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GENERAL SCHWARZ LEMMAS BETWEEN PSEUDO-HERMITIAN MANIFOLDS AND HERMITIAN MANIFOLDS*

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Abstract From the viewpoint of differential geometry, Schwarz lemmas of distance-decreasing type and volume-decreasing type can be obtained by the estimates of sum functions and product functions of all eigenvalues of holomorphic maps. This paper investigates general Schwarz lemmas by estimating partial sum functions and partial product functions of the eigenvalues of generalized holomorphic maps between pseudo-Hermitian manifolds and Hermitian manifolds.

Keywords General Schwarz lemmas, pseudo-Hermitian manifolds, Hermitian manifolds, CR maps, transversally holomorphic maps.

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1. Introduction

The Schwarz lemma is a principal tool in various branches of mathematics. Motivated by Pick and Ahlfors, differential geometric ideas blend into the study of Schwarz lemma, such as Chern, Lu, Yau, Chen-Cheng-Lu, Royden (cf. [3, 4, 10, 15, 18]). Essentially, these Schwarz-type lemmas can be classified as distance-decreasing property and volume-decreasing property. Suppose that M^{2m} and N^{2m} are two Kähler manifolds with fundamental forms ω_M and ω_N respectively. Let $f : M \to N$ be a holomorphic map. Then the distance-decreasing Schwarz lemma can be evaluated by estimating the density $\langle f^*\omega_N, \omega_M\rangle$ or the largest eigenvalue of $f^*\omega_N$; the volume-decreasing one is obtained by estimating the quotient of volume elements, i.e. $(f^*\omega_N)^m/\omega_M^m$. These two quantities are the sum and product of eigenvalues of $f^*\omega_N$ respectively. Recently, Ni [11, 12] investigated the partial sum and partial product of eigenvalues which leads general Schwarz-type lemmas; as an application, he reveals the relations between l-Ricci curvature and the rank of a holomorphic map between Kähler manifolds. These general Schwarz-type lemma have been generalized to holomorphic maps between Hermitian manifolds by the second author [16].

CR geometry is an odd-dimensional analogue of complex geometry. A CR manifold of hypersurface type which admits a positive definite pseudo-Hermitian structure is called a pseudo-Hermitian manifold. In particular, Sasakian manifolds who

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have strong connections with Kähler geometry are special pseudo-Hermitian manifolds. The classical Schwarz lemmas for the following generalized holomorphic maps (see Section 2 for definitions):

(1) CR maps from pseudo-Hermitian manifolds to Hermitian manifolds,

(2) (J^N, J)-holomorphic maps from Hermitian manifolds to pseudo-Hermitian manifolds,

(3) transversally holomorphic maps between two pseudo-Hermitian manifolds,

(4) CR maps between two pseudo-Hermitian manifolds,

have been established by the first author and his collaborators in [6,7]. In particular, they applied the maps of type (1) and type (4) to explore CR Carathéodory distance and CR Kobayashi hyperbolicity. The present paper is going to discuss the general Schwarz lemmas for types (1)–(3). Our main theorems are as follows:

Theorem 1.1. *Suppose that $(M^{2m+1}, HM, J, \theta)$ is a closed pseudo-Hermitian manifold and (N^{2n}, J^N) is a Hermitian manifold with fundamental form ω_N. Let $f : M \to N$ be a basic CR map and $1 \leq l \leq \min\{m, n\}$.*

(a) If the pseudo-Hermitian l-Ricci curvature of M bounded from below by $-k \leq 0$ and the holomorphic bisectional curvature of N bounded from above by $-K < 0$, then the sum of the l-largest eigenvalues of $f^*\omega_N$ is bounded by k/K.

(b) If the pseudo-Hermitian l-scalar curvature of M bounded from below by $-k \leq 0$ and the first l-Chern-Ricci curvature of N bounded from above by $-K < 0$, then the product of the l-largest eigenvalues of $f^*\omega_N$ is bounded by $\left(\frac{k}{lK}\right)^l$. In particular, if $k = 0$, then $\operatorname{rank}_{\mathbb{R}}(df) \leq 2l$.

Theorem 1.2. *Suppose that (N^{2n}, J^N) is a closed Hermitian manifold with fundamental form ω_N and $(M^{2m+1}, HM, J, \theta)$ is a Sasakian manifold. Let $f : M \to N$ be a (J^N, J)-holomorphic map and $1 \leq l \leq \min\{m, n\}$.*

(a) If the second l-Chern-Ricci curvature N bounded from below by $-k \leq 0$ and the pseudo-Hermitian bisectional curvature of M bounded from above by $-K < 0$, then the sum of the l-largest eigenvalues of f^*G_θ is bounded by k/K.

(b) If the l-Chern-scalar curvature of N bounded from below by $-k \leq 0$ and the pseudo-Hermitian l-Ricci curvature of M bounded from above by $-K < 0$, then the product of the l-largest eigenvalues of f^*G_θ is bounded by $\left(\frac{k}{lK}\right)^l$. In particular, if $k = 0$, then $\operatorname{rank}_{\mathbb{R}}(df) \leq 2l$.

Theorem 1.3. *Suppose that (M^{2m+1}, H, J, θ) is a closed pseudo-Hermitian manifold and $(\tilde{M}^{2n+1}, \tilde{H}, \tilde{J}, \tilde{\theta})$ is a Sasakian manifold. Let $f : M \to \tilde{M}$ be a transversally holomorphic map and $1 \leq l \leq \min\{m, n\}$.*

(a) If the pseudo-Hermitian l-Ricci curvature of M bounded from below by $-k \leq 0$ and the pseudo-Hermitian bisectional curvature of $\tilde{M}$ bounded from above by $-K < 0$, then the sum of the l-largest eigenvalues of $f^*G_{\tilde{\theta}}$ is bounded by k/K.

(b) If the pseudo-Hermitian l-scalar curvature of M bounded from below by $-k \leq 0$ and the pseudo-Hermitian l-Ricci curvature of $\tilde{M}$ bounded from above by $-K <$

0, then the product of the l-largest eigenvalues of $f^*G_{\tilde{\theta}}$ is bounded by $\left(\frac{k}{lK}\right)^l$.
In particular, if $k = 0$, then $\mathrm{rank}_{\mathbb{R}}(df) \leq 2l$.

The present paper will use the moving frame method to avoid the lack of compatible coordinates with CR structure. One can generalize Theorems 1.1–1.3 to complete noncompact domains by maximum principle and some suitable complex Hessian comparison theorems (cf. [6, 7]). Since the pseudo-Hermitian curvature tensor is symmetric bihermitian, then one can also weaken the pseudo-Hermitian bisectional curvature condition to the pseudo-Hermitian sectional curvature condition by Royden's method [15].

For a CR map f from a pseudo-Hermitian manifold (M, H, J, θ) to a pseudo-Hermitian manifold $(N, \tilde{H}, \tilde{J}, \tilde{\theta})$ which is type (4), its pull-back $f^*G_{\tilde{\theta}}$ on $T_{1,0}M$ is diagonal and all eigenvalues are same. Hence its general Schwarz lemmas have been deduced in [7].

2. Preliminaries

In this section, we will briefly introduce the pseudo-Hermitian geometry (cf. [8] for details).

A CR manifold (M, HM, J) is a real odd-dimensional C^∞ manifold with an 1-codimension subbundle HM of TM and an integrable almost complex structure $J \in \mathrm{Aut}(HM)$. Set

$$T_{1,0}M = \{X \in HM \mid JX = \sqrt{-1}X\} \quad \text{and} \quad T_{0,1}M = \overline{T_{1,0}M}. \tag{2.1}$$

The integrable condition is equivalent to $\big[\Gamma(T_{1,0}M), \Gamma(T_{1,0}M)\big] \subset \Gamma(T_{1,0}M)$. A **pseudo-Hermitian manifold**, denoted by (M, HM, J, θ) or $(M, T_{1,0}M, \theta)$, is an orientable CR manifold with a positive pseudo-Hermitian structure θ which satisfies that $HM = \mathrm{Ker}\,\theta$ and its related Levi form

$$L_\theta(X, \bar{Y}) == -\sqrt{-1}d\theta(X, \bar{Y}) \quad \text{for any } X, Y \in T_{1,0}M \tag{2.2}$$

is positive definite.

On a pseudo-Hermitian manifold (M, HM, J, θ), there is a unique globally defined vector field ξ (called the Reeb vector field), with $\xi \lrcorner \theta = 1$ and $\xi \lrcorner d\theta = 0$. It leads the decomposition $TM = HM \oplus \mathbb{R}\xi$ which extends the almost complex structure J to an endomorphism of TM by requiring $J\xi = 0$. Let $\pi_H : TM \to HM$ be the natural projection and

$$G_\theta(X, Y) = d\theta(\pi_H(X), J\pi_H(Y)) \quad \text{for any } X, Y \in TM \tag{2.3}$$

which is J-invariant and symmetric. The Webster metric $g_\theta = G_\theta + \theta \otimes \theta$ is Riemannian. In pseudo-Hermitian geometry, there is a canonical linear connection ∇ (cf. [8]), called Tanaka-Webster connection, which preserves the horizontal bundle HM, the almost complex structure J and the pseudo-Hermitian structure θ; moreover, its torsion satisfies

$$T_\nabla(X, Y) = 2d\theta(X, Y)\xi \quad \text{and} \quad T_\nabla(\xi, JX) + JT_\nabla(\xi, X) = 0. \tag{2.4}$$

The pseudo-Hermitian torsion τ of ∇ is a TM-valued 1-form defined by $\tau(X) = T_\nabla(\xi, X)$ for $X \in TM$. It induces a trace-free symmetric tensor field A given by

$$A(X,Y) = g_\theta(\tau(X), Y). \tag{2.5}$$

A pseudo-Hermitian manifold with vanishing pseudo-Hermitian torsion is Sasakian. Such manifolds are regarded as an odd-dimensional analogue of Kähler manifolds (cf. [2]).

The Levi form can lead a Hermitian metric on $T_{1,0}M$. Let $\{\xi_i\}$ be a local unitary frame of $T_{1,0}M$ and $\{\theta^i\}$ be its dual frame. The structure equations for the Tanaka-Webster connection can be expressed by

$$d\theta^i = \sum_j \theta^j \wedge \theta^i_j + \theta \wedge \tau^i, \quad \theta^i_j + \theta^{\bar{j}}_{\bar{i}} = 0, \quad d\theta^i_j = \sum_k \theta^k_j \wedge \theta^i_k + \Pi^i_j, \tag{2.6}$$

where

$$\Pi^i_j = 2\sqrt{-1}(\theta^i \wedge \tau_j + \theta_j \wedge \tau^i) + \sum_{k,l} R^i_{jk\bar{l}}\theta^k \wedge \theta^{\bar{l}} + \sum_l A^i_{\bar{l},j}\theta \wedge \theta^{\bar{l}} - \sum_k A_{kj,\bar{i}}\theta \wedge \theta^k. \tag{2.7}$$

Set $R_{\bar{i}jk\bar{l}} = R^i_{jk\bar{l}}$. The pseudo-Hermitian curvature tensor is symmetric bihermitian, that is

$$R_{\bar{i}jk\bar{l}} = R_{\bar{i}kj\bar{l}} \quad \text{and} \quad R_{\bar{i}jk\bar{l}} = R_{j\bar{i}\bar{l}k}. \tag{2.8}$$

Let R be the curvature tensor on $T_{1,0}M$ of Tanaka-Webster connection. As holomorphic bisectional curvature, one can define the pseudo-Hermitian bisectional curvature of $X = \sum_i X^i\xi_i, Y = \sum_i Y^i\xi_i \in T_{1,0}M$ by

$$H(X,Y) = \frac{R(\bar{X}, X, Y, \bar{Y})}{|X|^2\ |Y|^2} = \frac{\sum_{i,j,k,t} R_{\bar{i}jk\bar{t}}X^{\bar{i}}X^jY^kY^{\bar{t}}}{(\sum_i X^iX^{\bar{i}})(\sum_j Y^jY^{\bar{j}})}. \tag{2.9}$$

When $X = Y$, it becomes the pseudo-Hermitian sectional curvature (cf. [17]). As l-Ricci curvature in Kähler geometry, we can similarly define the pseudo-Hermitian l-Ricci curvature of an l-dimensional subspace $\Sigma_x \subset T_{1,0}M_x$ at $x \in M$ as follows:

$$\mathrm{Ric}_l(\Sigma_x, X, \bar{Y}) = \mathrm{trace}_{\Sigma_x} R(\bar{Y}, X, \cdot, \bar{\cdot}) = \mathrm{trace}_{\Sigma_x} R(\bar{\cdot}, \cdot, X, \bar{Y}). \tag{2.10}$$

If $\{\xi_i\}_{i=1}^l$ is a unitary frame of Σ_x, then

$$\mathrm{Ric}_l(\Sigma_x, X, \bar{Y}) = \sum_{i,j=1}^m \sum_{k=1}^l R_{\bar{i}jk\bar{k}}Y^{\bar{i}}X^j = \sum_{i,j=1}^m \sum_{k=1}^l R_{\bar{k}kj\bar{i}}X^jY^{\bar{i}} \tag{2.11}$$

where $\dim M = 2m+1$. We say the pseudo-Hermitian l-Ricci curvature bounded from below (above) by k if

$$\mathrm{Ric}_l(\Sigma_x, X, \bar{X}) \geq k|X|^2 \quad (\leq k|X|^2) \quad \text{for all } \Sigma_x \text{ and } X \in \Sigma_x. \tag{2.12}$$

Moreover, the pseudo-Hermitian l-scalar curvature on a l-dimensional subspace $\Sigma_x \subset T_{1,0}M_x$ at $x \in M$ is given by

$$S_l(\Sigma_x) = \sum_{i,j=1}^l R_{\bar{i}ij\bar{j}} \tag{2.13}$$

where $\{\xi_i\}_{i=1}^l$ is a unitary frame of Σ_x. Obviously, when $l = m$, Ric_l and S_l are exactly the classical pseudo-Hermitian Ricci curvature and pseudo-Hermitian scalar curvature respectively.

Suppose that (N, J^N, ω_N) is a Hermitian manifold with fundamental form ω_N. Let $\{\eta_\alpha\}$ be a local unitary frame of $T_{1,0}N$ and $\{\omega^\alpha\}$ be its dual. The structure equations of Chern connection are

$$d\omega^\alpha = \omega^\beta \wedge \omega_\beta^\alpha + \frac{1}{2} T^\alpha_{\beta\gamma} \omega^\beta \wedge \omega^\gamma,\ \omega_\beta^\alpha + \omega_{\bar\alpha}^{\bar\beta} = 0,\ d\omega_\beta^\alpha = \omega_\beta^\gamma \wedge \omega_\gamma^\alpha + R^N_{\bar\alpha\beta\gamma\bar\rho} \omega^\gamma \wedge \omega^{\bar\rho} \tag{2.14}$$

where T and R^N are the torsion tensor and curvature tensor of Chern connection. The holomorphic bisectional curvature of $Z = \sum_\alpha Z^\alpha \eta_\alpha, W = \sum_\alpha W^\alpha \eta_\alpha \in T_{1,0}N$ is given by

$$H^N(Z, W) = \frac{R^N(\bar Z, Z, W, \bar W)}{|Z|^2\ |W|^2} = \frac{\sum_{\alpha,\beta,\gamma,\rho} R^N_{\bar\alpha\beta\gamma\bar\rho} Z^{\bar\alpha} Z^\beta W^\gamma W^{\bar\rho}}{(\sum_\alpha Z^\alpha Z^{\bar\alpha})(\sum_\beta W^\beta W^{\bar\beta})}. \tag{2.15}$$

For $l \le n = \dim_{\mathbb{C}} N$, let Σ_y be an l-dimensional subspace of $T_{1,0}N_y$ at $y \in N$ and $\{\eta_\alpha\}_{\alpha=1}^l$ be a unitary frame of Σ_y. The first l-Chern-Ricci curvature and second l-Chern-Ricci curvature of Σ_y are defined by

$$\mathrm{Ric}_l^{(1)}(\Sigma_y, Z, \bar W) = \mathrm{trace}_{\Sigma_y} R^N(\bar\cdot, \cdot, Z, \bar W) = \sum_{\alpha,\beta=1}^n \sum_{\gamma=1}^l R^N_{\bar\gamma\gamma\alpha\bar\beta} Z^\alpha W^{\bar\beta}, \tag{2.16}$$

$$\mathrm{Ric}_l^{(2)}(\Sigma_y, Z, \bar W) = \mathrm{trace}_{\Sigma_y} R^N(\bar W, Z, \cdot, \bar\cdot) = \sum_{\alpha,\beta=1}^n \sum_{\gamma=1}^l R^N_{\bar\beta\alpha\gamma\bar\gamma} Z^\alpha W^{\bar\beta}, \tag{2.17}$$

respectively; the l-Chern-scalar curvature of Σ_y is defined by

$$S_l^N(\Sigma_y) = \sum_{\alpha,\beta=1}^l R^N_{\bar\alpha\alpha\beta\bar\beta}. \tag{2.18}$$

In Kähler geometry, $\mathrm{Ric}_l^{(1)}$ and $\mathrm{Ric}_l^{(2)}$ are same; one can refer to the references [11–13] for more discussion of l-curvatures.

Next we will introduce the generalized holomorphic maps in pseudo-Hermitian geometry. In the birth of CR geometry, mathematicians found that the function theory of strictly pseudoconvex domains in $\mathbb{C}^n$ has strong connection with that of their boundaries which inspired the study of CR functions of CR manifolds (cf. [1]). The natural generalization of CR functions is CR maps.

Definition 2.1. Suppose that (M, HM, J) is a CR manifold and (N, J^N) is a complex manifold. A smooth map $f : M \to N$ is called **CR** if $df \circ J = J^N \circ df$ holds on HM.

It is notable that under a local holomorphic variables of N, the components of a CR map are CR functions. Assume that (M, HM, J, θ) is a pseudo-Hermitian manifold with Reeb vector field ξ. A map $f : M \to N$ is said to be **basic** if $df(\xi) = 0$. Due to the extension of J, basic CR maps from pseudo-Hermitian manifolds to complex manifolds are also called (J, J^N)-holomorphic maps which is an important object of Siu-type rigidity theorem (cf. [5, 9]).

Definition 2.2. Suppose that (N, J^N) is a complex manifold and (M, HM, J, θ) is a pseudo-Hermitian manifold. A smooth map $f : M \to N$ is called $\boldsymbol{(J^N, J)}$-**holomorphic** if $df \circ J^N = J \circ df$ holds on TN. Here J is the extended almost complex structure.

It is too strong to require a (J^N, J)-holomorphic map to be horizontal, that is $df(TN) \subset HM$, since it makes the image of f lying in a single fiber (cf. [6]).

Definition 2.3. Suppose that (M, H, J, θ) and $(\tilde{M}, \tilde{H}, \tilde{J}, \tilde{\theta})$ are two pseudo-Hermitian manifolds. Let $f : M \to \tilde{M}$ be a smooth map.

(1) f is said to be **CR** if $df(H) \subset \tilde{H}$ and $df \circ J = \tilde{J} \circ df$ holds on H;

(2) f is said to be **transversally holomorphic** if $df \circ J = \tilde{J} \circ df$ holds on TM.

Suppose that $f : (M, H, J, \theta) \to (\tilde{M}, \tilde{H}, \tilde{J}, \tilde{\theta})$ is a CR map between two pseudo-Hermitian manifolds. Due to the preservation of horizontal bundles and complex structure, we have

$$f^*\tilde{\theta} = \lambda\theta \quad \text{and then} \quad f^*G_{\tilde{\theta}} = \lambda G_\theta \tag{2.19}$$

where $\lambda = \tilde{\theta}(df(\xi))$. Hence, all eigenvalues of $f^*G_{\tilde{\theta}}$ are same. For CR maps between two pseudo-Hermitian manifolds, the analogous Schwarz lemma to Theorem 1.3 can be directly obtained by the method of Theorem 3.6 in [7] which is for noncompact case.

3. Proof of Main Theorems

In this section, since the proofs of Theorem 1.1–1.3 are similar, we will give the details of the first one and omit the latter two.

Suppose that $(M^{2m+1}, HM, J, \theta)$ is a pseudo-Hermitian manifold and (N^{2n}, J^N) is a Hermitian manifold with fundamental form ω_N. Let $f : M \to N$ be a CR map and $l \le \min\{m, n\}$ be a positive integer. Since the restriction $f^*\omega_N$ on $T_{1,0}M$ is positive semi-definite, then all of its eigenvalues values at $x \in M$ can be listed as follows:

$$\lambda_1(x) \ge \lambda_2(x) \ge \cdots \ge \lambda_m(x) \ge 0. \tag{3.1}$$

Hence the sum function and product function of l-largest eigenvalues are

$$\sigma_l(x) = \sum_{i=1}^{l} \lambda_i(x) \quad \text{and} \quad p_l(x) = \prod_{i=1}^{l} \lambda_i(x) \tag{3.2}$$

respectively. Let $G_l(M)$ be the Grassmann l-plane bundle of $T_{1,0}M$ and $\pi_G : G_l(M) \to M$ be the projection. Then we find that

$$\sigma_l(x) = \max_{\Sigma_x \in \pi_G^{-1}(x)} \operatorname{trace}_{\Sigma_x} f^*\omega_N \tag{3.3}$$

which leads the continuity of σ_l. Moreover, letting

$$E = \left\{ e \in \wedge^l T_{1,0}M \;\middle|\; |e| = 1 \right\} \tag{3.4}$$

and the natural projection $\pi_E : E \to M$, the product function can be reformulated by

$$p_l(x) = \max_{e \in \pi_E^{-1}(x)} \big(f^*\omega_N\big)^l(e, \bar{e}) \tag{3.5}$$

which implies the continuity of p_l.

Let $\{\xi_i\}$ be a local unitary frame of $T_{1,0}M$ and $\{\theta^i\}$ be its dual. Let $\{\eta_\alpha\}$ be a local unitary frame of $T_{1,0}N$ and $\{\omega^\alpha\}$ be its dual. Denote by $\{f^\alpha_A\}$, $\{f^\alpha_{AB}\}$ and $\{f^\alpha_{ABC}\}$ the components of df, ∇df and $\nabla^2 df$ under these frames respectively. In particular, we use the index "0" to denote the covariant derivative along the Reeb vector field ξ. Using the structure equations of Tanaka-Webster connection and Chern connection, the authors in [6] have deduced the commutation relations of high-order covariant derivatives for (J, J^N)-holomorphic maps which are just basic CR maps.

Lemma 3.1 (Equations (3.6), (3.7) and (3.10) in [6])**.** *Suppose that*

$$f : (M, HM, J, \theta) \to (N, J^N, \omega_N)$$

is a basic CR map. Then

$$f^\alpha_{\bar{i}} = 0, \quad f^\alpha_0 = 0, \tag{3.6}$$

$$f^\alpha_{i0} = f^\alpha_{i\bar{j}} = 0, \quad f^\alpha_{ij} = f^\alpha_{ji}, \tag{3.7}$$

$$f^\alpha_{ij\bar{k}} = \sum_{t=1}^m f^\alpha_t R^M_{\bar{t}j\bar{k}} - \sum_{\beta,\gamma,\rho=1}^n f^\beta_i f^\gamma_j f^{\bar{\rho}}_{\bar{k}} R^N_{\bar{\alpha}\beta\gamma\bar{\rho}}. \tag{3.8}$$

Now we can prove (a) of Theorem 1.1.

Proof of Theorem 1.1(a). Since σ_l is continuous, then there is at least one maximum point $x \in M$. Choose a suitable unitary frame $\{\xi_{i,x}\}$ of $T_{1,0}M_x$ such that $f^*\omega_N$ is diagonal; in other words,

$$\left(\sum_\alpha f^\alpha_i f^{\bar{\alpha}}_{\bar{j}}\right)(x) = \lambda_i(x)\delta_{i\bar{j}}. \tag{3.9}$$

Without loss of generality, $\lambda_i(x)$ is decreasing about i. In this proof, we always take the sum of $\alpha, \beta, \gamma, \ldots$ over $1, 2, \ldots, n$. Parallelly extend the frame $\{\xi_{i,x}\}$ to a neighborhood of x and denote by $\{\xi_i\}$. Set

$$\tilde{\sigma}_l = \sum_{i=1}^l f^*\omega_N(\xi_i, \xi_{\bar{i}}) = \sum_{i=1}^l \sum_\alpha f^\alpha_i f^{\bar{\alpha}}_{\bar{i}}. \tag{3.10}$$

On one hand, due to (3.3), $\tilde{\sigma}_l \le \sigma_l$ and $\tilde{\sigma}_l(x) = \sigma_l(x)$ which implies that x is still a local maximum point of $\tilde{\sigma}$. On the other hand, by the commutation relations for second-order covariant derivatives (see Lemma 3.1 in [14]), we know that

$$(\tilde{\sigma}_l)_{j\bar{k}} - (\tilde{\sigma}_l)_{\bar{k}j} = 2\sqrt{-1}\delta_{j\bar{k}}(\tilde{\sigma}_l)_0 = 2\sqrt{-1}\sum_{i=1}^l \sum_\alpha (f^\alpha_{i0} f^{\bar{\alpha}}_{\bar{i}} + f^\alpha_i f^{\bar{\alpha}}_{\bar{i}0})\delta_{j\bar{k}} = 0 \tag{3.11}$$

due to (3.7). Hence

$$(\tilde{\sigma}_l)_{j\bar{j}}(x) = \frac{1}{2}\sum_{i,j=1}^l \left[\sum_\alpha (f^\alpha_i f^{\bar{\alpha}}_{\bar{i}})_{j\bar{j}} + \sum_\alpha (f^\alpha_i f^{\bar{\alpha}}_{\bar{i}})_{\bar{j}j}\right](x) \le 0. \tag{3.12}$$

The commutation relation (3.8) leads the following calculation at x

$$\begin{aligned}\sum_{j=1}^{l}(\tilde{\sigma}_l)_{j\bar{j}} &= \sum_{i,j=1}^{l}\sum_{\alpha}(f^{\alpha}_{ij}f^{\bar{\alpha}}_{\bar{i}\bar{j}} + f^{\alpha}_{ij\bar{j}}f^{\bar{\alpha}}_{\bar{i}})\\ &= \sum_{i,j=1}^{l}\sum_{\alpha}f^{\alpha}_{ij}f^{\bar{\alpha}}_{\bar{i}\bar{j}} + \sum_{i,j=1}^{l}\sum_{\alpha}\sum_{k=1}^{m}f^{\alpha}_{k}f^{\bar{\alpha}}_{\bar{i}}R^{M}_{\bar{k}ij\bar{j}} - \sum_{i,j=1}^{l}\sum_{\alpha,\beta,\gamma,\rho}f^{\bar{\alpha}}_{\bar{i}}f^{\beta}_{i}f^{\gamma}_{j}f^{\bar{\rho}}_{\bar{j}}W^{\alpha}_{\beta\gamma\bar{\rho}}\\ &= \sum_{i,j=1}^{l}\sum_{\alpha}f^{\alpha}_{ij}f^{\bar{\alpha}}_{\bar{i}\bar{j}} + \sum_{i,j=1}^{l}\lambda_i(x)R^{M}_{\bar{i}ij\bar{j}} - \sum_{i,j=1}^{l}R^{N}\Big(df(\xi_{\bar{i}}), df(\xi_i), df(\xi_j), df(\xi_{\bar{j}})\Big).\end{aligned} \tag{3.13}$$

By the curvature assumptions, we know that

$$\sum_{i,j=1}^{l}\lambda_i(x)R^{M}_{\bar{i}ij\bar{j}} = \sum_{i=1}^{l}\lambda_i(x)\sum_{j=1}^{l}R^{M}_{\bar{i}ij\bar{j}} \geq -k\sum_{i=1}^{l}\lambda_i(x) = -k\sigma_l(x) \tag{3.14}$$

and

$$\begin{aligned}-\sum_{i,j=1}^{l}R^{N}\Big(df(\xi_{\bar{i}}), df(\xi_i), df(\xi_j), df(\xi_{\bar{j}})\Big) &\geq \sum_{i,j=1}^{l}K\big|df(\xi_i)\big|^2\cdot\big|df(\xi_j)\big|^2\\ &= K\sum_{i,j=1}^{l}\lambda_i\lambda_j = K\sigma_l^2(x)\end{aligned} \tag{3.15}$$

since f is basic CR. Using (3.14) and (3.15), the inequality (3.13) becomes

$$0 \geq -k\sigma_l(x) + K\sigma_l^2(x) \Longrightarrow \sigma_l(x) \leq \frac{k}{K}. \tag{3.16}$$

Hence the proof is finished by

$$\max_{M}\sigma_l = \sigma_l(x) \leq \frac{k}{K}. \qquad \square$$

Next we prove (b) of Theorem 1.1.

Proof of Theorem 1.1(b). Since p_l is continuous, then we can choose a maximum point $x \in M$ of p_l. Without loss of generality, assume that $p_l(x) > 0$. Choose a suitable frame $\{\xi_{i,x}\}$ of $T_{1,0}M_x$ such that $f^*\omega_N$ is diagonal; in other words,

$$\left(\sum_{\alpha}f^{\alpha}_{i}f^{\bar{\alpha}}_{\bar{j}}\right)(x) = \lambda_i(x)\delta_{i\bar{j}}. \tag{3.17}$$

Without loss of generality, $\lambda_i(x)$ is decreasing about i. Extend $\{\xi_{i,x}\}$ to a local frame $\{\xi_i\}$ in a neighborhood of x by parallel transformation of Tanaka-Webster connection. Due to (3.5), we know that

$$\tilde{p}_l \triangleq (f^*\omega_N)^l(\xi_1,\ldots,\xi_l,\xi_{\bar{1}},\ldots,\xi_{\bar{l}}) = \sum_{\alpha_1,\ldots,\alpha_l}\prod_{i=1}^{l}f^{\alpha_i}_{i}f^{\bar{\alpha}_i}_{\bar{i}} \leq p_l \tag{3.18}$$

which implies that x is still a maximum point of $\tilde{p}_l$. By (3.7) and Lemma 3.1 in [14],

$$\sum_{j=1}^{l}[(\tilde{p}_l)_{j\bar{j}} - (\tilde{p}_l)_{\bar{j}j}] = 2l\sqrt{-1}(\tilde{p}_l)_0 = 2l\sqrt{-1}\sum_{\alpha_1,\ldots,\alpha_l}\prod_{i=1}^{l}\left(f_{i0}^{\alpha_i} f_{\bar{i}}^{\bar{\alpha}_i} + f_i^{\alpha_i} f_{\bar{i}0}^{\bar{\alpha}_i}\right) = 0 \tag{3.19}$$

which states that $(\tilde{p}_l)_{j\bar{j}}$ is real and thus

$$\sum_{j=1}^{l}(\tilde{p}_j)_{j\bar{j}}(x) \leq 0. \tag{3.20}$$

Using (3.8), we can do the following calculation at x

$$\begin{aligned}
(\tilde{p}_l)_{j\bar{j}} &= \sum_{\alpha_I}\sum_{i_1}\left(f_{i_1 j}^{\alpha_{i_1}} f_{\bar{i}_1}^{\bar{\alpha}_{i_1}} \prod_{i\neq i_1} f_i^{\alpha_i} f_{\bar{i}}^{\bar{\alpha}_i}\right)_{\bar{j}} \\
&= \sum_{\alpha_I}\sum_{i_1}\left(f_{i_1 j\bar{j}}^{\alpha_{i_1}} f_{\bar{i}_1}^{\bar{\alpha}_{i_1}} + f_{i_1 j}^{\alpha_{i_1}} f_{\bar{i}_1\bar{j}}^{\bar{\alpha}_{i_1}}\right)\prod_{i\neq i_1} f_i^{\alpha_i} f_{\bar{i}}^{\bar{\alpha}_i} \\
&\quad + \sum_{\alpha_I}\sum_{i_1\neq i_2} f_{i_1 j}^{\alpha_{i_1}} f_{\bar{i}_1}^{\bar{\alpha}_{i_1}} f_{i_2}^{\alpha_{i_2}} f_{\bar{i}_2\bar{j}}^{\bar{\alpha}_{i_2}} \prod_{i\neq i_1,i_2} f_i^{\alpha_i} f_{\bar{i}}^{\bar{\alpha}_i} \\
&= \sum_{i_1}\sum_{\alpha_{i_1}}\frac{p_l}{\lambda_{i_1}}\left(\sum_k f_{\bar{i}_1}^{\bar{\alpha}_{i_1}} f_k^{\alpha_{i_1}} R^M_{\bar{k}i_1 j\bar{j}} - \sum_{\beta,\gamma,\rho} f_{\bar{i}_1}^{\bar{\alpha}_{i_1}} f_{i_1}^{\beta} f_j^{\gamma} f_{\bar{j}}^{\bar{\rho}} R^N_{\bar{\alpha}_{i_1}\beta\gamma\bar{\rho}} + f_{i_1 j}^{\alpha_{i_1}} f_{\bar{i}_1\bar{j}}^{\bar{\alpha}_{i_1}}\right) \\
&\quad + \sum_{\alpha_I}\sum_{i_1\neq i_2}\frac{p_l}{\lambda_{i_1}\lambda_{i_2}} f_{i_1 j}^{\alpha_{i_1}} f_{\bar{i}_1}^{\bar{\alpha}_{i_1}} f_{i_2}^{\alpha_{i_2}} f_{\bar{i}_2\bar{j}}^{\bar{\alpha}_{i_2}} \\
&= \sum_i\sum_\alpha\frac{p_l}{\lambda_i}\left(\sum_k f_{\bar{i}}^{\bar{\alpha}} f_k^{\alpha} R^M_{\bar{k}ij\bar{j}} - \sum_{\beta,\gamma,\rho} f_{\bar{i}}^{\bar{\alpha}} f_i^{\beta} f_j^{\gamma} f_{\bar{j}}^{\bar{\rho}} R^N_{\bar{\alpha}\beta\gamma\bar{\rho}} + f_{ij}^{\alpha} f_{\bar{i}\bar{j}}^{\bar{\alpha}}\right) \\
&\quad + \sum_{\alpha,\beta}\sum_{i\neq k}\frac{p_l}{\lambda_i\lambda_k} f_{ij}^{\alpha} f_{\bar{i}}^{\bar{\alpha}} f_k^{\beta} f_{\bar{k}\bar{j}}^{\bar{\beta}}
\end{aligned} \tag{3.21}$$

where $\alpha_I = (\alpha_1,\ldots,\alpha_l)$ and the sums about $i, j, k, \ldots$ are taken over $1, 2, \ldots, l$. The first term of (3.21) can be written as

$$\sum_i p_l R^M_{\bar{i}ij\bar{j}} - \sum_i \frac{p_l}{\lambda_i} R^N\left(df(\xi_{\bar{j}}), df(\xi_j), df(\xi_i), df(\xi_{\bar{j}})\right) + \sum_i\sum_\alpha \frac{p_l}{\lambda_i} f_{ij}^{\alpha} f_{\bar{i}\bar{j}}^{\bar{\alpha}}. \tag{3.22}$$

Since at x,

$$(\tilde{p}_l)_j = \sum_{\alpha_I}\sum_{i_1} f_{i_1 j}^{\alpha_{i_1}} f_{\bar{i}_1}^{\bar{\alpha}_{i_1}} \prod_{i\neq i_1} f_i^{\alpha_i} f_{\bar{i}}^{\bar{\alpha}_i} = \sum_i\sum_\alpha \frac{p_l}{\lambda_i} f_{ij}^{\alpha} f_{\bar{i}}^{\bar{\alpha}} \tag{3.23}$$

then the second term of (3.21) becomes

$$\sum_{i,\alpha}\frac{1}{\lambda_i} f_{ij}^{\alpha} f_{\bar{i}}^{\bar{\alpha}}\left[\sum_{k,\beta}\frac{p_l}{\lambda_k} f_k^{\beta} f_{\bar{k}\bar{j}}^{\bar{k}} - \sum_\beta \frac{p_l}{\lambda_i} f_i^{\beta} f_{\bar{i}\bar{j}}^{\bar{\beta}}\right] = \frac{1}{p_l}\left|(\tilde{p}_l)_j\right|^2 - \sum_i \frac{p_l}{\lambda_i^2}\sum_{\alpha,\beta} f_{\bar{i}}^{\bar{\alpha}} f_{ij}^{\alpha} f_i^{\beta} f_{\bar{i}\bar{j}}^{\bar{\beta}}. \tag{3.24}$$

By substituting (3.22) and (3.24) into (3.21), we have

$$\sum_j (\tilde{p}_l)_{j\bar{j}} = \sum_{i,j}\left[p_l R^M_{i\bar{i}j\bar{j}} - \frac{p_l}{\lambda_i} R^N\Big(df(\xi_{\bar{j}}), df(\xi_j), df(\xi_i), df(\xi_{\bar{j}})\Big)\right] + \sum_j \frac{1}{p_l}\big|(\tilde{p}_l)_j\big|^2$$
$$+ \sum_{i,j}\frac{p_l}{\lambda_i^2}\sum_{\alpha,\beta}\Big(f^\alpha_{ij} f^{\bar{\alpha}}_{\bar{i}\bar{j}} f^\beta_i f^{\bar{\beta}}_{\bar{i}} - f^{\bar{\alpha}}_{\bar{i}} f^\alpha_{ij} f^\beta_i f^{\bar{\beta}}_{\bar{i}\bar{j}}\Big). \tag{3.25}$$

To deal with the last term in the above equation, fix i, j and denote

$$a = (f^1_{ij}, \ldots, f^n_{ij}) \quad \text{and} \quad b = (f^1_i, \ldots, f^n_i). \tag{3.26}$$

By Schwarz inequality:

$$\big|a \cdot \bar{b}\big| \le |a| \cdot |b|, \tag{3.27}$$

we find

$$\left|\sum_\alpha f^\alpha_{ij} f^{\bar{\alpha}}_{\bar{i}}\right|^2 \le \left(\sum_\alpha f^\alpha_{ij} f^{\bar{\alpha}}_{\bar{i}\bar{j}}\right) \cdot \left(\sum_\beta f^\beta_i f^{\bar{\beta}}_{\bar{i}}\right) \tag{3.28}$$

where

$$\text{left side } = \sum_\alpha f^\alpha_{ij} f^{\bar{\alpha}}_{\bar{i}} \cdot \overline{\sum_\beta f^\beta_{ij} f^{\bar{\beta}}_{\bar{i}}} = \sum_{\alpha,\beta} f^\alpha_{ij} f^{\bar{\alpha}}_{\bar{i}} f^{\bar{\beta}}_{\bar{i}\bar{j}} f^\beta_i, \tag{3.29}$$

$$\text{right side } = \sum_{\alpha,\beta} f^\alpha_{ij} f^{\bar{\alpha}}_{\bar{i}\bar{j}} f^\beta_i f^{\bar{\beta}}_{\bar{i}} \tag{3.30}$$

which implies that the last term in (3.25) is nonnegative. Since $p_l(x) > 0$ and

$$\langle df(\xi_i), df(\xi_{\bar{j}})\rangle = \sum_\alpha f^\alpha_i f^{\bar{\alpha}}_{\bar{j}} = \lambda_i \delta_{i\bar{j}}, \quad \text{at} \quad x \in M, \tag{3.31}$$

then

$$\left\{\frac{df(\xi_{i,x})}{\sqrt{\lambda_i}},\ i = 1, \ldots, l\right\} \tag{3.32}$$

is a unitary frame of l-dimensional subspace $df(\Sigma_x) \subset T_{1,0}N_{f(x)}$, where Σ_x is an l-dimensional space formed by $\{\xi_{i,x}\}_{i=1}^l$. By geometric inequality and curvature assumptions, we find

$$\begin{aligned}
&-\sum_{i,j}\frac{p_l}{\lambda_i} R^N\Big(df(\xi_{\bar{i}}), df(\xi_i), df(\xi_j), df(\xi_{\bar{j}})\Big)\\
&= -p_l \sum_{i,j} R^N\left(\frac{df(\xi_{\bar{i}})}{\sqrt{\lambda_i}}, \frac{df(\xi_i)}{\sqrt{\lambda_i}}, df(\xi_j), df(\xi_{\bar{j}})\right)\\
&= -p_l \sum_j \mathrm{Ric}^{(1)}_l\big(df(\Sigma_x), df(\xi_j)\big)\\
&\ge K p_l \sum_j \lambda_j \ge K l p_l^{1+\frac{1}{l}}.
\end{aligned} \tag{3.33}$$

Applying (3.20), (3.25) and (3.33) with the curvature assumption on M, we have

$$0 \geq \sum_j (\tilde{p}_l)_{j\bar{j}} \geq -p_l k + Klp_l^{1+\frac{1}{l}} \tag{3.34}$$

at x. The proof is finished by

$$\max_M p_l = p_l(x) \leq \left(\frac{k}{lK}\right)^l. \tag{3.35}$$

□

The key ingredient for Theorem 1.1 is the commutation relations (Lemma 3.1) for CR maps under suitable local frames. The similar results for (J^N, J)-holomorphic maps and transversally holomorphic maps have been deduced in [6,7] which will be listed at the end of the paper. The rest of the proofs of Theorem 1.2 and Theorem 1.3 is left to the reader.

Lemma 3.2 (cf. Equations (3.21) and (3.25) in [6]). *Suppose that* (N^{2n}, J^N, ω_N) *is a Hermitian manifold and* (M^{2m+1}, H, J, θ) *is a pseudo-Hermitian manifold. Let* $f : N \to M$ *be a* (J^N, J)*-holomorphic map. Then*

$$f^i_{\alpha\beta} = f^i_{\beta\alpha} + \sum_{\gamma=1}^n f^i_\gamma T^\gamma_{\alpha\beta}, \quad f^i_{\alpha\bar{\beta}} = -\sum_{j=1}^m f^0_\alpha f^{\bar{j}}_{\bar{\beta}} A_{\bar{i}\bar{j}}, \tag{3.36}$$

$$\begin{aligned} f^i_{\alpha\beta\bar{\gamma}} - f^i_{\alpha\bar{\gamma}\beta} = &\sum_{\rho=1}^n f^i_\rho R^N_{\bar{\rho}\alpha\beta\bar{\gamma}} - \sum_{j,k,t=1}^m f^j_\alpha f^k_\beta f^{\bar{t}}_{\bar{\gamma}} R^M_{\bar{i}jk\bar{t}} \\ &- \sum_{j,k=1}^m f^j_\alpha (A_{\bar{i}\bar{k},j} f^0_\beta f^{\bar{k}}_{\bar{\gamma}} + A_{kj,\bar{i}} f^0_{\bar{\gamma}} f^k_\beta), \end{aligned} \tag{3.37}$$

where f^α_{AB} *and* f^α_{ABC} *are components of* ∇df *and* $\nabla^2 df$ *under some local unitary frames* $\{\eta_\alpha\}$ *and* $\{\xi_i\}$ *of* $T_{1,0}N$ *and* $T_{1,0}M$.

Lemma 3.3 (cf. Equations (2.11) and (2.14) in [7]). *Suppose that* (M^{2m+1}, H, J, θ) *is a pseudo-Hermitian manifold and* $(\tilde{M}^{2n+1}, \tilde{H}, \tilde{J}, \tilde{\theta})$ *is a Sasakian manifold. Let* $f : M \to \tilde{M}$ *be a transversally holomorphic map. Then*

$$f^\alpha_{ij} = f^\alpha_{ji}, \quad f^\alpha_{i\bar{j}} = f^\alpha_{i0} = 0, \tag{3.38}$$

$$f^\alpha_{ij\bar{k}} - f^\alpha_{i\bar{k}j} = \sum_{t=1}^m f^\alpha_t R_{\bar{t}ij\bar{k}} - \sum_{\beta,\gamma,\rho=1}^n f^\beta_i f^\gamma_j f^{\bar{\rho}}_{\bar{k}} \tilde{R}_{\bar{\alpha}\beta\gamma\bar{\rho}}, \tag{3.39}$$

where R *and* $\tilde{R}$ *are pseudo-Hermitian curvature of* M *and* $\tilde{M}$ *respectively,* f^α_{AB} *and* f^α_{ABC} *are components of* ∇df *and* $\nabla^2 df$ *under some local unitary frames* $\{\xi_i\}$ *and* $\{\eta_\alpha\}$ *of* $T_{1,0}M$ *and* $T_{1,0}\tilde{M}$.

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