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SOME SYSTEMS WITH C^1 REGULARITY AND ONLY NEGATIVE LYAPUNOV EXPONENTS

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Abstract In this paper, we prove that for a C^1 diffeomorphism preserving an ergodic measure μ with only negative Lyapunov exponents, the support set of μ is a periodic orbit. For a skew product system preserving an ergodic measure with only negative fiberwise exponents, whose fiber maps are C^1 diffeomorphisms, we get that for almost all fibers, the disintegration of this measure on fibers is supported on finitely many points.

Keywords Negative Lyapunov exponents, skew product systems, periodic point.

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1. Introduction

The study of the periodic points plays an important role in the study of the dynamical systems since periodic points provide enough information for the system. For instance, the cohomology classes of Hölder cocycles over hyperbolic systems are characterized by its information on periodic points (see some recent results for instance [2, 5, 7, 15] and the references therein.) Another topic is the periodic approximation, see for instance [3, 8, 9] and the references therein.

Classical results showed that for a locally maximal hyperbolic set, the periodic points are dense in the non-wandering set, see Katok and Hasselblatt's book [6]. For $C^{1+\alpha}$ diffeomorphisms on compact manifolds which preserve hyperbolic measures, the existence of periodic points also can be guaranteed by the closing lemma, see Katok and Hasselblatt's book [6]. Their proof based on Pesin theory for $C^{1+\alpha}$ diffeomorphisms. For C^1 diffeomorphisms with a hyperbolic ergodic measure, it is unknown whether the closing lemma still holds. But for C^1 diffeomorphisms with only negative Lyapunov exponents, we can still deduce the existence of the periodic orbits by the existence of the hyperbolic times.

For a skew product system preserving an ergodic measure with only negative fiberwise exponents, whose fiber maps are C^1 diffeomorphisms, we get that for almost all fibers, the disintegration of this measure on fibers is supported on finitely many points. We can not get a periodic point in this case since there might be no periodic points in the base space. For instance, if the base dynamic is the irrational rotation on the circle, then there is no periodic points for the skew product system.

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Such kinds of results can be found in some literatures with the $C^{1+\alpha}$ regularity in [6, 11], which utilized the powerful tool developed by Pesin [4] for the non-uniformly hyperbolic systems. Notice that in general such a tool is invalid for C^1 regularity.

Furthermore, for a C^1 skew product system preserving an ergodic measure with only negative fiberwise exponents and Anosov base dynamic, Kocsard and Potrie in [7] showed the existence of the periodic orbits. The Anosov closing lemma guarantees the existence of the periodic point for the base dynamics and the existence of only negative fiberwise exponents allows one to construct a contraction on the fiber of the periodic point.

Next we give the details of our setting and results.

Let $f : X \to X$ be a measure preserving transformation of a probability space (X, ν).

Definition 1.1. A measurable function $A : X \times \mathbb{Z} \to GL(n, \mathbb{R})$ is called a cocycle over f if there is a measurable function $A : X \to GL(n, \mathbb{R})$ such that for any $x \in X$, it holds that

$$A(x, m) = A(f^{m-1}(x))...A(x) \qquad \text{for } m > 0$$

and $A(x, 0) = Id$. If f is invertible, we further assume that it satisfies that

$$A(x, m) = A(f^m(x))^{-1}...A(f^{-1}(x))^{-1} \qquad \text{for } m < 0.$$

The map $A(x)$ is called a generator of the cocycle $A(x, n)$. Sometimes, we also refer this as a cocycle.

Next we recall the definition of the Lyapunov exponents and the Oseledec's Theorem.

Lemma 1.1 ([10]). *Let $f : X \to X$ be an invertible measure preserving transformation of a probability space (X, ν) and $A(x, n)$ a measurable cocycle over f. Suppose that*

$$\log^+ \|A^{\pm}(x)\| \in L^1(X, \nu),$$

then there exists an $f-$invariant measurable set $Y \subset X$ (i.e. $f(Y) = Y$) with $\nu(Y) = 1$, such that for each $x \in Y$,

(i) There is an invariant splitting of $\mathbb{R}^n$,

$$\mathbb{R}^n = H_1(x) \oplus H_2(x) \oplus ... \oplus H_{k(x)}(x).$$

(ii) Besides, there exist numbers $\lambda_1(x) < \lambda_2(x) < ... < \lambda_{k(x)}(x)$ such that for $v \in H_i(x)$, $v \neq 0$, the following limit exists

$$\lim_{n\to+\infty} \frac{1}{n} \log \|A(x, n)v\| = \lambda_i(x).$$

If ν is ergodic for f, then $\lambda_i(x)$ and $k(x)$ are independent of the choice of the point $x \in Y$. Thus we can denote them by $\lambda_1(\nu), \lambda_2(\nu), ..., \lambda_k(\nu)$ and we call them the *Lyapunov exponents* of the cocycle $A(x, n)$ with respect to the measure ν.

Recall that the *support set* of a measure μ on a measurable space X is defined as the collection of the points $x \in X$ with the property that for any neighborhood $U(x)$ of x, we have $\mu(U(x)) > 0$. Denote the support set of the measure μ by *supp* μ.

Next we state our main results.

Theorem 1.1. *Let M be a compact Riemannian manifold, $f : M \to M$ be a C^1 diffeomorphism and μ be an ergodic measure for f. If all the Lyapunov exponents with respect to the measure μ are negative, then the support of the measure μ is a periodic orbit and the periodic orbit is an attractor.*

Remark 1.1. This result was proved under the assumption that f is a $C^{1+\alpha}$ diffeomorphism in [4, 11]. They used Pesin theory. The author would like to thank the reviewer to point out that Pesin theory also holds for this case due to Theorem 3.11 of [1]. But here we give a simple proof without using Pesin theory.

Remark 1.2. If all the Lyapunov exponents with respect to the measure μ are positive, then we can instead consider f^{-1} to deduce the result. Thus the statement can be replaced by the condition that all the Lyapunov exponents with respect to the measure μ have the same sign. Besides, there are counter-examples for systems allowing zero Lyapunov exponents. For instance, the irrational rotation on the circle preserves the Haar measure and is ergodic, whose Lyapunov exponent is zero. But it has no periodic points and the support set of the Haar measure is the circle.

Suppose $(X, \mathcal{B}, \nu)$ is a complete probability space and $f : X \to X$ is an invertible transformation which is ergodic with respect to ν. Recall that *complete* means that any subset of a measurable set with zero measure is measurable. Let M be a smooth compact Riemannian manifold endowed with the Borel $\sigma-$algebra $\mathcal{B}(M)$ and $\varphi : X \to \text{Diff}^1(M)$ a map, namely for each $x \in X$, φ_x is a C^1 diffeomorphism on M. Assume that $X \times M$ is endowed with the product $\sigma-$algebra $\mathcal{B} \otimes \mathcal{B}(M)$. Define the skew product transformation $F : X \times M \to X \times M$ as

$$F(x, y) = (f(x), \varphi_x(y))$$

and assume that it is Borel measurable. Besides, suppose that F preserves an invariant ergodic measure μ on $X \times M$ such that $\pi_*\mu = \nu$, where $\pi : X \times M \to X$ is the projection.

Remark 1.3. The assumption that X is complete is to ensure that $\pi(B)$ is measurable for any measurable set $B \subset X \times M$. One can refer to Theorem 4.5 of [13] for this fact. The author would like to thank the reviewer to point out this fact.

Fix $x \in X$ and define the maps $\varphi_x^{(k)}, k \in \mathbb{Z}$, on M as

$$\varphi_x^{(k+1)} = \varphi_{f^k(x)} \circ \varphi_x^{(k)},$$

where $\varphi_x^{(0)}$ is the identity map. Hence for $k > 0$, $\varphi_x^{(k)} = \varphi_{f^{k-1}(x)} \circ ... \circ \varphi_x$. For $k < 0$, $\varphi_x^{(k)} = (\varphi_{f^k(x)})^{-1} \circ ... \circ (\varphi_{f^{-1}(x)})^{-1}$.

Since the tangent bundle to M is measurably trivial, the derivative map of φ along the M direction gives a cocycle $X \times M \times \mathbb{Z} \to GL(n, \mathbb{R})$, where $n = dimM$:

$$(x, p, k) \mapsto D_p\varphi_x^k.$$

If $\log^+ \|D\varphi\| \in L^1(X \times M, \mu)$, then Oseledec's Theorem and ergodicity imply that the Lyapunov exponents $\lambda_1 < \lambda_2 < ... < \lambda_\ell$ of this cocycle exist and are constant for $\mu - a.e.(x, p)$. We call these numbers the *fiberwise exponents* of F. For a more detailed description of the fiberwise exponents, one can also refer to Avila, Kocsard and Liu 's work in [2].

Theorem 1.2. *Under the above assumptions and further suppose that* $\{\varphi_x\}_{x\in X}$ *and* $\{\|D\varphi_x\|\}_{x\in X}$ *are equicontinuous respectively. Suppose that all the fiberwise exponents with respect to the measure* μ *are negative, then there exists a set* $S \subset X \times M$ *with* $\mu(S) = 1$ *and an integer* $k \geq 1$ *such that*

$$card(S \cap \{x\} \times M) = k$$

for every $x \in \pi(S)$.

Remark 1.4. This result was proved under the assumption that $\{\varphi_x\}_{x\in X}$ is a family of $C^{1+\alpha}$ diffeomorphisms in Ruelle and Wilkinson's [12]. In [12], the authors proved the results via Pesin theory which requires the integrability of $\log^+ \|D\varphi\|_\alpha$, where $\|\cdot\|_\alpha$ is the $\alpha-$Hölder norm. Also they don't need the equi-continuity assumption. Since Pesin theory is invalid for diffeomorphisms with only C^1 regularity, we need such a technical assumption. Theorem 1 can be viewed as a corollary of this theorem. For this we only need to take X as a single point.

2. Proof of the main results

In this section, we give the proof of our results.

2.1. Diffeomorphisms with only negative Lyapunov exponents

The main idea of this proof is the following: we use Birkhoff's ergodic theorem (see Walter's book [14]) to get a good estimate of the derivatives on a set with positive measure and use Poincáre's recurrence theorem to construct a contraction map so as to get a periodic orbit. The ergodicity of the measure yields the result.

Proof of Theorem 1.1. By the assumption, the largest Lyapunov exponent with respect to the measure μ is negative, i.e.

$$\lim_{n\to+\infty} \frac{1}{n} \int_M \log ||D_x f^n|| d\mu \triangleq \lambda < 0.$$

Thus there exists $N \in \mathbb{N}$ such that

$$\frac{1}{N} \int_M \log ||D_x f^N|| d\mu \leq \frac{\lambda}{2}.$$

Apply Birkhoff's ergodic theorem to $\log ||D_x f^N||$, there is $\overline{\varphi} \in L^1(M,\mu)$ such that for $\mu - a.e.$ $x \in M$, it holds that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log ||D_{f^{iN}(x)} f^N|| = \overline{\varphi}(x),$$

and

$$\int_M \overline{\varphi}(x) d\mu = \int_M \log ||D_x f^N|| d\mu.$$

Thus, we have

$$\int_M \overline{\varphi}(x) d\mu \leq \frac{N\lambda}{2}.$$

Hence there is a subset $A' \subset M$ with $\mu(A') > 0$ such that for any $x \in A'$, we have $\overline{\varphi}(x) \le \frac{N\lambda}{2}$ and hence

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log ||D_{f^{iN}(x)}f^N|| \le \frac{N\lambda}{2}.$$

By Egorov's theorem, there is a subset $A \subset A'$ with $\mu(A) > 0$ such that the convergence above is uniform for $x \in A$. Thus there exists $K \in \mathbb{N}$ such that for $n \ge K$, any $x \in A$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} \log ||D_{f^{iN}(x)}f^N|| \le \frac{N\lambda}{4},$$

i.e.

$$\prod_{i=0}^{n-1} ||D_{f^{iN}(x)}f^N|| \le e^{\frac{nN\lambda}{4}}.$$

Here we take K large enough such that $e^{\frac{KN\lambda}{4}} \le \frac{1}{4}$.

By the uniform continuity of $\log ||D_x f^N||$ on M, for $\varepsilon > 0$ such that $e^{\frac{KN\lambda}{4}}e^{K\varepsilon} \le \frac{1}{2}$ and $e^{\frac{N\lambda}{4}}e^{\varepsilon} \le 1$, there exists $\delta_1 > 0$ such that for any $x, y \in M$ with $d(x,y) \le \delta_1$, we have

$$e^{-\varepsilon} \le \frac{||D_x f^N||}{||D_y f^N||} \le e^{\varepsilon}.$$

By the uniform continuity of f^N, there exists $0 < \delta < \delta_1$ such that

$$diam(f^{iN}B(x,\delta)) \le \delta_1 \quad \text{for} \quad \text{any} \quad x \in M \quad \text{and} \quad 0 \le i \le K-1.$$

Thus for any $x, y \in M$ with $d(x,y) \le \delta$, we have

$$e^{-K\varepsilon} \le \frac{\prod_{i=0}^{K-1} ||D_{f^{iN}(y)}f^N||}{\prod_{i=0}^{K-1} ||D_{f^{iN}(x)}f^N||} \le e^{K\varepsilon}.$$

Hence, for any $x \in A$ and $y \in B(x,\delta)$, we have

$$\prod_{i=0}^{K-1} ||D_{f^{iN}(y)}f^N|| \le \prod_{i=0}^{K-1} ||D_{f^{iN}(x)}f^N||e^{K\varepsilon} \le e^{\frac{KN\lambda}{4}}e^{K\varepsilon} \le \frac{1}{2}.$$

This implies that $||D_y f^{KN}|| \le \frac{1}{2}$ for $y \in B(x,\delta)$. Thus

$$f^{KN}B(x,\delta) \subset B(f^{KN}(x), \frac{\delta}{2}) \subset B(f^{KN}(x), \delta_1).$$

By the choice of $\delta_1 > 0$, we have

$$\|D_{f^{KN}(y)}f^N\| \le \|D_{f^{KN}(x)}f^N\|e^{\varepsilon}.$$

Thus for $y \in B(x,\delta)$, we have

$$||D_y f^{(K+1)N}|| \le \prod_{i=0}^{K} ||D_{f^{iN}(y)}f^N|| \le \prod_{i=0}^{K} ||D_{f^{iN}(x)}f^N||e^{(K+1)\varepsilon}$$

$$\leq e^{\frac{(K+1)N\lambda}{4}} e^{(K+1)\varepsilon} \leq \frac{1}{2}.$$

This implies that $f^{(K+1)N}B(x,\delta) \subset B(f^{(K+1)N}(x), \frac{\delta}{2})$.

By the same argument, we get for any $n \geq K$, $f^{nN}B(x,\delta) \subset B(f^{nN}(x), \frac{\delta}{2})$ and for any $y \in B(x,\delta)$ and $n \geq K$ we have $||D_y f^{nN}|| \leq \frac{1}{2}$.

Fix $x \in A \cap supp\, \mu$. By the definition of the support set, we have $\mu(B(x, \frac{\delta}{8})) > 0$. By Poincáre's recurrence theorem, for $\mu - a.e.\ y \in B(x, \frac{\delta}{8})$, there exist integers $0 < n_1 < n_2 < ... < n_k < ...$ such that $f^{n_i N}(y) \in B(x, \frac{\delta}{8})$ for each $i = 1, 2, ...$ We choose $n_i \geq K$. Notice that $B(y, \frac{\delta}{2}) \subset B(x,\delta)$ and hence

$$f^{n_i N}B(y, \frac{\delta}{2}) \subset B(f^{n_i N}(y), \frac{\delta}{4}) \subset B(y, \frac{\delta}{2}) \subset B(x,\delta).$$

Thus $f^{n_i N}|_{B(y,\frac{\delta}{2})} : B(y, \frac{\delta}{2}) \to B(y, \frac{\delta}{2})$ is a contraction. By the contraction mapping theorem, there is a unique fixed point p for $f^{n_i N}$ in $B(y, \frac{\delta}{2})$. By the invariance of the measure μ, one can show that $\mu(\{p\}) = \mu(B(y, \frac{\delta}{2})) > 0$. By the ergodicity of the measure μ for f, we get that $\mu(Orb_f(p)) = 1$, which yields the theorem. □

2.2. Skew products with only negative fiberwise exponents

The proof is inspired by the work of [12] where $\{\varphi_x\}_{x\in X}$ is a family of $C^{1+\alpha}$ diffeomorphisms. But notice that they utilize the powerful tool of the non-uniform hyperbolicity theory to get a good estimate of the derivatives, which is invalid for our case. Instead, we analyze the top Lyapunov exponents to obtain the desired result. To get a uniform estimate of the derivatives, we need the equi-continuity assumptions on $\{\varphi_x\}_{x\in X}$ and $\{D\varphi_x\}_{x\in X}$. Notice that such a condition is unnecessary for $C^{1+\alpha}$ diffeomorphisms, since Pesin theory and Lusin's theorem can help us to get a uniform estimate on a ball of uniform size.

Proof of Theorem 1.2. By the assumption, the largest fiberwise exponent with respect to the measure μ is negative, i.e.

$$\lim_{n\to+\infty} \frac{1}{n} \int_{X\times M} \log ||D_y \varphi_x^{(n)}|| d\mu \triangleq \lambda < 0.$$

Thus there exists $N \in \mathbb{N}$ such that

$$\frac{1}{N} \int_{X\times M} \log ||D_y \varphi_x^{(N)}|| d\mu \leq \frac{\lambda}{2}.$$

Apply Birkhoff's ergodic theorem to $\log ||D_y \varphi_x^{(N)}||$, there is $\overline{\varphi} \in L^1(X \times M, \mu)$ such that for $\mu - a.e.(x,y) \in X \times M$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log ||D_{\varphi_x^{(iN)}(y)} \varphi_{f^{iN}(x)}^{(N)}|| = \overline{\varphi}(x,y)$$

and

$$\int_{X\times M} \overline{\varphi}(x,y) d\mu = \int_{X\times M} \log ||D_y \varphi_x^{(N)}|| d\mu \leq \frac{N\lambda}{2}.$$

Hence there is a subset $A' \subset X \times M$ with $\mu(A') > 0$ such that for any $(x, y) \in A'$, we have $\overline{\varphi}(x, y) \le \frac{N\lambda}{2}$ and hence

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log ||D_{\varphi_x^{(iN)}(y)} \varphi_{f^{iN}(x)}^{(N)}|| \le \frac{N\lambda}{2}.$$

By Egorov's theorem, there is a subset $A \subset A'$ with $\mu(A) > 0$ such that the convergence above is uniform for $(x, y) \in A$, thus there exists $K \in \mathbb{N}$ such that for any $(x, y) \in A$, $n \ge K$, we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \log ||D_{\varphi_x^{(iN)}(y)} \varphi_{f^{iN}(x)}^{(N)}|| \le \frac{N\lambda}{4},$$

i.e.

$$\prod_{i=0}^{n-1} ||D_{\varphi_x^{(iN)}(y)} \varphi_{f^{iN}(x)}^{(N)}|| \le e^{\frac{nN\lambda}{4}}. \tag{2.1}$$

Here we take K large enough such that $e^{\frac{KN\lambda}{4}} \le \frac{1}{4}$.

Fix $\varepsilon > 0$ such that $e^{\frac{N\lambda}{4}+\varepsilon} < 1$ and $e^{\frac{KN\lambda}{4}+\varepsilon K} \le \frac{1}{2}$. By the equicontinuity of $\{\log ||D_y \varphi_x^{(N)}||\}_{x\in X}$, there exists $\delta > 0$ such that for any $(x, y) \in X \times M$ and $(x, y') \in \{x\} \times B(y, \delta)$, we have $e^{-\varepsilon} \le \frac{||D_y \varphi_x^{(N)}||}{||D_{y'} \varphi_x^{(N)}||} \le e^{\varepsilon}$.

By the equicontinuity of $\{\varphi_x^{(N)}\}_{x\in X}$, we can choose $0 < \delta_1 \le \delta$ such that for any $(x, y) \in X \times M$ and $(x, y') \in \{x\} \times B(y, \delta_1)$, we have $d(\varphi_x^{(iN)}(y), \varphi_x^{(iN)}(y')) \le \delta$ for $0 \le i \le K - 1$. Thus

$$e^{-\varepsilon K} \le \frac{\prod_{i=0}^{K-1} ||D_{\varphi_x^{(iN)}(y')} \varphi_{f^{iN}(x)}^{(N)}||}{\prod_{i=0}^{K-1} ||D_{\varphi_x^{(iN)}(y)} \varphi_{f^{iN}(x)}^{(N)}||} \le e^{\varepsilon K}. \tag{2.2}$$

Next, we use mathematical induction to show that for $(x, y) \in A$ and $n \ge K$, it holds that

$$\varphi_x^{(nN)}(\{x\} \times B(y, \delta_1)) \subset \{f^{nN}(x)\} \times B(\varphi_x^{(nN)}(y), \frac{\delta_1}{2}) \tag{2.3}$$

and for $(x, y') \in \{x\} \times B(y, \delta_1)$, it holds that

$$\prod_{i=0}^{n-1} ||D_{\varphi_x^{(iN)}(y')} \varphi_{f^{iN}(x)}^{(N)}|| \le \prod_{i=0}^{n-1} ||D_{\varphi_x^{(iN)}(y)} \varphi_{f^{iN}(x)}^{(N)}|| e^{n\varepsilon} \le e^{(\frac{N\lambda}{4}+\varepsilon)n} \le \frac{1}{2}. \tag{2.4}$$

By (2.1) and (2.2), for $(x, y) \in A$ and $(x, y') \in \{x\} \times B(y, \delta_1)$, we have

$$\prod_{i=0}^{K-1} ||D_{\varphi_x^{(iN)}(y')} \varphi_{f^{iN}(x)}^{(N)}|| \le \prod_{i=0}^{K-1} ||D_{\varphi_x^{(iN)}(y)} \varphi_{f^{iN}(x)}^{(N)}|| e^{\varepsilon K} \le e^{\frac{KN\lambda}{4}+\varepsilon K} \le \frac{1}{2}.$$

Hence

$$\varphi_x^{(KN)}(\{x\} \times B(y, \delta_1)) \subset \{f^{KN}(x)\} \times B(\varphi_x^{(KN)}(y), \frac{\delta_1}{2}).$$

Assume that (2.3) and (2.4) hold for $n = j > K$, next we show that they also hold for $n = j + 1$.

For $(x,y)\in A$ and $(x,y')\in \{x\}\times B(y,\delta_1)$, by inductive assumption, we have

$$d(\varphi_x^{(jN)}(y),\varphi_x^{(jN)}(y'))\le \delta.$$

It follows that

$$e^{-\varepsilon}\le \frac{||D_{\varphi_x^{(jN)}(y)}\varphi^{(N)}_{f^{jN}(x)}||}{||D_{\varphi_x^{(jN)}(y')}\varphi^{(N)}_{f^{jN}(x)}||}\le e^{\varepsilon}.$$

By inductive assumption and (2.1),

$$\begin{aligned}\prod_{i=0}^{j}||D_{\varphi_x^{(iN)}(y')}\varphi^{(N)}_{f^{iN}(x)}|| &= ||D_{\varphi_x^{(jN)}(y')}\varphi^{(N)}_{f^{jN}(x)}||\prod_{i=0}^{j-1}||D_{\varphi_x^{(iN)}(y')}\varphi^{(N)}_{f^{iN}(x)}||\\ &\le ||D_{\varphi_x^{(jN)}(y)}\varphi^{(N)}_{f^{jN}(x)}||e^{\varepsilon}\prod_{i=0}^{j-1}||D_{\varphi_x^{(iN)}(y)}\varphi^{(N)}_{f^{iN}(x)}||e^{\varepsilon j}\\ &\le e^{(\frac{N\lambda}{4}+\varepsilon)(j+1)}\le \frac{1}{2}.\end{aligned}$$

Hence

$$\varphi_x^{((j+1)N)}(\{x\}\times B(y,\delta_1))\subset \{f^{(j+1)N}(x)\}\times B(\varphi_x^{((j+1)N)}(y),\frac{\delta_1}{2}).$$

This finishes the proof of (2.3) and (2.4).

Let $B'=\pi(A)$, then $\nu(B')=\pi_*\mu(B')=\mu(\pi^{-1}\pi(A))\ge\mu(A)>0$.

Lemma 2.1. *There is $B\subset B'$ with $\nu(B)>0$ such that for $\nu-a.e.$ $x\in B$, μ_x has an atom.*

Proof. Since $\mu(A)>0$, there is $B\subset\pi(A)$ with $\nu(B)>0$ such that for any $x\in B$, $\mu_x(A_x)\ge\mu(A)$, where $A_x=\{y\in M|(x,y)\in A\}$. Set $a=\mu(A)$.

Suppose $\mathcal{U}$ is a finite cover of M consisting of closed balls with diameters less than $\frac{\delta_1}{5}$. Denote the number of balls in $\mathcal{U}$ by C. Define $m(x)\triangleq \inf\sum_{i=1}^{k(x)} diam\ U_i$, where the infimum is taken over all the collections of closed sets with diameter less than $\frac{\delta_1}{5}$ and $k(x)\le C$ and also $\mu_x(\cup_{i=1}^{k(x)}U_i)\ge a$.

Denote $m=esssup_{x\in B}\ m(x)$. We show that $m=0$.

Otherwise $m>0$ and hence we can choose $J\ge K$ such that $C\Delta e^{(\frac{N\lambda}{4}+\varepsilon)J}<\frac{m}{2}$, where Δ is the diameter of M. We also assume that J is large enough such that $\Delta e^{(\frac{N\lambda}{4}+\varepsilon)J}\le\frac{\delta_1}{5}$. Denote

$$\tilde{B}\triangleq B\cap\cup_{j=0}^{+\infty}f^{jN}B.$$

By Poincáre's recurrence theorem, for $\nu-a.e.$ $x\in B$, there exist integers $n_1<n_2<...<n_k<...$ such that $f^{n_iN}(x)\in B$ for each i. Hence $\nu(\tilde{B})=\nu(B)$. For any $z\in\tilde{B}$, there exists $x\in B$ and $j\ge J$ such that $z=f^{jN}(x)$. Denote by $\mathcal{U}'$ the collection of elements in $\mathcal{U}$ which has non-empty intersection with A_x. We denote $\mathcal{U}'=\{U_1,...,U_{C'}\}$, where $C'\le C$ is the cardinality of $\mathcal{U}'$. By the invariance of the measure μ with respect to F, we have

$$\mu_{f^{jN}(x)}(\cup_{i=1}^{C'}\varphi_x^{(jN)}U_i)=(\varphi_x^{(jN)})_*\mu_x(\cup_{i=1}^{C'}\varphi_x^{(jN)}U_i)=\mu_x(\cup_{i=1}^{C'}U_i).$$

Notice that $\mathcal{U}'$ is a cover of A_x. Thus for $x\in B$, we have

$$\mu_{f^{jN}(x)}(\cup_{i=1}^{C'}\varphi_x^{(jN)}U_i)=\mu_x(\cup_{i=1}^{C'}U_i)\ge a.$$

For each $U_i \in \mathcal{U}'$, we have

$$diam(\varphi_x^{(jN)} U_i) \leq \Delta e^{(\frac{N\lambda}{4}+\varepsilon)J} \leq \frac{\delta_1}{5}.$$

Thus

$$m(z) \leq \sum_{i=1}^{C'} diam(\varphi_x^{(jN)} U_i) \leq C' \Delta e^{(\frac{N\lambda}{4}+\varepsilon)J} < \frac{m}{2}.$$

This implies that $m \leq \sup_{z\in\tilde{B}} m(z) < \frac{m}{2}$, which yields $m = 0$.

Thus for $\nu-$ $a.e.$ $x \in B$, $m(x) = 0$. For such an $x \in B$, there are a sequence of closed sets $\{V_i\}$ satisfying $\lim_{i\to+\infty} diam(V_i) = 0$ and $\mu_x(V_i) \geq \frac{a}{C}$ for each i. Take $z_i \in V_i$, then any accumulated point of $\{z_i\}$ is an atom for μ_x.

This finishes the proof of the lemma. □

For $x \in X$, set $d(x) = \sup_{y\in M} \mu_x(\{y\})$. Notice that $d(x)$ is $f-$invariant and is positive for $v - a.e.$ $x \in B$. Since f is ergodic with respect to the measure ν, we know that $d(x)$ is constant for $\nu - a.e.$ $x \in X$ and we denote this number by d. For such an x, there are only finite many points $y \in M$ such that $\mu_x(\{y\}) > \frac{d}{2}$. It follows that there exists $y \in M$ such that $\mu_x(\{y\}) = d$. Hence $S \triangleq \{(x,y) \in X \times M : \mu_x(\{y\}) = d\}$ is non-empty. Notice that S is $F-$invariant and

$$\mu(S) = \int_{X\times M} 1_S \, d\mu = \int_X \int_M 1_S \, d\mu_x d\nu \geq \int_X d \, d\nu = d.$$

By the ergodicity of the measure μ with respect to F, we have $\mu(S) = 1$. Thus for $\nu - a.e.$ x, $\mu_x(S) = 1$. Without loss of generality, we assume that for any $x \in \pi(S)$, it holds that $\mu_x(S) = 1$. Since for any $(x,y) \in S$, $\mu_x(\{y\}) = d$, we get that there exists $k \in \mathbb{N}$ such that for any $x \in \pi(S)$, $card(S \cap \{x\} \times M) = k$. □

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