. We suppose G is a inverse operator, then

$$S(t) = Gf(t), \quad a.e.$$

According to the elliptic regularity theorem of linear elliptic system, we can prove that

$$\|S\|_{L^2(0,t_0;H^2_{\text{per}}(\Omega))} \leq C_{t_0,\kappa}. \tag{2.41}$$

Combination of this estimate with (2.31) yield (2.32). □

Lemma 2.4. *For a sufficiently small constant t_0, there holds*

$$\|S_t\|_{L^2(0,t_0;(H^2_{\text{per}}(\Omega))')} \leq C_{t_0,\kappa}. \tag{2.42}$$

Proof. For any $\varphi \in L^2(0,t_0;H^2_{\text{per}}(\Omega))$, by (1.24), Lemma 2.2 and Lemma 2.3, we infer that

$$\begin{aligned}
&|\int_{Q_{t_0}} S_t \varphi d(\tau,x)| \\
&= |\int_{Q_{t_0}} \text{div}(m_\kappa(\widetilde{\hat{S}})\nabla(\hat{\psi}'(S) - \bar{\varepsilon}\cdot\sigma - \nu\Delta S))\varphi d(\tau,x)| \\
&= |\int_{Q_{t_0}} m_\kappa(\widetilde{\hat{S}})\nabla(\hat{\psi}'(S) - \bar{\varepsilon}\cdot\sigma - \nu\Delta S)\cdot\nabla\varphi d(\tau,x)| \\
&\leq C\int_0^{t_0} \|\nabla\hat{\psi}'(S) - \nabla(\bar{\varepsilon}\cdot\sigma) - \nu\nabla\Delta S\|\|\nabla\varphi\| d\tau \\
&\leq C\|\nabla\hat{\psi}'(S) - \nabla(\bar{\varepsilon}\cdot\sigma) - \nu\nabla\Delta S\|_{L^2(Q_{t_0})}\|\nabla\varphi\|_{L^2(Q_{t_0})} \\
&\leq C(\|\nabla\hat{\psi}'(S)\|_{L^2(Q_{t_0})} + \|\nabla\sigma\|_{L^2(Q_{t_0})} + \|\nabla\Delta S\|_{L^2(Q_{t_0})})\|\nabla\varphi\|_{L^2(Q_{t_0})} \\
&\leq C\|\nabla\varphi\|_{L^2(Q_{t_0})} \leq C\|\varphi\|_{L^2(0,t_0;H^2_{\text{per}}(\Omega))},
\end{aligned} \tag{2.43}$$

we thus get (2.42). □

Lemma 2.5 (Aubin-Lions, [18])**.** *Let B_0, B, B_1 be Banach spaces which satisfy that B_0, B_1 are reflexive and that $B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$, here $\hookrightarrow\hookrightarrow$ denotes compact embedding. For $0 \leq p_0, p_1 \leq \infty$, we define*

$$W = \{f | f \in L^{p_0}(0,T;B_0),\ f' = \frac{df}{dt} \in L^{p_1}(0,T;B_1)\},$$

(i) If $1 \leq p_0 < \infty$, $p_1 = 1$, then the embedding of W in $L^{p_0}(0,T;B)$ is compact.
(ii) If $p_0 = \infty$, $1 < p_1 \leq \infty$, then the embedding of W in $C([0,T];B)$ is compact.

We now prove that (u,σ,S) is a weak solution to approximate problem (2.1)–(2.7) for $t\in[0,t_0]$. In order to construct a sequence of approximate solutions of (2.1)–(2.7), we select a sequence of functions $(b^n,S_0^n)\in C_{\text{per}}^{\frac{\alpha}{4},\alpha}(Q_{t_0})\times C_{\text{per}}^{4+\alpha}(\Omega)$, such that for $n\to\infty$

$$\|b^n-b\|_{L^2(Q_{t_0})}+\|S_0^n-S_0\|_{H^1_{\text{per}}(\Omega)}\to 0, \tag{2.44}$$

here (b,S_0) are the functions given in Theorem 2.1. Using S^{n-1} instead of $\hat{S}$ as a given function, then (u^n,σ^n,S^n) be a sequence of approximate solutions of (2.1)–(2.7). Let $\eta=\frac{1}{n}$ in (2.19), by Lemmas 2.2–2.4, there exist constants C_{t_0} and $C_{t_0,\kappa}$ independent of n such that

$$\|S^n\|_{L^\infty(0,t_0;H^1_{\text{per}}(\Omega))}+\|u^n\|_{L^\infty(0,t_0;H^2_{\text{per}}(\Omega))}+\|\sigma^n\|_{L^\infty(0,t_0;H^1_{\text{per}}(\Omega))}\le C_{t_0}, \tag{2.45}$$
$$\|S^n\|_{L^2(0,t_0;H^3_{\text{per}}(\Omega))}+\|S^n_t\|_{L^2(0,t_0;(H^2_{\text{per}}(\Omega))')}\le C_{t_0,\kappa}. \tag{2.46}$$

We choose $P_0=P_1=2$, $B_0=H^3_{\text{per}}(\Omega)$, $B=C^{1+\alpha}_{\text{per}}(\Omega)$ or $H^2_{\text{per}}(\Omega)$, $B_1=(H^2_{\text{per}}(\Omega))'$, here $0<\alpha<\frac{1}{2}$. Estimate (2.46) implies that those spaces satisfy the assumptions of Lemma 2.5, then for $n\to\infty$

$$\|S^n-S\|_{L^2(0,t_0;C^{1+\alpha}_{\text{per}}(\Omega))}\to 0, \tag{2.47}$$
$$\|S^n-S\|_{L^2(0,t_0;H^2_{\text{per}}(\Omega))}\to 0. \tag{2.48}$$

By (2.20), (2.44) and (2.48), there is $(u,\sigma)\in L^2(0,t_0;H^2_{\text{per}}(\Omega)\times H^1_{\text{per}}(\Omega))$ such that

$$\|u^n-u\|_{L^2(0,t_0;H^2_{\text{per}}(\Omega))}+\|\sigma^n-\sigma\|_{L^2(0,t_0;H^1_{\text{per}}(\Omega))}\to 0. \tag{2.49}$$

Then, estimates (2.45)–(2.49) and the uniqueness of limit imply

$$S\in L^\infty(0,t_0;H^1_{\text{per}}(\Omega))\cap L^2(0,t_0;H^3_{\text{per}}(\Omega)),\quad S_t\in L^2(0,t_0;(H^2_{\text{per}}(\Omega))'), \tag{2.50}$$
$$u\in L^\infty(0,t_0;H^2_{\text{per}}(\Omega)),\quad \sigma\in L^\infty(0,t_0;H^1_{\text{per}}(\Omega)). \tag{2.51}$$

From (2.46) we can choose a subsequence such that

$$\nabla\Delta S^n \rightharpoonup \nabla\Delta S\quad \text{weakly in } L^2(Q_{t_0}). \tag{2.52}$$

(1.11) and (1.24) yield

$$\begin{aligned}|m_\kappa(\widetilde{S^{n-1}})-m_\kappa(S)|&=|m(\widetilde{S^{n-1}})+\kappa-m(S)-\kappa|=|m(\widetilde{S^{n-1}})-m(S)|\\&\le M_1|\widetilde{S^{n-1}}-S|,\end{aligned} \tag{2.53}$$

combination of this inequality with the property of mollifier and (2.48) imply

$$\|\widetilde{S^{n-1}}-S\|_{L^2(Q_{t_0})}\le\|\widetilde{S^{n-1}}-S^{n-1}\|_{L^2(Q_{t_0})}+\|S^{n-1}-S\|_{L^2(Q_{t_0})}\to 0, \tag{2.54}$$

therefore

$$\|m_\kappa(\widetilde{S^{n-1}})-m_\kappa(S)\|_{L^2(Q_{t_0})}\le M_1\|\widetilde{S^{n-1}}-S\|_{L^2(Q_{t_0})}\to 0, \tag{2.55}$$

from (2.47), we also have

$$\|m_\kappa(\widetilde{S^{n-1}})-m_\kappa(S)\|_{L^2(0,t_0;L^\infty(\Omega))}\le M_1\|\widetilde{S^{n-1}}-S\|_{L^2(0,t_0;L^\infty(\Omega))}\to 0. \tag{2.56}$$

Invoking that $\nabla\hat{\psi}'(S^n)=\hat{\psi}''(S^n)\nabla S^n$, from (2.31) and (2.34) we infer that $\nabla\hat{\psi}'(S^n)\in L^2(Q_{t_0})$, then there is a function $\chi\in L^2(Q_{t_0})$, such that

$$\nabla\hat{\psi}'(S^n)\rightharpoonup\chi\quad\text{weakly in } L^2(Q_{t_0}),\tag{2.57}$$

we need to prove that $\chi=\nabla\hat{\psi}'(S)$. When $4r-2\le 6$, that is $r\le 2$, by (1.15), (2.22), (2.47) and Hölder's inequality, we conclude that

$$\begin{aligned}\|\hat{\psi}''(S^n)-\hat{\psi}''(S)\|^2_{L^2(Q_{t_0})}&=\int_{Q_{t_0}}|\hat{\psi}''(S^n)-\hat{\psi}''(S)|^2d(\tau,x)\\&=\int_{Q_{t_0}}|\hat{\psi}'''(\xi)(S^n-S)|^2d(\tau,x)\\&\le C\int_{Q_{t_0}}(|\xi|^{2r-1}+1)^2|S^n-S|^2d(\tau,x)\\&\le C\int_0^{t_0}(\|\xi\|^{4r-2}_{L^{4r-2}(\Omega)}+1)\|S^n-S\|^2_{L^\infty(\Omega)}d\tau\\&\le C\int_0^{t_0}(\|\xi\|^{4r-2}_{H^1_{\text{per}}(\Omega)}+1)\|S^n-S\|^2_{L^\infty(\Omega)}d\tau\\&\le C\int_0^{t_0}\|S^n-S\|^2_{L^\infty(\Omega)}d\tau\to 0,\end{aligned}\tag{2.58}$$

where, ξ is valued in S and S^n, that is

$$\hat{\psi}''(S^n)\to\hat{\psi}''(S)\quad\text{strongly in } L^2(Q_{t_0}).\tag{2.59}$$

Combination of the following convergence property

$$\nabla S^n\rightharpoonup\nabla S\quad\text{weakly in } L^2(Q_{t_0}),\tag{2.60}$$

we obtain

$$\hat{\psi}''(S^n)\nabla S^n\rightharpoonup\hat{\psi}''(S)\nabla S\quad\text{weakly in } L^1(Q_{t_0}),\tag{2.61}$$

then, (2.57) and (2.61) imply $\chi=\hat{\psi}''(S)\nabla S$, and

$$\nabla\hat{\psi}'(S^n)\rightharpoonup\nabla\hat{\psi}'(S)\quad\text{weakly in } L^2(Q_{t_0}).\tag{2.62}$$

It thus follows from (2.55) and (2.62) that

$$m_\kappa(\widetilde{S^{n-1}})\nabla\hat{\psi}'(S^n)\rightharpoonup m_\kappa(S)\nabla\hat{\psi}'(S)\quad\text{weakly in } L^1(Q_{t_0}).\tag{2.63}$$

Similarly, by

$$\nabla T^n\rightharpoonup\nabla T\quad\text{weakly in } L^2(Q_{t_0}),\tag{2.64}$$

(2.52) and (2.55) we deduce that

$$m_\kappa(\widetilde{S^{n-1}})\nabla T^n\rightharpoonup m_\kappa(S)\nabla T\qquad\text{weakly in } L^1(Q_{t_0}),\tag{2.65}$$

$$m_\kappa(\widetilde{S^{n-1}})\nabla\Delta S^n\rightharpoonup m_\kappa(S)\nabla\Delta S\quad\text{weakly in } L^1(Q_{t_0}).\tag{2.66}$$

In summary, approximate problem (2.1)–(2.7) has local solutions for a sufficiently small constant t_0. Next, we derive uniform a-priori estimates for a arbitrarily positive constant T_e, which implies the global solutions of approximate problem (2.1)–(2.7) are obtained.

2.2. Global solutions

Lemma 2.6. *There hold for any* $t \in [0, T_e]$

$$\|S\|_{L^\infty(0,T_e;H^1_{\rm per}(\Omega))} + \|S_t\|_{L^2(0,T_e;(H^2_{\rm per}(\Omega))')} \le C, \tag{2.67}$$

$$\|u\|_{L^\infty(0,T_e;H^2_{\rm per}(\Omega))} + \|\sigma\|_{L^\infty(0,T_e;H^1_{\rm per}(\Omega))} \le C, \tag{2.68}$$

$$\int_{Q_{T_e}} m_\kappa(S)|\nabla(\psi_S - \nu\Delta S)|^2 d(\tau, x) \le C, \tag{2.69}$$

$$\int_{Q_{T_e}} m_\kappa(S)|\nabla\Delta S|^2 d(\tau, x) \le C, \tag{2.70}$$

$$\|S\|_{L^2(0,T_e;H^3_{\rm per}(\Omega))} \le C_\kappa. \tag{2.71}$$

Proof. We consider the following total free energy density

$$\psi^*(\varepsilon, S, \nabla S) = \frac{\nu}{2}|\nabla S|^2 + \psi(\varepsilon, S),$$

where

$$\psi(\varepsilon, S) = \frac{1}{2}(D(\varepsilon - \bar{\varepsilon}S)) \cdot (\varepsilon - \bar{\varepsilon}S) + \hat{\psi}(S).$$

Differentiating $\psi^*(\varepsilon, S, \nabla S)$ with respect to t, integrating by parts with respect to x over Ω, and substituting (2.3) into the resulting equation, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \psi^*(\varepsilon, S, \nabla S)dx &= \int_\Omega \nu\nabla S \cdot \nabla S_t + \psi_\varepsilon \cdot \varepsilon_t + \psi_S S_t dx \\
&= \int_\Omega (\psi_S - \nu\Delta S)S_t dx + \int_\Omega \psi_\varepsilon \cdot \varepsilon_t dx \\
&= \int_\Omega (\psi_S - \nu\Delta S)\mathrm{div}(m_\kappa(S)\nabla(\psi_S - \nu\Delta S))dx + \int_\Omega \psi_\varepsilon \cdot \varepsilon_t dx \\
&= -\int_\Omega \nabla(\psi_S - \nu\Delta S) \cdot m_\kappa(S)\nabla(\psi_S - \nu\Delta S)dx + \int_\Omega \psi_\varepsilon \cdot \varepsilon_t dx \\
&= -\int_\Omega m_\kappa(S)|\nabla(\psi_S - \nu\Delta S)|^2 dx + \int_\Omega \psi_\varepsilon \cdot \varepsilon_t dx.
\end{aligned} \tag{2.72}$$

The symmetry of σ implies

$$\int_\Omega \psi_\varepsilon \cdot \varepsilon_t dx = \int_\Omega \sigma \cdot \varepsilon_t dx = \int_\Omega \sigma \cdot \nabla u_t dx = -\int_\Omega \mathrm{div}\sigma \cdot u_t dx = \int_\Omega b \cdot u_t dx. \tag{2.73}$$

Integration of (2.72) with respect to t from 0 to T_e, by (2.73) we know

$$\begin{aligned}
&\int_\Omega \psi^*(\varepsilon, S, \nabla S)(t, x)dx + \int_{Q_{T_e}} m_\kappa(S)|\nabla(\psi_S - \nu\Delta S)|^2 d(\tau, x) \\
&= \int_\Omega \psi^*(\varepsilon, S, \nabla S)(0, x) + \int_{Q_{T_e}} b \cdot u_t d(\tau, x).
\end{aligned} \tag{2.74}$$

Noting that $S_0 \in H^1_{\rm per}(\Omega))$, the elliptic regularity theory implies $u_0 \in H^2_{\rm per}(\Omega)$, then

$$|\int_\Omega \psi^*(S, \nabla S)(0, x)| = |\int_\Omega \frac{\nu}{2}|\nabla S_0|^2 + \frac{1}{2}(D(\varepsilon(u_{0x}) - \bar{\varepsilon}S_0)) \cdot (\varepsilon(u_{0x}) - \bar{\varepsilon}S_0) + \hat{\psi}(S_0)dx|$$

$$\begin{aligned}
&\le \int_\Omega \frac{\nu}{2}|\nabla S_0|^2 + \frac{C}{2}|\varepsilon(u_{0x}) - \bar{\varepsilon} S_0|^2 + C(|S_0|^{2r+2} + 1)dx \\
&\le C(\|\nabla S_0\|^2 + \|\varepsilon(u_{0x})\|^2 + \|S_0\|^2 + \|S_0\|^{2r+2}_{L^{2r+2}(\Omega)} + 1) \\
&\le C(\|S_0\|^2_{H^1_{\mathrm{per}}(\Omega)} + \|u_0\|^2_{H^2_{\mathrm{per}}(\Omega)} + \|S_0\|^2_{H^1_{\mathrm{per}}(\Omega)} + \|S_0\|^{2r+2}_{H^1_{\mathrm{per}}(\Omega)} + 1) \\
&\le C. \qquad (2.75)
\end{aligned}$$

we know

$$\begin{aligned}
\int_{Q_{T_e}} b \cdot u_t d(\tau, x) &= \int_\Omega b \cdot u|_0^t dx - \int_{Q_{T_e}} b_t \cdot u d(\tau, x) \\
&= \int_\Omega b(t) \cdot u(t) dx - \int_\Omega b(0) \cdot u(0) dx - \int_{Q_{T_e}} b_t \cdot u d(\tau, x), \qquad (2.76)
\end{aligned}$$

by the assumption of b

$$|\int_\Omega b(0) \cdot u(0) dx| \le C. \qquad (2.77)$$

We let $b = (\int_{x_0(t)}^x b dy)_x$, $\int_\Omega b dy = 0$, it follows from $b \in L^\infty(0, T_e; L^2(\Omega))$, $b_t \in L^2(Q_{T_e})$, the Poincaré inequality and the Young inequality that

$$\begin{aligned}
|\int_\Omega b(t) \cdot u(t) dx| &= |\int_\Omega (\int_{x_0(t)}^x b dy) \nabla u dx| \le \|\int_{x_0(t)}^x b dy\| \|\nabla u\| \\
&\le \mu \|\nabla u\|^2 + C_\mu \|\int_{x_0(t)}^x b dy\|^2 \le \mu \|\nabla u\|^2 + C_\mu, \qquad (2.78)
\end{aligned}$$

and

$$\begin{aligned}
|\int_{Q_{T_e}} b_t \cdot u d(\tau, x)| &= |\int_0^{T_e} \int_\Omega (\int_{x_0(t)}^x b dy)_t \nabla u dx d\tau| \\
&\le \|(\int_{x_0(t)}^x b dy)_t\|_{L^2(Q_{T_e})} \|\nabla u\|_{L^2(Q_{T_e})} \le C \|\nabla u\|_{L^2(Q_{T_e})} \\
&= C \int_0^{T_e} \|\nabla u\|^2 d\tau. \qquad (2.79)
\end{aligned}$$

Invoking that $|\varepsilon(\nabla u)|^2 \ge \frac{1}{2}|\nabla u|^2$, then (2.74)–(2.79) yield

$$\begin{aligned}
&\int_\Omega \psi^*(\varepsilon, S, \nabla S)(t, x) dx + \int_{Q_{T_e}} m_\kappa(S) |\nabla(\psi_S - \nu \Delta S)|^2 d(\tau, x) \\
&\le C + \mu \|\varepsilon\|^2 + C \int_0^{T_e} \|\varepsilon\|^2 d\tau, \qquad (2.80)
\end{aligned}$$

let μ be sufficiently small, from (1.12) we deduce that

$$\begin{aligned}
&\int_\Omega (\frac{\nu}{2}|\nabla S|^2 + C S^{2r+2} + \frac{1}{2}(D(\varepsilon - \bar{\varepsilon} S)) \cdot (\varepsilon - \bar{\varepsilon} S)) dx \\
&+ \int_{Q_{T_e}} m_\kappa(S) |\nabla(\psi_S - \nu \Delta S)|^2 d(\tau, x) \le C + C \int_0^{T_e} \|\varepsilon\|^2 d\tau. \qquad (2.81)
\end{aligned}$$

By

$$\|\varepsilon\|^2 \le 2\|\varepsilon - \bar{\varepsilon}S\|^2 + 2\|\bar{\varepsilon}S\|^2 \le \int_\Omega (CS^{2r+2} + \frac{1}{2}(D(\varepsilon - \bar{\varepsilon}S)) \cdot (\varepsilon - \bar{\varepsilon}S))dx, \quad (2.82)$$

thus

$$\begin{aligned}
&\int_\Omega (\frac{\nu}{2}|\nabla S|^2 + CS^{2r+2} + \frac{1}{2}(D(\varepsilon - \bar{\varepsilon}S)) \cdot (\varepsilon - \bar{\varepsilon}S))dx \\
&+ \int_{Q_{T_e}} m_\kappa(S)|\nabla(\psi_S - \nu\Delta S)|^2 d(\tau, x) \\
\le& C\int_{Q_{T_e}} (S^{2r+2} + \frac{1}{2}(D(\varepsilon - \bar{\varepsilon}S)) \cdot (\varepsilon - \bar{\varepsilon}S))d(\tau, x) \\
\le& C\int_{Q_{T_e}} (\frac{\nu}{2}|\nabla S|^2 + CS^{2r+2} + \frac{1}{2}(D(\varepsilon - \bar{\varepsilon}S)) \cdot (\varepsilon - \bar{\varepsilon}S))d(\tau, x). \quad (2.83)
\end{aligned}$$

Applying the Gronwall inequality in the integral form we obtain

$$\begin{aligned}
&\int_\Omega (\frac{\nu}{2}|\nabla S|^2 + CS^{2r+2} + \frac{1}{2}(D(\varepsilon - \bar{\varepsilon}S)) \cdot (\varepsilon - \bar{\varepsilon}S))dx \\
&+ \int_{Q_{T_e}} m_\kappa(S)|\nabla(\psi_S - \nu\Delta S)|^2 d(\tau, x) \le C, \quad (2.84)
\end{aligned}$$

then (2.69) is proved. Owing to $r \ge \frac{1}{2}$, we have $S \in L^\infty(0, T_e; H^1_{\mathrm{per}}(\Omega))$, by (2.20) we get (2.68). For all $\varphi \in L^2(0, t_0; H^2_{\mathrm{per}}(\Omega))$, we infer from (1.24) and (2.69) that

$$\begin{aligned}
|\int_{Q_{T_e}} S_t \varphi d(\tau, x)| &= |\int_{Q_{T_e}} m_\kappa(S)\nabla(\psi_S - \nu\Delta S) \cdot \nabla\varphi d(\tau, x)| \\
&\le \int_{Q_{T_e}} |m_\kappa(S)^{\frac{1}{2}}\nabla(\psi_S - \nu\Delta S)||m_\kappa(S)^{\frac{1}{2}}\nabla\varphi| d(\tau, x) \\
&\le C\int_{Q_{T_e}} m_\kappa(S)|\nabla(\psi_S - \nu\Delta S)|^2 d(\tau, x) + \int_{Q_{T_e}} m_\kappa(S)|\nabla\varphi|^2 d(\tau, x) \\
&\le C + C\int_{Q_{T_e}} |\nabla\varphi|^2 d(\tau, x) \\
&\le C(\|\varphi\|_{L^2(0, T_e; H^2_{\mathrm{per}}(\Omega))} + 1),
\end{aligned}$$

thus $S_t \in L^2(0, t_0; (H^2_{\mathrm{per}}(\Omega))')$. Similar to the proof of (2.32), taking into account (2.67) and (2.68), we obtain (2.71) and

$$\int_{Q_{T_e}} |\nabla\hat{\psi}'(S)|^2 d(\tau, x) \le C. \quad (2.85)$$

In order to prove (2.70). One obtains from (2.69) and the inequality $(a+b)^2 \le 2a^2 + 2b^2$ that

$$\begin{aligned}
\nu^2 \int_{Q_{T_e}} m_\kappa(S)|\nabla\Delta S|^2 d(\tau, x) \le& 2\int_{Q_{T_e}} m_\kappa(S)|\nabla(\psi_S - \nu\Delta S)|^2 d(\tau, x) \\
&+ 2\int_{Q_{T_e}} m_\kappa(S)|\nabla\psi_S|^2 d(\tau, x)
\end{aligned}$$

$$\leq C + 2\int_{Q_{T_e}} m_\kappa(S)|\nabla\psi_S|^2 d(\tau,x), \tag{2.86}$$

here

$$\begin{aligned}\int_{Q_{T_e}} m_\kappa(S)|\nabla\psi_S|^2 d(\tau,x) &= \int_{Q_{T_e}} m_\kappa(S)|\nabla\hat{\psi}'(S) - \nabla(\bar{\varepsilon}\cdot\sigma)|^2 d(\tau,x)\\ &\leq \int_{Q_{T_e}} (M_0+1)|\nabla\hat{\psi}'(S) - \nabla(\bar{\varepsilon}\cdot\sigma)|^2 d(\tau,x)\\ &\leq C\int_{Q_{T_e}} |\nabla\hat{\psi}'(S)|^2 + C\int_{Q_{T_e}} |\nabla(\bar{\varepsilon}\cdot\sigma)|^2 d(\tau,x)\\ &\leq C, \end{aligned}\tag{2.87}$$

we used (2.85). Thus (2.70) is obtained. □

Therefore, we complete the proof of the existence of weak solutions global in time to approximate problem (2.1)–(2.7).

Lemma 2.7. *For any $t\in[0,T_e]$, there holds the following energy inequality*

$$\begin{aligned}&\|\nabla S(t)\|^2 + \nu\int_{Q_{T_e}} m_\kappa(S)|\nabla\Delta S|^2 d(\tau,x)\\ \leq&\|\nabla S_0\|^2 + \int_{Q_{T_e}} m_\kappa(S)\nabla\hat{\psi}'(S)\cdot\nabla\Delta S d(\tau,x)\\ &-\int_{Q_{T_e}} m_\kappa(S)\nabla(\bar{\varepsilon}\cdot\sigma)\cdot\nabla\Delta S d(\tau,x).\end{aligned}\tag{2.88}$$

Proof. Let (u^n,σ^n,S^n) be the sequence of asymptotic solution constructed in the proof of Theorem 2.1, we replace the $\hat{S}$ in (2.14) by S^{n-1}

$$S_t^n = \operatorname{div}(m_\kappa(\widetilde{S^{n-1}})\nabla(\hat{\psi}'(S^n) - \bar{\varepsilon}\cdot\sigma^n - \nu\Delta S^n)). \tag{2.89}$$

Multiplying (2.89) by $-\Delta S^n$ and integrating the resulting equation over Q_{T_e}, we have

$$\begin{aligned}&\|\nabla S^n(t)\|^2 + \nu\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})|\nabla\Delta S^n|^2 d(\tau,x)\\ =&\|\nabla S_0^n\|^2 + \int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla\hat{\psi}'(S^n)\cdot\nabla\Delta S^n d(\tau,x)\\ &-\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla(\bar{\varepsilon}\cdot\sigma^n)\cdot\nabla\Delta S^n d(\tau,x),\end{aligned}$$

then taking $\liminf_{n\to\infty}$

$$\begin{aligned}&\liminf_{n\to\infty}\|\nabla S^n(t)\|^2 + \nu\liminf_{n\to\infty}\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})|\nabla\Delta S^n|^2 d(\tau,x)\\ =&\liminf_{n\to\infty}\|\nabla S_0^n\|^2 + \liminf_{n\to\infty}\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla\hat{\psi}'(S^n)\cdot\nabla\Delta S^n d(\tau,x)\\ &-\liminf_{n\to\infty}\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla(\bar{\varepsilon}\cdot\sigma^n)\cdot\nabla\Delta S^n d(\tau,x).\end{aligned}\tag{2.90}$$

We are going to prove

$$\begin{aligned}
&\|\nabla S(t)\|^2 + \nu \int_{Q_{T_e}} m_\kappa(S)|\nabla\Delta S|^2 d(\tau,x) \\
\le &\liminf_{n\to\infty} \|\nabla S^n(t)\|^2 + \nu \liminf_{n\to\infty} \int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})|\nabla\Delta S^n|^2 d(\tau,x), \qquad (2.91)\\
&\liminf_{n\to\infty} \|\nabla S_0^n\|^2 + \liminf_{n\to\infty} \int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla\hat\psi'(S^n)\cdot\nabla\Delta S^n d(\tau,x) \\
&- \liminf_{n\to\infty} \int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla(\bar\varepsilon\cdot\sigma^n)\cdot\nabla\Delta S^n d(\tau,x) \\
= &\|\nabla S_0\|^2 + \int_{Q_{T_e}} m_\kappa(S)\nabla\hat\psi'(S)\cdot\nabla\Delta S d(\tau,x) - \int_{Q_{T_e}} m_\kappa(S)\nabla(\bar\varepsilon\cdot\sigma)\cdot\nabla\Delta S d(\tau,x).
\end{aligned} \tag{2.92}$$

In order to prove (2.91). From (2.22) and the lower semi-continuity of the L^p-norm

$$\|\nabla S(t)\|^2 \le \liminf_{n\to\infty} \|\nabla S^n(t)\|^2. \tag{2.93}$$

It follows from (2.31) and (2.34) that $\int_{Q_{T_e}} |\nabla\hat\psi'(S^n)|^2 d(\tau,x) \le C$, then (2.24) and the inequality $(a+b)^2 \le 2a^2 + 2b^2$ imply

$$\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})|\nabla\Delta S^n|^2 d(\tau,x) \le C, \tag{2.94}$$

thus there is a function $\zeta \in L^2(Q_{T_e})$ satisfies

$$m_\kappa(\widetilde{S^{n-1}})^{\frac12}\nabla\Delta S^n \rightharpoonup \zeta \quad \text{weakly in } L^2(Q_{T_e}), \tag{2.95}$$

we now prove $\zeta = m_\kappa(S)^{\frac12}\nabla\Delta S$. Applying estimate (1.11) and the inequality $\sqrt{|x|} - \sqrt{|y|} \le \sqrt{|x-y|},\ x, y \in \mathbb{R}$

$$\begin{aligned}
|m_\kappa(\widetilde{S^{n-1}})^{\frac12} - m_\kappa(S)^{\frac12}| = |m(\widetilde{S^{n-1}})^{\frac12} - m(S)^{\frac12}| &\le |m(\widetilde{S^{n-1}}) - m(S)|^{\frac12} \\
&\le M_1^{\frac12}|\widetilde{S^{n-1}} - S|^{\frac12},
\end{aligned} \tag{2.96}$$

then

$$|m_\kappa(\widetilde{S^{n-1}})^{\frac12} - m_\kappa(S)^{\frac12}|^4 \le M_1^2|\widetilde{S^{n-1}} - S|^2,$$

integrating the above inequality over Q_{T_e}, from (2.48) and (2.53) we infer

$$\|m_\kappa(\widetilde{S^{n-1}})^{\frac12} - m_\kappa(S)^{\frac12}\|^4_{L^4(Q_{T_e})} \le M_1^2\|\widetilde{S^{n-1}} - S\|^2_{L^2(Q_{T_e})} \to 0, \tag{2.97}$$

that is

$$m_\kappa(\widetilde{S^{n-1}})^{\frac12} \to m_\kappa(S)^{\frac12} \quad \text{strongly in } L^4(Q_{T_e}). \tag{2.98}$$

From (2.52) we know

$$\nabla\Delta S^n \rightharpoonup \nabla\Delta S \quad \text{weakly in } L^2(Q_{T_e}), \tag{2.99}$$

then

$$m_\kappa(\widetilde{S^{n-1}})^{\frac12}\nabla\Delta S^n \rightharpoonup m_\kappa(S)^{\frac12}\nabla\Delta S \quad \text{weakly in } L^{\frac43}(Q_{T_e}), \tag{2.100}$$

thus, (2.95) and (2.100) yield $\zeta = m_\kappa(S)^{\frac{1}{2}}\nabla\Delta S$, and

$$m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\nabla\Delta S^n \rightharpoonup m_\kappa(S)^{\frac{1}{2}}\nabla\Delta S \quad \text{weakly in } L^2(Q_{T_e}), \tag{2.101}$$

we infer from the lower semi-continuity of the L^p-norm that

$$\nu\int_{Q_{T_e}} m_\kappa(S)|\nabla\Delta S|^2 d(\tau,x) \le \nu\liminf_{n\to\infty}\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})|\nabla\Delta S^n|^2 d(\tau,x), \tag{2.102}$$

therefore, (2.91) is derived.

For the proof of (2.92). By (2.44) we have

$$\liminf_{n\to\infty}\|\nabla S_0^n\|^2 = \|\nabla S_0\|^2. \tag{2.103}$$

We now study the convergence of $\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla\hat{\psi}'(S^n)\cdot\nabla\Delta S^n d(\tau,x)$.

$$\begin{aligned}
&\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla\hat{\psi}'(S^n)\cdot\nabla\Delta S^n d(\tau,x)\\
=&\int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}} m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\hat{\psi}''(S^n)\nabla S^n\cdot\nabla\Delta S^n d(\tau,x)\\
=&\int_{Q_{T_e}} (m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}} - m_\kappa(S)^{\frac{1}{2}}) m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\hat{\psi}''(S^n)\nabla S^n\cdot\nabla\Delta S^n d(\tau,x)\\
&+\int_{Q_{T_e}} m_\kappa(S)^{\frac{1}{2}} m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}(\hat{\psi}''(S^n)-\hat{\psi}''(S))\nabla S^n\cdot\nabla\Delta S^n d(\tau,x)\\
&+ m_\kappa(S)^{\frac{1}{2}} m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\hat{\psi}''(S)(\nabla S^n - \nabla S)\cdot\nabla\Delta S^n d(\tau,x)\\
&+ m_\kappa(S)^{\frac{1}{2}} m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\hat{\psi}''(S)\nabla S\cdot\nabla\Delta S^n d(\tau,x)\\
:=&J_{11}+J_{12}+J_{13}+J_{14}.
\end{aligned} \tag{2.104}$$

For J_{11}, the Gagliardo-Nirenberg inequality implies

$$\|\nabla S^n\|_{L^4(\Omega)} \le C\|\nabla\Delta S^n\|^{\frac{3}{8}}\|\nabla S^n\|^{\frac{5}{8}} + C\|\nabla S^n\|. \tag{2.105}$$

From (2.22), (2.56) and Hölder's inequality, when $8r \le 6$, $r \le \frac{3}{4}$

$$\begin{aligned}
|J_{11}| =& |\int_{Q_{T_e}} (m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}} - m_\kappa(S)^{\frac{1}{2}}) m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\hat{\psi}''(S^n)\nabla S^n\cdot\nabla\Delta S^n d(\tau,x)|\\
\le& C\int_{Q_{T_e}} |m_\kappa(\widetilde{S^{n-1}}) - m_\kappa(S)|^{\frac{1}{2}} m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}(|S^n|^{2r}+1)|\nabla S^n||\nabla\Delta S^n| d(\tau,x)\\
\le& C\int_0^{T_e} \|m_\kappa(\widetilde{S^{n-1}}) - m_\kappa(S)\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\nabla\Delta S^n\|(\|S^n\|_{L^{8r}(\Omega)}^{2r}+1)\\
&\times\|\nabla S^n\|_{L^4(\Omega)} d\tau\\
\le& C(\int_0^{T_e}\|m_\kappa(\widetilde{S^{n-1}}) - m_\kappa(S)\|_{L^\infty(\Omega)}^2 d\tau)^{\frac{1}{4}}(\int_0^{T_e}\|m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\nabla\Delta S^n\|^2 d\tau)^{\frac{1}{2}}\\
&\times\sup(\|S^n\|_{H^1_{\mathrm{per}}(\Omega)}^{2r}+1)(\int_0^{T_e}\|\nabla S^n\|_{L^4(\Omega)}^4 d\tau)^{\frac{1}{4}}
\end{aligned}$$

$$\leq C(\int_0^{T_e}\|m_\kappa(\widetilde{S^{n-1}})-m_\kappa(S)\|^2_{L^\infty(\Omega)}d\tau)^{\frac{1}{4}}(\int_0^{T_e}(\|\nabla\Delta S^n\|^{\frac{3}{2}}\|\nabla S^n\|^{\frac{5}{2}}+\|\nabla S^n\|^4)d\tau)^{\frac{1}{4}}$$
$$\leq C(\int_0^{T_e}\|m_\kappa(\widetilde{S^{n-1}})-m_\kappa(S)\|^2_{L^\infty(\Omega)}d\tau)^{\frac{1}{4}}\to 0. \tag{2.106}$$

For the convergence of J_{12}. From the following interpolation inequality

$$\|f\|_{L^\infty(\Omega)}\leq C\|\Delta f\|^{\frac{1}{2}}\|\nabla f\|^{\frac{1}{2}}+C\|f\|$$

and (2.48), we have

$$\int_0^{T_e}\|S^n-S\|^4_{L^\infty(\Omega)}d\tau\leq C\int_0^{T_e}\|\Delta S^n-\Delta S\|^2d\tau\to 0. \tag{2.107}$$

Applying the Gagliardo-Nirenberg inequality to obtaion

$$\|\nabla S^n\|_{L^{\frac{12}{5}}(\Omega)}\leq C\|\nabla\Delta S^n\|^{\frac{1}{8}}\|\nabla S^n\|^{\frac{7}{8}}+C\|\nabla S^n\|. \tag{2.108}$$

Then from (2.107), (2.108) and Hölder's inequality, when $24r-12\leq 6$, $r\leq\frac{3}{4}$

$$\begin{aligned}
|J_{12}|&=|\int_{Q_{T_e}}m_\kappa(S)^{\frac{1}{2}}m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}(\hat\psi''(S^n)-\hat\psi''(S))\nabla S^n\cdot\nabla\Delta S^n d(\tau,x)|\\
&=|\int_{Q_{T_e}}m_\kappa(S)^{\frac{1}{2}}m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\hat\psi'''(\xi)(S^n-S)\nabla S^n\cdot\nabla\Delta S^n d(\tau,x)|\\
&\leq C(M_0+1)^{\frac{1}{2}}\int_{Q_{T_e}}(|\xi|^{2r-1}+1)|S^n-S||\nabla S^n||m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\nabla\Delta S^n|d(\tau,x)\\
&\leq C\int_0^{T_e}(\|\xi\|^{2r-1}_{L^{24r-12}(\Omega)}+1)\|S^n-S\|_{L^\infty(\Omega)}\|\nabla S^n\|_{L^{\frac{12}{5}}(\Omega)}\|m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\nabla\Delta S^n\|d\tau\\
&\leq C\sup(\|\xi\|^{2r-1}_{H^1_{\mathrm{per}}(\Omega)}+1)(\int_0^{T_e}\|S^n-S\|^4_{L^\infty(\Omega)}d\tau)^{\frac{1}{4}}\\
&\quad\times(\int_0^{T_e}\|\nabla S^n\|^4_{L^{\frac{12}{5}}(\Omega)}d\tau)^{\frac{1}{4}}(\int_0^{T_e}\|m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\nabla\Delta S^n\|^2d\tau)^{\frac{1}{2}}\\
&\leq C(\int_0^{T_e}\|S^n-S\|^4_{L^\infty(\Omega)}d\tau)^{\frac{1}{4}}(\int_0^{T_e}(\|\nabla\Delta S^n\|^{\frac{1}{2}}\|\nabla S^n\|^{\frac{7}{2}}+C\|\nabla S^n\|^4)d\tau)^{\frac{1}{4}}\\
&\leq C(\int_0^{T_e}\|S^n-S\|^4_{L^\infty(\Omega)}d\tau)^{\frac{1}{4}}\to 0,
\end{aligned} \tag{2.109}$$

here, ξ is a suitable number between S and S^n.

To study J_{13}, by virtue of (2.47)

$$\|\nabla S^n-\nabla S\|_{L^2(0,T_e;L^\infty(\Omega))}\to 0.$$

From (2.22), (2.94) and Hölder's inequality, when $4r\leq 6$, $r\leq\frac{3}{2}$

$$\begin{aligned}
|J_{13}|&\leq CM_0\int_{Q_{T_e}}|m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}||\nabla\Delta S^n|(|S|^{2r}+1)|\nabla S^n-\nabla S|d(\tau,x)\\
&\leq C\int_0^{T_e}\|m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\nabla\Delta S^n\|(\|S\|^{2r}_{L^{4r}(\Omega)}+1)\|\nabla S^n-\nabla S\|_{L^\infty(\Omega)}d\tau
\end{aligned}$$

$$\leq C(\int_0^{T_e} \|m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\nabla\Delta S^n\|^2 d\tau)^{\frac{1}{2}} \sup(\|S\|^{2r}_{H^1_{\mathrm{per}}(\Omega)}+1)$$
$$\times (\int_0^{T_e} \|\nabla S^n - \nabla S\|^2_{L^\infty(\Omega)} d\tau)^{\frac{1}{2}}$$
$$\leq C(\int_0^{T_e} \|\nabla S^n - \nabla S\|^2_{L^\infty(\Omega)} d\tau)^{\frac{1}{2}} \to 0. \tag{2.110}$$

For the proof of $J_{14} \to 0$. The estimates (1.20), (2.31) and (2.34) imply $m_\kappa(S)^{\frac{1}{2}}\hat{\psi}''(S)\nabla S$ belongs to $L^2(Q_{T_e})$ then can be regarded as a test function, from (2.101) we infer that $m_\kappa(\widetilde{S^{n-1}})^{\frac{1}{2}}\nabla\Delta S^n \rightharpoonup m_\kappa(S)^{\frac{1}{2}}\nabla\Delta S$ weakly in $L^2(Q_{t_0})$, thus $J_{14} \to 0$.

From the foregoing, we have that

$$\liminf_{n\to\infty} \int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla\hat{\psi}'(S^n)\cdot\nabla\Delta S^n d(\tau,x) = \int_{Q_{T_e}} m_\kappa(S)\nabla\hat{\psi}'(S)\cdot\nabla\Delta S d(\tau,x). \tag{2.111}$$

In a similar way we obtain

$$\liminf_{n\to\infty} \int_{Q_{T_e}} m_\kappa(\widetilde{S^{n-1}})\nabla(\bar{\varepsilon}\cdot\sigma^n)\cdot\nabla\Delta S^n d(\tau,x) = \int_{Q_{T_e}} m_\kappa(S)\nabla(\bar{\varepsilon}\cdot\sigma)\cdot\nabla\Delta S d(\tau,x). \tag{2.112}$$

Consequently, Lemma 2.7 is proved. At the same time we complete the proof of Theorem 2.1. □

3. Existence of weak solutions to degenerate problem

In this section we are devoted to prove the existence of weak solutions to degenerate problem (1.1)–(1.7). We first construct a-priori estimates independent of κ for the solutions $(u^\kappa, \sigma^\kappa, S^\kappa)$ of approximate problem (2.1)–(2.7), it can be seen from the proof of Theorem 2.1, the bounds on the right hand side of (2.67)–(2.70) independent of κ, then there exists a constant C independent of κ, such that

$$\|S^\kappa\|_{L^\infty(0,T_e;H^1_{\mathrm{per}}(\Omega))} + \|S^\kappa_t\|_{L^2(0,T_e;(H^2_{\mathrm{per}}(\Omega))')} \leq C, \tag{3.1}$$
$$\|u^\kappa\|_{L^\infty(0,T_e;H^2_{\mathrm{per}}(\Omega))} + \|\sigma^\kappa\|_{L^\infty(0,T_e;H^1_{\mathrm{per}}(\Omega))} \leq C, \tag{3.2}$$
$$\int_{Q_{T_e}} m_\kappa(S^\kappa)|\nabla(\psi_{S^\kappa} - \nu\Delta S^\kappa)|^2 d(\tau,x) \leq C, \tag{3.3}$$
$$\int_{Q_{T_e}} m_\kappa(S^\kappa)|\nabla\Delta S^\kappa|^2 d(\tau,x) \leq C. \tag{3.4}$$

Lemma 3.1. *There is a constant C independent of κ such that*

$$\|\Delta S^\kappa\|_{L^2(Q_{T_e})} \leq C, \tag{3.5}$$
$$\|S^\kappa\|_{L^2(0,T_e;H^2_{\mathrm{per}}(\Omega))} \leq C. \tag{3.6}$$

Proof. We denote

$$g_\kappa(S) = -\int_S^A \frac{dl}{m_\kappa(l)}, \quad G_\kappa(S) = -\int_S^A g_\kappa(l)dl,$$

where, $A > \max|S^\kappa|$ for all small κ. Then

$$G'_\kappa(S) = g_\kappa(S),$$
$$G''_\kappa(S) = g'_\kappa(S) = \frac{1}{m_\kappa(S)}.$$

We know $(u^\kappa, \sigma^\kappa, S^\kappa)$ satisfies

$$S_t^\kappa = \operatorname{div}(m_\kappa(S^\kappa)\nabla(\hat{\psi}'(S^\kappa) - \bar{\varepsilon}\cdot\sigma^\kappa - \nu\Delta S^\kappa)). \tag{3.7}$$

We multiply the above equality by $g_\kappa(S)$, and integrate with respect to x over Ω

$$\begin{aligned}
\frac{d}{dt}\int_\Omega G_\kappa(S^\kappa)dx &= \int_\Omega m_\kappa(S^\kappa)\nabla(\hat{\psi}'(S^\kappa) - \bar{\varepsilon}\cdot\sigma^\kappa - \nu\Delta S^\kappa)\cdot\nabla g_\kappa(S^\kappa)dx \\
&= \int_\Omega m_\kappa(S^\kappa)\nabla(\hat{\psi}'(S^\kappa) - \bar{\varepsilon}\cdot\sigma^\kappa - \nu\Delta S^\kappa)\cdot\frac{1}{m_\kappa(S^\kappa)}\nabla S^\kappa dx \\
&= \int_\Omega \nabla(\hat{\psi}'(S^\kappa) - \bar{\varepsilon}\cdot\sigma^\kappa - \nu\Delta S^\kappa)\cdot\nabla S^\kappa dx \\
&= -\int_\Omega (\hat{\psi}'(S^\kappa) - \bar{\varepsilon}\cdot\sigma^\kappa - \nu\Delta S^\kappa)\Delta S^\kappa dx,
\end{aligned} \tag{3.8}$$

then we have

$$\frac{d}{dt}\int_\Omega G_\kappa(S^\kappa)dx + \nu\int_\Omega |\Delta S^\kappa|^2 dx \tag{3.9}$$

$$\begin{aligned}
&= -\int_\Omega (\hat{\psi}'(S^\kappa) - \bar{\varepsilon}\cdot\sigma^\kappa)\Delta S^\kappa dx \\
&\le \int_\Omega |\hat{\psi}'(S^\kappa)\Delta S^\kappa|dx + \int_\Omega |\bar{\varepsilon}\cdot\sigma^\kappa\Delta S^\kappa|dx \\
&\le \frac{\nu}{4}\int_\Omega |\Delta S^\kappa|^2dx + C\int_\Omega |\hat{\psi}'(S^\kappa)|^2dx + \frac{\nu}{4}\int_\Omega |\Delta S^\kappa|^2dx + C\int_\Omega |\bar{\varepsilon}\cdot\sigma^\kappa|^2dx \\
&\le \frac{\nu}{2}\int_\Omega |\Delta S^\kappa|^2dx + C\int_\Omega |\hat{\psi}'(S^\kappa)|^2dx + C\int_\Omega |\bar{\varepsilon}\cdot\sigma^\kappa|^2dx,
\end{aligned} \tag{3.10}$$

when $4r+2 \le 6$, $r \le 1$, we infer from (3.1) and (3.2) that

$$\begin{aligned}
\int_\Omega |\hat{\psi}'(S^\kappa)|^2dx &\le C\int_\Omega (|S^\kappa|^{2r+1}+1)^2dx \le C\int_\Omega (|S^\kappa|^{4r+2}+1)dx \\
&\le C(\|S^\kappa\|^{4r+2}_{L^{4r+2}(\Omega)}+1) \le C(\|S^\kappa\|^{4r+2}_{H^1_{\mathrm{per}}(\Omega)}+1) \le C,
\end{aligned} \tag{3.11}$$

and

$$\int_\Omega |\bar{\varepsilon}\cdot\sigma^\kappa|^2dx \le C\|\sigma^\kappa\|^2_{L^2(\Omega)} \le C\|\sigma^\kappa\|^2_{H^1_{\mathrm{per}}(\Omega)} \le C. \tag{3.12}$$

It follows from (3.9)–(3.12) that

$$\frac{d}{dt}\int_\Omega G_\kappa(S^\kappa)dx + \frac{\nu}{2}\int_\Omega |\Delta S^\kappa|^2 dx \le C.$$

Integration of above inequality with respect to t from 0 to T_e

$$\int_\Omega G_\kappa(S^\kappa(t))dx + \frac{\nu}{2}\int_{Q_{T_e}} |\Delta S^\kappa|^2 d(\tau,x) \le \int_\Omega G_\kappa(S_0)dx + C, \tag{3.13}$$

for initial value, from the assumption of S_0 we know

$$|\int_\Omega G_\kappa(S_0)dx| \le C. \tag{3.14}$$

Therefore, (3.1) and (3.5) yield (3.6). □

We are going to prove the existence of weak solutions to degenerate problem (1.1)–(1.7).

We choose

$$P_0 = P_1 = 2, \quad B_0 = H^2_{\mathrm{per}}(\Omega), \quad B = C^\alpha_{\mathrm{per}}(\Omega) \text{ or } H^1_{\mathrm{per}}(\Omega), \quad B_1 = (H^2_{\mathrm{per}}(\Omega))',$$

here, $\alpha \in (0, \frac{1}{2})$. The estimates (3.1) and (3.6) imply those spaces satisfy the assumptions of Lemma 2.5. Hence, there exists a subsequence of S^κ, not relabeled, satisfies

$$\|S^\kappa - S\|_{L^2(0,T_e;C^\alpha_{\mathrm{per}}(\Omega))} \to 0, \tag{3.15}$$

$$\|S^\kappa - S\|_{L^2(0,T_e;H^1_{\mathrm{per}}(\Omega))} \to 0. \tag{3.16}$$

Lemma 3.2. *There holds*

$$\nabla\Delta S \in L^2(G_S), \quad m(S)^{\frac{1}{2}}\nabla\Delta S \in L^2(G_S). \tag{3.17}$$

Proof. We denote

$$G_\delta = \{(t,x) \in Q_{T_e};\ m(S(t,x)) > \delta\}, \quad \delta > 0. \tag{3.18}$$

For any fixed $\delta > 0$, we choose $\kappa_0(\delta) > 0$, such that $(t,x) \in G_\delta$, $0 < \kappa < \kappa_0(\delta)$

$$m_\kappa(S^\kappa) \ge \frac{\delta}{2}. \tag{3.19}$$

By (3.4) we have

$$\int_{G_\delta} m_\kappa(S^\kappa)|\nabla\Delta S^\kappa|^2 d(\tau,x) \le C, \tag{3.20}$$

then

$$\int_{G_\delta} |\nabla\Delta S^\kappa|^2 d(\tau,x) \le \frac{C}{\delta}. \tag{3.21}$$

Thus, we choose a subsequence of $\nabla\Delta S^\kappa$, not relabeled, satisfies

$$\nabla\Delta S^\kappa \rightharpoonup \nabla\Delta S \quad \text{weakly in } L^2(G_\delta). \tag{3.22}$$

From (3.20) we know there is a function $\chi \in L^2(G_\delta)$, such that

$$m_\kappa(S^\kappa)^{\frac{1}{2}}\nabla\Delta S^\kappa \rightharpoonup \chi \quad \text{weakly in } L^2(G_\delta), \tag{3.23}$$

we now prove $\chi = m(S)^{\frac{1}{2}}\nabla\Delta S$. Applying the inequalities $\sqrt{|x|} - \sqrt{|y|} \le \sqrt{|x-y|}$, $x, y \in \mathbb{R}$ and $(a+b)^2 \le 2a^2 + 2b^2$

$$\begin{aligned}|m_\kappa(S^\kappa)^{\frac{1}{2}} - m(S)^{\frac{1}{2}}|^4 &\le |m_\kappa(S^\kappa) - m(S)|^2 \le 2|m(S^\kappa) - m(S)|^2 + 2|\kappa|^2 \\ &\le 2M_1^2|S^\kappa - S|^2 + 2\kappa^2. \end{aligned}\tag{3.24}$$

Integration of above inequality over G_δ

$$\int_{G_\delta} |m_\kappa(S^\kappa)^{\frac{1}{2}} - m(S)^{\frac{1}{2}}|^4 d(\tau,x) \le 2M_1^2 \int_{G_\delta} |S^\kappa - S|^2 d(\tau,x) + 2\int_{G_\delta} \kappa^2 d(\tau,x),$$

from (3.16), when $\kappa \to 0$

$$\|m_\kappa(S^\kappa)^{\frac{1}{2}} - m(S)^{\frac{1}{2}}\|^4_{L^4(G_\delta)} \le C\|S^\kappa - S\|^2_{L^2(G_\delta)} + 2\kappa^2|G_\delta| \to 0, \tag{3.25}$$

where, $|G_\delta|$ denotes the measure of G_δ, that is

$$m_\kappa(S^\kappa)^{\frac{1}{2}} \to m(S)^{\frac{1}{2}} \quad \text{strongly in } L^4(G_\delta), \tag{3.26}$$

which together with (3.22) imply

$$m_\kappa(S^\kappa)^{\frac{1}{2}}\nabla\Delta S^\kappa \rightharpoonup m(S)^{\frac{1}{2}}\nabla\Delta S \quad \text{weakly in } L^{\frac{4}{3}}(G_\delta), \tag{3.27}$$

by (3.23) and the uniqueness of limit we get that $\chi = m(S)^{\frac{1}{2}}\nabla\Delta S$, and

$$m_\kappa(S^\kappa)^{\frac{1}{2}}\nabla\Delta S^\kappa \rightharpoonup m(S)^{\frac{1}{2}}\nabla\Delta S \quad \text{weakly in } L^2(G_\delta), \tag{3.28}$$

then $\int_{G_\delta} m(S)|\nabla\Delta S|^2 d(\tau,x) \le C$, by the arbitrariness of δ we complete the proof of Lemma 3.2. □

Lemma 3.3. *When $\kappa \to 0$*

$$\int_{Q_{T_e}} S^\kappa \varphi_t d(\tau,x) \to \int_{Q_{T_e}} S\varphi_t d(\tau,x), \tag{3.29}$$

$$\int_\Omega S_0^\kappa(x)\varphi(0,x)dx \to \int_\Omega S_0(x)\varphi(0,x)dx, \tag{3.30}$$

$$\int_{Q_{T_e}} m_\kappa(S^\kappa)\nabla\hat{\psi}'(S^\kappa)\cdot\nabla\varphi d(\tau,x) \to \int_{Q_{T_e}} m(S)\nabla\hat{\psi}'(S)\cdot\nabla\varphi d(\tau,x), \tag{3.31}$$

$$\int_{Q_{T_e}} m_\kappa(S^\kappa)\nabla(\bar{\varepsilon}\cdot\sigma^\kappa)\cdot\nabla\varphi d(\tau,x) \to \int_{Q_{T_e}} m(S)\nabla(\bar{\varepsilon}\cdot\sigma)\cdot\nabla\varphi d(\tau,x), \tag{3.32}$$

$$\int_{Q_{T_e}} m_\kappa(S^\kappa)\nabla\Delta S^\kappa\cdot\nabla\varphi d(\tau,x) \to \int_{G_S} m(S)\nabla\Delta S\cdot\nabla\varphi d(\tau,x). \tag{3.33}$$

Proof. (3.29) and (3.30) is obvious. Similar to the derivation of (2.62), we obtain

$$\nabla\hat{\psi}'(S^\kappa) \rightharpoonup \nabla\hat{\psi}'(S), \quad \text{weakly in } L^2(Q_{T_e}). \tag{3.34}$$

Since

$$|m_\kappa(S^\kappa) - m(S)| \leq |m(S^\kappa) - m(S) + \kappa| \leq M_1|S^\kappa - S| + \kappa, \tag{3.35}$$

from (3.16), one obtains for $\kappa \to 0$ that

$$m_\kappa(S^\kappa) \to m(S) \quad \text{strongly in } L^2(Q_{T_e}), \tag{3.36}$$

and

$$m_\kappa(S^\kappa) \to m(S) \ \text{ strongly in } L^2(0, T_e; L^\infty(\Omega)), \tag{3.37}$$

then (3.34) and (3.36) result in

$$m_\kappa(S^\kappa)\nabla\hat{\psi}'(S^\kappa) \rightharpoonup m(S)\nabla\hat{\psi}'(S) \quad \text{weakly in } L^1(Q_{T_e}), \tag{3.38}$$

we thus arrive at (3.31).

According to (3.2) and the weak compactness lemma, there exists a subsequence of $\nabla\sigma^\kappa$, not relabeled, and a limit function $\nabla\sigma \in L^2(Q_{T_e})$, such that

$$\nabla\sigma^\kappa \rightharpoonup \nabla\sigma \ \text{ weakly in } L^2(Q_{T_e}), \tag{3.39}$$

(3.36) and (3.39) yield

$$m_\kappa(S^\kappa)\nabla(\bar{\varepsilon}\cdot\sigma^\kappa) \rightharpoonup m(S)\nabla(\bar{\varepsilon}\cdot\sigma) \ \text{ weakly in } L^1(Q_{T_e}), \tag{3.40}$$

then we obtain (3.32). We now prove (3.33), for any given $\delta > 0$

$$\begin{aligned}
&|\int_{Q_{T_e}} m_\kappa(S^\kappa)\nabla\Delta S^\kappa\cdot\nabla\varphi d(\tau,x) - \int_{G_S} m(S)\nabla\Delta S\cdot\nabla\varphi d(\tau,x)| \\
\leq&|\int_{Q_{T_e}\setminus G_\delta} m_\kappa(S^\kappa)\nabla\Delta S^\kappa\cdot\nabla\varphi d(\tau,x)| + |\int_{G_S\setminus G_\delta} m(S)\nabla\Delta S\cdot\nabla\varphi d(\tau,x)| \\
&+ |\int_{G_\delta} m_\kappa(S^\kappa)\nabla\Delta S^\kappa\cdot\nabla\varphi d(\tau,x) - \int_{G_\delta} m(S)\nabla\Delta S\cdot\nabla\varphi d(\tau,x)|,
\end{aligned} \tag{3.41}$$

here, $0 < \kappa < \kappa_0(\delta)$. From (3.20) we know

$$\begin{aligned}
|\int_{Q_{T_e}\setminus G_\delta} m_\kappa(S^\kappa)\nabla\Delta S^\kappa\cdot\nabla\varphi d(\tau,x)| &\leq \sup|\nabla\varphi||\int_{Q_{T_e}\setminus G_\delta} m_\kappa(S^\kappa)\nabla\Delta S^\kappa d(\tau,x)| \\
&\leq C(\delta+1)\sup|\nabla\varphi|,
\end{aligned} \tag{3.42}$$

and

$$|\int_{G_S\setminus G_\delta} m(S)\nabla\Delta S\cdot\nabla\varphi d(\tau,x)| \leq C\delta\sup|\nabla\varphi|, \tag{3.43}$$

since

$$\begin{aligned}
&|\int_{G_\delta} m_\kappa(S^\kappa)\nabla\Delta S^\kappa\cdot\nabla\varphi d(\tau,x) - \int_{G_\delta} m(S)\nabla\Delta S\cdot\nabla\varphi d(\tau,x)| \\
\leq&\int_{G_\delta}|m_\kappa(S^\kappa)-m(S)||\nabla\Delta S^\kappa||\nabla\varphi|d(\tau,x)+\int_{G_\delta}|m(S)||\nabla\Delta S^\kappa-\nabla\Delta S||\nabla\varphi|d(\tau,x) \\
:=&I_1+I_2,
\end{aligned} \tag{3.44}$$

where, (3.16) and (3.22) result in

$$\begin{aligned}
I_1 &= \int_{G_\delta} |m_\kappa(S^\kappa) - m(S)||\nabla\Delta S^\kappa||\nabla\varphi| d(\tau,x) \\
&\le \sup|\nabla\varphi| \int_{G_\delta} |m_\kappa(S^\kappa) - m(S)||\nabla\Delta S^\kappa| d(\tau,x) \\
&\le \sup|\nabla\varphi| \int_{G_\delta} |m(S^\kappa) - m(S) + \kappa||\nabla\Delta S^\kappa| d(\tau,x) \\
&\le M_1 \sup|\nabla\varphi| \int_{G_\delta} (|S^\kappa - S| + \kappa)|\nabla\Delta S^\kappa| d(\tau,x) \\
&\le M_1 \sup|\nabla\varphi|(\|S^\kappa - S\|_{L^2(G_\delta)} + \|\kappa\|_{L^2(G_\delta)})\|\nabla\Delta S^\kappa\|_{L^2(G_\delta)} \\
&\le CM_1 \sup|\nabla\varphi|(\|S^\kappa - S\|_{L^2(G_\delta)} + \|\kappa\|_{L^2(G_\delta)}) \to 0,
\end{aligned} \tag{3.45}$$

for I_2, owing to

$$\begin{aligned}
\int_{G_\delta} |m(S)\nabla\varphi|^2 d(\tau,x) &= \int_{G_\delta} |m(S)|^2|\nabla\varphi|^2 d(\tau,x) \le \sup|\nabla\varphi|^2 \int_{G_\delta} |m(S)|^2 d(\tau,x) \\
&\le \sup|\nabla\varphi|^2 \int_{G_\delta} (M_0+1)^2 d(\tau,x) \le C,
\end{aligned} \tag{3.46}$$

then $m(S)\nabla\varphi \in L^2(G_\delta)$. By $\nabla\Delta S^\kappa \rightharpoonup \nabla\Delta S$ weakly in $L^2(G_\delta)$, we make use of the definition of weak convergence to conclude that $I_2 \to 0$, therefore

$$\left| \int_{G_\delta} m_\kappa(S^\kappa)\nabla\Delta S^\kappa \cdot \nabla\varphi d(\tau,x) - \int_{G_\delta} m(S)\nabla\Delta S \cdot \nabla\varphi d(\tau,x)\right| \to 0. \tag{3.47}$$

From (3.41)–(3.47) and the arbitrariness of δ, (3.33) is derived. □

Consequently, (u,σ,S) is the weak solution to degenerate problem (1.1)–(1.7) which has the regularity properties stated in Theorem 1.1. The proof of Theorem 1.1 is complete.

4. Numerical experiments

In this section we perform a series of numerical simulations to describe the process of spinodal decomposition in binary alloys by applying the above model in two-dimensional space. We choose a degenerate mobility $m(S) = (\sin(-\frac{\pi S}{2}) + 1)\cdot(1 - \cos\frac{\pi S}{2})$ and a positive mobility $m = 1$, then we compare the results. The chemical free energy density is represented by

$$\hat{\psi}(S) = S^2(1-S)^2,$$

which is phenomenological double-well potential. The following nondimensional parameters are used: the gradient energy coefficient $\nu = 1$; the misfit strain

$$\bar{\varepsilon} = \begin{pmatrix} 0.01 & 0 \\ 0 & -0.01 \end{pmatrix};$$

the dimensionless elastic modulus tensor of material

$$D = \begin{pmatrix} 1400 & 600 & 0 \\ 600 & 1400 & 0 \\ 0 & 0 & 400 \end{pmatrix};$$

and the number of grid points is $N_x = N_y = 128$.

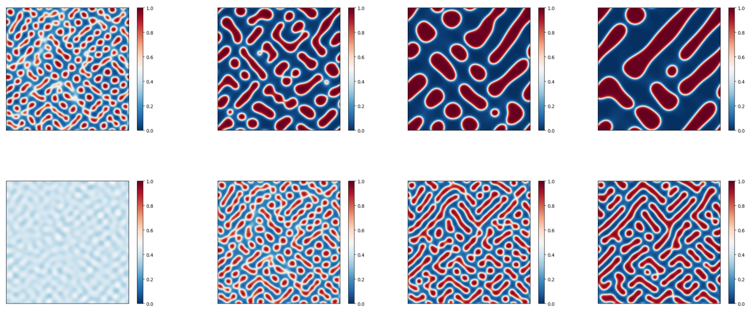

Figure 1. Evolution of the concentration during the spinodal decomposition with positive mobility $m = 1$ at dimensionless time $t^* = 80, 500, 2000, 4000$ (the top row) and degenerate mobility $m(S) = (\sin(-\frac{\pi S}{2}) + 1) \cdot (1 - \cos\frac{\pi S}{2})$ at dimensionless time $t^* = 500, 1200, 2000, 4000$ (the bottom row).

As can be seen from Figure 1, we choose the average composition of binary alloy $S_0 = 0.4$, then add a small perturbation to this component, with the increase of aging time, the system generates two new phases spontaneously with different components but the same structure. The red region and the blue region represent the two new phases with different components, and the other regions represent the transitional phase interfaces. In order to minimize the interfacial energy of the system, coarsening of the new phases is achieved by absorbing small particles into large particles, we can find easily from the figures on the top row that the size of new phases increase and the number of new phases decrease, this implies that the coarsening process is going on. Generally speaking, the variable mobility will reduce the diffusion of the system, it is obvious that the coarsening behaviour does not occur between the new phase of the same type when the mobility $m(S)$ is degenerate at the minimum points of chemical free energy density $\hat{\psi}(S)$. Comparing the figures on the top row with the figures at the bottom row we can easily discover that the coarsening of the new phases are easier to achieve when the mobility is positive. The existence of elastic strain energy caused by the difference of lattice parameters between the parent and new phases increases the resistance of phase transition, in order to minimize the elastic strain energy so as to minimize the total free energy, the new phases are arranged regularly along the elastic soft direction in the process of microstructure evolution.

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