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KAMAL TRANSFORM AND ULAM STABILITY OF DIFFERENTIAL EQUATIONS

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Abstract In the growth of the field of functional-differential equations and their Ulam stability, many researchers have utilized various methods to prove the Ulam stability of functional and differential equations. Hyers method and the fixed-point method are remarkably applied by many researchers to investigate the Ulam stability of functional and differential equations. In this research work, we propose a new method for investigating the Ulam stability of linear differential equations by using Kamal transform.

Keywords Kamal transform, generalized Hyers-Ulam stability, differential equation.

MSC(2010) 44A10, 39B82, 34A40, 26D10.

1. Introduction and preliminaries

Very often instead of a functional equation, we consider a functional inequality and one can ask the following question: When can one assert that the solutions of the inequality lie near to the solutions of the equation? A definition of stability in the case of homomorphisms between groups was suggested by a problem posed by Ulam [30] in 1940: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d\,(.,.)$. Given $\epsilon > 0$, does there exists a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d\,(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d\,(h)(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, if a mapping is almost homomorphism then there is a true homomorphism near it with small error as much as possible. If the answer is affirmative, we would call that the equation $H(x * y) = H(x) \diamond H(y)$ of homomorphism is stable.

In 1941, Hyers [8] was the first mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam for the cases where $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be Banach spaces. The result of Hyers is stated in the succeeding celebrated theorem.

Theorem 1.1. *Assume that $\mathcal{G}$ and $\mathcal{H}$ are Banach spaces. If a mapping $g : \mathcal{G} \to \mathcal{H}$ fulfills the inequality*

$$\|g(u + v) - g(u) - g(v)\| \leq \epsilon \tag{1.1}$$

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for some $\epsilon > 0$ *and for all* $u, v \in \mathcal{G}$*, then the limit*

$$\mathcal{A}(u) = \lim_{n\to\infty} \frac{1}{2^n} g(2^n u) \tag{1.2}$$

is the unique additive mapping such that

$$\|g(u) - \mathcal{A}(u)\| \leq \epsilon \tag{1.3}$$

for all $u \in \mathcal{G}$.

Based on the above said outcome, one can finalize that the additive functional equation $g(u+v) = g(u) + g(v)$ has Hyers-Ulam stability on $(\mathcal{X}, \mathcal{Y})$. In the above Theorem 1.1, an additive function $\mathcal{A}$ is created directly from the given function g which also fulfills (1.3) and it is most dominant technique to investigate the stability of several functional equations. Hyers theorem was indiscriminated by Aoki [2] in 1950 for additive mappings. See [1, 14–16, 19] for more information on functional equations and their stability.

The generalization of Ulam's problem was recently proposed by replacing functional euations with differential equations. Obloza seems to be the first author who proved the Ulam stability of differential equation in [23]. Thereafter, Alsina and Ger [3] published their papers, which handle the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$. The result obtained by Alsina and Ger was generalized by Takahasi *et al.* [27] to the case of the complex Banach space valued differential equation. Recently, many researchers have investigated the Hyers-Ulam stability of ordinary differential equations (see [5–7, 9–13, 17, 18, 20–22, 24–26, 28, 29]. Alqifiary and Jung [4] have proved the generalized Hyers-Ulam stability of linear differential equations by applying the Laplace transform method.

In this paper, our main aim is to study the Hyers-Ulam stability of the first order homogeneous linear differential equation of the form

$$p'(t) + \mu p(t) = 0 \tag{1.4}$$

and the non-homogeneous linear differential equation

$$p'(t) + \mu p(t) = r(t) \tag{1.5}$$

by applying Kamal transform method, where μ is a scalar, $p(t)$ and $r(t)$ are continuously differentiable functions.

Throughout this paper, $\mathbb{F}$ denotes the real field $\mathbb{R}$ or complex field $\mathbb{C}$.

A function $f : (0, \infty) \to \mathbb{F}$ is of exponential order if there exist constants $A, B \in \mathbb{R}$ such that $|f(t)| \leq Ae^{tB}$ for all $t > 0$. For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order, let us consider the set $\mathcal{A}$

$$\mathcal{A} = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0 \ |f(t)| < Me^{|t|/k_j},\ t \in (-1)^j \times [0, \infty) \right\}$$

where the constant M must be finite, while τ_1 and τ_2 may be infinite. The Kamal transform is defined by

$$G(u) = \mathcal{K}\{f(t)\} = \int_0^\infty f(t) e^{-t/u} dt, \qquad t \geq 0, \quad u \in (-\tau_1, \tau_2)$$

where the variable u in the Kamal transform is used to factor the variable t in the argument of the function f, specially for $f(t)$ in $\mathcal{A}$.

Definition 1.1 (Convolution of two functions)**.** Let f and g be Lebesgue integrable functions on $(-\infty, +\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(x) = \int_{-\infty}^{\infty} f(t)\ g(x-t)\ dt$$

exists. This integral defines a function h on S called the convolution of f and g. We also write $h = f * g$ to denote this function.

Now, we give the definitions of Hyers-Ulam stability and generalized Hyers-Ulam stability of the differential equations (1.4) and (1.5).

Definition 1.2. The linear differential equation (1.4) is said to have the Hyers-Ulam stability if there exists a constant $K > 0$ satisfying the following property: If for every $\epsilon > 0$, there exists a continuously differentiable function $p(t)$ satisfying the inequality $|p'(t) + \mu p(t)| \leq \epsilon$, then there exists some $q : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.4) and $|p(t) - q(t)| \leq K\epsilon$ for all $t > 0$. We call such K as the Hyers-Ulam stability constant for the differential equation (1.4).

Definition 1.3. We say that the non-homogeneous linear differential equation (1.5) has the Hyers-Ulam stability if there exists a continuously differentiable function $p(t)$ satisfying the following condition: If for every $\epsilon > 0$, there exists a positive constant K such that $|p'(t) + \mu p(t) - r(t)| \leq \epsilon$, then there exists a solution $q : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.5) and $|p(t) - q(t)| \leq K\epsilon$ for all $t > 0$. We call such K as the Hyers-Ulam stability constant for the differential equation (1.5).

Definition 1.4. We say that the homogeneous linear differential equation (1.4) has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ satisfying the following property: For every $\epsilon > 0$ and a continuously differentiable function $p(t)$, if there exists $\phi : (0, \infty) \to (0, \infty)$ satisfying the inequality $|p'(t) + \mu p(t)| \leq \phi(t)\epsilon$, then there exists some $q : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.4) and $|p(t) - q(t)| \leq K\ \phi(t)\epsilon$ for all $t > 0$. We call such K as the generalized Hyers-Ulam stability constant for the differential equation (1.4).

Definition 1.5. The differential equation (1.5) is said to have the generalized Hyers-Ulam stability if there exists a positive constant K satisfying the following condition: For every $\epsilon > 0$, there exists a continuously differentiable function $x(t)$, if $\phi : (0, \infty) \to (0, \infty)$ satisfies the inequality $|p'(t) + \mu p(t) - r(t)| \leq \phi(t)\epsilon$, then there exists a solution $q : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.5) and $|p(t) - q(t)| \leq K\ \phi(t)\epsilon$ for all $t > 0$. We call such K as the generalized Hyers-Ulam stability constant for the differential equation (1.5).

2. Hyers-Ulam stability of differential equations

In this section, we prove the Hyers-Ulam stability of the homogeneous and non-homogeneous linear differential equations (1.4) and (1.5) by using Kamal transform. Firstly, we prove the Hyers-Ulam stability of the first order homogeneous differential equation (1.4) by using Kamal transform method.

Theorem 2.1. *Let μ be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that if $p : (0, \infty) \to \mathbb{F}$ is a continuously differentiable function satisfying the inequality*

$$|p'(t) + \mu p(t)| \leq \epsilon \tag{2.1}$$

for all $t > 0$, then there exists a solution $q : (0, \infty) \to \mathbb{F}$ of the differential equation (1.4) such that $|p(t) - q(t)| \leq K\epsilon$ for all $t > 0$.

Proof. Assume that $p(t)$ is a continuously differentiable function satisfying the inequality (2.1). Let us define a function $z : (0, \infty) \to \mathbb{F}$ such that $z(t) =: p'(t) + \mu p(t)$ for all $t > 0$. In view of (2.1), we have $|z(t)| \leq \epsilon$. Now, taking Kamal transform to $z(t)$, we have

$$Z(u) = \mathcal{K}\{z(t)\} = \mathcal{K}\{p'(t) + \mu p(t)\} = \mathcal{K}\{p'(t)\} + \mu\, \mathcal{K}\{p(t)\} = \frac{P(u)}{u} - p(0) + \mu P(u).$$

Thus

$$\mathcal{K}\left\{p(t)\right\} = P(u) = \frac{uZ(u)}{1 + \mu u} + \frac{up(0)}{1 + \mu u}. \tag{2.2}$$

Set $q(t) = e^{-\mu t} p(0)$. Then $p(0) = q(0)$. Taking Kamal transform to $q(t)$, we get

$$\mathcal{K}\left\{q(t)\right\} = Q(u) = \frac{up(0)}{1 + \mu u}. \tag{2.3}$$

Thus $\mathcal{K}\left\{p'(t) + \mu p(t)\right\} = \mathcal{K}\left\{p'(t)\right\} + \mu \mathcal{K}\left\{p(t)\right\} = \dfrac{Q(u)}{u} - q(0) + \mu Q(u)$. Using (2.3), we have $\mathcal{K}\{p'(t) + \mu\ p(t)\} = 0$. Since $\mathcal{K}$ is a one-to-one operator, $q'(t) + \mu q(t) = 0$. Hence $q(t)$ is a solution of the differential equation (1.4). So $G(u) = \dfrac{u}{1 + \mu u}$. Then the equality $\mathcal{K}\left\{g(t)\right\} = \dfrac{u}{1 + \mu u}$ implies that $g(t) = \mathcal{K}^{-1}\left\{\dfrac{u}{1 + \mu u}\right\}$. Moreover, by (2.2) and (2.3), we obtain

$$\begin{aligned}\mathcal{K}\left\{p(t)\right\} - \mathcal{K}\left\{q(t)\right\} = P(u) - Q(u) = \frac{uZ(u)}{1 + \mu u} &= Z(u)G(u) = \mathcal{K}\left\{z(t)\right\}\mathcal{K}\left\{g(t)\right\} \\ &= \mathcal{K}\left\{p(t) - q(t)\right\} = \mathcal{K}\left\{z(t) * g(t)\right\},\end{aligned}$$

which gives $p(t) - q(t) = z(t) * g(t)$. Taking modulus on both sides, we have

$$|p(t) - q(t)| = |z(t) * g(t)| = \left|\int_{-\infty}^{\infty} z(t)\ g(t - x)\ dx\right| \leq |z(t)|\ \left|\int_{-\infty}^{\infty} q(t - x)\ dx\right| \leq K\epsilon.$$

Here $K = \left|\int_{-\infty}^{\infty} q(t - x)\ dx\right|$ and the integral exists for each value of t. Thus the homogeneous linear differential equation (1.4) has the Hyers-Ulam stability. □

Theorem 2.2. *Let μ be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that if $p : (0, \infty) \to \mathbb{F}$ is a continuously differentiable function satisfying the inequality*

$$|p'(t) + \mu p(t) - r(t)| \leq \epsilon \tag{2.4}$$

for all $t > 0$, then there exists a solution $q : (0, \infty) \to \mathbb{F}$ of the non-homogeneous differential equation (1.5) such that $|p(t) - q(t)| \leq K\epsilon$ for all $t > 0$.

Now, we prove the Hyers-Ulam stability of the first order non-homogeneous differential equation (1.5) by using Kamal transform method.

Proof. Assume that $p(t)$ is a continuously differentiable function satisfying the inequality (2.4). Let us define a function $z : (0, \infty) \to \mathbb{F}$ such that $z(t) =: p'(t) +$

$\mu p(t) - r(t)$ for all $t > 0$. In view of (2.4), we have $|z(t)| \le \epsilon$. Now, taking Kamal transform to $z(t)$, we have $\mathcal{K}\{z(t)\} = \mathcal{K}\{p'(t) + \mu p(t) - r(t)\}$ and

$$Z(u) = \mathcal{K}\{p'(t)\} + \mu \ \mathcal{K}\{p(t)\} - \mathcal{K}\{r(t)\} = \frac{P(u)}{u} - p(0) + \mu P(u) - R(u).$$

Thus

$$\mathcal{K}\left\{z(t)\right\} = Z(u) = \frac{u}{1+\mu u} Z(u) + \frac{up(0)}{1+\mu u} + \frac{u}{1+\mu u} R(u). \tag{2.5}$$

Set $q(t) = e^{-\mu t} p(0) + (r(t) * q(t))$. Then $p(0) = q(0)$. Taking Kamal Transform to $q(t)$, we obtain

$$\mathcal{K}\left\{q(t)\right\} = Q(u) = \frac{up(0)}{1+\mu u} + R(u)G(u). \tag{2.6}$$

Thus $\mathcal{K}\left\{p'(t) + \mu p(t)\right\} = \mathcal{K}\left\{p'(t)\right\} + \mu \mathcal{K}\left\{p(t)\right\} = \dfrac{Q(u)}{u} - q(0) + \mu Q(u)$. Using (2.6), we have $\mathcal{K}\{p'(t) + \mu \ p(t)\} = R(u) = \mathcal{K}\left\{r(t)\right\}$. Since $\mathcal{K}$ is a one-to-one operator, $q'(t) + \mu q(t) = r(t)$. Hence $q(t)$ is a solution of the differential equation (1.5). Let $G(u) = \dfrac{u}{1+\mu u}$. Then the equality $\mathcal{K}\left\{g(t)\right\} = \dfrac{u}{1+\mu u}$ implies that $g(t) = \mathcal{K}^{-1}\left\{\dfrac{u}{1+\mu u}\right\}$. Moreover, by (2.5) and (2.6), we obtain

$$\begin{aligned} \mathcal{K}\left\{p(t)\right\} - \mathcal{K}\left\{q(t)\right\} = P(u) - Q(u) = \frac{uZ(u)}{1+\mu u} &= Z(u)G(u) = \mathcal{K}\left\{z(t)\right\}\mathcal{K}\left\{g(t)\right\} \\ &= \mathcal{K}\left\{p(t) - q(t)\right\} = \mathcal{K}\left\{z(t) * g(t)\right\}, \end{aligned}$$

which gives $p(t) - q(t) = z(t) * g(t)$. Taking modulus on both sides, we have

$$|p(t) - q(t)| = |z(t) * g(t)| = \left|\int_{-\infty}^{\infty} z(t) \ g(t-x) \ dx\right| \le |z(t)| \ \left|\int_{-\infty}^{\infty} q(t-x) \ dx\right| \le K\epsilon.$$

Here $K = \left|\int_{-\infty}^{\infty} q(t-x) \ dx\right|$ and the integral exists for each value of t. Thus the homogeneous linear differential equation (1.5) has the Hyers-Ulam stability. □

3. Generalized Hyers-Ulam stability of differential equations

In this scetion, we prove the generalized Hyers-Ulam stability of the differential equations (1.4) and (1.5). Firstly, we prove the generalized Hyers-Ulam stability of the first order non-homogeneous differential equation (1.4) by using Kamal transform method.

Theorem 3.1. *Let μ be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that if $p : (0, \infty) \to \mathbb{F}$ is a continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ is an integrable function satisfying*

$$|p'(t) + \mu p(t)| \le \phi(t)\epsilon \tag{3.1}$$

for all $t > 0$, then there exists a solution $q : (0, \infty) \to \mathbb{F}$ of the homogeneous differential equation (1.4) such that $|p(t) - q(t)| \le K \ \phi(t)\epsilon$ for all $t > 0$.

Proof. Assume that $p(t)$ is a continuously differentiable function satisfying the inequality (3.1). Let us define a function $z : (0,\infty) \to \mathbb{F}$ such that $z(t) =: p'(t) + \mu p(t)$ for all $t > 0$. In view of (3.1), we have $|z(t)| \le \phi(t)\epsilon$. Now, taking Kamal Transform to $z(t)$, we have

$$\mathcal{K}\{p(t)\} = P(u) = \frac{uZ(u)}{1+\mu u} + \frac{up(0)}{1+\mu u}. \tag{3.2}$$

Set $q(t) = e^{-\mu t}p(0)$. Then $p(0) = q(0)$. Taking Kamal transform to $q(t)$, we obtain

$$\mathcal{K}\{q(t)\} = Q(u) = \frac{up(0)}{1+\mu u}. \tag{3.3}$$

Thus $\mathcal{K}\{p'(t) + \mu p(t)\} = \mathcal{K}\{p'(t)\} + \mu\mathcal{K}\{p(t)\} = \frac{Q(u)}{u} - q(0) + \mu Q(u)$. Using (3.3), we have $\mathcal{K}\{p'(t) + \mu\ p(t)\} = 0$. Since $\mathcal{K}$ is a one-to-one operator, $q'(t) + \mu q(t) = 0$. Hence $q(t)$ is a solution of the differential equation (1.4). So $G(u) = \frac{u}{1+\mu u}$. Then the equality $\mathcal{K}\{g(t)\} = \frac{u}{1+\mu u}$ implies that $g(t) = \mathcal{K}^{-1}\left\{\frac{u}{1+\mu u}\right\}$. Moreover, by (3.2) and (3.3), we obtain

$$\begin{aligned}\mathcal{K}\{p(t)\} - \mathcal{K}\{q(t)\} = P(u) - Q(u) = \frac{uZ(u)}{1+\mu u} &= Z(u)G(u) = \mathcal{K}\{z(t)\}\mathcal{K}\{g(t)\} \\ &= \mathcal{K}\{p(t) - q(t)\} = \mathcal{K}\{z(t) * g(t)\},\end{aligned}$$

which gives $p(t) - q(t) = z(t) * g(t)$. Similar to the proof of Theorem 2.1, we have $|p(t) - q(t)| \le K\phi(t)\epsilon$, where $K = \left|\int_{-\infty}^{\infty} q(t-x)\ dx\right|$, the integral exists for each value of t and $\phi(t)$ is an integrable function. Thus the differential equation (1.4) has the generalized Hyers-Ulam stability. □

Now, we prove the Hyers-Ulam stability of the non-homogeneous linear differential equation (1.5) by using Kamal transform method.

Theorem 3.2. *Let μ be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that if $p : (0,\infty) \to \mathbb{F}$ is a continuously differentiable function and $\phi : (0,\infty) \to (0,\infty)$ is an integrable function satisfying the condition*

$$|p'(t) + \mu p(t) - r(t)| \le \phi(t)\epsilon \tag{3.4}$$

for all $t > 0$, then there exists a solution $q : (0,\infty) \to \mathbb{F}$ of the non-homogeneous differential equation (1.5) such that $|p(t) - q(t)| \le K\ \phi(t)\epsilon$ for all $t > 0$.

Proof. Assume that $p(t)$ is a continuously differentiable function satisfying the inequality (3.4). Let us define a function $z : (0,\infty) \to \mathbb{F}$ such that $z(t) =: p'(t) + p(t) - r(t)$ for all $t > 0$. In view of (3.4), we have $|z(t)| \le \phi(t)\epsilon$. Now, taking Kamal Transform to $z(t)$, we have

$$\mathcal{K}\{z(t)\} = Z(u) = \frac{u}{1+\mu u}Z(u) + \frac{up(0)}{1+\mu u} + \frac{u}{1+\mu u}R(u). \tag{3.5}$$

Set $q(t) = e^{-\mu t}p(0) + (r(t) * q(t))$. Then $p(0) = q(0)$. Taking Kamal Transform to $q(t)$, we get

$$\mathcal{K}\{q(t)\} = Q(u) = \frac{up(0)}{1+\mu u} + R(u)G(u). \tag{3.6}$$

Thus $\mathcal{K}\{p'(t)+\mu p(t)\} = \mathcal{K}\{p'(t)\}+\mu\mathcal{K}\{p(t)\} = \frac{Q(u)}{u} - q(0) + \mu Q(u)$. Using (3.6), we have $\mathcal{K}\{p'(t)+\mu\ p(t)\} = R(u) = \mathcal{K}\{r(t)\}$. Since $\mathcal{K}$ is a one-to-one operator, $q'(t)+\mu q(t) = r(t)$. Hence $q(t)$ is a solution of the differential equation (1.5). So $G(u) = \frac{u}{1+\mu u}$. Then the equality $\mathcal{K}\{g(t)\} = \frac{u}{1+\mu u}$ implies that $g(t) = \mathcal{K}^{-1}\left\{\frac{u}{1+\mu u}\right\}$. Moreover, by (3.5) and (3.6), we obtain

$$\mathcal{K}\{p(t)\} - \mathcal{K}\{q(t)\} = P(u) - Q(u) = \frac{uZ(u)}{1+\mu u} = Z(u)G(u) = \mathcal{K}\{z(t)\}\mathcal{K}\{g(t)\}$$
$$\Rightarrow \mathcal{K}\{p(t)-q(t)\} = \mathcal{K}\{z(t)*g(t)\},$$

which gives $p(t)-q(t) = z(t)*g(t)$.

The rest of the proof is similar to the proof of Theorem 3.1. □

4. An example

Consider the non-homogeneous differential equation

$$p'(t)+p(t) = 5\sin t, \qquad p(0) = 1. \tag{4.1}$$

Using Theorem 2.2, we have $|p'(t)+p(t)-5\sin t| \le \epsilon$, where p is a continuously differentiable function. Let $z(t) = p'(t)+p(t)-5\sin t$ for all $t>0$. Then we have $|z(t)| \le \epsilon$. Now, taking Kamal Transform to $z(t)$, we get

$$Z(u) = \frac{P(u)}{u} - p(0) + P(u) - 5\left(\frac{u^2}{1+u^2}\right),$$
$$P(u) = \frac{u}{1+u}\left[Z(u)+1+\frac{5u^2}{1+u^2}\right].$$

Let $G(u) = \frac{u}{(1+u)}$. Then we have $\mathcal{K}\{g(t)\} = \frac{u}{(1+u)}$. Then we have a solution function $y(t) = e^{-t}p(0) + [(5\sin t)*g(t)]$ with $p(0) = q(0)$ and taking Kamal transform to $q(t)$, we get

$$\mathcal{K}\{q(t)\} = Q(u) = \frac{u}{(1+u)} + \frac{5u^2}{1+u^2}Q(u).$$

Also, $\mathcal{K}\{p'(t)+p(t)\} = 5\mathcal{K}\{\sin t\}$. Since $\mathcal{K}$ is a one-to-one operator, $p'(t)+p(t) = 5\sin t$. Hence $q(t)$ is a solution of the differential equation (4.1). Then by Theorem 2.2, we obtain that $|p(t)-q(t)| \le K\epsilon$. Hence the non-homogeneous differential equation (4.1) has the Hyers-Ulam stability.

5. Conclusion

In this paper, we have proposed a new method by using Kamal transform, alternatively, the existing Hyers method and the fixed-point method. Using the prosed new method, we have investigated the Hyers-Ulam stability and the generalized Hyers-Ulam stability of linear differential equations with a suitable example.

Declarations

Availablity of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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