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# THE MODIFIED ASSOR-LIKE METHOD FOR SADDLE POINT PROBLEMS

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**Abstract** In this paper, we established the modified accelerated symmetric SOR-like (MASSOR) method for solving the large sparse saddle point systems of linear equations. The convergence of the MASSOR method for solving saddle point problems is analyzed. Numerical examples are presented to show the effectiveness of the proposed method.

**Keywords** Saddle point problem, MASSOR-Like method, convergence analysis, numerical examples.

**MSC(2010)** 65F10, 65F50.

## 1. Introduction

Consider the following iterative solutions of saddle point problems of the form

$$\mathscr{A}u = \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} := b, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ is a matrix of full column rank, $p \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$ with $m \geq n$, $B^{\mathrm{T}}$ denotes the transpose of the matrix $B$. This kind of linear systems arise in many of scientific and engineering applications including mixed finite element approximation of elliptic PDEs [1, 38], Stokes equations and Navier-Stokes equations [21–23, 25], computational fluid dynamics [24], weighted least-squares problems [32, 46], the electronic networkss [13], a Karush-Kuhn-Tucker (KKT) system [18, 30] and so on.

Usually, the coefficient matrices $A, B$ of (1.1) are large and sparse in applications, iterative methods become more effective because of storage requirements and preservation sparsity. For solving the saddle point problem (1.1), Golub, Wu and Yuan [26] have proposed the SOR-like method by applying the splitting $\mathscr{A} = D - L - U$, where

$$D = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ B^{\mathrm{T}} & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -B \\ 0 & Q \end{pmatrix},$$

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with $Q \in \mathbb{R}^{n\times n}$ being nonsingular symmetric. Thus, the SOR-like iteration takes the following form

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} - \omega A^{-1}(By^{(k)} - p), \\ y^{(k+1)} = y^{(k)} + \omega Q^{-1}(B^{\mathrm{T}}x^{(k+1)} - q). \end{cases}$$

By taking advantages of the same splitting above, Darvishi and Hessari [19] studied SSOR method. Then the SSOR-like iteration takes the following form

$$\begin{cases} x^{(k+\frac{1}{2})} = (1-\omega)x^{(k)} - \omega A^{-1}(By^{(k)} - p), \\ y^{(k+\frac{1}{2})} = y^{(k)} + \omega Q^{-1}(B^{\mathrm{T}}x^{(k+\frac{1}{2})} - q), \\ y^{(k+1)} = y^{(k+\frac{1}{2})} + \frac{\omega}{1-\omega} Q^{-1}(B^{\mathrm{T}}x^{(k+\frac{1}{2})} - q), \\ x^{(k+1)} = (1-\omega)x^{(k+\frac{1}{2})} - \omega A^{-1}(By^{(k+1)} - p). \end{cases}$$

Moreover, Darvishi and Hessari [20] considered the splitting $\mathscr{A} = D - L - U$, where

$$D = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ B^{\mathrm{T}} & \beta Q \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -B \\ 0 & (1-\beta)Q \end{pmatrix},$$

with $\beta \in \mathbb{R}$, and presented a modified SSOR method:

$$\begin{cases} x^{(k+\frac{1}{2})} = (1-\omega)x^{(k)} - \omega A^{-1}(By^{(k)} - p), \\ y^{(k+\frac{1}{2})} = y^{(k)} + \frac{\omega}{1-\beta\omega} Q^{-1}(B^{\mathrm{T}}x^{(k+\frac{1}{2})} - q), \\ y^{(k+1)} = y^{(k+\frac{1}{2})} + \frac{\omega}{1-\omega+\beta\omega} Q^{-1}(B^{\mathrm{T}}x^{(k+\frac{1}{2})} - q), \\ x^{(k+1)} = (1-\omega)x^{(k+\frac{1}{2})} - \omega A^{-1}(By^{(k+1)} - p). \end{cases}$$

Recently, Saberi and Najafi [36] considered the splitting $\mathscr{A} = D - L - U$, where

$$D = \begin{pmatrix} \alpha A & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ B^{\mathrm{T}} & \beta Q \end{pmatrix}, \quad U = \begin{pmatrix} (1-\alpha)A & -B \\ 0 & (1-\beta)Q \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ and established the NMSSOR method. In addition, a number of efficient iterative methods have been studied in the literature, such as the Uzawa-type methods [10, 11, 47], the SOR-like methods and its variants [15, 27, 33, 34, 40, 41, 48], the Hermitian and skew-Hermitian splitting (HSS) methods and its variants [2–9], the preconditioned Krylov subspace methods [12, 14, 16, 17, 29, 31, 39] and so on.

In this paper, by taking advantages of the same splitting [28, 35, 37], we establish the modified accelerated symmetric SOR-like (MASSOR) method for solving (1.1).

The remainder of this paper is organized as follows. In Section 2, we introduce the modified ASSOR-like (MASSOR) method for solving saddle point problems. The convergence of the MASSOR method is studied in Section 3. In Section 4, some numerical experiments are performed to illustrate the effectiveness of the proposed method. Finally, some brief conclusions are given in Section 5.

## 2. Modified ASSOR method

For the coefficient matrix of (1.1), we consider the following matrix splitting

$$\mathscr{A} = D - L - U, \tag{2.1}$$

where $D$ is the block diagonal matrix and $L, U$ are full block lower-upper triangular matrices, with

$$D = \begin{pmatrix} \alpha A & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} -A & 0 \\ B^{\mathrm{T}} & \beta Q \end{pmatrix}, \quad U = \begin{pmatrix} \alpha A & -B \\ 0 & (1-\beta)Q \end{pmatrix}$$

Here $Q \in \mathbb{R}^{n\times n}$ is a nonsingular symmetric matrix, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$.

Let $b = (p^{\mathrm{T}}, -q^{\mathrm{T}})^{\mathrm{T}}$ and $u^{(i)} = ((x^{(i)})^{\mathrm{T}}, (y^{(i)})^{\mathrm{T}})^{\mathrm{T}}$ be the $i$th approximate solution of the system (1.1), $\omega \neq 0$. By using the forward SOR method, we get

$$(D - \omega L)u^{(k+\frac{1}{2})} = \big((1-\omega)D + \omega U\big)u^{(k)} + \omega b.$$

That is

$$u^{(k+\frac{1}{2})} = \mathcal{L}_{\alpha,\beta,\omega} u^{(k)} + \omega(D-\omega L)^{-1} b, \tag{2.2}$$

where

$$\begin{aligned} \mathcal{L}_{\alpha,\beta,\omega} &= (D-\omega L)^{-1}\big((1-\omega)D + \omega U\big) \\ &= \begin{pmatrix} \frac{\alpha}{\alpha+\omega} I_m & -\frac{\omega}{\alpha+\omega} A^{-1}B \\ \frac{\alpha\omega}{(\alpha+\omega)(1-\beta\omega)} Q^{-1}B^{\mathrm{T}} & I_n - \frac{\omega^2}{(\alpha+\omega)(1-\beta\omega)} Q^{-1}B^{\mathrm{T}}A^{-1}B \end{pmatrix}. \end{aligned}$$

Similar to the backward SOR method, we have

$$(D - \omega U)u^{(k+1)} = \big((1-\omega)D + \omega L\big)u^{(k+\frac{1}{2})} + \omega b,$$

i.e.,

$$u^{(k+1)} = \mathcal{U}_{\alpha,\beta,\omega} u^{(k+\frac{1}{2})} + \omega(D-\omega U)^{-1} b, \tag{2.3}$$

where

$$\begin{aligned} \mathcal{U}_{\alpha,\beta,\omega} &= (D-\omega U)^{-1}\big((1-\omega)D + \omega L\big) \\ &= \begin{pmatrix} \frac{\alpha-\alpha\omega-\omega}{\alpha(1-\omega)} I_m - \frac{\omega^2}{\alpha(1-\omega)(1-\omega+\beta\omega)} A^{-1}BQ^{-1}B^{\mathrm{T}} & -\frac{\omega}{\alpha(1-\omega)} A^{-1}B \\ \frac{\omega}{1-\omega+\beta\omega} Q^{-1}B^{\mathrm{T}} & I_n \end{pmatrix}. \end{aligned}$$

Note that

$$(D-\omega L) = \begin{pmatrix} (\alpha+\omega)A & 0 \\ -\omega B^T & (1-\beta\omega)Q \end{pmatrix}, \ (D-\omega U) = \begin{pmatrix} \alpha(1-\omega)A & \omega B \\ 0 & (1-\omega+\beta\omega)Q \end{pmatrix}.$$

Since $A$ is SPD and Q is nonsingular, therefore

$$\begin{aligned} \det(D-\omega L) &= (\alpha+\omega)^m (1-\beta\omega)^n \det(A)\det(Q) \neq 0, \\ \det(D-\omega U) &= \alpha^m (1-\omega)^m (1-\omega+\beta\omega)^n \det(A)\det(Q) \neq 0 \end{aligned}$$

if and only if $(\alpha+\omega)(1-\beta\omega)\neq 0$ and $\alpha(1-\omega)(1-\omega+\beta\omega)\neq 0$ , i.e.

$$\alpha \neq 0,\ \omega \neq \{1, -\alpha, \frac{1}{\beta}, \frac{1}{1-\beta}\}. \tag{2.4}$$

From equations (2.2) and (2.3), we get the modified ASSOR (MASSOR) iterative method as follows:

$$u^{(k+1)} = \mathcal{H}_{\alpha,\beta,\omega}u^{(k)} + \mathcal{M}^{-1}_{\alpha,\beta,\omega}b, \tag{2.5}$$

with

$$\mathcal{H}_{\alpha,\beta,\omega} = \mathcal{L}_{\alpha,\beta,\omega}\mathcal{U}_{\alpha,\beta,\omega} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where

$$\begin{aligned}
H_{11} &= \frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}I_m - \frac{\omega^2(2-\omega)}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}A^{-1}BQ^{-1}B^{\mathrm{T}},\\
H_{12} &= -\frac{\omega(2-\omega)}{(1-\omega)(\alpha+\omega)}A^{-1}B\\
&\quad + \frac{\omega^3(2-\omega)}{\alpha(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}A^{-1}BQ^{-1}B^{\mathrm{T}}A^{-1}B,\\
H_{21} &= \frac{\alpha\omega(2-\omega)}{(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}Q^{-1}B^{\mathrm{T}},\\
H_{22} &= I_n - \frac{\omega^2(2-\omega)}{(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}Q^{-1}B^{\mathrm{T}}A^{-1}B,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_{\alpha,\beta,\omega} &= \frac{1}{\omega(2-\omega)}(D-\omega L)D^{-1}(D-\omega U)\\
&= \frac{1}{\omega(2-\omega)}\begin{pmatrix} (\alpha+\omega)(1-\omega)A & \frac{\omega(\alpha+\omega)}{\alpha}B \\ -\omega(1-\omega)B^{\mathrm{T}} & (1-\beta\omega)(1-\omega+\beta\omega)Q - \frac{\omega^2}{\alpha}B^{\mathrm{T}}A^{-1}B \end{pmatrix}.
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{N}_{\alpha,\beta,\omega} &= \mathcal{M}_{\alpha,\beta,\omega} - \mathscr{A}\\
&= \frac{1}{\omega(2-\omega)}\begin{pmatrix} (\alpha-\alpha\omega-\omega)A & -\frac{\omega(\alpha-\alpha\omega-\omega)}{\alpha}B \\ \omega B^{\mathrm{T}} & (1-\beta\omega)(1-\omega+\beta\omega)Q - \frac{\omega^2}{\alpha}B^{\mathrm{T}}A^{-1}B \end{pmatrix},
\end{aligned}$$

then the MASSOR method can also be induced by the splitting

$$\mathscr{A} = \mathcal{M}_{\alpha,\beta,\omega} - \mathcal{N}_{\alpha,\beta,\omega},$$

the matrix $\mathcal{M}_{\alpha,\beta,\omega}$ can be regarded as the preconditioner of the linear system (1.1) and $\mathcal{H}_{\alpha,\beta,\omega} = \mathcal{M}^{-1}_{\alpha,\beta,\omega}\mathcal{N}_{\alpha,\beta,\omega}$ is the iteration matrix of the MASSOR method.

Therefore, the MASSOR method has the following algorithmic form.

**The MASSOR method.** Let Q be a nonsingular symmetric matrix. Given initial vectors $x^{(0)} \in \mathbb{R}^m$ and $y^{(0)} \in \mathbb{R}^n$ , and three relaxed parameters $\alpha, \beta, \omega$ satisfied

(2.4). For $k = 0, 1, 2, \ldots$ until the iteration sequence $\{((x^{(k)})^{\mathrm{T}}, (y^{(k)})^{\mathrm{T}})^{\mathrm{T}}\}$ converges, compute

$$\begin{cases} x^{(k+\frac{1}{2})} = \frac{\alpha}{\alpha+\omega} x^{(k)} - \frac{\omega}{\alpha+\omega} A^{-1}(By^{(k)} - p), \\ y^{(k+\frac{1}{2})} = y^{(k)} + \frac{\omega}{1-\beta\omega} Q^{-1}(B^{\mathrm{T}} x^{(k+\frac{1}{2})} - q), \\ y^{(k+1)} = y^{(k+\frac{1}{2})} + \frac{\omega}{1-\omega+\beta\omega} Q^{-1}(B^{\mathrm{T}} x^{(k+\frac{1}{2})} - q), \\ x^{(k+1)} = \frac{\alpha-\alpha\omega-\omega}{\alpha(1-\omega)} x^{(k+\frac{1}{2})} - \frac{\omega}{\alpha(1-\omega)} A^{-1}(By^{(k+1)} - p). \end{cases} \tag{2.6}$$

It is obvious that when $\beta = \frac{1}{2}$, the MASSOR method reduces to the ASSOR method [35]. In fact, the MASSOR method is a special case of the SSOR-like methods [43, 44].

# 3. Convergence of the MASSOR method

In this section, the convergence conditions of the MASSOR method are obtained. Firstly, we gave some lemmas, which are useful for our discussions.

**Lemma 3.1** (Young [45]). *Consider the quadratic equation $x^2 - bx + c = 0$, where $b$ and $c$ are real numbers. Both roots of the equation are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

**Lemma 3.2.** *If $\lambda$ is an eigenvalue of $\mathcal{H}_{\alpha,\beta,\omega}$, then $\lambda \neq 1$.*

**Proof.** Let $\lambda = 1$ be an eigenvalue of iteration matrix $\mathcal{H}_{\alpha,\beta,\omega}$ and $z = (x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}}$ be the corresponding eigenvector, i.e.

$$\mathcal{H}_{\alpha,\beta,\omega} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

equivalently,

$$\begin{cases} x = -A^{-1}By, \\ \alpha Q^{-1}B^{\mathrm{T}}x = \omega Q^{-1}B^{\mathrm{T}}A^{-1}By. \end{cases} \tag{3.1}$$

It follows from $\omega \neq -\alpha$, $Q$ is nonsingular and B is full column rank that $x = 0$ and $y = 0$, which contradicts that $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}}$ is an eigenvector of the iteration matrix $\mathcal{H}_{\alpha,\beta,\omega}$. Therefore, $\lambda \neq 1$. □

**Theorem 3.1.** *Suppose that $\lambda$ and $\mu$ satisfy*

$$\lambda\omega^2(2-\omega)^2\mu = (\lambda-1)(1-\beta\omega)(1-\omega+\beta\omega)\big((\alpha-\alpha\omega-\omega) - \lambda(1-\omega)(\alpha+\omega)\big). \tag{3.2}$$

*If $\lambda$ is an eigenvalue of $\mathcal{H}_{\alpha,\beta,\omega}$ such that $\lambda \neq \frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}$ and $\omega \neq \{\frac{1}{\beta}, \frac{1}{1-\beta}\}$, then $\mu$ is a nonzero eigenvalue of the matrix $Q^{-1}B^{\mathrm{T}}A^{-1}B$. Conversely, if $\mu$ is an eigenvalue of $Q^{-1}B^{\mathrm{T}}A^{-1}B$, then $\lambda$ is an eigenvalue of the matrix $\mathcal{H}_{\alpha,\beta,\omega}$.*

**Proof.** Let $\lambda$ be an eigenvalue of iteration matrix $\mathcal{H}_{\alpha,\beta,\omega}$ and $z = (x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}}$ be the corresponding eigenvector. Then we get

$$\mathcal{H}_{\alpha,\beta,\omega} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},$$

equivalently,

$$\begin{cases} \alpha\big((\alpha-\alpha\omega-\omega)-\lambda(\alpha+\omega)(1-\omega)\big)Ax=\omega\big(\lambda(\alpha+\omega)+(\alpha-\alpha\omega-\omega)\big)By, \\ \omega\big(1+\lambda(1-\omega)\big)B^\top x=(\lambda-1)\big((1-\beta\omega)(1-\omega+\beta\omega)Q-\frac{\omega^2}{\alpha}B^\top A^{-1}B\big)y. \end{cases} \tag{3.3}$$

From the first equation, it follows from the assumption that

$$x=\frac{\lambda\omega(\alpha+\omega)+\omega(\alpha-\alpha\omega-\omega)}{\alpha(\alpha-\alpha\omega-\omega)-\lambda\alpha(1-\omega)(\alpha+\omega)}A^{-1}By.$$

Taking the place of $x$ in the second equation yields

$$\lambda\omega^2(2-\omega)^2Q^{-1}B^{\mathrm{T}}A^{-1}By=(\lambda-1)(1-\beta\omega)(1-\omega+\beta\omega)\big((\alpha-\alpha\omega-\omega)-\lambda(1-\omega)(\alpha+\omega)\big)y,$$

it follows from (3.2) that

$$Q^{-1}B^{\mathrm{T}}A^{-1}By=\mu y.$$

Thus $\mu$ is an eigenvalue of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ and $y$ is the corresponding eigenvector. Moreover, if $\mu=0$ then $Q^{-1}B^{\mathrm{T}}A^{-1}B$ is a singular matrix, which is impossible. So $\mu$ is a nonzero eigenvalue of $Q^{-1}B^{\mathrm{T}}A^{-1}B$.

We can prove the second assertion by reversing the process. □

**Lemma 3.3.** *Suppose that $\mathcal{H}_{\alpha,\beta,\omega}$ is the iteration matrix of ASSOR method, $\alpha,\beta,\omega$ satisfy (2.4) and $m\geqslant n$. Then*

**(a)** *If $m>n$, then $\lambda=\frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}\neq 0$ is an eigenvalue of $\mathcal{H}_{\alpha,\beta,\omega}$ at least with multiplicity $m-n$.*

**(b)** *If $m=n$, then $\lambda=\frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}\neq 0$ is not an eigenvalue of $\mathcal{H}_{\alpha,\beta,\omega}$.*

**Proof.** See [35]. □

Then, we have the following convergence results.

**Theorem 3.2.** *Let $A\in\mathbb{R}^{m\times m}$ be symmetric positive definite and $B\in\mathbb{R}^{m\times n}$ be of full column rank. Assume that $\alpha,\beta$ and $\omega$ satisfy $\alpha\omega(\alpha+\omega)(1-\beta\omega)(2-\omega)(1-\omega+\beta\omega)(1-\omega)\neq 0$, $Q\in\mathbb{R}^{n\times n}$ is nonsingular symmetric such that all eigenvalues $\mu$ of $Q^{-1}B^TA^{-1}B$ are real number, let $\mu_{\max}=\max\,\{\mu\}$, $\mu_{\min}=\min\{\mu\}$. Then*

**(1)** *If $\mu_{\min}>0$, the MASSOR method is convergent if the following conditions hold:*

- *For $\omega\in(0,1)\cup(2,+\infty)$, it holds $\alpha>\frac{\omega^2}{2(1-\omega)}$; for $\omega\in(-\infty,0)\cup(1,2)$, it holds $\alpha<\frac{\omega^2}{2(1-\omega)}$;*
- *$(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)>0$;*
- *$\frac{\omega^2(2-\omega)^2\mu_{\max}}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}<2\big(1+\frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}\big)$.*

**(2)** *if $\mu_{\max}<0$, the MASSOR method is convergent if the following conditions hold:*

- *For $\omega\in(0,1)\cup(2,+\infty)$, it holds $\alpha>\frac{\omega^2}{2(1-\omega)}$; for $\omega\in(-\infty,0)\cup(1,2)$, it holds $\alpha<\frac{\omega^2}{2(1-\omega)}$;*
- *$(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)<0$;*

- $\frac{\omega^2(2-\omega)^2\mu_{\min}}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)} < 2\big(1 + \frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}\big)$.

**Proof.** Making use of Theorem 3.1 and by some algebra, we obtain

$$\lambda^2-\lambda\left(1+\frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}-\frac{\omega^2(2-\omega)^2\mu}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}\right)+\frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}=0.$$

By Lemma 3.1, $|\lambda| < 1$ if and only if

$$\left|\frac{\alpha - \alpha\omega - \omega}{(1-\omega)(\alpha+\omega)}\right| < 1, \tag{3.4}$$

and

$$\left|1+\frac{\alpha - \alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}-\frac{\omega^2(2-\omega)^2\mu}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}\right| < 1 + \frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}. \tag{3.5}$$

Firstly, it follows from (3.4) that

$$0 < \frac{\omega(2-\omega)}{(1-\omega)(\alpha+\omega)} < 2,$$

noticing $\omega \neq \{1, -\alpha\}$, the above inequality is equivalent to

$$\begin{cases} \omega(2-\omega) > 0, \\ (1-\omega)(\alpha+\omega) > 0, \\ \omega(2-\omega) < 2(1-\omega)(\alpha+\omega), \end{cases} \quad \text{or} \quad \begin{cases} \omega(2-\omega) < 0, \\ (1-\omega)(\alpha+\omega) < 0, \\ \omega(2-\omega) > 2(1-\omega)(\alpha+\omega). \end{cases}$$

By some algebra we get

$$\begin{cases} 0 < \omega < 1, \quad \alpha > \frac{\omega^2}{2(1-\omega)}, \\ 1 < \omega < 2, \quad \alpha < \frac{\omega^2}{2(1-\omega)}, \end{cases} \quad \text{or} \quad \begin{cases} \omega < 0, \quad \alpha < \frac{\omega^2}{2(1-\omega)}, \\ \omega > 2, \quad \alpha > \frac{\omega^2}{2(1-\omega)}, \end{cases}$$

i.e.,

$$\begin{cases} \omega \in (0,1) \cup (2, +\infty), \\ \alpha > \frac{\omega^2}{2(1-\omega)}, \end{cases} \quad \text{or} \quad \begin{cases} \omega \in (-\infty, 0) \cup (1, 2), \\ \alpha < \frac{\omega^2}{2(1-\omega)}. \end{cases} \tag{3.6}$$

Secondly, it follows from (3.5) that

$$0 < \frac{\omega^2(2-\omega)^2\mu}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)} < 2\left(1 + \frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}\right). \tag{3.7}$$

Therefore, if $\mu_{min} > 0$, then (3.7) holds if

$$(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega) > 0,$$

and

$$0 < \frac{\omega^2(2-\omega)^2\mu}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)} \leqslant \frac{\omega^2(2-\omega)^2\mu_{\max}}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}$$

$$< 2\left(1 + \frac{\alpha - \alpha\omega - \omega}{(1-\omega)(\alpha+\omega)}\right).$$

And if $\mu_{\max} < 0$, then (3.7) holds if

$$(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega) < 0,$$

and

$$0 < \frac{\omega^2(2-\omega)^2\mu}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)} < \frac{\omega^2(2-\omega)^2\mu_{\min}}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}$$
$$< 2\left(1 + \frac{\alpha - \alpha\omega - \omega}{(1-\omega)(\alpha+\omega)}\right).$$

Together this with (3.6), thus the desired results hold. □

According to Lemma 3.3, we obtain the following results.

**Corollary 3.1.** *Let $A$ be symmetric positive definite and $B$ be of full column rank, and let $Q$ be nonsingular symmetric. If $\mu$ is an eigenvalue of $Q^{-1}B^{\mathrm{T}}A^{-1}B$, then the eigenvalues of the matrix $\mathcal{H}_{\alpha,\beta,\omega}$ are given by $\lambda = \frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)}$ and $\lambda = \frac{1}{2}(t \pm \sqrt{t^2-4h})$, where $t = 1 + \frac{\alpha-\alpha\omega-\omega}{(1-\omega)(\alpha+\omega)} - \frac{\omega^2(2-\omega)^2\mu}{(1-\omega)(\alpha+\omega)(1-\beta\omega)(1-\omega+\beta\omega)}$ and $h = \frac{(\alpha-\alpha\omega-\omega)}{(1-\omega)(\alpha+\omega)}$.*

**Remark 3.1.** The assumption that all eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are real in Theorem 3.2 is reasonable. According to Theorem 3.2, the MASSOR method is not only suitable to the case that all the eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are positive but also to the case that all the eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are negative.

## 4. Numerical examples

In this section, we give some examples to compare the performance of the MASSOR method, SSOR [19] and MSSOR [42] methods.

All the computations are show in MATLAB 2017a [version 9.2.0.538062] on a personal computer with 3.20 GHz central processing unit (Intel(R) Core(TM) i5-6500 CPU) and 16.00G memory. We report the number of iterations (denoted by 'IT'), elapsed CPU time in seconds ( denoted by 'CPU'). All iteration processes are terminated when the current residuals satisfy $RES < 10^{-9}$, where

$$RES = \mathrm{norm}((x^{(k)})^{\mathrm{T}} - (x^{(k-1)})^{\mathrm{T}}, (y^{(k)})^{\mathrm{T}} - (y^{(k-1)})^{\mathrm{T}})^{\mathrm{T}},$$

with $\{((x^{(k)})^{\mathrm{T}}, (y^{(k)})^{\mathrm{T}})^{\mathrm{T}}\}$ being the current approximate solution, or the number of iteration steps $k_{\max} \leqslant 2000$. In actual computation, we choose the right-hand-side vector $(p^{\mathrm{T}}, q^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{m+n}$ such that the exact solution of (1.1) is $((x_*)^{\mathrm{T}}, (y_*)^{\mathrm{T}})^{\mathrm{T}} = (1, 1, \ldots, 1)^{\mathrm{T}}$ and the initial vector was set to the zero vector.

**Example 4.1** ( [3]). Consider the saddle point problem (1.1), where

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2}, \quad B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times p^2}$$

and

$$T = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{p \times p}, \quad F = \frac{1}{h} tridiag(-1, 1, 0) \in \mathbb{R}^{p \times p}$$

with $\otimes$ being the Kronecker product symbol and $h = \frac{1}{1+p}$ being the discretization mesh-size.

In this example, we let $m = 2p^2$ and $n = p^2$, i.e., the total number of variables is $m + n = 3p^2$. We consider three cases: $p = 8, 16$ and $p = 24$ for solving (1.1). Table 1 lists the minimum and maximum eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ for different preconditioned matrix $Q$.

**Table 1.** The minimum $\mu_{\min}$ and the maximum $\mu_{\max}$ eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$.

| $m$ | | 128 | 512 | 1152 |
|---|---|---|---|---|
| $n$ | | 64 | 256 | 576 |
| $m+n$ | | 192 | 768 | 1728 |
| Case I | $\mu_{\min}$ | 1.6e-3 | 4.4e-4 | 2.0e-4 |
| ($Q = B^{\mathrm{T}}B$) | $\mu_{\max}$ | 4.3e-2 | 4.0e-2 | 3.9e-2 |
| Case II | $\mu_{\min}$ | -4.3e-2 | -4.0e-2 | -3.9e-2 |
| ($Q = -B^{\mathrm{T}}B$) | $\mu_{\max}$ | -1.6e-3 | -4.4e-4 | -2.0e-4 |
| Case III | $\mu_{\min}$ | 1.82 | 9.9e-2 | 6.9e-2 |
| ($Q = \mathrm{tridiag}(B^{\mathrm{T}}A^{-1}B)$) | $\mu_{\max}$ | 1.25 | 1.25 | 1.25 |

Tables 2–4 show the numerical results and the parameters are chosen to be the experimentally found optimal ones that minimize the total number of iteration steps for those methods. By observing the numerical results, it is not difficult to find that the number of iterations and CPU time of the MASSOR method are much lesser than those of the SSOR and MSSOR methods when the experimentally optimal parameter is employed.

**Table 2.** Numerical results of the MASSOR, NMSSOR and MSSOR methods for Case I.

| $m$ | | 128 | 512 | 1152 |
|---|---|---|---|---|
| $n$ | | 64 | 256 | 576 |
| $m+n$ | | 192 | 768 | 1728 |
| SSOR | $\omega_*$ | 0.978 | 0.979 | 0.980 |
| | IT | 288 | 731 | 1513 |
| | CPU | 0.073 | 0.911 | 4.128 |
| MSSOR | $\omega_*$ | 1.5 | 1.8 | 1.8 |
| | $\beta_*$ | 0.65 | 0.45 | 0.55 |
| | IT | 150 | 163 | 413 |
| | CPU | 0.041 | 0.204 | 1.121 |
| MASSOR | $\omega_*$ | 1.5 | 1.8 | 1.8 |
| | $\alpha_*$ | -5.66 | -3.28 | -4.75 |
| | $\beta_*$ | 0.65 | 0.45 | 0.55 |
| | IT | 121 | 153 | 337 |
| | CPU | 0.033 | 0.192 | 0.813 |

**Table 3.** Numerical results of the MASSOR, NMSSOR and MSSOR methods for Case II.

| | | | | |
|---|---|---|---|---|
| $m$ | | 128 | 512 | 1152 |
| $n$ | | 64 | 256 | 576 |
| $m+n$ | | 192 | 768 | 1728 |
| SSOR | $\omega_*$ | 1.023 | 1.021 | 1.020 |
| | IT | 218 | 730 | 1512 |
| | CPU | 0.088 | 1.173 | 5.273 |
| MSSOR | $\omega_*$ | 1.5 | 1.8 | 1.8 |
| | $\beta_*$ | 0.68 | 0.43 | 0.56 |
| | IT | 130 | 585 | 358 |
| | CPU | 0.057 | 0.890 | 1.244 |
| MASSOR | $\omega_*$ | 1.5 | 1.8 | 1.8 |
| | $\alpha_*$ | 0.68 | 0.43 | 0.56 |
| | $\beta_*$ | -5.223 | -5.731 | -4.432 |
| | IT | 106 | 468 | 291 |
| | CPU | 0.046 | 0.735 | 1.009 |

**Table 4.** Numerical results of the MASSOR, NMSSOR and MSSOR methods for Case III.

| | | | | |
|---|---|---|---|---|
| $m$ | | 128 | 512 | 1152 |
| $n$ | | 64 | 256 | 576 |
| $m+n$ | | 192 | 768 | 1728 |
| SSOR | $\omega_*$ | 0.552 | 0.439 | 0.380 |
| | IT | 52 | 180 | 338 |
| | CPU | 0.014 | 0.138 | 0.548 |
| MSSOR | $\omega_*$ | 0.54 | 0.55 | 0.60 |
| | $\beta_*$ | 0.65 | 0.70 | 0.65 |
| | IT | 67 | 142 | 191 |
| | CPU | 0.018 | 0.107 | 0.305 |
| MASSOR | $\omega_*$ | 0.54 | 0.55 | 0.60 |
| | $\alpha_*$ | 2.23 | 3.13 | 4.85 |
| | $\beta_*$ | 0.58 | 0.63 | 0.66 |
| | IT | 52 | 111 | 128 |
| | CPU | 0.013 | 0.085 | 0.218 |

# 5. Conclusions

In this paper,we have discussed a modified accelerated symmetric SOR-Like (MASSOR) method to solve saddle point problem (1.1). The convergence of the proposed method have been given in this context. However, the determination of optimum values of the parameters needs further study.

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