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BIFURCATIONS AND EXACT TRAVELLING WAVE SOLUTIONS FOR A NEW INTEGRABLE NONLOCAL EQUATION*

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Abstract By using the method of dynamical systems, we consider the dynamical behavior of travelling wave solutions for a new integrable nonlocal equation. All possible exact explicit travelling wave solutions under different parameter conditions are given, including solitary wave solutions, periodic wave solutions and pseudo-peakon wave solutions.

Keywords Bifurcation, exact travelling wave solution, solitary wave solution, periodic wave solution, pseudo-peakon, integrable nonlocal equation.

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1. Introduction

Nowadays nonlocal integrable equations and the dynamics of them have gained the attention of the scholars [1, 2, 18, 19]. The first nonlocal model in integrable system was the celebrated PT-symmetric nonlinear Schrödinger (NLS) equation, which was introduced by Ablowitz and Musslimani [3]. Afterwards, a variety of nonlocal integrable equations with reverse space and/or time coupling have been established [4, 7, 8, 17, 20, 21].

As an integrable generalization of the NLS equation, Fokas and Lenells proposed [6, 9]

$$iu_t - \nu u_{tx} + \gamma u_{xx} + \sigma |u|^2 (u + i\nu u_x) = 0, \tag{1.1}$$

where ν, γ are real parameters and $\sigma = \pm 1$. This local equation is also called Fokas-Lenells (FL) equation, which is related to NLS equation from the perspective of bi-Hamiltonian. In the case of $\nu = 0$, FL equation may be reduced to the regular NLS equation and $\sigma\gamma = \pm 1$ determines whether the above equation is focused or defocused. It is important that the FL equation models nonlinear pulse propagation in monochrome optical fibers when certain higher-order nonlinear effects are taken into account [10].

Very recently, one of the authors has presented the following the reverse-space-time nonlocal nonlinear equation [22]:

$$q_{xt} + q - 2iq(x,t)q(-x,-t)q_x = 0, \tag{1.2}$$

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here q is a complex-valued function of x and t. It is noted that Eq.(1.2) has been changed from its original version Eq.(1.1) by a gauge transformation for convenience [10]. Compared with the local nonlinear FL equation, the nonlinear term $q(x,t)q^*(x,t)q_x(x,t)$ of the Eq.(1.1) is replaced by $q(x,t)q(-x,-t)q_x(x,t)$, which reflects the space-time reverse nonlocal coupling between $q(x,t)$ and $q(-x,-t)$. And it can be derived from a special reduction of the negative flow for the KN-hierarchy which means this new equation is also integrable. Furthermore, we have derived the Lax pair and obtained different kinds of exact solutions including bright/dark solitons, kink solutions, periodic solutions and mixed type solutions by using the Darboux transformation.

In [11], we considered the dynamical behavior and bifurcations to the local FL equation (1.1), and made a complete work to the exact travelling wave system. To the best of our knowledge, the dynamical behavior of travelling wave solutions for newly integrable nonlocal FL equation above was not studied before.

In our present work, the nonlocal FL equation is investigated to obtain the exact travelling wave solutions and their bifurcations depending on the parameter group of system. Now we are interested in analyzing the travelling wave solution with the form

$$q(x,t) = \phi(x-ct)e^{i(kx-\Omega t+\theta(x-ct))} = \phi(\xi)e^{i(kx-\Omega t+\theta(\xi))}, \tag{1.3}$$

where $\xi = x - ct$ and k, Ω, c are real parameters.

It follows from (1.3) that

$$q(x,t)q(-x,-t) = \phi(\xi)\phi(-\xi)e^{i(\theta(\xi)+\theta(-\xi))}. \tag{1.4}$$

If $\phi(\xi) = \phi(-\xi)$ and $\theta(-\xi) = -\theta(\xi)$, then $q(x,t)q(-x,-t) = \phi^2(\xi)$. In this case, the exact travelling wave solutions for the nonlocal equation is consistent with the local equation.

Substituting (1.3) into Eq.(1.2) and separating the real part and imaginary part, we have

$$\begin{aligned} &-c\phi_{\xi\xi} + \phi[k\Omega + (\Omega + kc)\theta_\xi + c\theta_\xi^2] + \phi + 2\phi^3(k + \theta_\xi) = 0, \\ &c\phi\theta_{\xi\xi} + 2c\phi_\xi\theta_\xi + (\Omega + kc)\phi_\xi + 2\phi^2\phi_\xi = 0. \end{aligned} \tag{1.5}$$

Multiply the second of these equations by ϕ and integrate the resulting equation, one find

$$2c\phi^2\theta_\xi + (\Omega + kc)\phi^2 + \phi^4 - g = 0, \tag{1.6}$$

where g is an integral constant. Solving for θ_ξ, we have

$$\theta_\xi = \frac{g - (\Omega + kc)\phi^2 - \phi^4}{2c\phi^2}. \tag{1.7}$$

Substituting (1.7) into the first equation of system (1.5), we obtain

$$4c^2\phi^3\phi_{\xi\xi} = g^2 + \phi^4(a_0 + 2a_1\phi^2 - 3\phi^4), \tag{1.8}$$

where $a_0 = 2g + 4c(k\Omega + 1) - (\Omega + ck)^2, a_1 = 2(ck - \Omega)$.

Eq. (1.8) is equivalent to the following planar dynamical system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{g^2 + \phi^4(a_0 + 2a_1\phi^2 - 3\phi^4)}{4c^2\phi^3}, \tag{1.9}$$

with the first integral

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{g^2 - a_0\phi^4 - a_1\phi^6 + \phi^8}{8c^2\phi^2} = h, \tag{1.10}$$

where h is the energy constant.

When we obtain an exact solution $\phi(\xi)$ of system (1.9), it follows from (1.7) that

$$\begin{aligned}\theta(\xi) &= \int_0^\xi \frac{(g - (\Omega + kc)\phi^2(\xi) - \phi^4)d\xi}{2c\phi^2} \\ &= -\left(\frac{ck+\Omega}{2c}\right)\xi - \frac{1}{2c}\left[\int_0^\xi \phi^2(\xi)d\xi - g\int_0^\xi \frac{d\xi}{\phi^2(\xi)}\right].\end{aligned} \tag{1.11}$$

By using the formula (1.3), it follows an exact travelling wave solution of Eq.(1.2).

When we take $g = 0$, system (1.8) becomes

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\phi}{4c^2}(-3\phi^4 + 2a_1\phi^2 + a_0), \tag{1.12}$$

which is a polynomial system and the first integral is as follows,

$$H_0(\phi, y) = \frac{1}{2}y^2 + \frac{1}{8c^2}(\phi^6 - a_1\phi^4 - a_0\phi^2) = h. \tag{1.13}$$

When $g \neq 0$, system (1.8) is a singular travelling wave system of the first class as named in [12–16] having the singular straight line $\phi = 0$. It is interesting that there exist periodic peakons and pseudo-peakons which have smooth wave profiles, because in a neighborhood of the singular straight line $\phi = 0$, there exist two "time scale" of wave variables, such that cusp wave profiles appear.

We consider the case $g = 0$ in this paper. It is necessary to find all possible exact solutions for system (1.12). In the following, we use the method of dynamical systems to investigate the dynamics of solutions of system (1.12) and to give the parametric representations of travelling wave solutions of equation (1.2) with the form (1.3) in the two-parameter plane (a_0, a_1), where we assume that the parameter c is fixed and $c > 0$.

The main result of this paper can be formulated in the following.

Theorem 1.1. *Considering the exact solutions of Eq.*(1.2) *with the form* (1.3)*, we have system* (1.9) *and formula* (1.11)*. For the integral constant* $g = 0$*, in system* (1.9)*, Eq.*(1.2) *has 12 different exact explicit travelling wave solutions with the form* (1.3)*, where* $\phi(\xi)$ *and* $\theta(\xi)$ *are given by* (3.4a)*,* (3.4*b*)*–*(3.15a)*,* (3.15*b*)*, respectively.*

The rest of this paper is organized as follows. In section 2, we investigate the bifurcations of phase portraits of system (1.12). In section 3, we discuss the existence of the exact travelling wave solutions and give all possible exact parametric representations for these solutions of $\phi(\xi)$ and $\theta(\xi)$ under different parameter conditions.

2. Bifurcations of phase portraits of system (1.12)

Defining $f_0(\phi) = \phi(a_0 + 2a_1\phi^2 - 3\phi^4)$. If ϕ_j is a zero point of $f_0(\phi)$, that is $f_0(\phi_j) = 0$, then $E_j(\phi_j, 0)$ is an equilibrium point of system (1.12).

Obviously,

$$f_0(\phi) = \phi(-3\phi^4 + 2a_1\phi^2 + a_0) \equiv \phi(-3\psi^2 + 2a_1\psi + a_0),$$
$$f_0'(\phi) = -15\phi^4 + 6a_1\phi^2 + a_0,$$

where $\psi = \phi^2$. Under the conditions of $\Delta = a_1^2 + 3a_0 > 0$ and $a_1 > 0, a_0 < 0$, the function $f_0(\phi)$ has two positive real roots $\phi_{1,2} = \sqrt{\frac{1}{3}(a_1 \mp \sqrt{\Delta})}$. When $\Delta > 0$ and $a_0 > 0$, the function $f_0(\phi)$ has only one positive real root ϕ_2. Thus, system (1.12) has at most two equilibrium points $E_1(\phi_1, 0)$ and $E_2(\phi_2, 0)$ in the positive $\phi-$axis.

Let $M(\phi_j, 0)$ be the coefficient matrix of the linearized system of (1.12) at an equilibrium point $(\phi_j, 0)$ and $J(\phi_j, 0) = \det M(\phi_j, 0)$. We have

$$J(0,0) = \det M(0,0) = -\frac{a_0}{4c^2}, \quad J(\phi_{1,2}, 0) = \det M(\phi_{1,2}, 0) = -\frac{1}{4c^2} f_0'(\phi_{1,2}).$$

By the theory of planar dynamical systems, we know that for a planar integrable system, if $J < 0$, the equilibrium point is a saddle point; if $J > 0$, the equilibrium point is a center point or a node point; if $J = 0$ and the Poincaré index of the equilibrium point is 0, then the equilibrium point is a cusp.

Let $h_0 = H_0(0,0) = 0$,

$$h_1 = H_0(\phi_1, 0) = \frac{(-a_1^2 + a_1\sqrt{\Delta} - 6a_0)\phi_1^2}{72c^2}, \; h_2 = H_0(\phi_2, 0) = -\frac{(a_1^2 + a_1\sqrt{\Delta} + 6a_0)\phi_2^2}{72c^2}.$$

It is easy to show that when $a_0 < 0, a_1 = 2\sqrt{-a_0}$, we have $h_1 = \frac{(-a_0)^{\frac{3}{2}}}{54c^2}$ and $h_2 = 0$.

We see from the above discussion that for a fixed $c > 0$, in the $(a_0, a_1)-$parameter plan, there exist the bifurcation curves: $a_0 = 0, L_1 : a_1 = 2\sqrt{-a_0}$ and $L_2 : a_1 = \sqrt{-3a_0}$, which partition the $(a_0, a_1)-$parameter plan into four regions $(I) - (IV)$ (see Fig.1 (a)). By qualitative analysis, we have seven phase portraits of system (1.12) under different parameter conditions which is shown in Fig.1 (b)–(h).

3. Some exact travelling wave solutions of equation (1.2)

It follows from (1.13) that

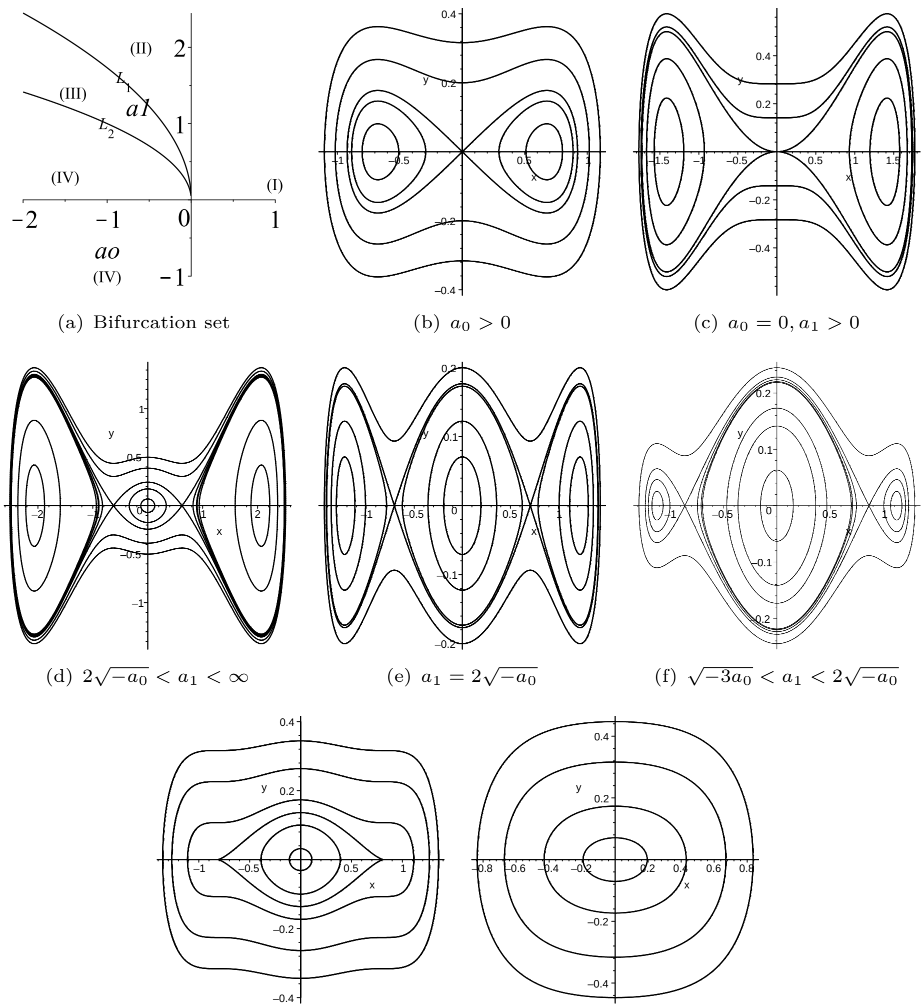
$$y^2 = 2h + \frac{1}{4c^2}(-\phi^6 + a_1\phi^4 + a_0\phi^2). \tag{3.1}$$

Hence, taking the initial condition $\phi(0) = \phi_0$, by using the first equation of (1.12), we have

$$\begin{aligned} \frac{\xi}{c} &= \int_{\phi_0}^{\phi} \frac{2d\phi}{\sqrt{8c^2h + a_0\phi^2 + a_1\phi^4 - \phi^6}} \\ &= \int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{\psi[8c^2h + a_0\psi + a_1\psi^2 - \psi^3]}} \equiv \int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{G(\psi)}}, \end{aligned} \tag{3.2}$$

where $\psi = \phi^2$. In addition, letting $g = 0$, we can get the following solution from (1.11),

$$\theta(\xi) = \int_0^{\xi} \frac{(-(\Omega + kc)\phi^2 - \phi^4)d\xi}{2c\phi^2} = -\left(\frac{ck + \Omega}{2c}\right)\xi - \frac{1}{2c}\left[\int_0^{\xi} \phi^2(\xi)d\xi\right]. \tag{3.3}$$



(a) Bifurcation set (b) $a_0 > 0$ (c) $a_0 = 0, a_1 > 0$

(d) $2\sqrt{-a_0} < a_1 < \infty$ (e) $a_1 = 2\sqrt{-a_0}$ (f) $\sqrt{-3a_0} < a_1 < 2\sqrt{-a_0}$

(g) $a_1 = \sqrt{-3a_0}$ (h) $a_1 < \sqrt{-3a_0}$

Figure 1. Bifurcations of phase portraits of system (1.12). (a) Bifurcation set of parameter group (a_0, a_1). (b) $(a_0, a_1) \in (I)$. (c) $a_0 = 0, a_1 > 0$. (d) $(a_0, a_1) \in (II)$. (e) $(a_0, a_1) \in (L_1)$. (f) $(a_0, a_1) \in (III)$. (g) $(a_0, a_1) \in (L_2)$. (h) $(a_0, a_1) \in (IV)$.

Obviously, when we have the exact solution of $\phi(\xi)$ and $\theta(\xi)$, by using (1.3), we can obtain an exact travelling wave solution of Eq.(1.2).

3.1. The case $a_0 > 0$, i.e., $(a_0, a_1) \in (I)$ (see Fig.1 (b))

(i) Corresponding to the level curves defined by $H_0(\phi, y) = h$ where $h \in (h_2, 0)$, there exist two families of periodic solutions of system (1.12). Now, the polynomial $G(\psi) = (\psi_a - \psi)(\psi - \psi_b)\psi(\psi - \psi_d)$ where $\psi_a > \psi_b > 0, \psi_d < 0$.

From (3.2), we can obtain the following parametric representations of the two families of periodic solutions of equation (1.12):

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\frac{\psi_b}{1-\alpha_1^2\mathrm{sn}^2(2\Omega_1\xi,k)}\right)^{\frac{1}{2}}, \tag{3.4a}$$

where $\alpha_1^2 = \frac{\psi_a-\psi_b}{\psi_a}, k^2 = \frac{(\psi_a-\psi_b)(-\psi_d)}{\psi_a(\psi_b-\psi_d)}, \Omega_1 = \frac{\sqrt{\psi_a(\psi_b-\psi_d)}}{4c}$.

According to the formulas (3.3) and (3.4a), it is easy to get that

$$\begin{aligned}\theta(\xi) &= -\left(\frac{ck+\Omega}{2c}\right)\xi - \frac{\psi_b}{2c}\int_0^{\xi}\frac{d\xi}{1-\alpha_1^2\mathrm{sn}^2(\Omega_1\xi,k)} \\ &= -\left(\frac{ck+\Omega}{2c}\right)\xi - \frac{\psi_b}{2c\Omega_1}\Pi(\arcsin(\mathrm{sn}(\Omega_1\xi),\alpha_1^2,k).\end{aligned} \tag{3.4b}$$

(ii) Corresponding to the level curves defined by $H_0(\phi,y)=0$, there exist two homoclinic orbits of system (1.10). Now, the polynomial $G(\psi) = \psi^2(a_0+a_1\psi-\psi^2) = \psi^2(\psi_M-\psi)(\psi-\psi_m)$, where $\psi_M = \frac{1}{2}(a_1+\sqrt{\Delta_2}) > \psi_2, \psi_m = \frac{1}{2}(a_1-\sqrt{\Delta_2}) < 0, \Delta_2 = a_1^2+4a_0$.

By (3.2), we can obtain the following parametric representations of the homoclinic orbits,

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\frac{2a_0}{\sqrt{\Delta_2}\cosh(\omega_0\xi)+a_1}\right)^{\frac{1}{2}}, \tag{3.5a}$$

where $\omega_0 = \frac{\sqrt{a_0}}{c}$.

It follows from (3.3) and (3.5a) that

$$\begin{aligned}\theta(\xi) &= -\left(\frac{ck+\Omega}{2c}\right)\xi - \frac{a_0}{c}\int_0^{\xi}\frac{d\xi}{\sqrt{\Delta_2}\cosh(\omega_0\xi)+a_1} \\ &= -\left(\frac{ck+\Omega}{2c}\right)\xi - \frac{\sqrt{a_0}}{c\omega_0}\arctan\left(\sqrt{\frac{\sqrt{\Delta_2}-a_1}{\sqrt{\Delta_2}+a_1}}\tanh\left(\frac{1}{2}\omega_0\xi\right)\right).\end{aligned} \tag{3.5b}$$

(iii) Corresponding to the level curves defined by $H_0(\phi,y)=h$ where $h\in(0,\infty)$, there exists a family of periodic orbits enclosing three equilibrium points of system (1.12). For a fixed $h\in(0,\infty)$, the polynomial $G(\psi) = (\psi_a-\psi)\psi[(\psi-b_1)^2+a_1^2]$.

Hence, from (3.2), we can get the parametric representations of the family of periodic solutions of equation (1.12) as follows:

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\sqrt{\frac{\psi_a B_1}{A_1-B_1}}\left(\frac{1+\alpha_2}{1+\alpha_2\mathrm{cn}(\Omega_2\xi,k)}-1\right)^{\frac{1}{2}}, \tag{3.6a}$$

where $A_1^2 = (\psi_a-b_1)^2+a_1^2,\ B_1^2 = b_1^2+a_1^2,\ \Omega_2 = \frac{\sqrt{A_1B_1}}{c},\ k^2 = \frac{\psi_a^2-(A_1-B_1)^2}{4A_1B_1}, \alpha_2 = \frac{A_1-B_1}{A_1+B_1}, K(k)$ is the complete elliptic integral of the first kind, $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$ are the Jacobian elliptic functions (see [5]).

From (3.3) and (3.6a), it is easy to get that

$$\theta(\xi) = -\left(\frac{ck+\Omega}{2c}+\frac{\psi_aB_1}{2c(A_1-B_1)}\right)\xi - \frac{\psi_aB_1(1+\alpha_2)}{2c(A_1-B_1)}\int_0^{\xi}\frac{d\xi}{1+\alpha_2\mathrm{cn}(\Omega_2\xi,k)}$$

$$= -\left(\frac{ck+\Omega}{2c} + \frac{\psi_a B_1}{2c(A_1-B_1)}\right)\xi - \frac{\psi_a B_1(1+\alpha_2)}{2c\Omega_2(A_1-B_1)}$$
$$\times\left[\frac{1}{1-\alpha_2^2}\Pi\left(\arccos(\mathrm{cn}(\Omega_2\xi,k)), \frac{\alpha_2^2}{\alpha_2^2-1}, k\right) - \alpha_2 f_1\right], \qquad (3.6b)$$

where $f_1 = \sqrt{\frac{1-\alpha_2^2}{k^2+(1-k^2)\alpha_2^2}}\arctan\left(\sqrt{\frac{k^2+(1-k^2)\alpha_2^2}{1-\alpha_2^2}}\frac{\mathrm{sn}(\Omega_2\xi,k)}{\mathrm{dn}(\Omega_2\xi,k)}\right)$, $\Pi(\cdot,\alpha,k)$ is the elliptic integral of the third kind (see [5]).

3.2. The case $a_1 > 0, a_0 = 0$ (see Fig.1 (c))

These refer to the case that the origin is a non-hyperbolic saddle point.

(i) Corresponding to the level curves defined by $H_0(\phi,y)=0$, there exist two homoclinic orbits of system (1.12) with $G(\psi)=\psi^3(a_1-\psi)$. Therefore, (3.2) gives rise to the following parametric representations of the two homoclinic orbits:

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\frac{a_1}{1+\frac{a_1^2}{4c^2}\xi^2}\right)^{\frac{1}{2}}. \qquad (3.7a)$$

Under the condition of $a_1>0$, it follows from (3.3) and (3.7a) that

$$\theta(\xi) = -\left(\frac{ck+\Omega}{2c}\right)\xi - \frac{a_1}{2c}\int_0^\xi \frac{d\xi}{1+\frac{a_1^2}{4c^2}\xi^2}$$
$$= -\left(\frac{ck+\Omega}{2c}\right)\xi - \arctan\left(\frac{a_1}{2c}\xi\right). \qquad (3.7b)$$

(ii) Corresponding to the level curves defined by $H_0(\phi,y)=h$ where $h\in(h_2,0)$, there exist two families of periodic orbits of equation (1.12). The parametric representation of the periodic wave solutions are the same as (3.4a).

(iii) For $h\in(0,\infty)$, corresponding to the level curves defined by $H_0(\phi,y)=h$, there exist a family of periodic orbits of system (1.12). The parametric representation of the periodic wave solutions are the same as (3.6a).

3.3. The case $2\sqrt{-a_0} < a_1 < \infty, a_0 < 0, h_2 < 0 < h_1$, i.e., $(a_0,a_1)\in(II)$ (see Fig.1 (d))

(i) Corresponding to the level curves defined by $H_0(\phi,y)=h$ where $h\in(h_2,0)$, there exist two families of periodic orbits of equation (1.12),which enclose the equilibrium points $(\pm\phi_2,0)$, respectively. Therefore, these periodic wave solutions have the same parametric representations as (3.4a).

(ii) Corresponding to the level curves defined by $H_0(\phi,y)=0$, there exist two periodic orbits of equation (1.12), which enclose the equilibrium points $(\pm\phi_2,0)$, respectively. Now the polynomial $G(\xi)=\psi^2(a_0+a_1\psi-\psi^2)$. The parametric representations of the periodic orbits are given by

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\frac{2|a_0|}{a_1-\sqrt{a_1^2+4a_0}\sin(\omega_1\xi)}\right)^{\frac{1}{2}}, \qquad (3.8a)$$

where $\omega_1 = \frac{\sqrt{|a_0|}}{c}$.

It follows from (3.3) and (3.8a) that

$$\begin{aligned}\theta(\xi) &= -\left(\frac{ck+\Omega}{2c}\right)\xi - \frac{|a_0|}{c}\int_0^{\xi}\frac{d\xi}{a_1-\sqrt{a_1^2+4a_0}\sin(\omega_1\xi)} \\ &= -\left(\frac{ck+\Omega}{2c}\right)\xi - \frac{\sqrt{|a_0|}}{c\omega_1}\arctan\left(\frac{a_1\tan(\frac{1}{2}\omega_1\xi)-\sqrt{a_1^2+4a_0}}{2\sqrt{|a_0|}}\right).\end{aligned} \tag{3.8b}$$

(iii) Corresponding to the level curves defined by $H_0(\phi, y) = h$ where $h \in (0, h_1)$, there exist three families of periodic orbits of equation (1.12), which enclose the equilibrium points $(\pm\phi_2, 0)$ and $(0, 0)$, respectively.

For the first case, two families of periodic orbits enclose the equilibrium points $(\pm\phi_2, 0)$, here, the polynomial $G(\psi) = (\psi_a - \psi)(\psi - \psi_b)(\psi - \psi_c)\psi$. Hence, from (3.2), we can obtain the parametric representations of these periodic orbits are as follows:

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\psi_c + \frac{\psi_b-\psi_c}{1-\alpha_3^2\mathrm{sn}^2(\Omega_3\xi, k)}\right)^{\frac{1}{2}}, \tag{3.9a}$$

where $\alpha_3^2 = \frac{\psi_a-\psi_b}{\psi_a-\psi_c}, \Omega_3 = \frac{\sqrt{(\psi_a-\psi_c)\psi_b}}{2c}, k^2 = \frac{(\psi_a-\psi_b)\psi_c}{(\psi_a-\psi_c)\psi_b}$.

According to equations (3.3) and (3.9a), we can get that

$$\begin{aligned}\theta(\xi) &= -\left(\frac{ck+\Omega}{2c}+\frac{\psi_c}{2c}\right)\xi - \frac{\psi_b-\psi_c}{2c}\int_0^{\xi}\frac{d\xi}{1-\alpha_3^2\mathrm{sn}^2(\Omega_3\xi, k)} \\ &= -\left(\frac{ck+\Omega}{2c}+\frac{\psi_c}{2c}\right)\xi - \frac{\psi_b-\psi_c}{2c\Omega_3}\Pi(\arcsin(\mathrm{sn}(\Omega_3\xi, k), \alpha_3^2, k).\end{aligned} \tag{3.9b}$$

The other case, the family of periodic orbits enclose the equilibrium points $(0, 0)$, now, the polynomial $G(\psi) = (\psi_a - \psi)(\psi_b - \psi)(\psi_c - \psi)\psi$. Thus, the parametric representation of the periodic orbits can be given by:

$$\phi(\xi) = \sqrt{\psi(\xi)} = \left(\psi_b - \frac{\psi_b-\psi_c}{1-\alpha_4^2\mathrm{sn}^2(\Omega_3\xi, k)}\right)^{\frac{1}{2}}, \tag{3.10a}$$

where $\alpha_4^2 = \frac{\psi_c}{\psi_b}, \Omega_3 = \frac{\sqrt{(\psi_a-\psi_c)\psi_b}}{2c}, k^2 = \frac{(\psi_a-\psi_b)\psi_c}{(\psi_a-\psi_c)\psi_b}$.

By the formulas (3.3) and (3.10a), we have

$$\begin{aligned}\theta(\xi) &= -\left(\frac{ck+\Omega}{2c}+\frac{\psi_b}{2c}\right)\xi + \frac{\psi_b-\psi_c}{2c}\int_0^{\xi}\frac{d\xi}{1-\alpha_4^2\mathrm{sn}^2(\Omega_3\xi, k)} \\ &= -\left(\frac{ck+\Omega}{2c}+\frac{\psi_b}{2c}\right)\xi + \frac{\psi_b-\psi_c}{2c\Omega_3}\Pi(\arcsin(\mathrm{sn}(\Omega_3\xi, k), \alpha_4^2, k).\end{aligned} \tag{3.10b}$$

(iv) Corresponding to the level curves defined by $H_0(\phi, y) = h_1$, there exist two homoclinic orbits which enclose the centers $(\pm\phi_2, 0)$ and a heteroclinic loop which enclose the center $(0, 0)$. For the two homoclinic orbits, the polynomial $G(\psi) = (\psi_M - \psi)(\psi - \psi_1)^2\psi$. Therefore, we can obtain the following parametric representations of the two corresponding solitary wave solutions,

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\psi_1 + \frac{2(\psi_M-\psi_1)\psi_1}{\psi_M\cosh(\omega_2\xi)-(\psi_M-2\psi_1)}\right)^{\frac{1}{2}}, \tag{3.11a}$$

where $\omega_2 = \frac{\sqrt{(\psi_M-\psi_1)\psi_1}}{c}$.

By using (3.3) and (3.11a), we can see that

$$\begin{aligned}\theta(\xi) &= -\left(\frac{ck+\Omega}{2c}+\frac{\psi_1}{2c}\right)\xi - \frac{(\psi_M-\psi_1)\psi_1}{c}\int_0^\xi \frac{d\xi}{\psi_M\cosh(\omega_2\xi)-(\psi_M-2\psi_1)} \\ &= -\left(\frac{ck+\Omega}{2c}+\frac{\psi_1}{2c}\right)\xi - \arctan\left(\sqrt{\frac{\psi_M-\psi_1}{\psi_1}}\tanh\left(\frac{1}{2}\omega_2\xi\right)\right).\end{aligned} \tag{3.11b}$$

For the upper heteroclinic orbits, it follows from (1.13) that

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\psi_1 - \frac{2\psi_1(\psi_M-\psi_1)}{\psi_M\cosh(\omega_2\xi)+(\psi_M-2\psi_1)}\right)^{\frac{1}{2}}, \tag{3.12a}$$

for $\xi \in (0,\infty), (-\infty,0)$,

respectively.

From (3.12a) and (3.3), we have

$$\begin{aligned}\theta(\xi) &= -\left(\frac{ck+\Omega}{2c}+\frac{\psi_1}{2c}\right)\xi + \frac{(\psi_M-\psi_1)\psi_1}{c}\int_0^\xi \frac{d\xi}{\psi_M\cosh(\omega_2\xi)+(\psi_M-2\psi_1)} \\ &= -\left(\frac{ck+\Omega}{2c}+\frac{\psi_1}{2c}\right)\xi + \arctan\left(\sqrt{\frac{\psi_1}{\psi_M-\psi_1}}\tanh\left(\frac{1}{2}\omega_2\xi\right)\right).\end{aligned} \tag{3.12b}$$

(v) When $h \in (h_1,\infty)$, corresponding to the level curves defined by $H_0(\phi,y)=h$, there exists a global family of periodic orbits of equation (1.12). We can get that the parametric representation of the periodic orbits are the same as (3.6a).

3.4. The case $a_1 = 2\sqrt{-a_0}, a_0 < 0, h_2 = 0 < h_1 = \frac{(-a_0)^{\frac{3}{2}}}{54c^2}$ i.e., $(a_0,a_1) \in (L_1)$ (see Fig.1 (e))

In this case, we have that $\psi_1 = \frac{1}{3}\sqrt{-a_0}, \psi_2 = \sqrt{-a_0}$.

(i) Corresponding to the level curves defined by $H_0(\phi,y)=h$ where $h \in (0,h_1)$, there exist three families of periodic orbits of equation (1.12), which enclose the equilibrium points $(\pm\phi_2,0)$ and $(0,0)$, respectively. The periodic wave solutions of the system possess the same parametric representations as (3.9a) and (3.10a).

(ii) Corresponding to the level curves defined by $H_0(\phi,y)=h_2$, there exist two homoclinic orbits which enclose the centers $(\pm\phi_2,0)$ and a heteroclinic loop which enclose the center $(0,0)$. Now, the polynomial $G(\psi) = \left(\frac{4}{3}\sqrt{|a_0|}-\psi\right)(\psi - \frac{1}{3}\sqrt{|a_0|})^2\psi$. Therefore, for the two homoclinic orbits, we can obtain the following solitary wave solutions,

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\frac{1}{3}\sqrt{|a_0|}+\frac{\sqrt{|a_0|}}{2\cosh(\omega_{20}\xi)-1}\right)^{\frac{1}{2}}, \tag{3.13a}$$

where $\omega_{20} = \frac{|a_0|}{3c}$.

It follows from (3.13a) and (3.3) that

$$\theta(\xi) = -\left(\frac{ck+\Omega}{2c} + \frac{\sqrt{|a_0|}}{6c}\right)\xi - \frac{\sqrt{|a_0|}}{2c}\int_0^\xi \frac{d\xi}{2\cosh(\omega_{20}\xi)-1}$$
$$= -\left(\frac{ck+\Omega}{2c} + \frac{\sqrt{|a_0|}}{6c}\right)\xi - \sqrt{\frac{3}{|a_0|}}\arctan\left(\sqrt{3}\tanh\left(\frac{1}{2}\omega_{20}\xi\right)\right). \tag{3.13b}$$

For the upper heteroclinic orbits, we can obtain

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\frac{1}{3}\sqrt{|a_0|} - \frac{\sqrt{|a_0|}}{2\cosh(\omega_{20}\xi)+1}\right)^{\frac{1}{2}}, \text{ for } \xi \in (0,\infty) \text{ or } (-\infty,0), \tag{3.14a}$$

respectively.

It follows from (3.12a) and (3.1) that

$$\theta(\xi) = -\left(\frac{ck+\Omega}{2c} + \frac{\sqrt{|a_0|}}{6c}\right)\xi + \frac{\sqrt{|a_0|}}{4c}\int_0^\xi \frac{d\xi}{2\cosh(\omega_{20}\xi)-1}$$
$$= -\left(\frac{ck+\Omega}{2c} + \frac{\sqrt{|a_0|}}{6c}\right)\xi + \sqrt{\frac{3}{|a_0|}}\arctan\left(\frac{1}{\sqrt{3}}\tanh\left(\frac{1}{2}\omega_{20}\xi\right)\right). \tag{3.14b}$$

(iii) Corresponding to the level curves defined by $H_0(\phi, y) = h$, where $h \in (h_1, \infty)$, there exists a global family of periodic orbits of equation (1.12). The system has a series of periodic wave solutions which has the same parametric representation as (3.4a).

3.5. The case $2\sqrt{|a_0|} > a_1 > \sqrt{3|a_0|}, a_0 < 0$, i.e., $(a_0, a_1) \in (III), 0 < h_2 < h_1$ (see Fig.1 (f))

(i) Corresponding to the level curves defined by $H_0(\phi, y) = h$ where $h \in (0, h_2]$, there exists a family of periodic orbits of equation (1.12), which enclose the origin $(0, 0)$. The system has a series of periodic wave solutions which has the same parametric representation as (3.6a).

(ii) Corresponding to the level curves defined by $H_0(\phi, y) = h$ where $h \in (h_2, h_1)$, there exist three families of periodic orbits of equation (1.12), which enclose the equilibrium points $(\pm\phi_2, 0)$ and $(0, 0)$, respectively. The parametric representations of the periodic wave solutions are the same as (3.9a) and (3.10a).

(iii) Corresponding to the level curves defined by $H_0(\phi, y) = h_1$, there exist two homoclinic orbits which enclose the centers $(\pm\phi_2, 0)$ and a heteroclinic loop which encloses the center $(0, 0)$. The parametric representations of the wave solutions are the same as (3.11a) and (3.12a).

(iv) Corresponding to the level curves defined by $H_0(\phi, y) = h$ where $h \in (h_1, \infty)$, there exists a global family of periodic orbits of equation (1.12), The system has a series of periodic wave solutions which has the same parametric representation as (3.6a).

3.6. The case $a_1 = \sqrt{-3a_0}, a_0 < 0$, i.e., $(a_0, a_1) \in (L_2)$ (see Fig.1 (g))

Note that, we have $h_1 = h_2 = \frac{\sqrt{3}|a_0|^{\frac{3}{2}}}{72c^2}, \psi_1 = \psi_2 = \frac{\sqrt{3|a_0|}}{3}$ in this case.

(i) Corresponding to the level curves defined by $H_0(\phi, y) = h$ where $h \in (0, h_1)$, there exists a family of periodic orbits of equation (1.12), which enclose the origin $(0, 0)$ as a center. The corresponding periodic wave solution has the same parametric representation as (3.6a).

(ii) Corresponding to the level curves defined by $H_0(\phi, y) = h_1$, there exists a heteroclinic loop, which enclose the center $(0, 0)$. Now, $G(\psi) = \left(\frac{\sqrt{3|a_0|}}{3} - \psi\right)^3 \psi$. The upper heteroclinic orbit corresponds a kink solution which has the following parametric representation

$$\phi(\xi) = \pm\sqrt{\psi(\xi)} = \pm\left(\frac{\sqrt{3|a_0|}}{3} - \frac{\sqrt{3|a_0|}}{3 + \frac{|a_0|}{4c^2}\xi^2}\right)^{\frac{1}{2}}, \text{ for } \xi \in (0, \infty) \text{ or } (-\infty, 0) \quad (3.15a)$$

respectively.

It follows from (3.13a) and (3.1) that

$$\begin{aligned}\theta(\xi) &= -\left(\frac{ck + \Omega}{2c} + \frac{\sqrt{3|a_0|}}{6c}\right)\xi + \frac{\sqrt{3|a_0|}}{4c}\int_0^\xi \frac{d\xi}{3 + \frac{|a_0|}{4c^2}\xi^2} \\ &= -\left(\frac{ck + \Omega}{2c} + \frac{\sqrt{3|a_0|}}{6c}\right)\xi + 6c \arctan\left(\frac{\sqrt{|a_0|}}{\sqrt{3}c}\xi\right).\end{aligned} \quad (3.15b)$$

(iii) Corresponding to the level curves defined by $H_0(\phi, y) = h$ where $h \in (h_1, \infty)$, there exists a global family of periodic orbits of equation (1.12). These corresponding periodic wave solutions has the same parametric representations as (3.6a).

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