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DYNAMICS OF A GENERALIZED LORENZ-LIKE CHAOS DYNAMICAL SYSTEMS*

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Abstract In this work, a new seven-parameter Lorenz-like chaotic system is presented and discussed by combining nonlinear dynamical systems theory with computer simulation. The existence of the ultimate bound set and global exponential attractive set of this chaotic system is proved by using Lyapunov's direct method. A family of analytic mathematical expression of the ultimate bound sets and global exponential attractive sets involving two parameters are obtained, respectively. Meanwhile, the volumes of the ultimate bound set and global exponential attractive set are obtained, respectively. Numerical simulations are conducted which validates the correctness of the proposed theoretical analysis.

Keywords Lyapunov exponents, Lyapunov-like function, global stability, global attractivity.

MSC(2010) 34D06, 34H10.

1. Introduction

In recent years, chaos theory and chaotic dynamical systems have garnered much attention from many disciplines [9,10,15,20,23,33] since the famous Lorenz chaotic system in 1963 [9]. Then, some other new chaotic systems have been discovered

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and widely studied, such as, Rossler system [27], hyperchaos Lorenz system [33], Chua system [20], Shimizu-Morioka system [10], Chen system [2], Lu system [15], Rabinovich system [1, 7, 26, 35, 39], Glukhovsky-Dolzhansky system [11] and Lorenz-Stenflo system [28]. In particular, chaotic dynamical systems have been paid more attention due to its potential applications in biological systems [36], secure communication [22], chemical reactions, electrical engineering, chaotic circuits and so on [3–8, 16–19, 21, 24, 29–31, 34, 36–38, 40–42, 47].

Boundedness is an important aspect in the study of chaotic dynamical systems [12, 13] which can be applied to analyze the Lyapunov dimension of chaotic attractors [14], chaos control and chaos synchronization [17]. The bounds of the famous Lorenz system was firstly studied by Leonov et al. in [12, 13] due to the important scientific and engineering research value of Lorenz system. Inspired by Leonov' thought, Liao et al. have proposed the concept of the global exponential attractive set of chaotic systems and have studied the global exponential attractive sets of the Lorenz system [17]. But Liao et al. have not proved that the Chen system and the Lu system also have the global exponential attractive set due to the complex algebraic structure of the Chen system and the Lu system [17]. Recently, Zhang et al. give new results on ultimate bound on the trajectories of the Lorenz system which contain the existing (Liao and his collaborators) results as special cases [49]. It is mentioned in the articles [25, 46] that how to get the bounds of the Chen system and the Lu system is an important yet nontrivial open problem. Recently, Zhang and his collaborators have made major progress in studying the bounds of the Chen system and the Lu system [43–45, 48].

In this paper, a new Lorenz-like chaotic system is presented via theoretical analysis and numerical simulations. The main contributions of this paper as follows. Firstly, a family of the ultimate bound sets and globally exponential attractive sets involving two parameters for the new Lorenz-like chaotic system have been obtained via Lyapunov's direct method, respectively. Secondly, the rate of the trajectories going from the exterior of the attractive set to the interior of the attractive set is also obtained. Finally, numerical simulations are conducted which validates the correctness of the proposed theoretical analysis.

The organization of this paper is as follows. The new seven-parameter Lorenz-like chaotic system is presented in Section 2. The boundedness of the generalized Lorenz-like chaotic system is given in Section 3. Global attractivity of the generalized Lorenz-like chaotic system has been investigated in Section 4. Section 5 draws conclusion and the expectation for the future work.

2. System model

The generalized Lorenz-like system is the seven-parameter family of differential equations given by:

$$\begin{cases} \dfrac{dx}{dt} = hy - ax - dyz, \\ \dfrac{dy}{dt} = cx - ky - exz, \\ \dfrac{dz}{dt} = xy - bz, \end{cases} \tag{2.1}$$

where x, y and z are real variables; a, b, k are positive real parameters and c, d, e, h are real parameters of system (2.1). Information about the chaotic attractors by calculating Lyapunov exponents and Lyapunov dimension can be referred to [5, 8, 32]. The local finite-time Lyapunov exponents of the dynamical system (2.1) are calculated numerically for $a = 10, b = \frac{8}{3}, c = 30, d = 0.01, e = 1, k = 1, h = 10$ with the initial state $(x_0, y_0, z_0) = (0.5, 2, 0.3)$ and time interval $[0, 10000]$. In this paper, all the simulations are carried out by using the fourth-order Runge-Kutta method with $h = 0.001$. When $a = 10, b = \frac{8}{3}, c = 30, d = 0.01, e = 1, k = 1, h = 10$, the local finite-time Lyapunov exponents of system (2.1) are $\lambda_{LE_1} = 3.0834, \lambda_{LE_2} = 6.8186, \lambda_{LE_3} = -23.5686$. The Lyapunov dimension of system (2.1) is $D_L = 2.063$. When $a = 10, b = \frac{8}{3}, c = 30, d = 0.01, e = 1, k = 1, h = 10$, chaotic attractors of system (2.1) are shown in figures 1-4.

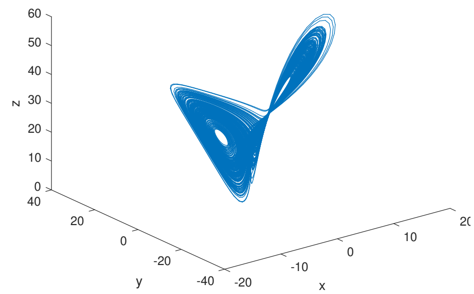

Figure 1. Chaotic attractor for $a = 10, b = \frac{8}{3}, c = 30, d = 0.01, e = 1, k = 1, h = 10$ in the xOyz space.

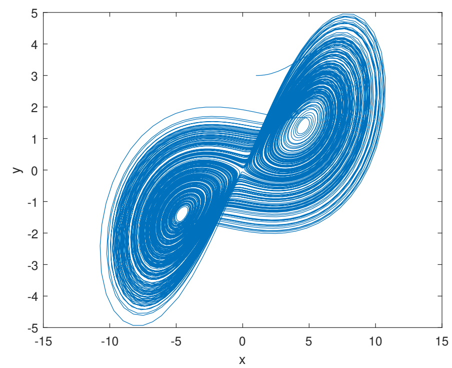

Figure 2. Chaotic attractor of system (2.1) in the xOy plane.

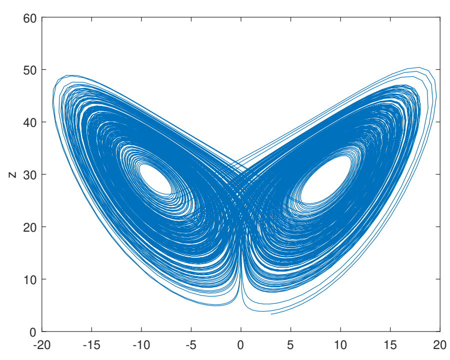

Figure 3. Chaotic attractor of system (2.1) in the xOz plane.

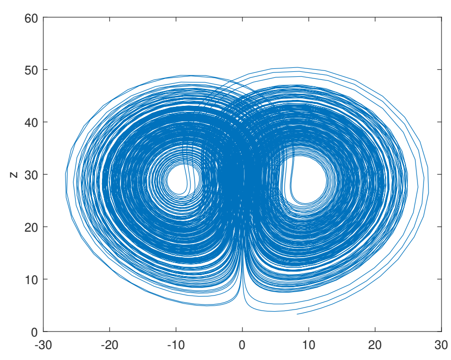

Figure 4. Chaotic attractor of system (2.1) in the yOz plane.

In the following part, we will discuss the ultimate boundedness and global attractivity of system (2.1) via Lyapunov's direct method.

3. Boundedness

In this section, we will discuss the boundedness of the generalized Lorenz-like system (2.1). The boundedness of system (2.1) is described by Theorem 3.1.

Theorem 3.1. *For any $\lambda > 0, \tau > 0$ satisfying the condition of $\lambda d + \tau e > 0$, such that*

$$\Omega_{\lambda,\tau} = \left\{ (x,y,z) \middle| \lambda x^2 + \tau y^2 + (\lambda d + \tau e)\left(z - \frac{\lambda h + \tau c}{\lambda d + \tau e}\right)^2 \le R^2_{\lambda,\tau} \right\}, \tag{3.1}$$

is the ultimate bound set of system (2.1)*, where*

$$R^2_{\lambda,\tau} = \begin{cases} \dfrac{b^2(\lambda h + \tau c)^2}{4k\,(b-k)\,(\lambda d + \tau e)}, & a \ge k, b \ge 2k, \\ \dfrac{b^2(\lambda h + \tau c)^2}{4a\,(b-a)\,(\lambda d + \tau e)}, & k \ge a, b \ge 2a, \\ \dfrac{(\lambda h + \tau c)^2}{\lambda d + \tau e}, & b < 2k, b < 2a. \end{cases}$$

Proof. Define the Lyapunov-like function

$$V_{\lambda,\tau}(X) = V_{\lambda,\tau}(x,y,z) = \lambda x^2 + \tau y^2 + (\lambda d + \tau e)\left(z - \frac{\lambda h + \tau c}{\lambda d + \tau e}\right)^2, \tag{3.2}$$

where $\lambda > 0, \tau > 0$ satisfying the condition of $\lambda d + \tau e > 0$.

Then, the derivative of $V_{\lambda,\tau}(X)$ is

$$\begin{aligned} &\left.\frac{dV_{\lambda,\tau}(X)}{dt}\right|_{(2.1)} \\ =&2\lambda x \frac{dx}{dt} + 2\tau y \frac{dy}{dt} + 2(\lambda d + \tau e)\left(z - \frac{\lambda h + \tau c}{\lambda d + \tau e}\right)\frac{dz}{dt} \\ =&2\lambda x (hy - ax - dyz) + 2\tau y (cx - ky - exz) + 2(\lambda d + \tau e)\left(z - \frac{\lambda h + \tau c}{\lambda d + \tau e}\right)(xy - bz) \\ =&-2a\lambda x^2 - 2k\tau y^2 - 2b(\lambda d + \tau e) z^2 + 2b(\lambda h + \tau c) z. \end{aligned}$$

We can get the conclusion that the surface Γ_0 defined by

$$\Gamma_0 = \left\{ (x,y,z) \middle| \frac{a\lambda x^2}{\frac{b(\lambda h + \tau c)^2}{4(\lambda d + \tau e)}} + \frac{k\tau y^2}{\frac{b(\lambda h + \tau c)^2}{4(\lambda d + \tau e)}} + \frac{b(\lambda d + \tau e)\left[z - \frac{\lambda h + \tau c}{2(\lambda d + \tau e)}\right]^2}{\frac{b(\lambda h + \tau c)^2}{4(\lambda d + \tau e)}} = 1 \right\} \tag{3.3}$$

is an ellipsoid for $\forall \lambda > 0, \tau > 0, \lambda d + \tau e > 0$. Outside Γ_0, $\frac{dV_{\lambda,\tau}(X)}{dt} < 0$, while inside Γ_0, $\frac{dV_{\lambda,\tau}(X)}{dt} > 0$. Thus, the ultimate boundedness for system (2.1) can only be reached on Γ_0. Since the $V_{\lambda,\tau}(X)$ is a continuous function and Γ_0 is a bounded closed set, then the continuous function (3.2) can reach its maximum value $\max\limits_{X \in \Gamma_0} V_{\lambda,\tau}(X) = R^2_{\lambda,\tau}$ on the bounded closed set Γ_0 (3.3). $\left\{ X \middle| V_{\lambda,\tau}(X) \le \max\limits_{X \in \Gamma_0} V_{\lambda,\tau}(X) = R^2_{\lambda,\tau} \right\}$

contains the solutions of the system (2.1). In order to work out $\max\limits_{X\in\Gamma_0} V_{\lambda,\tau}(X) = R^2_{\lambda,\tau}$, we can construct the following optimization problem:

$$\begin{cases} \max \ V_{\lambda,\tau}(X) = \max\left\{\lambda x^2 + \tau y^2 + (\lambda d + \tau e)\left(z - \dfrac{\lambda h + \tau c}{\lambda d + \tau e}\right)^2\right\}, \\ s.t. \dfrac{a\lambda x^2}{\frac{b(\lambda h+\tau c)^2}{4(\lambda d+\tau e)}} + \dfrac{k\tau y^2}{\frac{b(\lambda h+\tau c)^2}{4(\lambda d+\tau e)}} + \dfrac{b(\lambda d + \tau e)\left[z - \frac{\lambda h+\tau c}{2(\lambda d+\tau e)}\right]^2}{\frac{b(\lambda h+\tau c)^2}{4(\lambda d+\tau e)}} = 1. \end{cases} \tag{3.4}$$

The above optimization problem (3.4) is equivalent to

$$\begin{cases} \max \ V_{\lambda,\tau}(X) = \max\left\{\lambda x^2 + \tau y^2 + (\lambda d + \tau e)\left(z - \dfrac{\lambda h + \tau c}{\lambda d + \tau e}\right)^2\right\}, \\ s.t. \dfrac{\lambda x^2}{\frac{b(\lambda h+\tau c)^2}{4a(\lambda d+\tau e)}} + \dfrac{\tau y^2}{\frac{b(\lambda h+\tau c)^2}{4k(\lambda d+\tau e)}} + \dfrac{\left[z\sqrt{\lambda d + \tau e} - \frac{\lambda h+\tau c}{2\sqrt{\lambda d+\tau e}}\right]^2}{\frac{(\lambda h+\tau c)^2}{4(\lambda d+\tau e)}} = 1. \end{cases} \tag{3.5}$$

In order to solve the above optimization problem (3.5), let us denote

$$\sqrt{\lambda}x = x_1, \ \sqrt{\tau}y = y_1, \ \sqrt{\lambda d + \tau e}z = z_1.$$

Then problem (3.5) becomes

$$\begin{cases} \max \ V_{\lambda,\tau}(X) = \max\left\{x_1^2 + y_1^2 + \left(z_1 - \dfrac{\lambda h + \tau c}{\sqrt{\lambda d + \tau e}}\right)^2\right\}, \\ s.t. \ \dfrac{x_1^2}{\frac{b(\lambda h+\tau c)^2}{4a(\lambda d+\tau e)}} + \dfrac{y_1^2}{\frac{b(\lambda h+\tau c)^2}{4k(\lambda d+\tau e)}} + \dfrac{\left(z_1 - \frac{\lambda h+\tau c}{2\sqrt{\lambda d+\tau e}}\right)^2}{\frac{(\lambda h+\tau c)^2}{4(\lambda d+\tau e)}} = 1. \end{cases} \tag{3.6}$$

According to optimization method, we can get the optimal solution of (3.6) as follows

$$\max_{X\in\Gamma_0} V_{\lambda,\tau}(X) = R^2_{\lambda,\tau} = \begin{cases} \dfrac{b^2(\lambda h + \tau c)^2}{4k(b-k)(\lambda d + \tau e)}, & a \geq k, b \geq 2k, \\ \dfrac{b^2(\lambda h + \tau c)^2}{4a(b-a)(\lambda d + \tau e)}, & k \geq a, b \geq 2a, \\ \dfrac{(\lambda h + \tau c)^2}{\lambda d + \tau e}, & b < 2k, b < 2a. \end{cases}$$

Finally, we can easily show that (3.1) is the ultimate bound and positively invariant set of system (2.1). This completes the proof. □

Remark 3.1. We can get a series of ultimate bound sets and positively invariant sets for the generalized Lorenz-like system (2.1) by Theorem 3.1.
i)Let us take $\lambda = 1$ in Theorem 3.1, then we can get

$$\Omega_{1,\tau} = \left\{ (x,y,z) | \, x^2 + \tau y^2 + (d + \tau e)\left(z - \frac{h + \tau c}{d + \tau e}\right)^2 \leq l^2, \forall \tau > 0\right\}$$

is the ultimate bound set of system (2.1), where

$$l^2 = \begin{cases} \dfrac{b^2(h+\tau c)^2}{4k\left(b-k\right)\left(d+\tau e\right)}, & a \geq k, b \geq 2k, \\ \dfrac{b^2(h+\tau c)^2}{4a\left(b-a\right)\left(d+\tau e\right)}, & k \geq a, b \geq 2a, \\ \dfrac{(h+\tau c)^2}{d+\tau e}, & b < 2k, b < 2a. \end{cases}$$

ii) Let us take $\tau = 1$ in Theorem 3.1, then we can get

$$\Omega_{\lambda,1} = \left\{ (x,y,z) \middle| \, \lambda x^2 + y^2 + (\lambda d + e)\left(z - \frac{\lambda h + c}{\lambda d + e}\right)^2 \leq r^2, \forall \lambda > 0 \right\}$$

is the ultimate bound set of system (2.1), where

$$r^2 = \begin{cases} \dfrac{b^2(\lambda h+c)^2}{4k\left(b-k\right)\left(\lambda d+e\right)}, & a \geq k, b \geq 2k, \\ \dfrac{b^2(\lambda h+c)^2}{4a\left(b-a\right)\left(\lambda d+e\right)}, & k \geq a, b \geq 2a, \\ \dfrac{(\lambda h+c)^2}{\lambda d+e}, & b < 2k, b < 2a. \end{cases}$$

iii) Let us take $\lambda = 1, \tau = 1$ in Theorem 3.1, then we can get

$$\Omega_{1,1} = \left\{ (x,y,z) \middle| \, x^2 + y^2 + (d+e)\left(z - \frac{h+c}{d+e}\right)^2 \leq L^2 \right\}$$

is the ultimate bound set of system (2.1), where

$$L^2 = \begin{cases} \dfrac{b^2(h+c)^2}{4k\left(b-k\right)\left(d+e\right)}, & a \geq k, b \geq 2k, \\ \dfrac{b^2(h+c)^2}{4a\left(b-a\right)\left(d+e\right)}, & k \geq a, b \geq 2a, \\ \dfrac{(h+c)^2}{d+e}, & b < 2k, b < 2a. \end{cases}$$

When $a = 10, b = \frac{8}{3}, c = 30, d = 0.01, e = 1, k = 1, h = 10$, then we have the conclusions that

$$\Omega_{1,1} = \left\{ (x,y,z) \middle| \, x^2 + y^2 + 1.01(z - 39.60)^2 \leq 41.11^2 \right\}$$

is the ultimate bound set of system (2.1). In Fig.5, we show the bounds estimation for chaotic attractor of system (2.1) in xOyz space by $\Omega_{1,1}$.

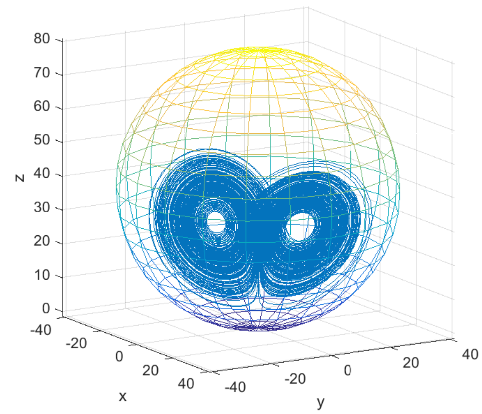

Figure 5. Bounds estimation for chaotic attractor of system (2.1) in xOyz space by $\Omega_{1,1}$.

iv) The volume of the ultimate bound set $\Omega_{\lambda,\tau}$ in Theorem 3.1 is $V\left(\Omega_{\lambda,\tau}\right) = \frac{\pi R_{\lambda,\tau}^3}{\sqrt{\lambda\tau(\lambda d+\tau e)}}$.

4. Global attractivity with exponential rate

For the rate estimation of the trajectories of system (2.1), we have the following conclusion. In the following section, we will investigate global attractivity of system (2.1) with exponential rate. We have the following Theorem 4.1.

Theorem 4.1. *Suppose that* $\forall\lambda > 0, \tau > 0$ *satisfying the condition of* $\lambda d + \tau e > 0$, *and let*

$$X\left(t\right) = \left(x\left(t\right), y\left(t\right), z\left(t\right)\right), L_{\lambda,\tau}^2 = \frac{b{(\lambda h + \tau c)}^2}{\theta\left(\lambda d + \tau e\right)}, \theta = \min\left(a, b, k\right) > 0.$$

Then we have the following exponential inequality

$$\left[V_{\lambda,\tau}\left(X\left(t\right)\right) - L_{\lambda,\tau}^2\right] \le \left[V_{\lambda,\tau}\left(X\left(t_0\right)\right) - L_{\lambda,\tau}^2\right] e^{-\theta(t-t_0)} \tag{4.1}$$

and

$$\Psi_{\lambda,\tau} = \left\{X \middle| V_{\lambda,\tau}\left(X\right) \le L_{\lambda,\tau}^2\right\} \tag{4.2}$$

is the global exponential attractive set of system (2.1), *i.e.,* $\overline{\lim_{t\to+\infty}} V_{\lambda,\tau}\left(X\left(t\right)\right) \le L_{\lambda,\tau}^2$.

Proof. Define the Lyapunov-like function

$$V_{\lambda,\tau}\left(X\right) = V_{\lambda,\tau}\left(x, y, z\right) = \lambda x^2 + \tau y^2 + \left(\lambda d + \tau e\right)\left(z - \frac{\lambda h + \tau c}{\lambda d + \tau e}\right)^2,$$

where $\lambda > 0, \tau > 0$ satisfying the condition of $\lambda d + \tau e > 0$.

Then, the derivative of $V_{\lambda,\tau}\left(X\right)$ is

$$\left.\frac{dV_{\lambda,\tau}\left(X\right)}{dt}\right|_{(2.1)} = 2\lambda x\frac{dx}{dt} + 2\tau y\frac{dy}{dt} + 2\left(\lambda d + \tau e\right)\left(z - \frac{\lambda h + \tau c}{\lambda d + \tau e}\right)\frac{dz}{dt}$$

$$
\begin{aligned}
&= 2\lambda x\left(hy - ax - dyz\right) + 2\tau y\left(cx - ky - exz\right) \\
&\quad + 2\left(\lambda d + \tau e\right)\left(z - \frac{\lambda h + \tau c}{\lambda d + \tau e}\right)\left(xy - bz\right) \\
&= -2a\lambda x^2 - 2k\tau y^2 - 2b\left(\lambda d + \tau e\right)z^2 + 2b\left(\lambda h + \tau c\right)z \\
&\le -a\lambda x^2 - k\tau y^2 - b\left(\lambda d + \tau e\right)z^2 + 2b\left(\lambda h + \tau c\right)z \\
&= -a\lambda x^2 - k\tau y^2 - b\left(\lambda d + \tau e\right)\left(z - \frac{\lambda h + \tau c}{\lambda d + \tau e}\right)^2 + b\frac{\left(\lambda h + \tau c\right)^2}{\lambda d + \tau e} \\
&\le -\theta V_{\lambda,\tau}\left(X\right) + b\frac{\left(\lambda h + \tau c\right)^2}{\lambda d + \tau e} \\
&= -\theta\left[V_{\lambda,\tau}\left(X\right) - L^2_{\lambda,\tau}\right].
\end{aligned}
$$

Thus, we have

$$\left[V_{\lambda,\tau}\left(X\left(t\right)\right) - L^2_{\lambda,\tau}\right] \le \left[V_{\lambda,\tau}\left(X\left(t_0\right)\right) - L^2_{\lambda,\tau}\right]e^{-\theta(t-t_0)}.$$

So,

$$\overline{\lim_{t\to+\infty}} V_{\lambda,\tau}\left(X\left(t\right)\right) \le L^2_{\lambda,\tau}.$$

Hence $\Psi_{\lambda,\tau} = \left\{X|\, V_{\lambda,\tau}\left(X\right) \le L^2_{\lambda,\tau}\right\}$ is the global exponential attractive set of system (2.1). This completes the proof. □

Remark 4.1. The volume of the global exponential attractive set $\Psi_{\lambda,\tau}$ in Theorem 4.1 is $V\left(\Psi_{\lambda,\tau}\right) = \frac{\pi L^3_{\lambda,\tau}}{\sqrt{\lambda\tau(\lambda d+\tau e)}}$.

5. Conclusions

By means of chaos dynamical systems theory, Lyapunov stability theory and inequality technique, a new Lorenz-like chaotic system is presented and discussed. The ultimate boundedness and global exponential attractivity of this chaotic system are obtained via Lyapunov's direct method, which is a challenging work but an important work in chaos dynamical systems. The bifurcation phenomenon, chaos control and synchronization will be considered in the future.

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