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A VARIATIONAL APPROACH FOR A PROBLEM INVOLVING A ψ-HILFER FRACTIONAL OPERATOR

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Abstract Boundary value problems driven by fractional operators has drawn the attention of several researchers in the last decades due to its applicability in several areas of Science and Technology. The suitable definition of the fractional derivative and its associated spaces is a natural problem that arise on the study of this kind of problem. A manner to avoid of such problem is to consider a general definition of fractional derivative. The purpose of this manuscript is to contribute, in the mentioned sense, by presenting the $\psi-$fractional spaces $\mathbb{H}_p^{\alpha,\beta;\psi}([0,T],\mathbb{R})$. As an application we study a problem, by using the Mountain Pass Theorem, which includes an wide class of equations.

Keywords ψ-fractional derivative space, variational structure, fractional differential equations, boundary value problem, mountain pass theorem.

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1. Introduction

In the last decades the Fractional Calculus has drawn the attention of several researchers due to some advantages with respect to the usual one which occurs for example in problems involving memory, see for instance [10, 15, 22, 24, 27]. An important fact is its applicability, see for example Sousa et. al. [30,31], where it is considered the fractional version of a mathematical model that describes, under certain conditions, the blood concentration of nutrients and its relation with the erythrocyte sedimentation. We also quote the references [5,7,11,14,16,17,23,26,35,38,39,43,46].

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with smooth boundary. Motivated from the usual Calculus, we have the development of the Sobolev spaces $W^{k,p}(\Omega)$ with $k \in \mathbb{N}$ and $p \geq 1$ and its applications. An important one is the Variational

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approach for the Dirichlet problem

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, whose main idea consists in to associate the solutions of the problem above with critical points of C^1 energy functional of the form

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \mathcal{H}(x,u)dx, u \in H_0^1(\Omega),$$

where $\mathcal{H}(x,t) = \int_0^t f(x,\xi)d\xi$ and $W_0^{1,2}(\Omega) := H_0^1(\Omega)$ denotes the functions of $W^{1,2}(\Omega)$ that are null on the boundary in the sense of the trace operator. There is a vast literature regarding such subject, thus we only mention some classical ones, see for instance [3,4].

Regarding the Variational approach in the fractional setting, the first paper is due to Jiao and Zhou [17], where the authors considered, by using the Mountain Pass Theorem, the boundary value problem

$$\begin{cases} {}_tD_T^\alpha\left({}_0D_t^\alpha u\left(t\right)\right) = \nabla F\left(t, u\left(t\right)\right), t \in [0,T] \text{ a.e,} \\ u\left(0\right) = u\left(T\right) = 0, \end{cases}$$

where ${}_tD_T^\alpha\left(\cdot\right)$ and ${}_0D_t^\alpha\left(\cdot\right)$ are the right and left derivative of order $0 < \alpha \leq 1$ in the Riemann-Lioville sense respectively, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function satisfying some assumptions. After this, several works with the classical variational arguments arose in the literature, see [1,13,32–34,36–38,41,44]. For example in [41] Ledesma obtained the existence of a solution for the problem

$$\begin{cases} {}_tD_T^\alpha\left({}_0D_t^\alpha u\left(t\right)\right) = f\left(t, u\left(t\right)\right), \ t \in [0,T], \\ u\left(0\right) = u\left(T\right) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (1/2, 1)$ and f is a function satisfying certain conditions. Regarding other related works see [6,8,12,18,20,21,40,42,45,47,48].

With the wide number of definitions of integrals and fractional derivatives, it is interesting to consider a general notion of fractional derivative of a function f with respect to another function. Such question was recently considered in Sousa & Oliveira [29], where the authors introduced the ψ-Hilfer fractional derivative and exhibited an wide class of examples. Thus from [29] it is natural to construct a suitable space and study its properties to consider, by using a variational approach, the problem

$$\begin{cases} {}^{\mathbf{H}}\mathbf{D}_{T-}^{\alpha,\beta;\psi}\left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\left(t\right)\right) = f\left(t, u\left(t\right)\right), \quad t \in [0,T], \\ I_{0+}^{\beta(1-\beta)}u\left(0\right) = I_{T-}^{\beta(1-\beta)}u\left(T\right) = 0, \end{cases} \tag{P}$$

where ${}^{\mathbf{H}}\mathbf{D}_{T-}^{\alpha,\beta;\psi}(\cdot)$, ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}(\cdot)$ are the right and left ψ-Hilfer fractional derivatives respectively of order $\alpha \in (1/2, 1]$ and type $0 \leq \beta \leq 1$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the conditions :

(f_1) $f \in C([0,T] \times \mathbb{R})$;

(f_2) (Ambrossetti-Rabinowitz condition) There is a constant $\mu > 2$ such that $0 < \mu \mathcal{H}(t,u) \leq u\ f(t,u)$ for every $t \in [0,T]$ and $u \in \mathbb{R} \setminus \{0\}$, where $\mathcal{H}(t,s) = \int_0^s f(t,\xi)\, d\xi$.

In what follows we describe in details the contributions of this work.

(i) It is presented a suitable space (denoted by $\mathbb{H}_p^{\alpha,\beta;\psi}([0,T],\mathbb{R})$) to study the problem (P).

(ii) Several important results are proved for the space $\mathbb{H}_p^{\alpha,\beta;\psi}([0,T],\mathbb{R})$ such as completeness, reflexivity and some embeddings. Such properties will be needed to consider a variational approach for (P).

(iii) A notion of weak solution for (P) is introduced and it is obtained the existence of a weak solution by using the classical Mountain Pass Theorem. To the best of our knowledge it is the first time that a Dirichlet problem with an operator which involves the $\psi-$Hilfer fractional derivative is studied in the literature. Moreover, the results of [41] are obtained for a larger class of equations.

The rest of the paper is organized as follows: Section 2 is devoted to present the fractional Riemann-Lioville integral with respect to another function, the $\psi-$Hilfer fractional derivative and some results that will be often used. In Section 3 the spaces $\mathbb{H}_p^{\alpha,\beta;\psi}([0,T],\mathbb{R})$ and examples are presented and several properties of such spaces are proved in Section 4. As an application of the mentioned results, it is proved in Section 5 the existence of solution for (P) by using the Mountain Pass Theorem.

2. Preliminaries

Let $[a,b]$ be a finite interval and $C[a,b], AC^n[a,b], C^n[a,b]$ be the spaces of continuous functions, $n-$times absolutely continuous functions, $n-$times continuously differentiable functions on $[a,b]$, respectively.

The space of the continuous functions f on $[a,b]$ with the norm defined by

$$\|f\|_{C[a,b]} = \max_{t \in [a,b]} |f(t)|.$$

On the order hand, we have $n-$times absolutely continuous given by

$$AC^n[a,b] = \{f : [a,b] \to \mathbb{R}; f^{(n-1)} \in AC[a,b]\}.$$

The weighted space $C_{\gamma;\psi}[a,b]$ is defined by

$$C_{\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{R}; (\psi(\cdot) - \psi(a))^\gamma f(\cdot) \in C[a,b]\}, 0 \leq \gamma < 1$$

with the norm

$$\|f\|_{C_{\gamma;\psi}[a,b]} = \|(\psi(\cdot) - \psi(a))^\gamma f\|_{C[a,b]} = \max_{t \in [a,b]} |(\psi(t) - \psi(a))^\gamma f(t)|.$$

The space $C^n_{\gamma;\psi}[a,b]$ is defined by

$$C^n_{\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{R}; f \in C^{(n-1)}[a,b]; f^{(n)} \in C_{\gamma;\psi}[a,b]\}, 0 \leq \gamma < 1$$

with the norm

$$\|f\|_{C^n_{\gamma;\psi}[a,b]} = \sum_{k=0}^{n-1} \|f^{(k)}\|_{C[a,b]} + \|f^{(n)}\|_{C_{\gamma;\psi}[a,b]}.$$

Definition 2.1 ([28, 29]). Let (a,b) $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Let ψ be an increasing and positive continuous function on $(a,b]$, having a continuous derivative ψ' on (a,b). The left and right-sided fractional integrals of a function u with respect to another function ψ on $[a,b]$ are defined by

$$\mathbf{I}^{\alpha;\psi}_{a+} u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)\left(\psi(x) - \psi(t)\right)^{\alpha-1} u(t)\, dt \tag{2.1}$$

and

$$\mathbf{I}^{\alpha;\psi}_{b-} u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)\left(\psi(t) - \psi(x)\right)^{\alpha-1} u(t)\, dt. \tag{2.2}$$

Lemma 2.1 ([28, 29]). *Let $\alpha > 0$ and $\delta > 0$. Then the following properties hold*

$$\mathbf{I}^{\alpha;\psi}_{a+}\ \mathbf{I}^{\delta;\psi}_{a+} u(x) = \mathbf{I}^{\alpha+\delta;\psi}_{a+} u(x) \tag{2.3}$$

and

$$\mathbf{I}^{\alpha;\psi}_{b-}\ \mathbf{I}^{\delta;\psi}_{b-} u(x) = \mathbf{I}^{\alpha+\delta;\psi}_{b-} u(x). \tag{2.4}$$

Definition 2.2 ([28, 29]). Consider that $\psi'(x) \neq 0$ $(-\infty \leq a < x < b \leq \infty)$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann-Liouville derivatives of a function u with respect to ψ of order α correspondent to the Riemann-Liouville, are defined by

$$\begin{aligned} \mathcal{D}^{\alpha;\psi}_{a+} u(x) &= \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathbf{I}^{n-\alpha;\psi}_{a+} u(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x \psi'(t)\left(\psi(x) - \psi(t)\right)^{n-\alpha-1} u(t)\, dt \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \mathcal{D}^{\alpha;\psi}_{b-} u(x) &= \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathbf{I}^{n-\alpha;\psi}_{b-} u(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_x^b \psi'(t)\left(\psi(t) - \psi(x)\right)^{n-\alpha-1} u(t)\, dt, \end{aligned} \tag{2.6}$$

where $n = [\alpha] + 1$.

Definition 2.3 ([28, 29]). Let $\alpha > 0$, $n \in \mathbb{N}$, $I = [a,b]$ with $-\infty \leq a < b \leq \infty$, $u, \psi \in C^n([a,b], \mathbb{R})$ two functions such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The left ψ-Caputo fractional derivative of u of order α is given by

$${}^C D^{\alpha;\psi}_{a+} u(x) = \mathbf{I}^{n-\alpha;\psi}_{a+} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n u(x) \tag{2.7}$$

and the right ψ-Caputo fractional derivative of u by

$$^{C}D_{b-}^{\alpha;\psi}u\left(x\right)=\mathbf{I}_{b-}^{n-\alpha;\psi}\left(-\frac{1}{\psi'\left(x\right)}\frac{d}{dx}\right)^{n}u\left(x\right) \tag{2.8}$$

where $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$ and $n=\alpha$ for $\alpha\in\mathbb{N}$.

Definition 2.4 ([28, 29]). Let $n-1<\alpha<n$ with $n\in\mathbb{N}$, $I=[a,b]$ is the interval such that $-\infty\leq a<b\leq\infty$ and $u,\psi\in C^{n}([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(x)\neq 0$, for all $x\in I$. The ψ-Hilfer fractional derivative left-sided and right-sided $^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ and $^{\mathbf{H}}\mathbf{D}_{b-}^{\alpha,\beta;\psi}(\cdot)$ of function of order α and type $0\leq\beta\leq 1$, are defined by

$$^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi}u\left(x\right)=\mathbf{I}_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'\left(x\right)}\frac{d}{dx}\right)^{n}\mathbf{I}_{a+}^{(1-\beta)(n-\alpha);\psi}u\left(x\right) \tag{2.9}$$

and

$$^{\mathbf{H}}\mathbf{D}_{b-}^{\alpha,\beta;\psi}u\left(x\right)=\mathbf{I}_{b-}^{\beta(n-\alpha);\psi}\left(-\frac{1}{\psi'\left(x\right)}\frac{d}{dx}\right)^{n}\mathbf{I}_{b-}^{(1-\beta)(n-\alpha);\psi}u\left(x\right). \tag{2.10}$$

The ψ-Hilfer fractional derivative as above defined, can be written in the following form

$$^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi}u\left(x\right)=\mathbf{I}_{a+}^{\gamma-\alpha;\psi}\mathcal{D}_{a+}^{\gamma;\psi}u\left(x\right) \tag{2.11}$$

and

$$^{\mathbf{H}}\mathbf{D}_{b-}^{\alpha,\beta;\psi}u\left(x\right)=\mathbf{I}_{b-}^{\gamma-\alpha;\psi}\mathcal{D}_{b-}^{\gamma;\psi}u\left(x\right), \tag{2.12}$$

with $\gamma=\alpha+\beta\left(n-\alpha\right)$ and $\mathbf{I}_{a+}^{\gamma-\alpha;\psi}(\cdot)$, $\mathcal{D}_{a+}^{\gamma;\psi}(\cdot)$, $\mathbf{I}_{b-}^{\gamma-\alpha;\psi}(\cdot)$, $\mathcal{D}_{b-}^{\gamma;\psi}(\cdot)$ are defined in (2.1), (2.2), (2.5), and (2.6).

Theorem 2.1 ([28, 29]). *If $u\in C_{\gamma,\psi}^{n}[a,b]$, $n-1<\alpha<n$ and $0\leq\beta\leq 1$, then*

$$\mathbf{I}_{a+}^{\alpha;\psi}\ {}^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi}u\left(x\right)=u\left(x\right)-\sum_{k=1}^{n}\frac{\left(\psi\left(x\right)-\psi\left(a\right)\right)^{\gamma-k}}{\Gamma\left(\gamma-k+1\right)}u_{\psi}^{[n-k]}\mathbf{I}_{a+}^{(1-\beta)(n-\alpha);\psi}u\left(a\right)$$

and

$$\mathbf{I}_{b-}^{\alpha;\psi}\ {}^{\mathbf{H}}\mathbf{D}_{b-}^{\alpha,\beta;\psi}u\left(x\right)=u\left(x\right)-\sum_{k=1}^{n}\frac{\left(-1\right)^{k}\left(\psi\left(b\right)-\psi\left(x\right)\right)^{\gamma-k}}{\Gamma\left(\gamma-k+1\right)}u_{\psi}^{[n-k]}\mathbf{I}_{b-}^{(1-\beta)(n-\alpha);\psi}u\left(b\right).$$

In what follows we consider the integration by parts rule for ψ-Riemann-Liouville fractional integral and for the ψ-Hilfer fractional derivative.

By Almeida [2], we know that the relation

$$\int_{a}^{b}\left(\mathbf{I}_{a+}^{\alpha;\psi}u\left(t\right)\right)\theta\left(t\right)\mathrm{dt}=\int_{a}^{b}u\left(t\right)\psi'\left(t\right)\mathbf{I}_{b-}^{\alpha;\psi}\left(\frac{\theta\left(t\right)}{\psi'\left(t\right)}\right)\mathrm{dt} \tag{2.13}$$

is valid. Now we present the integration by parts rule for the ψ-Hilfer fractional derivative, which plays a key role in the variational formulation of problem (P).

Theorem 2.2. *Let $\psi(\cdot)$ be an increasing and positive continuos function on $[a,b]$, having a continuous derivative $\psi'(\cdot) \neq 0$ on (a,b). If $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$, then*

$$\int_a^b \left({}^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi} u(t)\right) \theta(t)\,\mathrm{dt} = \int_a^b u(t)\,\psi'(t)\ {}^{\mathbf{H}}\mathbf{D}_{b-}^{\alpha,\beta;\psi} \left(\frac{\theta(t)}{\psi'(t)}\right) \mathrm{dt} \tag{2.14}$$

for any $u \in AC^1[a,b]$ and $\theta \in C^1[a,b]$ satisfying boundary conditions $u(a) = 0 = u(b)$.

Proof. In fact, using the Eq.(2.4), Eq.(2.12) and Theorem 2.1, yields

$$\begin{aligned}
&\int_a^b u(t)\,\psi'(t)\ {}^{\mathbf{H}}\mathbf{D}_{b-}^{\alpha,\beta;\psi} \left(\frac{\theta(t)}{\psi'(t)}\right) \mathrm{dt} \\
&= \int_a^b u(t)\,\psi'(t)\,\mathbf{I}_{b-}^{1-\alpha;\psi}\ D_{b-}^{1;\psi} \left(\frac{\theta(t)}{\psi'(t)}\right) \mathrm{dt} \\
&= \int_a^b \psi'(t) \left[\mathbf{I}_{a+}^{\alpha;\psi}\ {}^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi} u(t) + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} d_j\right] \mathbf{I}_{b-}^{1-\alpha;\psi}\ D_{b-}^{1;\psi} \left(\frac{\theta(t)}{\psi'(t)}\right) \mathrm{dt} \\
&\qquad \left(\text{where } d_j = \left(\frac{1}{\psi'(t)}\frac{d}{\mathrm{dt}}\right) \mathbf{I}_{b-}^{(1-\beta)(1-\alpha);\psi} u(a)\right) \\
&= \int_a^b \psi'(t)\,\mathbf{I}_{a+}^{\alpha;\psi}\ {}^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi} u(t)\,\mathbf{I}_{b-}^{1-\alpha;\psi}\ D_{b-}^{1;\psi} \left(\frac{\theta(t)}{\psi'(t)}\right) \mathrm{dt} \\
&\quad + \frac{d_j}{\Gamma(\gamma)} \int_a^b \psi'(t)\,(\psi(t)-\psi(a))^{\gamma-1}\,\mathbf{I}_{b-}^{1-\gamma;\psi}\ D_{b-}^{1;\psi} \left(\frac{\theta(t)}{\psi'(t)}\right) \mathrm{dt} \\
&= \int_a^b \mathbf{I}_{a+}^{\alpha;\psi}\ {}^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi} u(t)\,\mathbf{I}_{b-}^{-\alpha;\psi} \left(\frac{\theta(t)}{\psi'(t)}\right) \mathrm{dt} \\
&= \int_a^b \left({}^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi} u(t)\right) \theta(t)\,\mathrm{dt}.
\end{aligned}$$

□

Now we introduce more notations and some necessary definitions. Let X be a real Banach space, $\Phi \in C^1(X,\mathbb{R})$, which means that Φ is a continuously Fréchet-differentiable functional defined on X. Recall that $\Phi \in C^1(X,\mathbb{R})$ is said to satisfy the (PS) condition if any sequence $(u_k)_{k\in\mathbb{N}} \in X$, for which $(\Phi(u_k))_{k\in\mathbb{N}}$ is bounded and $\Phi'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in X.

Theorem 2.3 ([4], Mountain Pass Theorem)**.** *Let X be a real Banach space and $\Phi \in C^1(X,\mathbb{R})$ satisfying Palais-Smale condition. Suppose that*

(i) $\Phi(0) = 0$,

(ii) there is $\rho > 0$ and $\sigma > 0$ such that $\Phi(z) \geq \sigma$ for all $z \in X$ with $\|z\| = \rho$,

(iii) there exists z_1 in X with $\|z_1\| \geq \rho$ such that $\Phi(z_1) < \sigma$. Then Φ possesses a critical value $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{z\in[0,1]} \Phi(\gamma(z)) \tag{2.15}$$

where $\Gamma = \{\gamma \in C([0,T],X);\gamma(0) = 0,\ \gamma(1) = z_1\}$.

3. ψ-fractional derivative spaces $\mathbb{H}_p^{\alpha,\beta;\psi}([0,T],\mathbb{R})$

In this section we present the abstract spaces that will be used to study (P) in the variational framework.

Let $1 \leq p < \infty, T > 0$. Consider the Banach space $L^p([0,T],\mathbb{R})$ of functions on $[0,T]$ with values in $\mathbb{R}$ endowed with the norm

$$\|u\|_p = \left(\int_0^T |u(t)|^p \, dt\right)^{1/p}$$

and $L^\infty([0,T],\mathbb{R})$ is the Banach space of essentially bounded functions from $[0,T]$ into $\mathbb{R}$ equipped with the norm

$$\|u\|_\infty = ess \sup_{t\in[0,T]} |u(t)|.$$

Let $\varphi \in C_0^\infty[0,T]$, multiplying (P) by φ and integrating over $[0,T]$ we have

$$\int_0^T {}^{\mathbf{H}}\mathbf{D}_{T-}^{\alpha,\beta;\psi}\left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t)\right)\varphi(t)dt = \int_0^T f(t,u(t))\varphi(t)dt. \tag{3.1}$$

By Theorem 2.2, we get

$$\int_0^T {}^{\mathbf{H}}\mathbf{D}_{T-}^{\alpha,\beta;\psi}\left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t)\right)\varphi(t)dt = \int_0^T \psi'(t){}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t){}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\left(\frac{\varphi(t)}{\psi'(t)}\right)dt.$$

If

$${}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\left(\frac{\varphi(t)}{\psi'(t)}\right) = \frac{1}{\psi'(t)}{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\varphi(t) \ \ \forall t \in [0,T], \tag{3.2}$$

then, (3.1) can be rewritten as

$$\int_0^T {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t){}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\varphi(t)dt = \int_0^T f(t,u(t))\varphi(t)dt. \tag{3.3}$$

Motivated by this equality we introduce the following $\psi-$fractional spaces

Definition 3.1. Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and $1 < p < \infty$. The Left-sided $\psi-$fractional derivative space $\mathbb{H}_p^{\alpha,\beta;\psi} := \mathbb{H}_p^{\alpha,\beta;\psi}([0,T],\mathbb{R})$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ which is given by

$$\begin{aligned}\mathbb{H}_p^{\alpha,\beta;\psi} &= \left\{\begin{array}{c} u \in L^p([0,T],\mathbb{R});\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u \in L^p([0,T],\mathbb{R}), \\ \mathbf{I}_{0+}^{\beta(\beta-1)}u(0) = \mathbf{I}_{T-}^{\beta(\beta-1)}u(T) = 0 \end{array}\right\} \\ &= \overline{C_0^\infty([0,T],\mathbb{R})}\end{aligned} \tag{3.4}$$

with the following norm

$$\|u\|_{\mathbb{H}_p^{\alpha,\beta;\psi}} = \left(\|u\|_{L^p}^p + \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{L^p}^p\right)^{1/p} \tag{3.5}$$

where ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative with $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$.

4. Variational strucuture

The goal of this section is to prove some abstract results for the space $\mathbb{H}_p^{\alpha,\beta;\psi}$.

Proposition 4.1. *Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and $1 < p < \infty$. The fractional derivative space $\mathbb{H}_p^{\alpha,\beta;\psi}$ is a reflexive and separable Banach space.*

Proof. In fact, since $L^p([0,T],\mathbb{R})$ is reflexive and separable, the cartesian product space $L^p([0,T],\mathbb{R}) \times L^p([0,T],\mathbb{R})$ with respect to the norm

$$\|v\|_{L_2^p} = \left(\sum_{i=1}^{2} \|v_i\|_{L^p}^p\right)^{1/p} \tag{4.1}$$

where $v = (v_1, v_2) \in (L^p([0,T],\mathbb{R}))^2$ is also reflexive and separable.

Consider the space $\Omega = \left\{\left(u, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u\right) : u \in \mathbb{H}_p^{\alpha,\beta;\psi}\right\}$ which is a closed subset of $(L^p([0,T],\mathbb{R}))^2$ as $\mathbb{H}_p^{\alpha,\beta;\psi}$ is closed. Therefore, Ω is also reflexive and separable Banach space with respect to the norm (4.1) for $v = (v_1, v_2) \in \Omega$.

We form the operator $\mathcal{A} : \mathbb{H}_p^{\alpha,\beta;\psi} \to \Omega$ given by $\mathcal{A}(u) =: \left(u, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u\right)$, $u \in \mathbb{H}_p^{\alpha,\beta;\psi}$. Thus it follows that $\|u\|_{\mathbb{H}_p^{\alpha,\beta;\psi}} = \|\mathcal{A}u\|_{L_2^p}$, which means that the operator $\mathcal{A} : u \to \left(u, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u\right)$ is a isometric isomorphic mapping and the space $\mathbb{H}_p^{\alpha,\beta;\psi}$ is isometric to the space Ω. Thus $\mathbb{H}_p^{\alpha,\beta;\psi}$ is a reflexive and separable Banach space and this completes the proof. □

The proof of Lemma 4.1 can be obtained by adapting the arguments of [16, Lemma 3.1].

Lemma 4.1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$ and suppose that ψ' is increasing in $[0,T]$. Then, for any $f \in L^p([0,T],\mathbb{R})$, we have*

$$\left\| I_{0+}^{\alpha;\psi} f \right\|_{L^p([0,t])} \leq \frac{(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p([0,t])} \tag{4.2}$$

for $t \in [0,T]$.

Proof. For $p = 1$, since ψ' is increasing, we get

$$\begin{aligned}
\|I_{0+}^{\alpha;\psi} f\|_{L^1([0,t])} &= \int_0^t \left| \frac{1}{\Gamma(\alpha)} \int_0^\xi \psi'(s)[\psi(\xi) - \psi(s)]^{\alpha-1} u(s) ds \right| d\xi \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\xi \psi'(\xi)[\psi(\xi) - \psi(s)]^{\alpha-1} |u(s)| ds d\xi \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t |u(s)| \int_s^t \psi'(\xi)[\psi(\xi) - \psi(s)]^{\alpha-1} d\xi ds \\
&\leq \frac{(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \|u\|_{L^1([0,t])}.
\end{aligned} \tag{4.3}$$

Suppose that $1 < p < \infty$ and $g \in L^q([0,T],\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Since ψ' is increasing and

$$\psi(\xi) - \psi(s) = \int_s^\xi \psi'(\sigma) d\sigma,$$

then we have

$$[\psi'(\xi)(\xi-s)]^{\alpha-1} \leq \left(\int_s^{\xi} \psi'(\sigma)d\sigma\right)^{\alpha-1} \leq [\psi'(s)(\xi-s)]^{\alpha-1}.$$

So, by doing the change of variable $\sigma=\xi-s$ we get

$$\begin{aligned}
&\left|\int_0^t g(\xi)\int_0^{\xi}\psi'(s)(\psi(\xi)-\psi(s))^{\alpha-1}f(s)\,dsd\xi\right| \\
&\leq \left|\int_0^t\int_0^{\xi} g(\xi)[\psi'(s)]^{\alpha}(\xi-s)^{\alpha-1}g(\xi)f(s)dsd\xi\right| \\
&= \left|\int_0^t\int_{\sigma}^{t}[\psi'(\xi)]^{\alpha}\sigma^{\alpha-1}f(\xi-\sigma)g(\xi)d\xi d\sigma\right| \\
&= \int_0^t[\psi'(t)]^{\alpha}\sigma^{\alpha-1}\int_{\sigma}^{t}|f(\xi-\sigma)||g(\xi)|d\xi d\sigma \leq \frac{[\psi'(t)t]^{\alpha}}{\alpha}\|f\|_{L^p([0,t])}\|g\|_{L^q([0,t])}
\end{aligned}$$

for $t\in[0,T]$.

Now, consider the functional $\mathbf{H}_{\xi*_{\psi}f}$ for any fixed $t\in[0,T]$, given by

$$\mathbf{H}_{\xi*_{\psi}f}(g)=\int_0^t\left[\int_0^{\xi}\psi'(s)(\psi(\xi)-\psi(s))^{\alpha-1}f(s)\,ds\right]g(\xi)\,d\xi. \tag{4.4}$$

According to (4.4), it is obvious that $\mathbf{H}_{\xi*_{\psi}f}\in(L^q([0,T],\mathbb{R}))^*$ where $(L^q([0,T],\mathbb{R}))^*$ denotes the dual space of $L^q([0,T],\mathbb{R})$. Therefore, by inequalities (4.4) and (4.4) and Riesz representation theorem, there exists $h\in L^p([0,T],\mathbb{R})$ such that

$$\int_0^t h(\xi)g(\xi)\,d\xi=\int_0^t\left[\int_0^{\xi}\psi'(s)(\psi(\xi)-\psi(s))^{\alpha-1}f(s)\,ds\right]g(\xi)\,d\xi \tag{4.5}$$

and

$$\|h\|_{L^p([0,t])}\leq\frac{(\psi'(t)t)^{\alpha}}{\alpha}\|f\|_{L^p([0,t])} \tag{4.6}$$

for all $g\in L^q([0,T],\mathbb{R})$.

Hence, we have by (4.5)

$$\frac{1}{\Gamma(\alpha)}h(\xi)=\frac{1}{\Gamma(\alpha)}\int_0^{\xi}\psi'(s)(\psi(\xi)-\psi(s))^{\alpha-1}f(s)\,ds=I_{0+}^{\alpha;\psi}$$

for $\xi\in[0,t]$, which means that

$$\left\|I_{0+}^{\alpha;\psi}f\right\|_{L^p([0,t])}=\frac{1}{\Gamma(\alpha)}\|h\|_{L^p([o,t])}\leq\frac{(\psi'(t)t)^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p([0,t])}. \tag{4.7}$$

Combining (4.3) and (4.7), we obtain inequality (4.2). □

Proposition 4.2. *Consider $0<\alpha\leq1$, $0\leq\beta\leq1$, $[\psi'(t)]^q\leq\psi'(t)$ for all $t\in[0,T]$ and all $q\geq1$ with $1<p\leq\infty$. For all $u\in\mathbb{H}_p^{\alpha,\beta;\psi}$, if $\alpha>1/p$ it holds that $\mathbf{I}_{0+}^{\alpha;\psi}\left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t)\right)=u(t)$. Moreover, the inclusion $\mathbb{H}_p^{\alpha,\beta;\psi}\subset C([0,T],\mathbb{R})$ holds.*

Proof. Consider $\frac{1}{p}+\frac{1}{q}=1$, $0\leq t_1<t_2\leq T$ and $u\in L^p([0,T],\mathbb{R})$. Using the Hölder inequality and the fact that $\alpha>1/p$ we have

$$\begin{aligned}
&\left|\mathbf{I}_{0+}^{\alpha;\psi}u(t_1)-\mathbf{I}_{0+}^{\alpha;\psi}u(t_2)\right| \\
&=\frac{1}{\Gamma(\alpha)}\left|\begin{array}{l}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}u(s)\,ds\\ -\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}u(s)\,ds\\ +\int_{t_1}^{t_2}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}u(s)\,ds\end{array}\right| \\
&\leq\frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}\psi'(s)^q\left[(\psi(t_1)-\psi(s))^{\alpha-1}-(\psi(t_2)-\psi(s))^{\alpha-1}\right]^q ds\right)^{\frac{1}{q}} \\
&\quad\times\left(\int_0^{t_1}|u(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\quad+\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}\psi'(s)^q(\psi(t_2)-\psi(s))^{(\alpha-1)q}\,ds\right)^{\frac{1}{q}}\left(\int_{t_1}^{t_2}|u(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\leq\frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}\psi'(s)\left[(\psi(t_1)-\psi(s))^{(\alpha-1)q}-(\psi(t_2)-\psi(s))^{(\alpha-1)q}\right]ds\right)^{\frac{1}{q}}\|u\|_{L^p} \\
&\quad+\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{(\alpha-1)q}\,ds\right)^{\frac{1}{q}}\|u\|_{L^p} \\
&=\frac{1}{\Gamma(\alpha)}\left(\frac{(\psi(t_1)-\psi(0))^{(\alpha-1)q+1}}{(\alpha-1)q+1}-\frac{(\psi(t_2)-\psi(0))^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right)^{\frac{1}{q}}\|u\|_{L^p} \\
&\quad+\frac{1}{\Gamma(\alpha)}\left(\frac{(\psi(t_2)-\psi(t_1))^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right)^{\frac{1}{q}}\|u\|_{L^p} \\
&\leq\frac{2(\psi(t_2)-\psi(t_1))^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)[(\alpha-1)q+1]^{\frac{1}{q}}}\|u\|_{L^p}.
\end{aligned}\tag{4.8}$$

Applying (4.8), we obtain the continuity of $\mathbf{I}_{0+}^{\alpha;\psi}\left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t)\right)$ in $[0,T]$. From Theorem 2.1 we have

$$\mathbf{I}_{0+}^{\alpha;\psi}\left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t)\right)=u(t)+C(\psi(t)-\psi(0))^{\gamma-1}\tag{4.9}$$

$t\in[0,T]$.

Since $u(0)=0$ and $\mathbf{I}_{0+}^{\alpha;\psi}\left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t)\right)$ is continuous in $[0,T]$. Thus it follows that $C=0$, which implies $\mathbf{I}_{0+}^{\alpha;\psi}\left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t)\right)=u(t)$. The result is proved. □

Remark 4.1. In the case that $1-\alpha\geq 1/p$, for $u\in\mathbb{H}_p^{\alpha,\beta;\psi}$, we also have $\mathbf{I}_{0+}^{\alpha;\psi}\left({}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u(t)\right)=u(t)$. In fact, set $f(t)=\mathbf{I}_{0+}^{1-\alpha;\psi}u(t)$. According to Theorem 2.1 we only need to prove that $f(0)=\left[\mathbf{I}_{0+}^{1-\alpha;\psi}u(t)\right]_{t=0}=0$.

In the next result we prove that $\left[\mathbf{I}_{0+}^{1-\alpha;\psi}u(t)\right]_{t=0}=0$ in the case $1-\alpha\geq 1/p$.

Lemma 4.2. *Let* $0<\alpha<1$, $0\le\beta\le 1$, $u\in\mathbb{H}_p^{\alpha,\beta;\psi}\left([0,T],\mathbb{R}\right)$.
Then $\left[\mathbf{I}_{0+}^{1-\alpha;\psi}u\left(t\right)\right]_{t=0}=0$.

Proof. Let $u\in\mathbb{H}_p^{\alpha,\beta;\psi}\left([0,T],\mathbb{R}\right)$, then there is $\varphi_n\in C_0^{\infty}([0,T],\mathbb{R})$ such that

$$\|u-\varphi_n\|_{\mathbb{H}_p^{\alpha,\beta;\psi}}\to 0 \text{ as } n\to+\infty,$$

from where

$$\|u-\varphi_n\|_{L^p([0,t])}\to 0 \text{ as } n\to+\infty.$$

Hence, by Lemma 4.1

$$\begin{aligned}\|\mathbf{I}_{0+}^{1-\alpha;\psi}u\|_{L^p([0,t])}&\le\|\mathbf{I}_{0+}^{1-\alpha;\psi}(u-\varphi_n)\|_{L^p([0,t])}+\|\mathbf{I}_{0+}^{1-\alpha;\psi}\varphi_n\|_{L^p([0,t])}\\&\le\frac{(\psi(t)-\psi(0))^{1-\alpha}}{\Gamma(2-\alpha)}\|u-\varphi_n\|_{L^p([0,t])}+\frac{(\psi(t)-\psi(0))^{1-\alpha}}{\Gamma(2-\alpha)}\|\varphi_n\|_{L^p([0,t])}\\&\to 0 \text{ as } n\to+\infty \text{ and } t\to 0^+.\end{aligned}$$

Therefore

$$\left[\mathbf{I}_{0+}^{1-\alpha;\psi}u(t)\right]_{t=0}=0.$$

□

Proposition 4.3. *Let* $0<\alpha\le 1$, $0\le\beta\le 1$ *and* $1<p<\infty$. *If* $1-\alpha\ge 1/p$ *or* $\alpha>1/p$, *we have*

$$\|u\|_{L^p}\le\frac{\left(\psi\left(T\right)-\psi\left(0\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{L^p},\tag{4.10}$$

for all $u\in\mathbb{H}_p^{\alpha,\beta;\psi}$. *Moreover, if* $\alpha>1/p$ *and* $\dfrac{1}{p}+\dfrac{1}{q}=1$, *then*

$$\|u\|_{\infty}\le\frac{\left(\psi\left(T\right)-\psi\left(0\right)\right)^{\alpha-1/p}}{\Gamma\left(\alpha\right)\left(\left(\alpha-1\right)q+1\right)^{1/q}}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{L^p},\tag{4.11}$$

where $\|u\|_{\infty}=\sup\limits_{t\in[0,T]}|u\left(t\right)|$.

Proof. In order to obtain (4.10) and (4.11) it will be proved that

$$\left\|\mathbf{I}_{0+}^{\alpha;\psi}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{L^p}\le\frac{\left(\psi\left(T\right)-\psi\left(0\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{L^p}\tag{4.12}$$

for $1-\alpha>1$ or $\alpha>1/p$ and

$$\left\|\mathbf{I}_{0+}^{\alpha;\psi}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{L^p}\le\frac{\left(\psi\left(T\right)-\psi\left(0\right)\right)^{\alpha-1/p}}{\Gamma\left(\alpha\right)\left(\left(\alpha-1\right)q+1\right)^{1/q}}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{L^p}\tag{4.13}$$

for $\alpha>1/p$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Since ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\in L^p\left([0,T],\mathbb{R}\right)$ it follows from Lemma 4.1 that

$$\left\|\mathbf{I}_{0+}^{\alpha;\psi}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{L^p}\le\frac{\left(\psi\left(T\right)-\psi\left(0\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|_{L^p}.$$

Suppose that $\alpha > 1/p$. Choose $q > 0$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Using the Hölder inequality, we have for all $u \in \mathbb{H}_p^{\alpha,\beta;\psi}$ that

$$\begin{aligned}\left| \mathbf{I}_{0+}^{\alpha;\psi}\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right| &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^T \psi'(s) \left(\psi(T) - \psi(s)\right)^{(\alpha-1)q} ds \right)^{1/q} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right\|_{L^p} \\ &= \frac{\left(\psi(T) - \psi(0)\right)^{\alpha-1/p}}{\Gamma(\alpha)\left[q(\alpha-1)+1\right]^{1/q}} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right\|_{L^p}.\end{aligned}$$

□

According to inequality (4.10) the norms in $\mathbb{H}_p^{\alpha,\beta;\psi}$

$$\|u\|_{\mathbb{H}_p^{\alpha,\beta;\psi}} = \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right\|_{L^p} \tag{4.14}$$

and (4.14) are equivalent.

Note that by choosing $p = 2$ in Definition (3.4) we have that the space $\mathbb{H}_2^{\alpha,\beta;\psi}$ becomes a Hilbert space when endowed with the norm (4.14) and the inner product

$$\|u\|_{\mathbb{H}_2^{\alpha,\beta;\psi}} = \int_0^T {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u(t)\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} v(t) dt$$

respectively.

Proposition 4.4. *Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and $1 < p < \infty$. Assume that $\alpha > 1/p$ and let $\{u_k\}$ be a sequence that converges weakly to u in $\mathbb{H}_p^{\alpha,\beta;\psi}$. Then, for a subsequence it holds that $u_k \to u$ in $C([0,T],\mathbb{R})$, i.e., $\|u - u_k\|_\infty = 0$ as $k \to \infty$.*

Proof. Recall that if $\alpha > 1/p$ then

$$\|u\|_\infty \leq \frac{\left(\psi(T) - \psi(0)\right)^{\alpha-1/p}}{\Gamma(\alpha)\left((\alpha-1)q+1\right)^{1/q}} \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right\|_{L^p} \tag{4.15}$$

and

$$\|u\|_{\mathbb{H}_p^{\alpha,\beta;\psi}} = \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right\|_{L^p} = \left(\int_0^T \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u(t) \right|^p dt \right)^{1/p}. \tag{4.16}$$

Thus from (4.15) and (4.16) it follows that if $u_k \to u$ in $\mathbb{H}_p^{\alpha,\beta;\psi}$, then $u_k \to u$ in $C([0,T],\mathbb{R})$. Since $u_k \rightharpoonup u$ in $\mathbb{H}_p^{\alpha,\beta;\psi}$, it follows that $u_k \rightharpoonup u$ in $C([0,T],\mathbb{R})$. In fact, for any $h \in (C([0,T],\mathbb{R}))^*$, if $u_k \to u$ in $\mathbb{H}_p^{\alpha,\beta;\psi}$ then $u_k \to u$ in $C([0,T],\mathbb{R})$ and thus $h(u_k) \to h(u)$. Therefore, $h \in \left(\mathbb{H}_p^{\alpha,\beta;\psi}\right)^*$, which means that $(C([0,T],\mathbb{R}))^* \subset \left(\mathbb{H}_p^{\alpha,\beta;\psi}\right)^*$.

Hence, if $u_k \rightharpoonup u$ in $\mathbb{H}_p^{\alpha,\beta;\psi}$, then for any $h \in (C([0,T],\mathbb{R}))^*$, we have $h \in \left(\mathbb{H}_p^{\alpha,\beta;\psi}\right)^*$ and thus $h(u_k) \to h(u)$ i.e., $u_k \to u$ in $C([0,T],\mathbb{R})$. By the Banach-Steinhaus theorem, $\{u_k\}$ is bounded in $\mathbb{H}_p^{\alpha,\beta;\psi}$ and hence, in $C([0,T],\mathbb{R})$.

We claim that $\{u_k\}$ is equi-uniformly continuous. Let $q > 0$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $0 \leq t_1 < t_2 \leq T$ and $u \in \mathbb{H}_p^{\alpha,\beta;\psi}$. The Hölder inequality combined with (4.8) we have

$$\left| \mathbf{I}_{0+}^{\alpha;\psi} u(t_1) - \mathbf{I}_{0+}^{\alpha;\psi} u(t_2) \right| \leq \frac{2\left(\psi(t_2) - \psi(t_1)\right)^{\alpha-1/p}}{\Gamma(\alpha)\left((\alpha-1)q+1\right)^{1/q}} \|u\|_{L^p}. \tag{4.17}$$

By applying (4.16) and (4.17) we have

$$\begin{aligned}\left|u_k\left(t_1\right)-u_k\left(t_2\right)\right| &\leq \frac{2\left(\psi\left(t_2\right)-\psi\left(t_1\right)\right)^{\alpha-1/p}}{\Gamma\left(\alpha\right)\left(\left(\alpha-1\right)q+1\right)^{1/a}}\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u_k\right\|_{L^p}\\ &\leq C\left(\psi\left(t_2\right)-\psi\left(t_1\right)\right)^{\alpha-1/p}, \qquad (4.18)\end{aligned}$$

where $C>0$ is a constant. Thus by using the Arzela-Ascoli theorem, $\{u_k\}$ is relatively compact in $C\left([0,T],\mathbb{R}\right)$. By the uniqueness of the weak limit in $C\left([0,T],\mathbb{R}\right)$, every uniformly convergent subsequence of $\{u_k\}$ converges uniformly on $[0,T]$ to u. □

Remark 4.2. Let $\frac{1}{p}<\alpha\leq 1$, if $u\in\mathbb{H}_p^{\alpha,\beta;\psi}$, then $u\in L^q\left[0,T\right]$, for $q\in[p,+\infty]$, in fact

$$\int_0^T\left|u\left(t\right)\right|^q dt\leq\|u\|_\infty^{q-p}\|u\|_{L^p}^p. \qquad (4.19)$$

In particular the embedding $\mathbb{H}_p^{\alpha,\beta;\psi}\hookrightarrow L^q\left([0,1]\right)$ is continuous for all $q\in[p,+\infty]$.

Below we point out some examples regarding the previous definition.

1. Taking the limit $\beta\to 1$ in (3.4), we have the fractional derivative space given by

$$\mathbb{H}_p^{\alpha;\psi}=\mathbb{H}_p^{\alpha,1;\psi}=\left\{\begin{array}{l}u\in L^p\left([0,T],\mathbb{R}\right);\ {}^{C}D_{0+}^{\alpha;\psi}u\in L^p\left([0,T],\mathbb{R}\right),\\ u\left(0\right)=u\left(T\right)=0\end{array}\right\}$$

   with the following norm

$$\|u\|_{\mathbb{H}_p^{\alpha;\psi}}=\left(\|u\|_{L^p}^p+\left\|{}^{C}D_{0+}^{\alpha;\psi}y\right\|_{L^p}^p\right)^{1/p},$$

   where ${}^{C}D_{0+}^{\alpha;\psi}\left(\cdot\right)$ is the ψ-Caputo fractional derivative with $0<\alpha\leq 1$.

2. Taking the limit $\beta\to 1$ in (3.4) and choosing $\psi\left(t\right)=t$, we have the fractional derivative space given by

$$\mathbb{H}_p^{\alpha}=\mathbb{H}_p^{\alpha,1;t}=\left\{\begin{array}{l}u\in L^p\left([0,T],\mathbb{R}\right);\ {}^{C}D_{0+}^{\alpha}u\in L^p\left([0,T],\mathbb{R}\right),\\ u\left(0\right)=u\left(T\right)=0\end{array}\right\}$$

   with the following norm

$$\|u\|_{\mathbb{H}_p^{\alpha}}=\left(\|u\|_{L^p}^p+\left\|{}^{C}D_{0+}^{\alpha}u\right\|_{L^p}^p\right)^{1/p},$$

   where ${}^{C}D_{0+}^{\alpha;\psi}\left(\cdot\right)$ is the Caputo fractional derivative with $0<\alpha\leq 1$.

3. Taking the limit $\beta\to 0$ in (3.4), we have the fractional derivative space given by

$$\mathbb{H}_p^{\alpha;\psi}=\mathbb{H}_p^{\alpha,0;\psi}=\left\{\begin{array}{l}u\in L^p\left([0,T],\mathbb{R}\right);\ {}^{RL}\mathcal{D}_{0+}^{\alpha;\psi}u\in L^p\left([0,T],\mathbb{R}\right),\\ u\left(0\right)=u\left(T\right)=0\end{array}\right\}$$

with the following norm

$$\|u\|_{\mathbb{H}_p^{\alpha;\psi}} = \left(\|u\|_{L^p}^p + \left\|{}^{RL}\mathcal{D}_{0+}^{\alpha;\psi}u\right\|_{L^p}^p\right)^{1/p},$$

where ${}^{RL}\mathcal{D}_{0+}^{\alpha;\psi}(\cdot)$ is the ψ-Riemann-Liouville fractional derivative with $0<\alpha\leq 1$.

4. Taking the limit $\beta \to 0$ in (3.4) and $\psi(t) = t$ we have the fractional derivative space given by

$$\mathbb{H}_p^{\alpha} = \mathbb{H}_p^{\alpha,0;t}([0,T],\mathbb{R}) = \left\{\begin{array}{l} u \in L^p([0,T],\mathbb{R});{}^{RL}\mathcal{D}_{0+}^{\alpha}u \in L^p([0,T],\mathbb{R}), \\ u(0) = u(T) = 0 \end{array}\right\}$$

with the following norm

$$\|y\|_{\mathbb{H}_p^{\alpha}} = \left(\|u\|_{L^p}^p + \left\|{}^{RL}\mathcal{D}_{0+}^{\alpha}u\right\|_{L^p}^p\right)^{1/p},$$

where ${}^{RL}D_{0+}^{\alpha;\psi}(\cdot)$ is the Riemann-Liouville fractional derivative with $0<\alpha\leq 1$.

5. Taking $\psi(t) = t$ in (3.4), we have the fractional derivative space given by

$$\mathbb{H}_p^{\alpha,\beta} = \mathbb{H}_p^{\alpha,\beta;t} = \left\{\begin{array}{l} u \in L^p([0,T],\mathbb{R});\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta}u \in L^p([0,T],\mathbb{R}), \\ \mathbf{I}_{0+}^{\beta(\beta-1)}u(0) = \mathbf{I}_{T-}^{\beta(\beta-1)}u(T) = 0 \end{array}\right\}$$

with the following norm

$$\|u\|_{\mathbb{H}_p^{\alpha;\beta}} = \left(\|u\|_{L^p}^p + \left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta}y\right\|_{L^p}^p\right)^{1/p},$$

where ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta}(\cdot)$ is the Hilfer fractional derivative with $0<\alpha\leq 1$ and $0\leq\beta\leq 1$.

6. Taking the limit $\beta \to 1$ in (3.4) and $\psi(t) = t^{\rho}$, we have the fractional derivative space given by

$$\mathbb{H}_p^{\alpha} = \mathbb{H}_p^{\alpha,1;t^{\rho}} = \left\{\begin{array}{l} u \in L^p([0,T],\mathbb{R});\ {}^{KC}D_{0+}^{\alpha}u \in L^p([0,T],\mathbb{R}), \\ u(0) = u(T) = 0 \end{array}\right\}$$

with the following norm

$$\|u\|_{\mathbb{H}_p^{\alpha}} = \left(\|u\|_{L^p}^p + \left\|{}^{KC}D_{0+}^{\alpha}u\right\|_{L^p}^p\right)^{1/p},$$

where ${}^{KC}D_{0+}^{\alpha}(\cdot)$ is the Caputo-Katugampola fractional derivative with $0<\alpha\leq 1$.

7. Since the ψ-Hilfer fractional derivative admits a vast class of fractional derivative as particular cases, from the choice of ψ and the limit with $\beta \to 0$ or $\beta \to 1$, it is possible to contract other fractional derivative spaces for their respective fractional derivative.

5. Fractional nonlinear Dirichlet problem

The goal of this section is to prove the existence of solution for the fractional nonlinear Dirichlet problem (P). The notion of solution that will be considered is given below.

Definition 5.1. A function $u \in \mathbb{H}_2^{\alpha,\beta;\psi}$ is a weak solution for (P) if

$$\int_0^T {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u(t)\ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} v(t)\, dt = \int_0^T f(t, u(t))\, v(t)\, dt$$

for all $v \in \mathbb{H}_2^{\alpha,\beta;\psi}$.

Consider the energy functional given by

$$\mathcal{A}(u) = \frac{1}{2}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u(t)\right|^2 dt - \int_0^T \mathcal{H}(t, u(t))\, dt, u \in \mathbb{H}_2^{\alpha,\beta;\psi}, \tag{5.1}$$

where $\mathcal{H}$ is the primitive of f, that is, $\mathcal{H}(t,s) = \int_0^s f(t,\xi)\, d\xi$. Following [25] we have $\mathcal{A} \in C^1\left(\mathbb{H}_2^{\alpha,\beta;\psi}, \mathbb{R}\right)$ with

$$\mathcal{A}'(u)\, v = \int_0^T {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u(t)^{\mathbf{H}}\, \mathbf{D}_{0+}^{\alpha,\beta;\psi} v(t)\, dt - \int_0^T f(t, u(t))\, v(t)\, dt,$$

for $u, v \in \mathbb{H}_2^{\alpha,\beta;\psi}$. Thus, the solutions of (P) are given by the critical points of $\mathcal{A}$.

The result of this section is provided below.

Theorem 5.1. *Let $1/2 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and suppose that f satisfy (f_1) and (f_2). Then problem (P) has a nontrivial weak solution $u \in \mathbb{H}_2^{\alpha,\beta;\psi}$.*

The next two results can be found in [41].

Lemma 5.1. *If f satisfies the condition (f_2), then for every $t \in [0,T]$ the following inequalities hold*

$$\mathcal{H}(t,u) \leq \mathcal{H}\left(t, \frac{u}{|u|}\right) |u|^{\mu}, \ \text{ if } 0 < |u| \leq 1 \tag{5.2}$$

and

$$\mathcal{H}(t,u) \geq \mathcal{H}\left(t, \frac{u}{|u|}\right) |u|^{\mu}, \ \text{ if } |u| \geq 1 \tag{5.3}$$

Lemma 5.2. *Suppose that f satisfies the condition (f_2). Let*

$$m = \inf\left\{\mathcal{H}(t,u)\, / t \in [0,T], |u| = 1\right\}.$$

Then

$$\int_0^T \mathcal{H}(t, \xi u(t))\, dt \geq m\, |\xi|^{\mu} \int_0^T |u(t)|^{\mu}\, dt - Tm,$$

for all $\xi \in \mathbb{R}\backslash\{0\}$ and $u \in \mathbb{H}_2^{\alpha,\beta;\psi}$.

Lemma 5.3. *Suppose that f satisfies the conditions $(f_1) - (f_2)$. The functional given by (5.1) satisfy the Palais-Smale condition.*

Proof. To prove that $\mathcal{A}$ satisfy the Palais-Smale condition let $\{u_k\}$ be a sequence in $\mathbb{H}_2^{\alpha,\beta;\psi}$ such that

$$|\mathcal{A}(u_k)| \leq M, \ \lim_{k\to\infty} \mathcal{A}'(u_k) = 0. \tag{5.4}$$

First it will be proved that $\{u_k\}$ is bounded. Note that

$$\mathcal{A}(u_k) = \frac{1}{2}\|u_k(t)\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}} - \int_0^T \mathcal{H}(t, u_k(t))\,dt$$

and

$$\mathcal{A}'(u_k)\,u_k = \frac{1}{2}\|u_k(t)\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}} - \int_0^T f(t, u_k(t))\,u_k(t)\,dt.$$

Then by (5.4) we get

$$\begin{aligned}\left|\mathcal{A}(u_k) - \frac{1}{\mu}A'(u_k)\,u_k\right| &\leq |\mathcal{A}(u_k)| + \left|\frac{1}{\mu}\mathcal{A}'(u_k)\right||u_k| \\ &\leq M\left(1 + \|u_k(t)\|_{\mathbb{H}_2^{\alpha,\beta;\psi}}\right).\end{aligned} \tag{5.5}$$

On the other hand we have from (f_2) that

$$\begin{aligned}&\mathcal{A}(u_k) - \frac{1}{\mu}\mathcal{A}'(u_k)\,u_k \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}} - \int_0^T \mathcal{H}(t, u_k(t))\,dt - \frac{1}{\mu}\int_0^T f(t, u_k(t))\,u_k(t)\,dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}}\end{aligned} \tag{5.6}$$

Then by (5.5) and (5.6) we obtain that

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}} \leq \mathcal{A}(u_k) - \frac{1}{\mu}\mathcal{A}'(u_k)\,u_k \leq M\left(1 + \|u_k\|_{\mathbb{H}_2^{\alpha,\beta;\psi}}\right).$$

Since $\mu > 2$ it follows that $\{u_k\}$ is bounded in $\mathbb{H}_2^{\alpha,\beta;\psi}$.

From Proposition 4.1 we have that $\mathbb{H}_2^{\alpha,\beta;\psi}$ is reflexive space. Thus going to a subsequence if necessary, we may assume that $u_k \rightharpoonup u$ in $\mathbb{H}_2^{\alpha,\beta;\psi}$. Therefore

$$\begin{aligned}&\langle \mathcal{A}'(u_k) - \mathcal{A}'(u), u_k - u\rangle \\ &= \langle \mathcal{A}'(u_k), u_k - u\rangle - \langle \mathcal{A}'(u), u_k - u\rangle \\ &\leq \|\mathcal{A}'(u_k)\|\,\|u_k - u\|_{\mathbb{H}_2^{\alpha,\beta;\psi}} - \langle \mathcal{A}'(u), u_k - u\rangle.\end{aligned} \tag{5.7}$$

Taking limit with $k \to \infty$ on both sides of the inequality (5.7) we get

$$\langle \mathcal{A}'(u_k) - \mathcal{A}'(u), u_k - u\rangle \to 0.$$

From Propositions 4.3 and 4.4, we get that u_k is bounded in $C([0,T])$ and we can also assume that

$$\lim_{k\to\infty} \|u_k - u\|_\infty = 0.$$

Therefore,

$$\int_0^T \left[f\left(t,u_k\left(t\right)\right)-f\left(t,u\left(t\right)\right)\right]\left(u_k\left(t\right)-u\left(t\right)\right)dt\to 0$$

as $k\to\infty$.

Note that

$$\begin{aligned}&\left\langle \mathcal{A}'\left(u_k\right)-\mathcal{A}'\left(u\right),u_k-u\right\rangle\\ &=\left\|u_k-u\right\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}}-\int_0^T \left[f\left(t,u_k\left(t\right)\right)-f\left(t,u\left(t\right)\right)\right]\left(u_k\left(t\right)-u\left(t\right)\right)dt.\end{aligned}$$

So $\left\|u_k-u\right\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}}\to 0$ as $k\to\infty$, that is, $\{u_k\}$ converges strongly to u in $\mathbb{H}_2^{\alpha,\beta;\psi}$. □

Proof of Theorem 5.1. The arguments consists in use Theorem 2.3. Note that ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}0=0$ and

$$\mathcal{A}\left(0\right)=\frac{1}{2}\int_0^T\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}0\right|^2dt-\int_0^T\mathcal{H}\left(t,0\right)dt=0.$$

By using (4.11) we obtain that

$$\left\|u\right\|_\infty\le \widetilde{C}\left\|u\right\|_{\mathbb{H}_2^{\alpha,\beta;\psi}}, \tag{5.8}$$

where $\widetilde{C}:=\dfrac{\left(\psi\left(T\right)-\psi\left(0\right)\right)^{\alpha-1/2}}{\Gamma\left(\alpha\right)\left(\alpha-1\right)^{1/2}}$.

Consider $\widetilde{C}_1=\dfrac{1}{\widetilde{C}}$. If $\left\|u\right\|_{\mathbb{H}_2^{\alpha,\beta;\psi}}\le\widetilde{C}_1$, then it follows from (5.2) and (5.8) that

$$\begin{aligned}\int_0^T\mathcal{H}\left(t,u\left(t\right)\right)dt&\le\int_0^T\mathcal{H}\left(t,\frac{u\left(t\right)}{\left|u\left(t\right)\right|}\right)\left|u\left(t\right)\right|^\mu dt\\ &\le\left\|u\right\|^\mu_{L^\mu}\int_0^T\mathcal{H}\left(t,\frac{u\left(t\right)}{\left|u\left(t\right)\right|}\right)dt\\ &\le\left[\frac{\left(\psi\left(T\right)-\psi\left(0\right)\right)^{\alpha-1/2}}{\Gamma\left(\alpha\right)\left(\alpha-1\right)^{1/2}}\right]^\mu M\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right\|^\mu_{L^2}\\ &=C^\mu M\left\|u\right\|^\mu_{L^2}. \end{aligned}\tag{5.9}$$

Then

$$\begin{aligned}\mathcal{A}\left(u\right)&=\frac{1}{2}\left\|u\right\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}}-\int_0^T\mathcal{H}\left(t,u\left(t\right)\right)dt\\ &\ge\frac{1}{2}\left\|u\right\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}}-MC^\mu\left\|u\right\|^\mu_{\mathbb{H}_2^{\alpha,\beta;\psi}}.\end{aligned}\tag{5.10}$$

If $\left\|u\right\|_{\mathbb{H}_2^{\alpha,\beta;\psi}}\le C_1$ we have

$$\mathcal{A}\left(u\right)\ge\frac{1}{2}\left\|u\right\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}}-MC^\mu\frac{1}{2}\left\|u\right\|^\mu_{\mathbb{H}_2^{\alpha,\beta;\psi}}$$

$$= \frac{C_1^2}{2} - MC^{\mu}C_1^{\mu}. \tag{5.11}$$

Consider $\rho < \min\left\{C_1, \left(\frac{1}{2MC^{\mu}}\right)^{1/\mu-2}\right\}$ and $\widetilde{\gamma} = \frac{\rho^2}{2} - MC^{\mu}\rho^{\mu}$. Therefore $\mathcal{A}(u) \geq \rho$ for $\|u\|_{\mathbb{H}_2^{\alpha,\beta;\psi}} = \rho$. Thus the first part of Theorem 2.3 is verified.

Fix $u \in \mathbb{H}_2^{\alpha,\beta;\psi} \backslash \{0\}$. Consider $\xi \in \mathbb{R} \backslash \{0\}$ and define the sets

$$A = \{t \in [0,T] /\ |\xi u(t)| \leq 1\},$$

$$B = \{t \in [0,T] /\ |\xi u(t)| \geq 1\}.$$

From Lemma 5.1 we obtain that

$$\begin{aligned}
\mathcal{A}(\xi u) &\leq \frac{\xi}{2}\|u\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}} - \int_A \mathcal{H}(t, \xi u(t))\,dt \\
&\leq \frac{\xi}{2}\|u\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}} - m\int_A |\xi|^{\mu}|u(t)|^{\mu}\,dt \\
&\leq \frac{\xi}{2}\|u\|^2_{\mathbb{H}_2^{\alpha,\beta;\psi}} - m\int_0^T |\xi|^{\mu}|u(t)|^{\mu}\,dt + mT.
\end{aligned}$$

For ξ large enough we have $\mathcal{A}(e) \leq 0$. Therefore $\mathcal{A}$ satisfies the second part of Theorem 2.3 which proves the result. □

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References

[1] R. P. Agarwal, M. B. Ghaemi and S. Saiedinezhad, *The Nehari manifold for the degenerate p-Laplacian quasilinear elliptic equations*, Adv. Math. Sci. Appl., 2010, 20(1), 37–50.

[2] R. Almeida, *Further properties of Osler's generalized fractional integrals and derivatives with respect to another function*, Rocky Mountain J. Math., 2019, 49(8), 2459–2493.

[3] A. Ambrosetti, *Critical Points and Nonlinear Variational Problems*, Bull. Soc. Math. France ,120, Memoire 49, 1992.

[4] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical points theory and applications*, J. Func. Anal., 1973, 14, 349–381.

[5] L. Bai, B. Dai and F. Li, *Solvability of second-order Hamiltonian systems with impulses via variational method*, Appl. Math. Comput., 2013, 219(14), 7542–7555.

[6] A. Benhassine, *Existence of infinitely many solutions for a class of fractional Hamiltonian systems*, J. Elliptic Parabolic Equ., 2019, 5(1), 105–123.

[7] A. Boucenna and T. Moussaoui, *Existence of a positive solution for a boundary value problem via a topological-variational theorem*, J. Fract. Calc. Appl., 2014, 5(3S), 1–9.

[8] L. Bourdin, *Existence of a weak solution for fractional Euler–Lagrange equations*, J. Math. Anal. Appl., 2013, 399(1), 239–251.

[9] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer Science & Business Media, 2010.

[10] J. M. Carcione and F. Mainardi, *On the relation between sources and initial conditions for the wave and diffusion equations*, Comput. Math. Appl., 2017, 73(6), 906–913.

[11] G. Chai and J. Chen, *Existence of solutions for impulsive fractional boundary value problems via variational method*, Boundary Value Probl., 2017, 2017(1), 1–120.

[12] G. Cruz, A. Mendez and C. E. Torres Ledesma, *Multiplicity of solutions for fractional Hamiltonian systems with Liouville-Weyl fractional derivatives*, Frac. Calc. Appl. Anal., 2015, 18(4), 875–890.

[13] M. Ferrara and A. Hadjian, *Variational approach to fractional boundary value problems with two control parameters*, Elect. J. Diff. Equ., 2015, 2015(138), 1–15.

[14] G. J. Fix and J. P. Roof, *Least squares finite-element solution of a fractional order two-point boundary value problem*, Comput. Math. Appl., 2004, 48(7), 1017–1033.

[15] H. Hassani, J. A. Tenreiro Machado, Z. Avazzadeh and E. Naraghirad, *Generalized shifted Chebyshev polynomials: Solving a general class of nonlinear variable order fractional PDE*, Commun. Nonlinear Sci. Numer. Simul., 2020, 85, 105229.

[16] F. Jiao and Y. Zhou, *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Comput. Math. Appl., 2011, 62(3), 1181–1199.

[17] F. Jiao and Y. Zhou, *Existence results for fractional boundary value problem via critical point theory*, Inter. J. Bifur. Chaos, 2012, 22(04), 1250086.

[18] M. Karkulik, *Variational formulation of time-fractional parabolic equations*, Comput. Math. Appl., 2018, 75(11), 3929–3938.

[19] A. A. Kilbas, O. I. Marichev and S. G. Samko, *Fractional integral and derivatives (theory and applications)*, 1993.

[20] Y. Li and B. Dai, *Existence and multiplicity of nontrivial solutions for Liouville–Weyl fractional nonlinear Schrödinger equation*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matematicas, 2018, 112(4), 957–967.

[21] W. Liu, M. Wang and T. Shen, *Analysis of a class of nonlinear fractional differential models generated by impulsive effects*, Boundary Value Probl., 2017, 2017(1), 175.

[22] F. Mainardi, *The two forms of fractional relaxation of distributed order*, J. Vibr. Control, 2007, 13(9), 1249–1268.

[23] N. Nyamoradi and S. Tersian, *Existence of solutions for nonlinear fractional order p-Laplacian differential equations via critical point theory*, Fract. Calc. Appl. Anal., 2019, 22(4), 945–967.

[24] M. D. Ortigueira and J. Tenreiro Machado, *On the properties of some operators under the perspective of fractional system theory*, Commun. Nonlinear Sci. Numer. Simul., 2020, 82, 105022.

[25] P. Rabinowitz, *Minimax method in critical point theory with applications to differential equations*, CBMS Amer. Math. Soc., 65, 1986.

[26] R. Servadei and E. Valdinoci, *Weak and viscosity solutions of the fractional Laplace equation*, Publicacions Matematiques, 2014, 58(1), 133–154.

[27] C. J. Silva and Delfim F. M. Torres, *Stability of a fractional HIV/AIDS model*, Math. Comput. Simul., 2019, 164, 180–190.

[28] J. V. Sousa and E. Capelas de Oliveira, *Leibniz type rule: ψ-Hilfer fractional operator*, Commun. Nonlinear Sci. Numer. Simul. 2019, 77, 305–311.

[29] J. V. Sousa and E. Capelas de Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul., 2018, 60, 72–91.

[30] J. V. Sousa, E. Capelas de Oliveira and L. A. Magna, *Fractional calculus and the ESR test*, AIMS Math., 2017, 2(4), 692–705.

[31] J. V. Sousa, M. N. N. dos Santos, L. A. Magna and E. Capelas de Oliveira, *Validation of a fractional model for erythrocyte sedimentation rate*, Comput. Appl. Math., 2018, 37(5), 6903–6919.

[32] H. Sun and Q. Zhang, *Existence of solutions for a fractional boundary value problem via the Mountain Pass method and an iterative technique*, Comput. Math. Appl., 2012, 64(10), 3436–3443.

[33] C. E. Torres Ledesma, *Boundary value problem with fractional p-Laplacian operator*, Adv. Nonlinear Anal., 2016, 5(2), 133–146.

[34] C. E. Torres Ledesma, *Existence and concentration of solutions for a non-linear fractional Schrödinger equation with steep potential well*, Commun. Pure Appl. Anal., 2016, 15, 535–547.

[35] C. E. Torres Ledesma, *Existence of solution for a general fractional advection–dispersion equation*, Anal. Math. Phys., 2019, 9(3), 1303–1318.

[36] C. E. Torres Ledesma, *Existence of solutions for fractional Hamiltonian systems with nonlinear derivative dependence in* $\mathbb{R}$, J. Fract. Calc. Appl., 2016, 7(2), 74–87.

[37] C. E. Torres Ledesma, *Existence of a solution for the fractional forced pendulum*, J. Appl. Math. Comput. Mechanics, 2014, 13(1), 125–142.

[38] C. E. Torres Ledesma, *Existence and symmetric result for Liouville-Weyl fractional nonlinear Schrödinger equation*, Commun. Nonlinear Sci. Numer. Simul., 2015, 27(1), 314–327.

[39] C. E. Torres Ledesma and N. Nyamoradi, *Impulsive fractional boundary value problem with p-Laplace operator*, J. Appl. Math. Comput., 2017, 55(1), 257–278.

[40] C. E. Torres Ledesma and O. Pichardo, *Multiplicity of Solutions for a Class of Perturbed Fractional Hamiltonian Systems*, Bull. Malaysian Math. Sci. Soc., 2020, 43, 3897–3922.

[41] C. E. Torres Ledesma, *Mountain pass solution for a fractional boundary value problem*, J. Frac. Cal. Appl., 2012, 5(1), 1–10.

[42] Y. Wang, L. Liu and Y. Wu, *Positive solutions for a class of fractional boundary value problem with changing sign non-linearity*, Nonlinear Anal., Theory, Meth. Appl., 2011, 74(17), 6434–6441.

[43] D. Wu, C. Li and P. Yuan, *Multiplicity Solutions for a Class of Fractional Hamiltonian Systems With Concave–Convex Potentials*, Mediterr. J. Math., 2018, 15(2), 35.

[44] Z. Xie, Y. Jin and C. Hou, *Multiple solutions for a fractional difference boundary value problem via variational approach*, Abst. Appl. Anal., 2012, 2012.

[45] J. Xu, D. O'Regan and K. Zhang, *Multiple solutions for a class of fractional Hamiltonian systems*, Frac. Calc. Appl. Anal., 2015, 18(1), 48–63.

[46] X. Zhang, L. Liu and Y. Wu, *Variational structure and multiple solutions for a fractional advection–dispersion equation*, Comput. Math. Appl., 2014, 68(1), 1794–1805.

[47] Y. Zhao, H. Chen and B. Qin, *Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods*, Appl. Math. Comput., 2015, 257, 417–427.

[48] Z. Zhang, and C. E. Torres Ledesma, *Solutions for a class of fractional Hamiltonian systems with a parameter*, J. Appl. Math. Comput., 2017, 54(1), 451–468.