Journal of Applied Analysis and Computation
Volume 11, Number 3, June 2021, 1652–1662
Website:http://www.jaac-online.com
DOI:10.11948/20200477

THE HOM-TWISTED SMASH PRODUCT BIALGEBRAS*

Jiafeng Lü[1], Wenying Yu[1] and Ling Liu[1,†]

Abstract Let (H, α_H) be a Hom-Hopf algebra and (A, α_A) be an (H, α_H)-Hom-bimodule algebra with the maps α_A, α_H bijective. Then in this paper, we first introduce the notion of Hom-twisted smash product $(A \star H, \alpha_A \star \alpha_H)$ and then study the conditions for the Hom-twisted smash product and tensor coproduct to form a Hom-bialgebra and a Hom-Hopf algebra. Furthermore, we give a non-trival example of Hom-twisted smash product Hopf algebra and a characterization of left $(A \star H, \alpha_A \star \alpha_H)$-Hom module.

Keywords Hom-bialgebra, Hom-twisted smash product, tensor coproduct.

MSC(2010) 16T05.

1. Introduction

Hom-type algebras appeared in the physics literature of the 1990's, when looking for quantum deformations of some algebras of vector fields, like Witt and Virasoro algebras ([2, 7]). It was observed that algebras obtained by deforming certain Lie algebras no longer satisfied the Jacobi identity, but a modified version of it involving a homomorphism. An axiomatization of this type of algebras (called Hom-Lie algebras) was given in [11] for the first time. Here the associativity was replaced by the Hom-associativity: $\alpha(a)(bc) = (ab)\alpha(c)$. The coalgebra counterpart and the related notions of Hom-bialgebra and Hom-Hopf algebra were introduced in [12] and some of their properties were described. The original definitions of Hom-bialgebra and Hom-Hopf algebra involved two different linear maps α and β, with α twisting the associativity condition and β the coassociativity condition. Afterwards, two directions of study were developed, one considering the class such that $\beta = \alpha$, which are still called Hom-bialgebras and Hom-Hopf algebras ([16]) and another one, initiated in [3], where the map α is assumed to be invertible and $\beta = \alpha^{-1}$ (these are called monoidal Hom-bialgebras and monoidal Hom-Hopf algebras). Since Hom-bialgebras and monoidal Hom-bialgebras are different concepts, it turns out that our definitions, formulae and results are also different from the ones in [8]. There is a growing literature on Hom and BiHom-type algebras, let us just mention the very recent papers [1, 4, 9, 10].

†The corresponding author. Email: ntliulin@zjnu.cn(L. Liu)
[1]Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, China
*The authors were supported by National Natural Science Foundation of China (Nos. 11801515, 12071441), the Natural Science Foundation of Zhejiang Province (No. LY20A010003) and the Foundation of Zhejiang Educational Committee (No. Y201942625).

The concept of twisted smash product algebra for H-bimodule algebra has been introduced in Wang and Li [15]. If A is an H-bimodule algebra, then one can establish a twisted smash product $A \star H$. The usual smash product [14], and Doi-Takeuchi's double algebra [5] are all special cases of that algebra. Moreover, Drinfeld's double [6] is also such a twisted smash product algebra $H^{*cop} \star H$, where H is a finite dimensional Hopf algebra.

The main purpose of this paper is to study the twisted smash products $A \star H$ on Hom-Hopf algebra and give the conditions for the Hom-twisted smash product algebra and tensor coproduct to form a Hom-bialgebra. Meanwhile, we should give some non-trvial examples of Hom-twisted smash product algebras.

2. Preliminaries

Throughout the paper, let k denote a fixed field. All vector spaces, tensor products, and homomorphisms are over k. We will use the Sweedler's notation for terminologies on coalgebras. Let C be a coalgebra, we write comultiplication $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$.

In this section, we recall the definitions of the Hom-algebras, Hom-coalgebras, Hom-modules, Hom-smash products and so on (see [12, 13]).

Definition 2.1. A Hom-associative algebra is a triple (A, μ, α_A), in which A is a linear space, $\alpha_A : A \to A$ and $\mu : A \otimes A \to A$ are linear maps, with notation $\mu(a \otimes b) = ab$ such that

$$\begin{aligned} &\alpha_A(a)(bc) = (ab)\alpha_A(c), \\ &\alpha_A(ab) = \alpha_A(a)\alpha_A(b), \end{aligned} \tag{2.1}$$

for all $a, b, c \in A$. We call α_A the structure map of (A, μ, α_A).

If $\eta : k \to A$ is a linear map, such that $1_A a = \alpha_A(a) = a 1_A$, $\alpha_A(1_A) = 1_A$, here we write $\eta(1_k) = 1_A$, then $(A, \mu_A, \eta, \alpha_A)$ is called a Hom-associative algebra with an unit 1_A.

Definition 2.2. A Hom-coassociative coalgebra is a triple (C, Δ, α_C) in which C is a linear space, $\alpha_C : C \to C$ and $\Delta : C \to C \otimes C$ are linear maps such that

$$\begin{aligned} &(\alpha_C \otimes \alpha_C) \circ \Delta = \Delta \circ \alpha_C, \\ &(\Delta \otimes \alpha_C) \circ \Delta = (\alpha_C \otimes \Delta) \circ \Delta. \end{aligned} \tag{2.2}$$

If $\varepsilon : C \to k$ is a linear map, such that $(\varepsilon \otimes Id) \circ \Delta = \alpha_C = (Id \otimes \varepsilon) \circ \Delta, \varepsilon \circ \alpha_C = \varepsilon$, then $(C, \Delta, \varepsilon, \alpha_C)$ is called a Hom-coassociative coalgebra with counit ε.

Definition 2.3. Let (A, μ_A, α_A) be a Hom-associative algebra, M a linear space and $\alpha_M : M \to M$ a linear map.
(i) A left (A, α_A)-Hom module structure on (M, α_M) consists of a linear map $A \otimes M \to M, a \otimes m \mapsto a \cdot m$, satisfying the conditions (for all $a, a' \in A$, $m \in M$)

$$\begin{aligned} &\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m), \\ &\alpha_A(a) \cdot (a' \cdot m) = (aa') \cdot \alpha_M(m). \end{aligned} \tag{2.3}$$

(ii) A right (A, α_A)-Hom module structure on (M, α_M) consists of a linear map

$M \otimes A \to M, m \otimes a \mapsto m \cdot a$, satisfying the conditions (for all $a, a^{'} \in A$, $m \in M$)

$$\begin{aligned} &\alpha_M(m \cdot a) = \alpha_M(m) \cdot \alpha_A(a), \\ &(m \cdot a) \cdot \alpha_A(a^{'}) = \alpha_M(m) \cdot (aa^{'}). \end{aligned} \tag{2.4}$$

Definition 2.4. A Hom-bialgebra is quadruple $(H, \mu_H, \Delta, \alpha_H)$, in which (H, μ_H, α_H) is a Hom-associative algebra, (H, Δ, α_H) is a Hom-coassociative coalgebra and moreover Δ is a morphism of Hom-associative algebras.

Thus, a Hom-bialgebra is a Hom-associative algebra (H, μ_H, α_H) endowed with comultiplication $\Delta : H \to H \otimes H$, with notation $\Delta(h) = h_1 \otimes h_2$, such that, for all $h, h^{'} \in H$, we have:

$$\Delta(hh^{'}) = h_1 h_1^{'} \otimes h_2 h_2^{'}, \tag{2.5}$$

If $(H, \mu_H, \eta, \alpha_H)$ is a Hom-associative algebra with an unit 1_H, $(H, \Delta, \varepsilon, \alpha_H)$ is a Hom-coassociative coalgebra with a counit ε, satisfying $\Delta(hh^{'}) = \Delta(h)\Delta(h^{'})$, $\Delta(1_H) = 1_H \otimes 1_H, \varepsilon(hh^{'}) = \varepsilon(h)\varepsilon(h^{'})$, then $(H, \mu_H, \eta, \Delta, \varepsilon, \alpha_H)$ is a Hom-bialgebra with unit and counit.

In fact, if there exists a morphism (called antipode) $S_H : H \to H$ such that

$$\begin{aligned} &S_H(h_1)h_2 = \varepsilon(h)1_H = h_1 S_H(h_2), \qquad (2.6) \\ &S_H \circ \alpha_H = \alpha_H \circ S_H, \end{aligned}$$

for all $h \in H$, then $(H, \mu_H, \eta, \Delta, \varepsilon, \alpha_H, S_H)$ is called a Hom-Hopf algebra.

Definition 2.5. Assume that $(H, \mu_H, \Delta_H, \alpha_H)$ is a Hom-bialgebra. A Hom-associative algebra (A, μ_A, α_A) is called a left (H, α_H)-Hom module algebra if (A, α_A) is a left (H, α_H)-Hom module, with action denoted by $H \otimes A \to A, h \otimes a \mapsto h \cdot a$, such that the following condition is satisfied:

$$\alpha_H^2(h) \cdot (aa^{'}) = (h_1 \cdot a)(h_2 \cdot a^{'}), \qquad \forall h \in H, a, a^{'} \in A. \tag{2.7}$$

Similarly we can define right (H, α_H)-Hom module algebra.

Definition 2.6. Let (A, α_A) be a Hom-algebra and (M, α_M) be a left and right (A, α_A)-Hom-module satisfying the following condition

$$(a \to m) \leftarrow \alpha_A(b) = \alpha_A(a) \to (m \leftarrow b), \tag{2.8}$$

for all $a, b \in A$ and $m \in M$, then we call (M, α_M) be an (A, α_A)-Hom-bimodule.

If (A, α_A) is both left (H, α_H)-Hom module algebra and right (H, α_H)-Hom module algebra, and (A, α_A) is an (H, α_H)-Hom-bimodule, then we call (A, α_A) is an (H, α_H)-Hom-bimodule algebra.

Definition 2.7. Let (A, α_A) be a left (H, α_H)-Hom module algebra. The Hom-smash product $(A\#H, \alpha_A\#\alpha_H)$ of (A, α_A) and (H, α_H) is defined as follows, for all $a, b \in A, h, k \in H$:

(1) as k-spaces, $A\#H = A \otimes H$,

(2) Hom-multiplication is given by

$$(a\#h)(b\#k) = a(\alpha_H^{-2}(h_1) \to \alpha_A^{-1}(b))\#\alpha_H^{-1}(h_2)k.$$

Note that $(A\#H, \alpha_A\#\alpha_H)$ is a Hom-algebra with the unit $1_A\#1_H$.

3. The Hom-twisted smash product bialgebras $A\star H$

In this section, we assume that the Hom-algebra is always unital. First we introduce the notion of a Hom-twisted smash product $(A\star H, \alpha_A\star\alpha_H)$ and give the Hom-smash product and Drinfeld's double as examples. Next we find a necessary and sufficient condition making it into a Hom-bialgebra with the tensor coproduct, generalizing the main constructions in [15]. Finally we give a non-trival example of Hom-twisted smash product Hopf algebra and a characterization of left $(A\star H, \alpha_A\star\alpha_H)$-Hom module.

Proposition 3.1. *Let (A,μ_A,α_A) be a Hom-algebra and $(H,\mu_H,\eta,\Delta,\varepsilon,\alpha_H,S_H)$ be a Hom-Hopf algebra with unit 1_H and counit ε. (A,α_A) is an (H,α_H)-bimodule algebra with the left (H,α_H)-Hom module action $\rightarrow$ and the right (H,α_H)-Hom module action $\leftarrow$. We define a Hom-twisted smash product $(A\star H,\alpha_A\star\alpha_H)$ with the multiplication on the vector space $A\otimes H$ as follows*

$$(a\otimes h)(b\otimes l)=a((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H\alpha_H^{-2}(h_2))\otimes\alpha_H^{-2}(h_{12})l,$$

for all $a,b\in A, h,l\in H$. The element $a\otimes h$ of $A\star H$ will usually be written as $a\star h$. Then $(A\star H,\alpha_A\star\alpha_H)$ is a Hom-algebra with the unit $1_A\star 1_H$.

Proof. We first compute $(1_A\star 1_H)(b\star l)=1_A((1_H\rightarrow\alpha_A^{-2}(b))\leftarrow S_H(1_H))\star 1_H l = 1_A b\star\alpha_H(l)=\alpha_A(b)\star\alpha_H(l)$, and similarly we get $(a\star h)(1_A\star 1_H)=\alpha_A(a)\star\alpha_H(h)$. Next for any $a\star h, b\star l, c\star g\in A\star H$, we have

$$\begin{aligned}
&[(a\star h)(b\star l)](\alpha_A(c)\star\alpha_H(g))\\
&=[a((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H\alpha_H^{-2}(h_2))\star\alpha_H^{-2}(h_{12})l](\alpha_A(c)\star\alpha_H(g))\\
&=[a((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H\alpha_H^{-2}(h_2))][(\alpha_H^{-6}(h_{1211})\alpha_H^{-4}(l_{11})\rightarrow\alpha_A^{-1}(c))\\
&\quad\leftarrow S_H(\alpha_H^{-4}(h_{122})\alpha_H^{-2}(l_2))]\star(\alpha_H^{-4}(h_{1212})\alpha_H^{-2}(l_{12}))\alpha_H(g),
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha_A(a)\star\alpha_H(h))[(b\star l)(c\star g)]\\
&=(\alpha_A(a)\star\alpha_H(h))[b((\alpha_H^{-4}(l_{11})\rightarrow\alpha_A^{-2}(c))\leftarrow S_H(\alpha_H^{-2}(l_2)))\star\alpha_H^{-2}(l_{12})g]\\
&=\alpha_A(a)[\alpha_H^{-3}(h_{11})\rightarrow\alpha_A^{-2}(b((\alpha_H^{-4}(l_{11})\rightarrow\alpha_A^{-2}(c))\leftarrow S_H(\alpha_H^{-2}(l_2))))\leftarrow S_H\alpha_H^{-1}(h_2)]\\
&\quad\star\alpha_H^{-1}(h_{12})(\alpha_H^{-2}(l_{12})g)\\
&=\alpha_A(a)[(\alpha_H^{2}(\alpha_H^{-5}(h_{11}))\rightarrow\alpha_A^{-2}(b((\alpha_H^{-4}(l_{11})\rightarrow\alpha_A^{-2}(c))\leftarrow S_H\alpha_H^{-2}(l_2))))\\
&\quad\leftarrow S_H(\alpha_H^{2}(\alpha_H^{-3}(h_2)))]\star\alpha_H^{-1}(h_{12})(\alpha_H^{-2}(l_{12})g)\\
&\overset{(2.7)}{=}\alpha_A(a)([(\alpha_H^{-5}(h_{111})\rightarrow\alpha_A^{-2}(b))(\alpha_H^{-5}(h_{112})\rightarrow((\alpha_H^{-6}(l_{11})\rightarrow\alpha_A^{-4}(c))\leftarrow S_H\alpha_H^{-4}(l_2)))]\\
&\quad\leftarrow S_H(\alpha_H^{2}(\alpha_H^{-3}(h_2))))\star\alpha_H^{-1}(h_{12})(\alpha_H^{-2}(l_{12})g)\\
&=\alpha_A(a)([(\alpha_H^{-5}(h_{111})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H(\alpha_H^{-3}(h_{22}))]((\alpha_H^{-5}(h_{112})\rightarrow((\alpha_H^{-6}(l_{11})\rightarrow\alpha_A^{-4}(c))\\
&\quad\leftarrow S_H\alpha_H^{-4}(l_2))))\leftarrow S_H(\alpha_H^{-3}(h_{21}))))\star\alpha_H^{-1}(h_{12})(\alpha_H^{-2}(l_{12})g)\\
&\overset{(2.3)}{=}\alpha_A(a)([(\alpha_H^{-5}(h_{111})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H(\alpha_H^{-3}(h_{22}))]((\alpha_H^{-6}(h_{112})\alpha_H^{-5}(l_{11})\rightarrow\alpha_A^{-2}(c))\\
&\quad\leftarrow S_H\alpha_H^{-3}(l_2))S_H(\alpha_H^{-4}(h_{21}))))\star\alpha_H^{-1}(h_{12})(\alpha_H^{-2}(l_{12})g)\\
&\overset{(2.4)}{=}[a((\alpha_H^{-5}(h_{111})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H(\alpha_H^{-3}(h_{22})))]((\alpha_H^{-5}(h_{112})\alpha_H^{-4}(l_{11})\rightarrow\alpha_A^{-1}(c))
\end{aligned}$$

$$\leftarrow S_H(\alpha_H^{-3}(h_{21})\alpha_H^{-2}(l_2))) \leftarrow (\alpha_H^{-2}(h_{12})\alpha_H^{-2}(l_{12}))\alpha_H(g).$$

From the coassociativity of (H, Δ, α_H),

$$\begin{aligned}
&(h_{11})_1 \otimes (h_{11})_2 \otimes h_{12} \otimes (h_2)_1 \otimes (h_2)_2\\
=\ &(\alpha_H^{-1}(h_{111}))_1 \otimes (\alpha_H^{-1}(h_{111}))_2 \otimes \alpha_H^{-1}(h_{112}) \otimes h_{12} \otimes \alpha_H(h_2)\\
\overset{(2.2)}{=}\ &(h_{11})_1 \otimes (h_{11})_2 \otimes \alpha_H^{-1}(h_{121}) \otimes \alpha_H^{-1}(h_{122}) \otimes \alpha_H(h_2)\\
=\ &h_{111} \otimes h_{112} \otimes \alpha_H^{-1}(h_{121}) \otimes \alpha_H^{-1}(h_{122}) \otimes \alpha_H(h_2)\\
\overset{(2.2)}{=}\ &\alpha_H(h_{11}) \otimes h_{121} \otimes \alpha_H^{-1}(h_{1211}) \otimes \alpha_H^{-2}(h_{1222}) \otimes \alpha_H(h_2)\\
=\ &\alpha_H(h_{11}) \otimes \alpha_H^{-1}(h_{1211}) \otimes \alpha_H^{-2}(h_{1212}) \otimes \alpha_H^{-1}(h_{122}) \otimes \alpha_H(h_2),
\end{aligned}$$

we get

$$\begin{aligned}
&h_{11} \otimes h_{1211} \otimes h_{1212} \otimes h_{122} \otimes h_2\\
=\ &\alpha_H^{-1}(h_{11})_1 \otimes \alpha_H(h_{11})_2 \otimes \alpha_H^2(h_{12}) \otimes \alpha_H(h_2)_1 \otimes \alpha_H^{-1}(h_2)_2
\end{aligned}$$

and it follows

$$[(a \star h)(b \star l)](\alpha_A(c) \star \alpha_H(g)) = (\alpha_A(a) \star \alpha_H(h))[(b \star l)(c \star g)].$$

Thus $(A \star H, \alpha_A \star \alpha_H)$ is a Hom-algebra. □

The following lemma is obvious.

Lemma 3.1. *Let (A, α_A) be an (H, α_H)-Hom-bimodule algebra, then there are two Hom-algebra isomorphisms $A \cong A \star 1_H$ via $a \mapsto a \star 1_H$ and $H \cong 1_A \star H$ via $h \mapsto 1_A \star h$. So we denote $ah = (a \star 1_H)(1_A \star h)$ and $ha = (1_A \star h)(a \star 1_H)$.*

As special cases of the Hom-twisted smash product, we get the following examples.

Example 3.1. Let (A, α_A) be a left (H, α_H)-Hom module algebra with the trivial right (H, α_H)-action, that is $a \leftarrow h = \alpha_A(a)\varepsilon(h)$. Then (A, α_A) is an (H, α_H)-Hom-bimodule algebra. The Hom-twisted smash product is actually a Hom-samsh product $(A\#H, \alpha_H\#\alpha_H)$ (see Definition 2.7).

Example 3.2. Let $(H, \mu_H, \eta, \Delta, \varepsilon, \alpha_H)$ be a finite dimensional Hom-Hopf algebra with a bijective antipode S_H. Then (H^*, α_H^*) is an (H, α_H)-Hom-bimodule algebra with module maps: $h \rightarrow f = \alpha_H^{*2}(f_1)\langle f_2, \alpha_H^{-1}(h)\rangle$, $f \leftarrow h = \alpha_H^{*2}(f_2)\langle f_1, S_H^{-2}\alpha_H^{-1}(h)\rangle$. The Drinfeld's double $D(H)$ (see [13]) is defined as a vector space $H^{*cop} \otimes H$ with the multiplication:

$$\begin{aligned}
(f \otimes a)(g \otimes b) &= \langle g_1, S_H^{-1}(a_{22})\rangle\langle g_{22}, a_1\rangle f g_{21} \otimes a_{21} b\\
&= f((\alpha_H^{-4}(a_{11}) \rightarrow \alpha_H^{*-2}(g)) \leftarrow S_H\alpha_H^{-2}(a_2)) \otimes \alpha_H^{-2}(a_{12})b,
\end{aligned}$$

for all $a, b \in H$ and $f, g \in H^*$. The unit is $\varepsilon \otimes 1_H$, $\alpha_{D(H)} = \alpha_H^* \otimes \alpha_H$.

Lemma 3.2. *Let $(A \star H, \alpha_A \star \alpha_H)$ be a Hom-twisted smash product, $a \star 1_H, 1_A \star h \in A \star H$. Then*

$$(a \star 1_H)(1_A \star h) = \alpha_A(a) \star \alpha_H(h), \tag{3.1}$$
$$(1_A \star h)(a \star 1_H) = ((\alpha_H^{-3}(h_{11}) \rightarrow \alpha_A^{-1}(a)) \leftarrow S\alpha_H^{-1}(h_2)) \star \alpha_H^{-1}(h_{12}). \tag{3.2}$$

Now we give the main result of the paper as follows.

Theorem 3.1. *Let (A, α_A) be a Hom-bialgebra and an (H, α_H)-Hom-bimodule algebra.*

(1) The Hom-twisted smash product algebra $(A \star H, \alpha_A \star \alpha_H)$ equipped with the tensor product Hom-coalgebra structure (i.e. $\Delta(a \star h) = (a_1 \star h_1) \otimes (a_2 \star h_2), \varepsilon(a \star h) = \varepsilon(a)\varepsilon(h)$) makes $(A \star H, \alpha_A \star \alpha_H)$ into a Hom-bialgebra, if the following conditions hold:

(a) $\varepsilon((\alpha_H^{-2}(h_1) \to \alpha_A^{-1}(a)) \leftarrow S_H\alpha_H^{-1}(h_2)) = \varepsilon(h)\varepsilon(a)$,

(b) $\Delta((\alpha_H^{-2}(h_1) \to \alpha_A^{-1}(a)) \leftarrow S_H\alpha_H^{-1}(h_2)) = ((\alpha_H^{-2}(h_{11}) \to \alpha_A^{-1}(a_1)) \leftarrow S_H\alpha_H^{-1}(h_{12})) \otimes((\alpha_H^{-2}(h_{21}) \to \alpha_A^{-1}(a_2)) \leftarrow S_H\alpha_H^{-1}(h_{22}))$,

(c) $(h_1 \to a) \otimes h_2 = (h_2 \to a) \otimes h_1$,

(d) $a \leftarrow S_H(h_1) \otimes h_2 = a \leftarrow S_H(h_2) \otimes h_1$,

for all $a \in A, h \in H$. Furthermore, if (A, α_A, S_A) is a Hom-Hopf algebra, then $(A \star H, \alpha_A \star \alpha_H)$ is also a Hom-Hopf algebra with the antipode $S_{A\star H}$ defined by

$$S_{A\star H}(a \star h) = (1_A \star S_H(\alpha_H^{-1}(h)))(S_A(\alpha_A^{-1}(a)) \star 1_H).$$

(2) If the right action of (H, α_H) on (A, α_A) satisfies the condition $\varepsilon_A(a \leftarrow h) = \varepsilon_A(a)\varepsilon_H(h)$, then the Hom-twisted smash product algebra $(A \star H, \alpha_A \star \alpha_H)$ equipped with the tensor product Hom-coalgebra structure, makes $(A \star H, \alpha_A \star \alpha_H)$ into a Hom-bialgebra if and only if conditions (a), (b), (c) and (d) in (1) hold.

Proof. (1) It is easy to check $(A \star H, \Delta_{A\star H}, \varepsilon_{A\star H}, \alpha_A \star \alpha_H)$ is a Hom-coalgebra. Taking $a \star h, b \star l \in A \star H$, we have

$$\begin{aligned}
&\Delta[(a \star h)(b \star l)] \\
=\ &\Delta[a((\alpha_H^{-4}(h_{11}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-2}(h_2)) \star \alpha_H^{-2}(h_{12})l] \\
\overset{(2.5)}{=}\ &[a_1((\alpha_H^{-4}(h_{11}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-2}(h_2))_1 \star \alpha_H^{-2}(h_{121})l_1] \\
&\otimes[a_2((\alpha_H^{-4}(h_{11}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-2}(h_2))_2 \star \alpha_H^{-2}(h_{122})l_2] \\
\overset{(2.2)}{=}\ &[a_1((\alpha_H^{-4}(h_{11}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-3}(h_{22}))_1 \star \alpha_H^{-1}(h_{12})l_1] \\
&\otimes[a_2((\alpha_H^{-4}(h_{11}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-3}(h_{22}))_2 \star \alpha_H^{-1}(h_{21})l_2] \\
\overset{(c)}{=}\ &[a_1((\alpha_H^{-4}(h_{12}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-3}(h_{22}))_1 \star \alpha_H^{-1}(h_{11})l_1] \\
&\otimes[a_2((\alpha_H^{-4}(h_{12}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-3}(h_{22}))_2 \star \alpha_H^{-1}(h_{21})l_2] \\
\overset{(d)}{=}\ &[a_1((\alpha_H^{-4}(h_{12}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-3}(h_{21}))_1 \star \alpha_H^{-1}(h_{11})l_1] \\
&\otimes[a_2((\alpha_H^{-4}(h_{12}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-3}(h_{21}))_2 \star \alpha_H^{-1}(h_{22})l_2] \\
=\ &[a_1((\alpha_H^{-5}(h_{211}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-4}(h_{212}))_1 \star h_1l_1] \\
&\otimes[a_2((\alpha_H^{-5}(h_{211}) \to \alpha_A^{-2}(b)) \leftarrow S_H\alpha_H^{-4}(h_{212}))_2 \star \alpha_H^{-1}(h_{22})l_2] \\
\overset{(b)}{=}\ &[a_1((\alpha_H^{-5}(h_{2111}) \to \alpha_A^{-2}(b_1)) \leftarrow S_H\alpha_H^{-4}(h_{2112})) \star h_1l_1] \\
&\otimes[a_2((\alpha_H^{-5}(h_{2121}) \to \alpha_A^{-2}(b_2)) \leftarrow S_H\alpha_H^{-4}(h_{2122})) \star \alpha_H^{-1}(h_{22})l_2] \\
=\ &[a_1((\alpha_H^{-4}(h_{112}) \to \alpha_A^{-2}(b_1)) \leftarrow S_H\alpha_H^{-2}(h_{12})) \star \alpha_H^{-2}(h_{111})l_1] \\
&\otimes[a_2((\alpha_H^{-3}(h_{21}) \to \alpha_A^{-2}(b_2)) \leftarrow S_H\alpha_H^{-3}(h_{221})) \star \alpha_H^{-2}(h_{222})l_2]
\end{aligned}$$

$$\begin{aligned}
&\overset{(c)(d)}{=} [a_1((\alpha_H^{-4}(h_{111}) \to \alpha_A^{-2}(b_1)) \leftarrow S_H\alpha_H^{-2}(h_{12})) \star \alpha_H^{-2}(h_{112})l_1] \\
&\quad \otimes[a_2((\alpha_H^{-3}(h_{21}) \to \alpha_A^{-2}(b_2)) \leftarrow S_H\alpha_H^{-3}(h_{222})) \star \alpha_H^{-2}(h_{221})l_2] \\
&= [a_1((\alpha_H^{-4}(h_{111}) \to \alpha_A^{-2}(b_1)) \leftarrow S_H\alpha_H^{-2}(h_{12})) \star \alpha_H^{-2}(h_{112})l_1] \\
&\quad \otimes[a_2((\alpha_H^{-4}(h_{211}) \to \alpha_A^{-2}(b_2)) \leftarrow S_H\alpha_H^{-2}(h_{22})) \star \alpha_H^{-2}(h_{212})l_2] \\
&= [(a_1 \star h_1)(b_1 \star l_1)] \otimes [(a_2 \star h_2)(b_2 \star l_2)] \\
&= [(a_1 \star h_1) \otimes (a_2 \star h_2)][(b_1 \star l_1) \otimes (b_2 \star l_2)] = \Delta(a \star h)\Delta(b \star l).
\end{aligned}$$

This shows that $\Delta_{A\star H}$ is an algebra map. By condition (a) it is easy to verify that $\varepsilon_{A\star H} = \varepsilon_A \otimes \varepsilon_H$ is also an algebra map. Now we show that $S_{A\star H}$ is the antipode of $A \star H$ as follows:

$$\begin{aligned}
&(a_1 \star h_1)S_{A\star H}(a_2 \star h_2) \\
&= (a_1 \star h_1)[(1_A \star S_H\alpha_H^{-1}(h_2))(S_A\alpha_A^{-1}(a_2) \star 1_H)] \\
&\overset{(2.1)}{=} [(\alpha_A^{-1}(a_1) \star \alpha_H^{-1}(h_1))(1_A \star S_H\alpha_H^{-1}(h_2))](S_A(a_2) \star 1_H) \\
&= [\alpha_A^{-1}(a_1)((\alpha_H^{-5}(h_{111}) \to 1_A) \leftarrow S_H\alpha_H^{-3}(h_{12})) \star \alpha_H^{-3}(h_{112})S_H\alpha_H^{-1}(h_2)](S_A(a_2) \star 1_H) \\
&= [\alpha_H^{-1}(a_1)1_A\varepsilon_H(h_{111})\varepsilon_H(h_{12}) \star \alpha_H^{-3}(h_{112})S_H\alpha_H^{-1}(h_2)](S_A(a_2) \star 1_H) \\
&= [a_1 \star \alpha_H^{-1}(h_1)S_H\alpha_H^{-1}(h_2)](S_A(a_2) \star 1_H) \\
&= (a_1 \star 1_H)(S_A(a_2) \star 1_H)\varepsilon_H(h) = (a_1S_A(a_2) \star 1_H)\varepsilon_H(h) \\
&= 1_A \star 1_H\varepsilon_A(a)\varepsilon_H(h) = 1_A \star 1_H\varepsilon_{A\star H}(a \star h).
\end{aligned}$$

Similarly one can get $S_{A\star H}(a_1 \star h_1)(a_2 \star h_2) = 1_A \star 1_H\varepsilon_{A\star H}(a \star h)$.

(2) ($\Leftarrow$) See (1).

($\Rightarrow$) Condition (a) is a consequence of $\varepsilon(h)\varepsilon(a) = \varepsilon_{A\star H}((1_A \star h)(a \star 1_H)) = \varepsilon((\alpha_H^{-3}(h_{11}) \to \alpha_A^{-1}(a)) \leftarrow S\alpha_H^{-1}(h_2))\varepsilon(\alpha_H^{-1}(h_{12})) = \varepsilon((\alpha_H^{-3}(h_{11}) \to \alpha_A^{-1}(a)) \leftarrow S_H\alpha_H^{-1}(h_2))\varepsilon(h_{12}) = \varepsilon((\alpha_H^{-2}(h_1) \to \alpha_A^{-1}(a)) \leftarrow S_H\alpha_H^{-1}(h_2))$.

Since $\Delta((1_A \star h)(a \star 1_H)) = \Delta(1_A \star h)\Delta(a \star 1_H)$, we get
$[((\alpha_H^{-3}(h_{11}) \to \alpha_A^{-1}(a)) \leftarrow S_H\alpha_H^{-1}(h_2))_1 \star \alpha_H^{-1}(h_{121})] \otimes [((\alpha_H^{-3}(h_{11}) \to \alpha_A^{-1}(a)) \leftarrow S_H\alpha_H^{-1}(h_2))_2 \star \alpha_H^{-1}(h_{122})] = [((\alpha_H^{-3}(h_{111}) \to \alpha_A^{-1}(a_1)) \leftarrow S_H\alpha_H^{-1}(h_{12})) \star \alpha_H^{-1}(h_{112})] \otimes [((\alpha_H^{-3}(h_{211}) \to \alpha_A^{-1}(a_2)) \leftarrow S_H\alpha_H^{-1}(h_{22})) \star \alpha_H^{-1}(h_{212})]$. $(*)$

Applying $Id_A \otimes \varepsilon_H \otimes Id_A \otimes \varepsilon_H$ to $(*)$, we obtain
$((\alpha_H^{-3}(h_{11}) \to \alpha_A^{-1}(a)) \leftarrow S_H\alpha_H^{-1}(h_2))_1 \star \varepsilon(h_{121}) \otimes ((\alpha_H^{-3}(h_{11}) \to \alpha_A^{-1}(a)) \leftarrow S_H\alpha_H^{-1}(h_2))_2 \star \varepsilon(h_{122}) = \Delta((\alpha_H^{-2}(h_1) \to \alpha_A^{-1}(a)) \leftarrow S_H\alpha_H^{-1}(h_2))$
and
$((\alpha_H^{-3}(h_{111}) \to \alpha_A^{-1}(a_1)) \leftarrow S_H\alpha_H^{-1}(h_{12})) \star \varepsilon(h_{112}) \otimes ((\alpha_H^{-3}(h_{211}) \to \alpha_A^{-1}(a_2)) \leftarrow S_H\alpha_H^{-1}(h_{22})) \star \varepsilon(h_{212})] = ((\alpha_H^{-2}(h_{11}) \to \alpha_A^{-1}(a_1)) \leftarrow S_H\alpha_H^{-1}(h_{12})) \otimes ((\alpha_H^{-2}(h_{21}) \to \alpha_A^{-1}(a_2)) \leftarrow S_H\alpha_H^{-1}(h_{22}))$.

It follows that condition (b) holds. Using the fact $\varepsilon_A(a \leftarrow h) = \varepsilon_A(a)\varepsilon_H(h)$ and condition (a), we have $\varepsilon_A(h \to a) = \varepsilon_A(a)\varepsilon_H(h)$. Hence we get $\varepsilon_A((h \to a) \leftarrow l) = \varepsilon_A(h \to a)\varepsilon_H(l) = \varepsilon_A(a)\varepsilon_H(h)\varepsilon_H(l)$.

Applying $\varepsilon_A \otimes Id_H \otimes Id_A \otimes Id_H$ to $(*)$, we have

$\alpha_H^{-1}(h_{121}) \otimes ((\alpha_H^{-2}(h_{11}) \to a) \leftarrow S_H(h_2)) \otimes \alpha_H^{-1}(h_{122}) = \alpha(h_1) \otimes ((\alpha_H^{-3}(h_{211}) \to \alpha_A^{-2}(a)) \leftarrow S_H\alpha_H^{-1}(h_{22})) \otimes \alpha_H^{-1}(h_{212})$. $(**)$

Applying $(Id_H \otimes \leftarrow)(Id_H \otimes Id_A \otimes S_H^2)$ to $(**)$, we obtain

$$\alpha_H(h_1) \otimes [((\alpha_H^{-3}(h_{211}) \to \alpha_A^{-2}(a)) \leftarrow S_H\alpha_H^{-1}(h_{22})) \leftarrow S_H^2\alpha_H^{-1}(h_{212})]$$

$$\begin{aligned}
&\overset{(2.8)}{=} \alpha_H(h_1)\otimes[(\alpha_H^{-2}(h_{211})\to\alpha^{-1}(a))\leftarrow(S_H\alpha_H^{-1}(h_{22})S_H^2\alpha_H^{-2}(h_{212}))]\\
&\overset{(2.2)}{=} h_{11}\otimes[(\alpha_H^{-1}(h_{12})\to\alpha_A^{-1}(a))\leftarrow(S_H\alpha_H^{-1}(h_{22})S_H^2\alpha_H^{-1}(h_{21}))]\\
&\overset{(2.6)}{=} h_{11}\otimes[(\alpha_H^{-1}(h_{12})\to\alpha_A^{-1}(a))\leftarrow 1_H\varepsilon(h_2)]\\
&= \alpha_H(h_1)\otimes(\alpha_H^{-1}(h_2)\to a),
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_H^{-1}(h_{121})\otimes[((\alpha_H^{-2}(h_{11})\to a)\leftarrow S_H(h_2))\leftarrow S_H^2\alpha_H^{-1}(h_{122})]\\
\overset{(2.8)}{=}\ &\alpha_H^{-1}(h_{121})\otimes[(\alpha_H^{-1}(h_{11})\to\alpha_A(a))\leftarrow(S_H(h_2)S_H^2\alpha_H^{-2}(h_{122}))]\\
\overset{(2.2)}{=}\ &h_{12}\otimes[(\alpha_H^{-1}(h_{11})\to\alpha_A(a))\leftarrow(S_H\alpha_H^{-1}(h_{22})S_H^2\alpha_H^{-1}(h_{21}))]\\
\overset{(2.6)}{=}\ &h_{12}\otimes[(\alpha_H^{-1}(h_{11})\to\alpha_A(a))\leftarrow 1_H\varepsilon(h_2)]\\
=\ &\alpha_H(h_2)\otimes(\alpha_H^{-1}(h_1)\to a).
\end{aligned}$$

Thus we have $(\alpha_H^{-1}(h_1)\to a)\otimes\alpha_H(h_2)=(\alpha_H^{-1}(h_2)\to a)\otimes\alpha_H(h_1)$, and this equals to $(h_1\to a)\otimes h_2=(h_2\to a)\otimes h_1$. This means condition (c) holds.

Applying $Id_A\otimes Id_H\otimes\varepsilon_A\otimes Id_H$ to $(*)$ and using (c), we have

$h_{11}\otimes((\alpha_H^{-2}(h_{12})\to a)\leftarrow S_H\alpha_H^{-1}(h_{22}))\otimes h_{21}=\alpha_H^{-1}(h_{111})\otimes((\alpha_H^{-3}(h_{112})\to a)\leftarrow S_H\alpha_H^{-1}(h_{12}))\otimes\alpha_H(h_2)$. $(***)$

Applying $(\to\otimes Id_H)(S_H\otimes Id_A\otimes Id_H)$ to $(***)$, we obtain

$$\begin{aligned}
&[S_H(h_{11})\to((\alpha_H^{-2}(h_{12})\to a)\leftarrow S_H\alpha_H^{-1}(h_{22}))]\otimes h_{21}\\
\overset{(2.8)}{=}\ &[S_H(h_{11})\to(\alpha_H^{-1}(h_{12})\to(a\leftarrow S_H(\alpha_H^{-2}(h_{22}))))]\otimes h_{21}\\
\overset{(2.3)}{=}\ &[(S_H\alpha_H^{-1}(h_{11})\alpha_H^{-1}(h_{12}))\to(\alpha_A(a)\leftarrow S_H\alpha_H^{-1}(h_{22}))]\otimes h_{21}\\
\overset{(2.6)}{=}\ &[\varepsilon(h_1)1_H\to(\alpha_A(a)\leftarrow S_H\alpha_H^{-1}(h_{22}))]\otimes h_{21}\\
=\ &(\alpha_A^2(a)\leftarrow S_H\alpha_H(h_2))\otimes\alpha_H(h_1),
\end{aligned}$$

and

$$\begin{aligned}
&[S_H\alpha_H^{-1}(h_{111})\to(\alpha_H^{-2}(h_{112})\to(a\leftarrow S_H\alpha_H^{-2}(h_{12}))]\otimes\alpha_H(h_2)\\
\overset{(2.3)}{=}\ &[(S_H\alpha_H^{-2}(h_{111})\alpha_H^{-2}(h_{112}))\to(\alpha_A(a)\leftarrow S_H\alpha_H^{-1}(h_{12}))]\otimes\alpha_H(h_2)\\
\overset{(2.6)}{=}\ &[\varepsilon(h_{11})(\alpha_A^2(a)\leftarrow S_H(h_{12}))]\otimes\alpha_H(h_2)\\
=\ &(\alpha_A^2(a)\leftarrow S_H\alpha_H(h_1))\otimes\alpha_H(h_2).
\end{aligned}$$

This means $(\alpha_A^2(a)\leftarrow S_H\alpha_H(h_1))\otimes\alpha_H(h_2)=(\alpha_A^2(a)\leftarrow S_H\alpha_H(h_2))\otimes\alpha_H(h_1)$, and this equals to $(a\leftarrow S_H(h_1))\otimes h_2=(a\leftarrow S_H(h_2))\otimes h_1$. Thus condition (d) holds. The proof is finished. □

Example 3.3. Let $H_4=sp\{1_H,g,x,gx\}$ and the automorphism α defined as: $H_4\to H_4,\alpha(1_H)=1_H,\alpha(g)=g,\alpha(x)=-x,\alpha(gx)=-gx$. Then (H_4,α) is a Hom-algebra with multiplication: $1_H1_H=1_H,1_Hg=g,1_Hx=-x,g^2=1_H,x^2=0,xg=-gx$, and (H_4,α) is a Hom-Hopf algebra with comultiplication, counit and antipode defined by

$$\Delta(1_H)=1_H\otimes 1_H,\Delta(x)=(-x)\otimes g+1_H\otimes(-x),$$

$$\Delta(g) = g \otimes g, \Delta(gx) = xg \otimes 1_H + g \otimes xg,$$
$$\varepsilon(1_H) = 1, \varepsilon(g) = 1, \varepsilon(x) = 0, \varepsilon(gx) = 0,$$
$$S_H(1_H) = 1_H, S_H(g) = g, S_H(x) = -gx, S_H(gx) = x.$$

Let $A = sp\{1_A, a\}$ be the group Hopf algebra with $a^2 = 1_A$ and $\Delta(a) = a \otimes a, S(a) = a = a^{-1}$. Then (A, Id_A) is a Hom-bialgebra.

Define left action $H \otimes A \to A$ as $h \cdot 1_A = \varepsilon(h)1_A, 1_H \cdot a = a, g \cdot a = a, x \cdot a = 0, (gx) \cdot a = 0$ and right action $A \otimes H \to A$ such that $1_A \cdot h = 1_A\varepsilon(h), a \cdot 1_H = a, a \cdot g = a, a \cdot x = 0, a \cdot (gx) = 0$. It is easy to check (A, Id_A) is an (H_4, α)-bimodule algebra.

Thus $(A \star H = \{1_A \otimes 1_H, 1_A \otimes x, 1_A \otimes g, 1_A \otimes gx, a \otimes 1_H, a \otimes x, a \otimes g, a \otimes gx\}, Id_A \otimes \alpha)$ is a Hom-twisted smash product Hopf algebra. Its multiplication is defined as follows:

$\cdot$	$1_A \otimes 1_H$	$1_A \otimes x$	$1_A \otimes g$	$1_A \otimes gx$	$a \otimes 1_H$	$a \otimes x$	$a \otimes g$	$a \otimes gx$
$1_A \otimes 1_H$	$1_A \otimes 1_H$	$1_A \otimes (-x)$	$1_A \otimes g$	$1_A \otimes (xg)$	$a \otimes 1_H$	$a \otimes (-x)$	$a \otimes g$	$a \otimes xg$
$1_A \otimes x$	0	0	0	0	0	0	0	0
$1_A \otimes g$	$1_A \otimes g$	$1_A \otimes gx$	$1_A \otimes 1_H$	$1_A \otimes x$	$a \otimes g$	$a \otimes gx$	$a \otimes 1_H$	$a \otimes x$
$1_A \otimes gx$	0	0	0	0	0	0	0	0
$a \otimes 1_H$	$a \otimes 1_H$	$a \otimes (-x)$	$a \otimes g$	$a \otimes xg$	$1_A \otimes 1_H$	$1_A \otimes (-x)$	$1_A \otimes g$	$1_A \otimes xg$
$a \otimes x$	0	0	0	0	0	0	0	0
$a \otimes g$	$a \otimes g$	$a \otimes gx$	$a \otimes 1_H$	$a \otimes x$	$1_A \otimes g$	$1_A \otimes gx$	$1_A \otimes 1_H$	$1_A \otimes x$
$a \otimes gx$	0	0	0	0	0	0	0	0

Its comultiplication, counit and antipode are defined as follows:

$$\Delta(1_A \otimes 1_H) = (1_A \otimes 1_H) \otimes (1_A \otimes 1_H), \varepsilon(1_A \otimes 1_H) = 1,$$
$$\Delta(1_A \otimes g) = (1_A \otimes g) \otimes (1_A \otimes g), \varepsilon(1_A \otimes g) = 1,$$
$$\Delta(1_A \otimes x) = (1_A \otimes (-x)) \otimes (1_A \otimes g) + (1_A \otimes 1_H) \otimes (1_A \otimes (-x)), \varepsilon(1_A \otimes x) = 0,$$
$$\Delta(1_A \otimes gx) = (1_A \otimes xg) \otimes (1_A \otimes 1_H) + (1_A \otimes g) \otimes (1_A \otimes xg), \varepsilon(1_A \otimes gx) = 0,$$
$$\Delta(a \otimes 1_H) = (a \otimes 1_H) \otimes (a \otimes 1_H), \varepsilon(a \otimes 1_H) = 0,$$
$$\Delta(a \otimes g) = (a \otimes g) \otimes (a \otimes g), \varepsilon(a \otimes g) = 0,$$
$$\Delta(a \otimes x) = (a \otimes (-x)) \otimes (a \otimes g) + (a \otimes 1_H) \otimes (a \otimes (-x)), \varepsilon(a \otimes x) = 0,$$
$$\Delta(a \otimes gx) = (a \otimes xg) \otimes (a \otimes 1_H) + (a \otimes g) \otimes (a \otimes xg), \varepsilon(a \otimes gx) = 0,$$
$$S(1_A \otimes 1_H) = 1_A \otimes 1_H, S(1_A \otimes g) = 1_A \otimes g,$$
$$S(1_A \otimes x) = 1_A \otimes xg, S(1_A \otimes gx) = 1_A \otimes x,$$
$$S(a \otimes 1_H) = a \otimes 1_H, S(a \otimes g) = a \otimes g,$$
$$S(a \otimes x) = a \otimes xg, S(a \otimes gx) = a \otimes x.$$

Definition 3.1. Let (H, α_H) be a Hom-bialgebra. A Hom-coalgebra (B, α_B) is called a left (H, α_H)-Hom module coalgebra if (B, α_B) is a left (H, α_H)-Hom module with action $\to$ obeying the following axioms:

$$\Delta(h \to b) = h_1 \to b_1 \otimes h_2 \to b_2, \quad \varepsilon_B(h \to b) = \varepsilon_H(h)\varepsilon_B(b),$$

for all $b \in B$ and $h \in H$.

If the right action is trivial, then condition (d) in Theorem 3.1 holds and conditions (a) and (b) are satisfied if and only if (A, α_A) is a left (H, α_H)-Hom module coalgebra. Thus we have:

Corollary 3.1. *Let (A,α_A) be a Hom-bialgebra and a left (H,α_H)-Hom-module algebra. Then the usual Hom-smash product $(A\#H,\alpha_A\#\alpha_H)$ equipped with the tensor product Hom-coalgebra structure makes $(A\#H,\alpha_A\#\alpha_H)$ into a Hom-bialgebra if and only if (A,α_A) is a left (H,α_H)-Hom module coalgebra and $(h_1\rightarrow a)\otimes h_2=(h_2\rightarrow a)\otimes h_1$ holds, for all $h\in H, a\in A$.*

Finally, we give a characterization of left $(A\star H,\alpha_A\star\alpha_H)$-Hom module.

Proposition 3.2. *Let (A,α_A) be an (H,α_H)-Hom-bimodule algebra and (M,γ) be a vector space over k. Then (M,γ) is a left $(A\star H,\alpha_A\star\alpha_H)$-Hom module if and only if (M,γ) is a left (A,α_A)-Hom module and a left (H,α_H)-Hom module such that*

$$h\cdot(a\cdot m)=((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-1}(a))\leftarrow S_H(\alpha_H^{-2}(h_2)))\cdot(\alpha_H^{-3}(h_{12})\cdot m), \tag{3.3}$$

for all $h\in H, a\in A$ and $m\in M$.

Proof. $(\Rightarrow)$ Let (M,γ) be a left $(A\star H,\alpha_A\star\alpha_H)$-Hom module with the module action $\rightharpoonup$. We define:

$$a\cdot m=(a\star 1_H)\rightharpoonup m,\qquad h\cdot m=(1_A\star h)\rightharpoonup m.$$

Then (M,γ) is both a left (A,α_A)-Hom module and a left (H,α_H)-Hom module by Lemma 3.1. Moreover,

$$\begin{aligned}
h\cdot(a\cdot m) &= (1_A\star h)\rightharpoonup((a\star 1_H)\rightharpoonup m)\\
&\overset{(2.3)}{=}[(1_A\star\alpha_H^{-1}(h))(a\star 1_H)]\rightharpoonup\gamma(m)\\
&\overset{(3.2)}{=}[((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-1}(a))\leftarrow S_H\alpha_H^{-2}(h_2))\star\alpha_H^{-2}(h_{12})]\rightharpoonup\gamma(m)\\
&\overset{(3.1)}{=}[(((\alpha_H^{-5}(h_{11})\rightarrow\alpha_A^{-2}(a))\leftarrow S_H\alpha_H^{-3}(h_2))\star 1_H)(1_A\star\alpha_H^{-3}(h_{12}))]\rightharpoonup\gamma(m)\\
&= (((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-1}(a))\leftarrow S_H\alpha_H^{-2}(h_2))\star 1_H)\rightharpoonup[(1_A\star\alpha_H^{-3}(h_{12}))\rightharpoonup m]\\
&= ((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-1}(a))\leftarrow S_H\alpha_H^{-2}(h_2))\cdot(\alpha_H^{-3}(h_{12})\cdot m).
\end{aligned}$$

$(\Leftarrow)$ Let $(a\star h)\rightharpoonup m=a\cdot(\alpha_H^{-1}(h)\cdot\gamma^{-1}(m))$. Then $(1_A\star 1_H)\rightharpoonup m=1_A\cdot(1_H\cdot\gamma^{-1}(m))=\gamma(m)$. For any $a\star h, b\star l\in A\star H$ and $m\in M$, we compute

$$\begin{aligned}
&[(a\star h)(b\star l)]\rightharpoonup\gamma(m)\\
&= [a((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H\alpha_H^{-2}(h_2))\star\alpha_H^{-2}(h_{12})l]\rightharpoonup\gamma(m)\\
&= [a((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H\alpha_H^{-2}(h_2))]\cdot((\alpha_H^{-3}(h_{12})\alpha_H^{-1}(l)]\cdot\gamma(m))\\
&\overset{(2.3)}{=}[a((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H\alpha_H^{-2}(h_2))]\cdot(\alpha_H^{-2}(h_{12})\cdot(\alpha_H^{-1}(l)\cdot\gamma^{-1}(m)))\\
&= \alpha_A(a)\cdot[((\alpha_H^{-4}(h_{11})\rightarrow\alpha_A^{-2}(b))\leftarrow S_H\alpha_H^{-2}(h_2))\cdot(\alpha_H^{-3}(h_{12})\cdot(\alpha_H^{-2}(l)\cdot\gamma^{-2}(m)))]\\
&\overset{(3.3)}{=}\alpha_A(a)\cdot[h\cdot(\alpha_A^{-1}(b)\cdot(\alpha_H^{-2}(l)\cdot\gamma^{-2}(m)))]\\
&= (\alpha_A(a)\star\alpha_H(h))\rightharpoonup(b\cdot(\alpha_H^{-1}(l)\cdot\gamma^{-1}(m)))\\
&= (\alpha_A(a)\star\alpha_H(h))\rightharpoonup((b\star l)\rightharpoonup m).
\end{aligned}$$

Thus (M,γ) is a left $(A\star H,\alpha_A\star\alpha_H)$-Hom-module. This finishes the proof. □

References

[1] V. Abramov and S. Silvestrov, *3-Hom-Lie algebras based on σ-derivation and involution*, Adv. Appl. Clifford Algebra, 2020, 30(3), 45, 13 pp.

[2] N. Aizawa and H. Sato, *q-deformation of the Virasoro algebra with central extension*, Phys. Lett. B, 1991, 256(2), 185–190.

[3] S. Caenepeel and I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra, 2011, 39(6), 2216–2240.

[4] Y. Chen, H. Zheng and L. Zhang, *Double Hom-associative algebra and double Hom-Lie bialgebra*, Adv. Appl. Clifford Algebra, 2020, 30(1), 8, 25 pp.

[5] Y. Doi and M. Takeuchi, *Multiplication alteration by two-cocycles in the quantum view*, Comm. Algebra, 1994, 22(14), 5715–5732.

[6] V. G. Drinfeld, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Berkeley, 1986.

[7] N. Hu, *q-Witt algebras, q-Lie algebras, q-holomorph structure and representations*, Algebra Colloq., 1999, 6(1), 51–70.

[8] L. Liu and Q. L. Guo, *On Twisted Smash Products of Monoidal Hom-Hopf Algebras*, Comm. Algebra, 2016, 44(10), 4140–4164.

[9] L. Liu, A. Makhlouf, C. Menini and F. Panaite, *BiHom-Novikov algebras and infinitesimal BiHom-bialgebras*, J. Algebra, 2020, 560, 1146–1172.

[10] T. Ma, H. Yang, L. Liu and Q. Chen, *On unified Hom-Yetter-Drinfeld categories*, J. Geom. Phys., 2019, 144(7), 81–107.

[11] A. Makhlouf and S. Silvestrov, *Hom-algebras structures*, J. Gen. Lie Theory Appl., 2008, 2(2), 52–64.

[12] A. Makhlouf and S. D. Silvestrov, *Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras, in Generalized Lie Theory in Mathematics, Physics and Beyond*, Springer Verlag, Berlin, 2009.

[13] A. Makhlouf and F. Panaite, *Hom-L-R-smash products, Hom-diagonal crossed products and the Drinfel'd double of a Hom-Hopf algebra*, J. Algebra, 2015, 441, 341–343.

[14] R. Molnar, *Semi-direct products of Hopf algebras*, J. Algebra, 1977, 47(1), 29–51.

[15] S. Wang and J. Li, *On twisted smash products for bimodule algebras and the Drinfeld double*, Comm. Algebra, 1998, 26(8), 2435–2444.

[16] D. Yau, *Hom-quantum groups I: Quasi-triangular Hom-bialgebras*, J. Phys. A: Math. Theor., 2012, 45(6), 065203.