Journal of Applied Analysis and Computation
Volume 11, Number 4, August 2021, 1731–1748
Website:http://www.jaac-online.com
DOI:10.11948/20200041

PERIODIC SOLUTION FOR SECOND-ORDER DAMPED NEUTRAL DIFFERENTIAL EQUATION VIA A FIXED POINT THEOREM OF LERAY-SCHAUDER TYPE*

Zhibo Cheng[1,†], Lisha Lv[1] and Feifan Li[1]

Abstract The aim of this paper is to show that a fixed point theorem of Leray-Schauder type can be applied to damped neutral differential equations. Using the positivity of Green's function, we prove the existence of a positive periodic solution for second-order damped neutral differential equation in the cases that sub-linearity, semi-linearity and super-linearity conditions.

Keywords Periodic solution, neutral equation, second-order, fixed point theorem of Leray-Schauder type.

MSC(2010) 34C25, 34K40.

1. Introduction

The main purpose of this paper is to consider the existence of a positive periodic solution for the following second-order damped neutral differential equation

$$(u(t) - c(t)u(t-\sigma))'' + b(t)u'(t) + a(t)u(t) = f(t, u(t-\sigma)), \tag{1.1}$$

where $c \in C^1(\mathbb{R}, \mathbb{R})$ is an ω-periodic function and $|c(t)| \neq 1$, σ is a constant and $0 \leq \sigma < \omega$, $a,\ b \in C(\mathbb{R}, (0, +\infty))$ are ω-periodic functions, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is an ω-periodic function about t, ω is a positive constant.

The existence of positive ω-periodic solutions has become one of the most important problems in the study of neutral differential equations and lots of work on the existence were obtained by applying the fixed point theorem in cones [13,15–17], Krasnoselskii's fixed point theorem [1–4,7,10], coincidence degree theory [6,11,12].

In 2007 Wu and Wang [17] investigated the following second-order neutral differential equation

$$(u(t) - cu(t-\sigma))'' + a(t)u(t) = \psi b(t) f(u(t - \sigma(t))), \tag{1.2}$$

where c is a constant and $c \in (-1, 0)$, ψ is a constant and $0 < \psi < 1$, $\sigma \in C(\mathbb{R}, \mathbb{R})$ is a positive ω-periodic function. The authors obtained equation (1.2) has

[†]The corresponding author. Email address: czb_1982@126.com (Z. Cheng)
[1]School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China
[*]Research is supported by National Natural Science Foundation of China (11501170), Technological innovation talents in universities and colleges in Henan Province (21HASTIT025) and Fundamental Research Funds for the Universities of Henan Province (NSFRF170302).

a positive ω-periodic solution by applications of the property of neutral operator $(A_1 u)(t) := u(t) - cu(t-\sigma)$. Afterward, Cheung et al. [7] in 2009 enlarged the rang of parameter c, they discussed the existence of a positive ω-periodic solution for the following second-order neutral differential equation

$$(u(t) - cu(t-\sigma(t)))'' + a(t)u(t) = f(t, u(t-\sigma(t))), \tag{1.3}$$

where $c \in (-1,1)$. Recently, applying the property of neutral operator $(A_1 u)(t)$ again, Cheng and Li [4] in 2018 studied the existence of a positive ω-periodic solution for a kind of second-order neutral differential equation

$$(u(t) - cu(t-\sigma))'' + a(t)u(t) = f(t, u(t-\delta(t))), \tag{1.4}$$

where $c \in \left(-\frac{m}{M+m}, \frac{m}{M+m}\right)$, here $m := \min\limits_{t\in[0,\omega]} a(t)$, and $M := \max\limits_{t\in[0,\omega]} a(t)$, $\delta \in C(\mathbb{R},\mathbb{R})$ is a positive ω-periodic function. Afterwards, the authors [4] proved the existence of a positive ω-periodic solution for equation (1.4) in the case that $c \in (-\infty,-1) \cup (1,+\infty)$ if the following condition holds:

(P_1) There exist two constants $r_1,\ R_1$ such that $0 < \frac{M}{m} r_1 < R_1$ and

$$M\left(1 - \frac{1}{c}\right) r_1 \le a(t)u(t-\sigma) - \frac{f(t, u(t-\delta(t)))}{c} \le m\left(1 - \frac{1}{c}\right) R_1,$$

for all $t \in [0,\omega]$ and $u \in [r_1, R_1]$. The above nonlinear term $f(t,u)$ of equations (1.2), (1.3) and (1.4) only satisfied sub-linearity condition, and condition (P_1) is relatively strong. In 2019, by application of a fixed point theorem of Leray-Schauder type, Cheng and Lv [14] proved the following second-order neutral differential equation

$$(u(t) - c(t)u(t-\sigma(t)))'' + a(t)u(t) = f(t, u(t-\delta(t))),$$

where $|c(t)| < \frac{m}{M+m}$.

We are mainly motivated by the recent work [3, 4, 7, 14, 17] and focus on equation (1.1), where the nonlinear term f may satisfy sub-linearity, semi-linearity and super-linearity conditions at infinity. The aim of this paper is to show that fixed point theorem of Leray-Schauder type can be applied to the damped neutral equations. Using the positivity of Green's function and the property of neutral operator $(\mathbf{A}u)(t) := u(t) - c(t)u(t-\sigma)$, we obtain the existence of a positive ω-periodic solution to equation (1.1) in the case that $|c(t)| \neq 1$.

Remark 1.1. In [4, 7, 17], the authors obtained that neutral equations have at least one positive periodic solution, and the range of parameter c is $(-1,1)$ [4], $(-\infty,-1) \cup (-\frac{m}{M+m}, \frac{m}{M+m}) \cup (1,+\infty)$ [7] or $(-1,0)$ [17], respectively. However, in this paper, we enlarge the range of parameter c, i.e. $c \in (-\infty,1) \cup (1,+\infty)$. Our new results generalize some recent results contained in [4, 7, 17].

Remark 1.2. It is worth mentioning that in [4, 7, 17], applying fixed point theorem in cones and Krasnoselskii's fixed point theorem, the authors obtained the existence of positive periodic solutions for equations (1), (1.3) and (1.4) in the case that sub-linearity condition. In this paper, we establish the existence of a positive periodic solution for equation (1.1) in the cases that sub-linearity, semi-linearity and super-linearity conditions. Therefore, our results can be more general.

2. Green's function

Define

$$C_\omega := \{u \in C(\mathbb{R},\mathbb{R}) : u(t+\omega) = u(t),\ \text{for } t \in \mathbb{R}\},$$

with norm $\|u\| := \max_{t\in[0,\omega]} |u(t)|$. Clearly, $(C_\omega, \|\cdot\|)$ is a Banach space.
Define

$$c_0 := \min_{t\in[0,\omega]} |c(t)|, \qquad c_\infty := \max_{t\in[0,\omega]} |c(t)|.$$

We consider the following nonhomogeneous linear differential equation

$$\begin{cases} v''(t) + b_0 v'(t) + a_0 v(t) = h(t), \\ v(0) = v(\omega),\ v'(0) = v'(\omega), \end{cases} \tag{2.1}$$

where b_0, a_0 are two constant, $b_0,\ a_0 > 0$ and $h \in C(\mathbb{R}, (0, +\infty))$ is an ω-periodic function. Equation (2.1) has an unique ω-periodic solution which can be written as

$$v(t) = \int_0^\omega G(t,s)h(s)ds,$$

where $G(t,s)$ is the Green's function for equation (2.1). In the following, we study that Green's function $G(t,s)$ is a positive for all $(t,s) \in [0,\omega] \times [0,\omega]$.

Case 1 $b_0 > 2\sqrt{a_0}$.

Lemma 2.1. *Assume that condition $b_0 > 2\sqrt{a_0}$ holds. Then the Green's function $G(t,s)$ is given by*

$$G(t,s) = \frac{1}{\lambda_2 - \lambda_1} \begin{cases} \frac{e^{\lambda_2(t-s)}}{1-e^{\lambda_2\omega}} - \frac{e^{\lambda_1(t-s)}}{1-e^{\lambda_1\omega}}, & 0 \le s \le t \le \omega, \\ \frac{e^{\lambda_2(t-s+\omega)}}{1-e^{\lambda_2\omega}} - \frac{e^{\lambda_1(t-s+\omega)}}{1-e^{\lambda_1\omega}}, & 0 \le t \le s \le \omega, \end{cases}$$

where λ_1 and λ_2 are the characteristic roots of the homogeneous equation

$$v''(t) + b_0 v'(t) + a_0 v(t) = 0,$$

that is,

$$\lambda_1 = \frac{-b_0 - \sqrt{b_0^2 - 4a_0}}{2}, \quad \lambda_2 = \frac{-b_0 + \sqrt{b_0^2 - 4a_0}}{2},$$

we note that $\lambda_1 < \lambda_2 < 0$. Moreover, the Green's function $G(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$.

Proof. This Lemma is proved in Ref. [8], for convenience of readers, we present the proof as following. Applying the method of variation of constant, we get the general solution of (2.1), which is the following form $v(t) = c_1(t)e^{\lambda_1 t} + c_2(t)e^{\lambda_2 t}$. Therefore, we arrive that

$$c_1'(t) = \frac{-1}{e^{\lambda_1 t}(\lambda_2 - \lambda_1)} h(t),\ c_2'(t) = \frac{1}{e^{\lambda_2 t}(\lambda_2 - \lambda_1)} h(t).$$

Since $v(t),\ v'(t)$ are periodic function, we obtain

$$c_1(t) = \frac{-e^{\lambda_1\omega}}{1-e^{\lambda_1\omega}} \int_t^\omega \frac{1}{e^{\lambda_1 s}(\lambda_2-\lambda_1)} h(s) - \frac{1}{1-e^{\lambda_1\omega}} \int_0^t \frac{1}{e^{\lambda_1 s}(\lambda_2-\lambda_1)} h(s)ds,$$

$$c_2(t) = \frac{e^{\lambda_2\omega}}{1-e^{\lambda_2\omega}}\int_t^\omega \frac{1}{e^{\lambda_2 s}(\lambda_2-\lambda_1)}h(s) + \frac{1}{1-e^{\lambda_2\omega}}\int_0^t \frac{1}{e^{\lambda_2 s}(\lambda_2-\lambda_1)}h(s)ds.$$

Therefore, the solution of (2.1) can be written as

$$\begin{aligned}
v(t) &= c_1(t)e^{\lambda_1 t} + c_2(t)e^{\lambda_2 t}\\
&= \frac{-e^{\lambda_1(t+\omega)}}{1-e^{\lambda_1\omega}}\int_t^\omega \frac{1}{e^{\lambda_1 s}(\lambda_2-\lambda_1)}h(s)ds - \frac{e^{\lambda_1 t}}{1-e^{\lambda_1\omega}}\int_0^t \frac{1}{e^{\lambda_1 s}(\lambda_2-\lambda_1)}h(s)ds\\
&\quad + \frac{e^{\lambda_2(t+\omega)}}{1-e^{\lambda_2\omega}}\int_t^\omega \frac{1}{e^{\lambda_2 s}(\lambda_2-\lambda_1)}h(s)ds + \frac{e^{\lambda_2 t}}{1-e^{\lambda_2\omega}}\int_0^t \frac{1}{e^{\lambda_2 s}(\lambda_2-\lambda_1)}h(s)ds\\
&= \int_t^\omega \frac{-e^{\lambda_1(t+\omega)}}{(1-e^{\lambda_1\omega})e^{\lambda_1 s}(\lambda_2-\lambda_1)}h(s)ds - \int_0^t \frac{e^{\lambda_1 t}}{(1-e^{\lambda_1\omega})e^{\lambda_1 s}(\lambda_2-\lambda_1)}h(s)ds\\
&\quad + \int_t^\omega \frac{e^{\lambda_2(t+\omega)}}{(1-e^{\lambda_2\omega})e^{\lambda_2 s}(\lambda_2-\lambda_1)}h(s)ds + \int_0^t \frac{e^{\lambda_2 t}}{(1-e^{\lambda_2\omega})e^{\lambda_2 s}(\lambda_2-\lambda_1)}h(s)ds\\
&= \int_0^t \left[\frac{e^{\lambda_2 t}}{(1-e^{\lambda_2\omega})e^{\lambda_2 s}(\lambda_2-\lambda_1)} - \frac{e^{\lambda_1 t}}{(1-e^{\lambda_1\omega})e^{\lambda_1 s}(\lambda_2-\lambda_1)}\right]h(s)ds\\
&\quad + \int_t^\omega \left[\frac{-e^{\lambda_1(t+\omega)}}{(1-e^{\lambda_1\omega})e^{\lambda_1 s}(\lambda_2-\lambda_1)} + \frac{e^{\lambda_2(t+\omega)}}{(1-e^{\lambda_2\omega})e^{\lambda_2 s}(\lambda_2-\lambda_1)}\right]h(s)ds\\
&= \int_0^t \left[\frac{e^{\lambda_2 t}}{(1-e^{\lambda_2\omega})e^{\lambda_2 s}(\lambda_2-\lambda_1)} - \frac{e^{\lambda_1 t}}{(1-e^{\lambda_1\omega})e^{\lambda_1 s}(\lambda_2-\lambda_1)}\right]h(s)ds\\
&\quad + \int_t^\omega \left[\frac{-e^{\lambda_1(t+\omega)}}{(1-e^{\lambda_1\omega})e^{\lambda_1 s}(\lambda_2-\lambda_1)} + \frac{e^{\lambda_2(t+\omega)}}{(1-e^{\lambda_2\omega})e^{\lambda_2 s}(\lambda_2-\lambda_1)}\right]h(s)ds\\
&= \int_0^t \left[\frac{1}{(\lambda_2-\lambda_1)}\left(\frac{e^{\lambda_2(t-s)}}{(1-e^{\lambda_2\omega})} - \frac{e^{\lambda_1(t-s)}}{(1-e^{\lambda_1\omega})}\right)\right]h(s)ds\\
&\quad + \int_t^\omega \left[\frac{1}{(\lambda_2-\lambda_1)}\left(\frac{e^{\lambda_2(t-s+\omega)}}{(1-e^{\lambda_2\omega})} - \frac{e^{\lambda_1(t-s+\omega)}}{(1-e^{\lambda_1\omega})}\right)\right]h(s)ds.
\end{aligned}$$

We get the Green's function of equation (2.1), that is,

$$G(t,s) = \frac{1}{\lambda_2-\lambda_1}\begin{cases} \frac{e^{\lambda_2(t-s)}}{1-e^{\lambda_2\omega}} - \frac{e^{\lambda_1(t-s)}}{1-e^{\lambda_1\omega}}, & 0 \le s \le t \le \omega,\\ \frac{e^{\lambda_2(t-s+\omega)}}{1-e^{\lambda_2\omega}} - \frac{e^{\lambda_1(t-s+\omega)}}{1-e^{\lambda_1\omega}}, & 0 \le t \le s \le \omega. \end{cases}$$

□

Case 2 $b_0 = 2\sqrt{a_0}$.

Lemma 2.2. *Assume that condition $b_0 = 2\sqrt{a_0}$ holds. Then the Green function $G(t,s)$ is given by*

$$G(t,s) = \frac{1}{e^{m\omega}-1}\begin{cases} e^{\sqrt{a_0}(t-s)}\left[\frac{\omega e^{\sqrt{a_0}\omega}}{e^{\sqrt{a_0}\omega}-1} + s - t\right], & 0 \le s \le t \le \omega,\\ e^{\sqrt{a_0}(\omega-s+t)}\left[\frac{\omega}{e^{\sqrt{a_0}\omega}-1} + s - t\right], & 0 \le t \le s \le \omega. \end{cases}$$

Moreover, the Green's function $G(t,s) > 0$ for all $(t,s) \in [0,\omega]\times[0,\omega]$.

Proof. Similar to the proof of Lemma 2.1. □

Case 3 $b_0 < 2\sqrt{a_0}$.

Lemma 2.3. *Assume that condition $b_0 < 2\sqrt{a_0}$ holds. Furthermore, suppose that the following inequality holds:*

$$a_0 < \left(\frac{\pi}{\omega}\right)^2 + \left(\frac{b_0}{2}\right)^2. \tag{2.2}$$

Then the Green function $G(t,s) > 0$ is given by

$$G(t,s) = \frac{1}{\beta\kappa}\begin{cases} e^{\alpha(\omega+t-s)}\sin(\beta(\omega+s-t)) + e^{\alpha(t-s)}\sin(\beta(t-s)), & 0\le s\le t\le \omega \\ e^{\alpha(\omega+t-s)}\sin(\beta(\omega+t-s)) + e^{\alpha(2\omega+t-s)}\sin(\beta(s-t)), & 0\le t\le s\le \omega, \end{cases}$$

where $\alpha := -\frac{b_0}{2}$, $\beta := \frac{\sqrt{4a_0-b_0^2}}{2}$, $\kappa := 1-2e^{\alpha\omega}\cos(\beta\omega)+e^{2\alpha\omega}$.

Proof. Similar to the proof of Lemma 2.1. □

Lemma 2.4 ([8, Appendix A]). *Assume that condition $b_0 < 2\sqrt{a_0}$ and* (2.2) *hold. Then $\int_0^\omega G(t,s)ds = \frac{1}{a_0}$. Moreover, the Green's function $G(t,s) > 0$ for all $(t,s) \in [0,\omega]\times[0,\omega]$.*

On the other hand, we consider the nonhomogeneous linear differential equation

$$\begin{cases} v''(t) + b(t)v'(t) + a(t)v(t) = h(t), \\ v(0) = v(\omega),\ v'(0) = v'(\omega). \end{cases} \tag{2.3}$$

Equation (2.3) has an unique ω-periodic solution which can be written as

$$v(t) = \int_0^\omega \tilde{G}(t,s)h(s)ds,$$

where $\tilde{G}(t,s)$ is the Green function of the equation (2.3).

Lemma 2.5 ([5, Lemma 2.2]). *Assume that the following condition holds:*
(A) There exist continuous ω-functions $a_1(t)$ and $a_2(t)$ such that $\int_0^\omega a_1(t)dt > 0$, $\int_0^\omega a_2(t)dt > 0$ and

$$a_1(t) + a_2(t) = b(t), \quad a_1'(t) + a_1(t)a_2(t) = a(t), \quad \textit{for } t \in R.$$

Then $\int_0^\omega \tilde{G}(t,s)ds = \frac{1}{a(t)}$. Moreover, $\tilde{G}(t,s) > 0$ for all $(t,s) \in [0,\omega]\times[0,\omega]$.

3. Positive periodic solution for equation (1.1) in the case that $|c(t)| < \frac{a_0 - b_\infty c'_\infty}{a_\infty + \delta b_\infty + a_0}$, where $\delta := \frac{\max_{t\in[0,\omega]}\left|\frac{\partial G(t,s)}{\partial t}\right|}{l}$, l is the minimum of $G(t,s)$ on $\mathbb{R}\times\mathbb{R}$

At first, we recall a fixed point theorem of Leray-Schauder type will be used in what following.

Lemma 3.1 ([9, Theorem 5]). *Let $\mathfrak{B}(0,r_1)$ (respectively, $\mathfrak{B}[0,r_1]$) be the open ball (respectively, the closed ball) in a Banach space $X = (X, \|\cdot\|)$ with center 0 and radius r_1. Suppose $\mathcal{A}, \mathcal{B} : X \to X$ are two operators satisfying the following conditions:*

(a) $\mathcal{A}$ *is a contraction;*

(b) $\mathcal{B}$ *is continuous and completely continuous.*

Then either

(i) $\exists\, u \in \mathfrak{B}[0, r_1]$ *with* $u = \mathcal{A}u + \mathcal{B}u$*;*
or

(ii) $\exists\, u \in \partial\mathfrak{B}[0, r_1]$ *and* $\lambda \in (0,1)$ *with* $u = \lambda\mathcal{A}(\frac{u}{\lambda}) + \lambda\mathcal{B}u$*.*

Lemma 3.2 ([18, Theorem 4.1])**.** *If* $|c(t)| < 1$ *for* $t \in \mathbb{R}$*, then the operator* $\mathbf{A}$ *has a continuous inverse* $\mathbf{A}^{-1}$ *on* C_ω*, satisfying*

$$\left|\left(\mathbf{A}^{-1}u\right)(t)\right| \le \frac{\|u\|}{1 - c_\infty}, \qquad \forall\, u \in C_\omega.$$

3.1. Equation (1.1) in the case that $c(t) \in (0, \frac{a_0 - b_\infty c'_\infty}{a_\infty + \delta b_\infty + a_0})$

Theorem 3.1. *Suppose* $c(t) \in \left(0, \frac{a_0 - b_\infty c'_\infty}{a_\infty + \delta b_\infty + a_0}\right)$ *for* $t \in \mathbb{R}$*,* $c'_\infty < \frac{a_0}{b_\infty}$ *and equation (2.2) hold. Furthermore, assume that there exists a constant* $r > 0$ *such that the following conditions are satisfied:*
(H_1) There exist continuous, non-negative functions $h(u)$ *and* $k(t)$ *such that*

$$0 \le f(t, u(t-\sigma)) \le k(t)h(u), \text{ for all } (t,u) \in [0,\omega] \times [0,r],$$

where $h(u)$ *is non-decreasing in* $[0, r]$*.*
(H_2) The following inequality holds

$$K^* < \frac{r(a_o - b_\infty c'_\infty - (a_0 + \delta b_\infty + a_\infty)c_\infty)}{a_0 h(r)},$$

where $K(t) := \int_0^\omega G(t,s)k(s)ds$*, and* $K^* := \max\limits_{t\in[0,\omega]} K(t)$*.*
Then equation (1.1) has at least one positive ω*-periodic solution* $u(t)$*.*

Proof. We Consider the family of equations of equation (1.1)

$$(u(t) - c(t)u(t-\sigma))'' + b(t)u'(t) + a(t)u(t) = \lambda f(t, u(t-\sigma)). \tag{3.1}$$

It can be written as the following form

$$\begin{aligned} &(u(t) - c(t)u(t-\sigma))'' + b(t)(u(t) - c(t)u(t-\sigma))' + a(t)(u(t) - c(t)u(t-\sigma)) \\ =&\lambda f(t, u(t-\sigma)) - a(t)c(t)u(t-\sigma) - b(t)(c(t)u(t-\sigma))'. \end{aligned} \tag{3.2}$$

Taking $v(t) = u(t) - c(t)u(t-\sigma)$, then equation (3.2) can be transformed into

$$v''(t) + b(t)v'(t) + a(t)v(t) = \lambda f(t, u(t-\sigma)) - a(t)c(t)u(t-\sigma) - b(t)(c(t)u(t-\sigma))'. \tag{3.3}$$

Then we consider

$$\begin{aligned} v''(t) + b(t)v'(t) + a(t)v(t) =& \lambda f(t, u(t-\sigma)) - a(t)c(t)u(t-\sigma) \\ & - b(t)c'(t)u(t-\sigma) - b(t)c(t)u'(t-\sigma). \end{aligned} \tag{3.4}$$

Consider the following equation

$$\begin{aligned}v''(t) + b_0 v'(t) + a_0 v(t) =& \lambda f(t, u(t-\sigma)) + (b_0 - b(t))v'(t) + (a_0 - a(t))v(t)\\ &- a(t)c(t)u(t-\sigma) - b(t)c'(t)u(t-\sigma) - b(t)c(t)u'(t-\sigma).\end{aligned}$$

Consider the family of equations

$$\begin{aligned}v''(t) + b_0 v'(t) + a_0 v(t) =& \lambda f(t, u(t-\sigma)) + (b_0 - b(t))v'(t) + (a_0 - a(t))v(t)\\ &+ a(t)H(v(t)) + b(t)Q(v(t)) + b(t)S(v(t)), \quad \lambda \in (0,1),\end{aligned} \tag{3.5}$$

where $H(v(t)) = -c(t)(\mathbf{A}^{-1}v)(t-\sigma) = -c(t)u(t-\sigma)$, $Q(v(t)) = -c'(t)u(t-\sigma)$, $S(v(t)) = -c(t)u'(t-\sigma)$.

Define

$$K := \{u \in X : u(t) > 0, \text{ for } t \in [0,\omega] \text{ and } \min_{t\in R} u(t) \geq \mu\|u\|, \ |u'(t)| \leq \delta u(t)\},$$

and

$$X = C^1_\omega := \{u \in C^1, \ u(t+\omega) = u(t), \ u'(t+\omega) = u'(t)\},$$

where $\mu := \frac{l}{L}$, L is the maximum of $G(t,s)$ on $\mathbb{R} \times \mathbb{R}$, $\|u\| := \max\{|u|_\infty, \ |u'|_\infty\}$. Define operators $\mathcal{T}, \mathcal{N} : \mathcal{K} \to C_\omega$ by

$$(\mathcal{T}f)(t) = \int_0^\omega G(t,s) f(s, u(s-\sigma)) ds, \tag{3.6}$$

$$(\mathcal{N}v)(t) = (b_0 - b(t))v'(t) + (a_0 - a(t))v(t) + a(t)H(v(t)) + b(t)Q(v(t)) + b(t)S(v(t)). \tag{3.7}$$

From equations (3.7) and (3.6), the solution for equation (3.5) can be written as

$$v(t) = \lambda(\mathcal{T}f)(t) + (\mathcal{T}\mathcal{N}v)(t).$$

In view of $c(t) \in \left(0, \frac{a_0 - b_\infty c'_\infty}{a_\infty + \delta b_\infty + a_0}\right)$ for $t \in \mathbb{R}$ and $\|\mathcal{T}\| \leq \frac{1}{a_0}$, applying Lemma 3.2, we get

$$\begin{aligned}\|\mathcal{T}\mathcal{N}\| &\leq \|\mathcal{T}\|\|\mathcal{N}\|\\ &\leq \frac{1}{a_0}\left(b_0 - b_0 + a_0 - a_0 + a_\infty \frac{c_\infty}{1-c_\infty} + b_\infty \frac{c'_\infty}{1-c_\infty} + b_\infty \delta \frac{c_\infty}{1-c_\infty}\right)\\ &\leq \frac{1}{a_0}\left(\frac{b_\infty c'_\infty + \delta b_\infty c_\infty + a_\infty c_\infty}{1-c_\infty}\right)\\ &\leq \frac{b_\infty c'_\infty + \delta b_\infty c_\infty + a_\infty c_\infty}{a_0 - a_0 c_\infty}\\ &< 1.\end{aligned} \tag{3.8}$$

Hence, we deduce

$$v(t) = \lambda(I - \mathcal{T}\mathcal{N})^{-1}(\mathcal{T}f)(t).$$

Define an operator $\mathcal{P} : \mathcal{K} \to C_\omega$ by

$$(\mathcal{P}f)(t) = (I - \mathcal{T}\mathcal{N})^{-1}(\mathcal{T}f)(t).$$

From inequality (3.8), we obtain

$$\begin{aligned}(\mathcal{P}f)(t) &= (I-\mathcal{T}\mathcal{N})^{-1}(\mathcal{T}f)(t)\\ &\le \frac{\|Tf\|}{1-\|\mathcal{T}\mathcal{N}\|}\\ &\le \frac{a_o - a_0 c_\infty}{a_o - b_\infty c'_\infty - (a_0+\delta b_\infty + a_\infty)c_\infty}\|Tf\|.\end{aligned} \tag{3.9}$$

Next, we consider the existence of periodic solutions for equation (1.1). Define operators $\mathcal{A},\ \mathcal{B}:\mathcal{K}\to C_\omega$ by

$$(\mathcal{A}u)(t) = c(t)u(t-\sigma), \qquad (\mathcal{B}u)(t) = \mathcal{P}(f(t,u(t-\sigma))).$$

From the above analysis, the existence of a positive ω-periodic solution for equation (3.5) is just a fixed point of the following operator equation

$$u = \lambda\mathcal{B}u + \lambda\mathcal{A}\left(\frac{u}{\lambda}\right) \quad \text{in } \mathcal{K}. \tag{3.10}$$

Define

$$\mathfrak{B}[0,r] := \{u\in C_\omega:\ 0\le u\le r,\ \text{for } t\in\mathbb{R}\}, \tag{3.11}$$

where r is defined in Theorem 3.1. Obviously, $\mathfrak{B}[0,r]$ is a bounded closed convex set in C_ω. For any $u\in\mathcal{K}\bigcap\mathfrak{B}$, and $t\in\mathbb{R}$, we arrive at

$$(\mathcal{A}u)(t+\omega) = c(t+\omega)u(t+\omega-\sigma) = c(t)u(t-\sigma) = (\mathcal{A}u)(t).$$

Besides,

$$(\mathcal{B}u)(t+\omega) = \mathcal{P}(f(t+\omega,u(t+\omega-\sigma))) = \mathcal{P}(f(t,u(t-\sigma))) = (\mathcal{B}u)(t),$$

which show that $(\mathcal{A}u)(t)$ and $(\mathcal{B}u)(t)$ are ω-periodic. For any $u_1,\ u_2\in\mathcal{K}\bigcap\mathfrak{B}$, we obtain

$$|(\mathcal{A}u_1)(t)-(\mathcal{A}u_2)(t)| = |c(t)u_1(t-\sigma)-c(t)u_2(t-\sigma)| \le c_\infty\|u_1-u_2\|. \tag{3.12}$$

In view of $c(t)\in\left(0,\frac{a_0-b_\infty c'_\infty}{a_\infty+\delta b_\infty+a_0}\right)$, we obtain that $\mathcal{A}$ is contractive. It is clear that $\mathcal{B}$ is completely continuous in [3, Theorem 3.1].

On the other hand, we claim that any fixed point u of equation (3.10) for any $\lambda\in(0,1)$ must satisfy $\|u\|\ne r$. Assume, by way of contradiction, that the above claim dues to not hold. Then, there exists a u of fixed point of equation (3.10) for some $\lambda\in(0,1)$ such that $\|u\|=r$. From inequality (3.9), conditions (H_1) and (H_2), we obtain

$$\begin{aligned}u(t) =& \lambda(\mathcal{B}u)(t) + \lambda\left(\mathcal{A}\left(\frac{u}{\lambda}\right)\right)(t)\\ =& \lambda\mathcal{P}(f(t,u(t-\sigma))) + \lambda\frac{1}{\lambda}c(t)u(t-\sigma)\\ =& \lambda\mathcal{P}(f(t,u(t-\sigma))) + c(t)u(t-\sigma)\\ \le& \frac{a_o - a_0 c_\infty}{a_o - b_\infty c'_\infty - (a_0+\delta b_\infty + a_\infty)c_\infty}\|\mathcal{T}f\| + c(t)u(t-\sigma)\\ \le& \frac{a_o - a_0 c_\infty}{a_o - b_\infty c'_\infty - (a_0+\delta b_\infty + a_\infty)c_\infty}\max_{t\in[0,\omega]}\int_0^\omega G(t,s)f(s,u(s-\sigma))ds + c_\infty r\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{a_o - a_0 c_\infty}{a_o - b_\infty c'_\infty - (a_0 + \delta b_\infty + a_\infty)c_\infty} \max_{t\in[0,\omega]} \int_0^\omega G(t,s)k(s)h(u(s))ds + c_\infty r \\
&\leq \frac{a_o - a_0 c_\infty}{a_o - b_\infty c'_\infty - (a_0 + \delta b_\infty + a_\infty)c_\infty} \cdot K^* h(r) + c_\infty r \\
&< r.
\end{aligned}
$$

Thus, $r = \|u\| < r$, this is a contradiction. Applying Lemma 3.1, we see that $u = \mathcal{A}u + \mathcal{B}u$ has a fixed point u in $\mathcal{K} \bigcap \mathfrak{B}$. Therefore, equation (1.1) has at least one positive ω-periodic solution $u(t)$. □

Corollary 3.1. *Assume $c(t) \in \left(0, \frac{a_0 - b_\infty c'_\infty}{a_\infty + \delta b_\infty + a_0}\right)$ for $t \in \mathbb{R}$, $c'_\infty < \frac{a_0}{b_\infty}$ and equation (2.2) hold. Furthermore, suppose that the nonlinear term f satisfies the following condition:*
(F_1) There exist a continuous positive ω-periodic function $d(t)$ and positive constants ρ, μ such that

$$f(t,u) = \mu d(t) u^\rho, \text{ for all } (t,u) \in [0,\omega] \times \mathbb{R}.$$

(i) If $\rho < 1$, then equation (1.1) has at least one positive ω-periodic solution for each $\mu > 0$.
(ii) If $\rho \geq 1$, then equation (1.1) has at least one positive ω-periodic solution for each $0 < \mu < \mu_1 := \sup\limits_{r>0} \frac{r^{1-\rho}(a_o - b_\infty c'_\infty - (a_0 + \delta b_\infty + a_\infty)c_\infty)}{\Phi^* a_0}$.

Proof. We apply Theorem 3.1. Take

$$k(t) = \mu d(t), \quad h(u) = u^\rho(t).$$

Then condition (H_1) is satisfied and the existence condition (H_2) becomes

$$\mu < \frac{r^{1-\rho}(a_o - b_\infty c'_\infty - (a_0 + \delta b_\infty + a_\infty)c_\infty)}{\Phi^* a_0},$$

for some $r > 0$, where $\Phi(t) := \int_0^\omega G(t,s)d(s)ds$, and $\Phi^* := \max\limits_{t\in[0,\omega]} \Phi(t)$. Therefore, equation (1.1) has at least one positive ω-periodic solution for

$$0 < \mu < \mu_1 := \frac{r^{1-\rho}(a_o - b_\infty c'_\infty - (a_0 + \delta b_\infty + a_\infty)c_\infty)}{\Phi^* a_0}.$$

Note that $\mu_1 = \infty$ if $\rho < 1$ and $\mu_1 < \infty$ if $\rho \geq 1$, we have (i) and (ii). □

Theorem 3.2. *Suppose $c(t) \in \left(0, \frac{a_0 - b_\infty c'_\infty}{a_\infty + \delta b_\infty + a_0}\right)$ for $t \in \mathbb{R}$, $c'_\infty < \frac{a_0}{b_0}$, conditions (H_1), (H_2) and (A) hold. Then equation (1.1) has at least one positive ω-periodic solution $u(t)$.*

Proof. We consider the equation (3.3)

$$v''(t)+b(t)v'(t)+a(t)v(t)=\lambda f(t,u(t-\sigma))-a(t)c(t)u(t-\sigma)-b(t)(c(t)u(t-\sigma))'. \quad (3.13)$$

Consider the following equation

$$
\begin{aligned}
&v''(t) + b(t)v'(t) + a(t)v(t) \\
=&\lambda(f(t,u(t-\sigma)) + a(t)H(v(t)) + b(t)Q(v(t)) + b(t)S(v(t)), \quad \lambda \in (0,1).
\end{aligned}
$$

Define operators $\tilde{T}, \tilde{\mathcal{N}} : \mathcal{K} \to C_\omega$ by

$$(\tilde{\mathcal{T}} f)(t) = \int_0^\omega \tilde{G}(t,s) f(s, u(s-\sigma)) ds, \tag{3.14}$$

$$(\tilde{\mathcal{N}} v)(t) = a(t) H(v(t)) + b(t) Q(v(t)) + b(t) S(v(t)). \tag{3.15}$$

From equations (3.14) and (3.15), the solution for equation (3.13) can be written as

$$v(t) = \lambda(\tilde{\mathcal{T}} f)(t) + (\tilde{\mathcal{T}} \tilde{\mathcal{N}} v)(t).$$

In view of $c(t) \in \left(0, \frac{a_0 - b_\infty c'_\infty}{a_\infty + \delta b_\infty + a_0}\right)$ for $t \in \mathbb{R}$ and $\|\tilde{\mathcal{T}}\| \leq \frac{1}{a_0}$, applying Lemma 3.2, we get

$$\begin{aligned}
\|\tilde{\mathcal{T}} \tilde{\mathcal{N}}\| &\leq \|\tilde{\mathcal{T}}\| \|\tilde{\mathcal{N}}\| \\
&\leq \frac{1}{a_0} \left(a_\infty \frac{c_\infty}{1 - c_\infty} + b_\infty \frac{c'_\infty}{1 - c_\infty} + b_\infty \delta \frac{c_\infty}{1 - c_\infty} \right) \\
&\leq \frac{1}{a_0} \left(\frac{b_\infty c'_\infty + \delta b_\infty c_\infty + a_\infty c_\infty}{1 - c_\infty} \right) \\
&\leq \frac{b_\infty c'_\infty + \delta b_\infty c_\infty + a_\infty c_\infty}{a_0 - a_0 c_\infty} \\
&< 1.
\end{aligned} \tag{3.16}$$

Hence, we deduce

$$v(t) = \lambda (I - \tilde{\mathcal{T}} \tilde{\mathcal{N}})^{-1} (\tilde{\mathcal{T}} f)(t).$$

Define an operator $\tilde{\mathcal{P}} : \mathcal{K} \to C_\omega$ by

$$(\tilde{\mathcal{P}} f)(t) = (I - \tilde{\mathcal{T}} \tilde{\mathcal{N}})^{-1} (\tilde{\mathcal{T}} f)(t). \tag{3.17}$$

From equation (3.16), we obtain

$$\begin{aligned}
(\tilde{\mathcal{P}} f)(t) &= (I - \tilde{\mathcal{T}} \tilde{\mathcal{N}})^{-1} (\tilde{\mathcal{T}} f)(t) \\
&\leq \frac{\|\tilde{\mathcal{T}} f\|}{I - \|\tilde{\mathcal{T}} \tilde{\mathcal{N}}\|} \\
&\leq \frac{a_o - a_0 c_\infty}{a_o - b_\infty c'_\infty - (a_0 + \delta b_\infty + a_\infty) c_\infty} \|\tilde{\mathcal{T}} f\|.
\end{aligned} \tag{3.18}$$

Next, we consider the existence of periodic solutions for equation (1.1). Define operator $\mathcal{B}' : \mathcal{K} \to C_\omega$ by

$$(\mathcal{B}' u)(t) = \tilde{\mathcal{P}}(f(t, u(t - \sigma))).$$

From the above analysis, the existence of a positive ω-periodic solution for equation (3.13) is just a fixed point of the following operator equation

$$u = \lambda \mathcal{B}' u + \lambda \mathcal{A} \left(\frac{u}{\lambda} \right) \quad \text{in } \mathcal{K}. \tag{3.19}$$

For any $u \in \mathcal{K} \bigcap \mathfrak{B}$, and $t \in \mathbb{R}$, we arrive at

$$(\mathcal{B}' u)(t + \omega) = \tilde{\mathcal{P}}(f(t + \omega, u(t + \omega - \sigma))) = \tilde{\mathcal{P}}(f(t, u(t - \sigma))) = (\mathcal{B}' u)(t),$$

which show that$(\mathcal{B}'u)(t)$ is ω-periodic. It is clear that $\mathcal{B}'$ is completely continuous in [3, Theorem 3.1].

On the other hand, we claim that any fixed point u of equation (3.19) for any $\lambda \in (0,1)$ must satisfy $\|u\| \neq r$. Assume, by way of contradiction, that the above claim dues to not hold. Then, there exists a u of fixed point of equation (3.19) for some $\lambda \in (0,1)$ such that $\|u\| = r$. From (H_1) and (H_2), we obtain

$$\begin{aligned}
u(t) =&\lambda(\mathcal{B}'u)(t) + \lambda\left(\mathcal{A}\left(\frac{u}{\lambda}\right)\right)(t)\\
=&\lambda\tilde{\mathcal{P}}(f(t,u(t-\sigma))) + \lambda\frac{1}{\lambda}c(t)u(t-\sigma)\\
=&\lambda\tilde{\mathcal{P}}(f(t,u(t-\sigma))) + c(t)u(t-\sigma)\\
\leq&\frac{a_o - a_0c_\infty}{a_o - b_\infty c'_\infty - (a_0+\delta b_\infty + a_\infty)c_\infty}\|\tilde{\mathcal{T}}f\| + c(t)u(t-\sigma)\\
\leq&\frac{a_o - a_0c_\infty}{a_o - b_\infty c'_\infty - (a_0+\delta b_\infty + a_\infty)c_\infty}\max_{t\in[0,\omega]}\int_0^\omega \tilde{G}(t,s)f(s,u(s-\sigma)))ds + c_\infty r\\
\leq&\frac{a_o - a_0c_\infty}{a_o - b_\infty c'_\infty - (a_0+\delta b_\infty + a_\infty)c_\infty}\max_{t\in[0,\omega]}\int_0^\omega \tilde{G}(t,s)k(s)h(u(s))ds + c_\infty r\\
\leq&\frac{a_o - a_0c_\infty}{a_o - b_\infty c'_\infty - (a_0+\delta b_\infty + a_\infty)c_\infty}\cdot K^*h(r) + c_\infty r\\
<&r.
\end{aligned}$$

Thus, $r = \|u\| < r$, this is a contradiction. Applying Lemma 3.1, we see that $u = \mathcal{A}u + \mathcal{B}'u$ has a fixed point u in $\mathcal{K}\bigcap\mathfrak{B}$. Therefore, equation (1.1) has at least one positive ω-periodic solution $u(t)$. □

Remark 3.1. It is clear that the conditions of Theorem 3.1 are relatively weaker than Theorem 3.2, because of the positiveness of $G(t,s)$ and $\tilde{G}(t,s)$.

3.2. Equation (1.1) in the case that $c(t) \in \left(-\frac{a_0 - b_\infty c'_\infty}{a_\infty+\delta b_\infty + a_0}, 0\right)$

Theorem 3.3. *Suppose $c(t) \in \left(-\frac{a_0 - b_\infty c'_\infty}{a_\infty+\delta b_\infty + a_0}, 0\right)$ for $t \in \mathbb{R}$, $c'_\infty < \frac{a_0}{b_\infty}$, equation (2.2) and (H_1) hold. Furthermore, assume that the following condition is satisfied:*
(H_3) There exists a constant $r > 0$ such that

$$K^* < \frac{r(a_o - b_\infty c'_\infty - (a_0+\delta b_\infty + a_\infty)c_\infty)}{a_o(1-c_\infty)h(r)}.$$

Then equation (1.1) has at least one positive ω-periodic solution $u(t)$.

Proof. We follow the same notations and use a similar method as in the proof of Theorem 3.1. We claim that any fixed point u of equation (3.10) for any $\lambda \in (0,1)$ must satisfy $\|u\| \neq r$. Assume, by way of contradiction, that the above claim dues to not hold. Then, there exists a u of fixed point of equation (3.10) for some $\lambda \in (0,1)$ such that $\|u\| = r$. From the conditions (H_1) and (H_3), we get

$$\begin{aligned}
u(t)=&\lambda(\mathcal{B}u)(t) + \lambda\left(\mathcal{A}\left(\frac{u}{\lambda}\right)\right)(t)\\
=&\lambda\mathcal{P}(f(t,u(t-\sigma))) + \lambda\frac{1}{\lambda}c(t)u(t-\sigma)
\end{aligned}$$

$$
\begin{aligned}
&=\lambda\mathcal{P}(f(t,u(t-\sigma)))+c(t)u(t-\sigma)\\
&\le\frac{a_o-a_0c_\infty}{a_o-b_\infty c'_\infty-(a_0+\delta b_\infty+a_\infty)c_\infty}\|\mathcal{T}f\|-|c(t)|u(t-\sigma)\\
&\le\frac{a_o-a_0c_\infty}{a_o-b_\infty c'_\infty-(a_0+\delta b_\infty+a_\infty)c_\infty}\max_{t\in[0,\omega]}\int_0^\omega G(t,s)f(s,u(s-\sigma)))ds-|c(t)|u(t-\sigma)\\
&\le\frac{a_o-a_0c_\infty}{a_o-b_\infty c'_\infty-(a_0+\delta b_\infty+a_\infty)c_\infty}\max_{t\in[0,\omega]}\int_0^\omega G(t,s)k(s)h(u(s))ds\\
&\le\frac{a_o-a_0c_\infty}{a_o-b_\infty c'_\infty-(a_0+\delta b_\infty+a_\infty)c_\infty}\cdot K^*h(r)\\
&<r.
\end{aligned}
$$

Thus, $r=\|u\|<r$, this is a contradiction. Applying Lemma 3.1, we see that $u=\mathcal{A}u+\mathcal{B}u$ has a fixed point u in $\mathcal{K}\bigcap\mathfrak{B}$. Therefore, equation (1.1) has at least one positive ω-periodic solution $u(t)$. □

By Theorem 3.3 and Corollary 3.1, we get the following conclusion.

Corollary 3.2. *Assume that* $c(t)\in\left(-\frac{a_0-b_\infty c'_\infty}{a_\infty+\delta b_\infty+a_0},0\right)$ *for* $t\in\mathbb{R}$, $c'_\infty<\frac{a_0}{b_\infty}$, *equation (2.2) and* (F_1) *hold, then we have*
(i) If $\rho<1$, *then equation* (1.1) *has at least one positive* ω*-periodic solution for each* $\mu>0$.
(ii) If $\rho\ge1$, *then equation* (1.1) *has at least one positive* ω*-periodic solution for each* $0<\mu<\mu_2:=\sup\limits_{r>0}\frac{r^{1-\rho}(a_o-b_\infty c'_\infty-(a_0+\delta b_\infty+a_\infty)c_\infty)}{a_o(1-c_\infty)\Phi^*}$.

Theorem 3.4. *Suppose* $c(t)\in\left(0,\frac{a_0-b_\infty c'_\infty}{a_\infty+\delta b_\infty+a_0}\right)$ *for* $t\in\mathbb{R}$, $c'_\infty<\frac{a_0}{b_0}$, *conditions* (H_1), (H_3), *and (A) hold. Then equation (1.1) has at least one positive* ω*-periodic solution* $u(t)$.

Proof. We follow the same notations and use a similar method as in the proof of Theorem 3.2. Next, we claim that any fixed point u of equation (3.19) for any $\lambda\in(0,1)$ must satisfy $\|u\|\neq r$. Assume, by way of contradiction, that the above claim dues to not hold. Then, there exists a fixed point u of equation (3.19) for some $\lambda\in(0,1)$ such that $\|u\|=r$. From equation (3.5), conditions (H_1) and (H_3), we get

$$
\begin{aligned}
u(t)&=\lambda(\mathcal{B}'u)(t)+\lambda\left(\mathcal{A}\left(\frac{u}{\lambda}\right)\right)(t)\\
&=\lambda\tilde{\mathcal{P}}(f(t,u(t-\sigma)))+\lambda\frac{1}{\lambda}c(t)u(t-\sigma)\\
&=\lambda\tilde{\mathcal{P}}(f(t,u(t-\sigma)))+c(t)u(t-\sigma)\\
&\le\frac{a_o-a_0c_\infty}{a_o-b_\infty c'_\infty-(a_0+\delta b_\infty+a_\infty)c_\infty}\|\tilde{\mathcal{T}}f\|-|c(t)|u(t-\sigma)\\
&\le\frac{a_o-a_0c_\infty}{a_o-b_\infty c'_\infty-(a_0+\delta b_\infty+a_\infty)c_\infty}\max_{t\in[0,\omega]}\int_0^\omega\tilde{G}(t,s)f(s,u(s-\sigma)))ds-|c(t)|u(t-\sigma)\\
&\le\frac{a_o-a_0c_\infty}{a_o-b_\infty c'_\infty-(a_0+\delta b_\infty+a_\infty)c_\infty}\max_{t\in[0,\omega]}\int_0^\omega\tilde{G}(t,s)k(s)h(u(s))ds\\
&\le\frac{a_o-a_0c_\infty}{a_o-b_\infty c'_\infty-(a_0+\delta b_\infty+a_\infty)c_\infty}\cdot K^*h(r)<r.
\end{aligned}
$$

Thus, $r = \|u\| < r$, this is a contradiction. Therefore, applying Lemma 3.1, we obtain that $u = \mathcal{A}u + \mathcal{B}'u$ has a fixed point u in $\mathcal{K} \bigcap \mathfrak{B}$. Hence, equation (1.1) has at least one positive ω-periodic solution $u(t)$. □

Remark 3.2. If $|c(t)| > 1$, from (3.12), we do not obtain that $\mathcal{A}$ is contractive. Therefore, the above method does not apply to the case that $|c(t)| > 1$. Next, we find another way to get over this problem.

4. Positive periodic solutions for equation (1.1) in the case that $|c(t)| > 1$

Now, we present our results for the existence of a positive ω-periodic solution for equation (1.1) in the case that $|c(t)| > 1$.

4.1. Equation (1.1) in the case that $c(t) \in (1, +\infty)$

Theorem 4.1. *Suppose $c(t) \in (1, +\infty)$ and condition (A) hold. Furthermore, assume that there exists a constant $r > 0$ such that the following conditions are satisfied:*
(H_4) There exist continuous, non-negative functions $h(u)$ and $k(t)$ such that

$$0 \le u(t-\sigma)(a(t)c(t) + b(t)c'(t) + \tilde{\delta}b(t)c(t)) - f(t, u(t-\sigma)) \le k(t)h(u),$$
$$\text{for all } (t,u) \in [0,\omega] \times [0,r],$$

where $\tilde{\delta} := \frac{\max_{t\in[0,\omega]}\left|\frac{\partial \tilde{G}(t,s)}{\partial t}\right|}{\tilde{l}}$, *$\tilde{l}$ is the minimum of $\tilde{G}(t,s)$ on $\mathbb{R} \times \mathbb{R}$ and $h(u)$ is non-decreasing in $[0, r]$.*
(H_5) The following condition holds

$$K^* < \frac{(c_0 - 1)r}{h(r)}.$$

Then equation (1.1) has at least one positive ω-periodic solution $u(t)$.

Proof. We consider equation (1.1), it can be written as the the following form

$$\begin{aligned} &(c(t)u(t-\sigma) - u(t))'' + b(t)(c(t)u(t-\sigma) - u(t))' + a(t)(c(t)u(t-\sigma) - u(t)) \\ =&a(t)c(t)u(t-\sigma) + b(t)(c(t)u(t-\sigma))' - f(t, u(t-\sigma)). \end{aligned} \tag{4.1}$$

Taking $v(t) = c(t)u(t-\sigma) - u(t)$, then equation (4.1) can be written as the following form

$$v''(t) + b(t)v'(t) + a(t)v(t) = a(t)c(t)u(t-\sigma) + b(t)(c(t)u(t-\sigma))' - f(t, u(t-\sigma)).$$

Consider the following equation

$$v''(t)+b(t)v'(t)+a(t)v(t)=\lambda\left(a(t)c(t)u(t-\sigma)+b(t)(c(t)u(t-\sigma))'-f(t,u(t-\sigma))\right). \tag{4.2}$$

We obtain

$$v(t) = \int_0^\omega \tilde{G}(t,s)\lambda\left(a(s)c(s)u(s-\sigma) + b(s)(c(s)u(s-\sigma))' - f(s, u(s-\sigma)))\right) ds.$$

If u is a positive ω-periodic solution of (4.1), then we have

$$c(t)u(t-\sigma)=v(t)+u(t),$$

that is

$$u(t-\sigma)=\frac{1}{c(t)}v(t)+\frac{1}{c(t)}u(t).$$

Then we have

$$\begin{aligned}u(t)=&\frac{1}{c(t+\sigma)}v(t+\sigma)+\frac{1}{c(t+\sigma)}u(t+\sigma)\\=&\lambda\frac{1}{c(t+\sigma)}\int_0^\omega\tilde{G}(t+\sigma,s)\left(a(s)c(s)u(s-\sigma)+b(s)(c(s)u(s-\sigma))'-f(s,u(s-\sigma))\right))ds\\&+\lambda\frac{1}{\lambda}\frac{1}{c(t+\sigma)}u(t+\sigma).\end{aligned}\tag{4.3}$$

Define operators $\tilde{\mathcal{A}},\tilde{\mathcal{B}}:\mathcal{K}\to C_\omega$ by

$$\begin{aligned}(\tilde{\mathcal{A}}u)(t)=&\frac{1}{c(t+\sigma)}u(t+\sigma),\\(\tilde{\mathcal{B}}u)(t)=&\frac{1}{c(t+\sigma)}\\&\times\int_0^\omega\tilde{G}(t+\sigma,s)\left(a(s)c(s)u(s-\sigma)+b(s)(c(s)u(s-\sigma))'-f(s,u(s-\sigma))\right))\,ds.\end{aligned}$$

By the above analysis, the existence of a positive ω-periodic of equation (4.2) is equivalent to the existence of solutions for the operator equation

$$u=\lambda\tilde{\mathcal{B}}u+\lambda\tilde{\mathcal{A}}\left(\frac{u}{\lambda}\right)\quad\text{in }\mathcal{K}.\tag{4.4}$$

Next, we conclude the existence of a positive ω-periodic for equation (1.1). For any $u\in\mathcal{K}\bigcap\mathfrak{B}$ be as in equation (3.11), and $t\in\mathbb{R}$, we have

$$(\tilde{\mathcal{A}}u)(t+\omega)=\frac{1}{c(t+\sigma+\omega)}u(t+\sigma+\omega)=\frac{1}{c(t+\sigma)}u(t+\sigma)=(\tilde{\mathcal{A}}u)(t).$$

Besides,

$$\begin{aligned}&(\tilde{\mathcal{B}}u)(t+\omega)\\=&\frac{1}{c(t+\sigma+\omega)}\\&\times\int_0^\omega\tilde{G}(t+\sigma+\omega),s)(a(s)c(s)u(s-\sigma)+b(s)(c(s)u(s-\sigma))'-f(s,u(s-\sigma))))ds\\=&\frac{1}{c(t+\sigma)}\int_0^\omega\tilde{G}(t+\sigma),s)(a(s)c(s)u(s-\sigma)+b(s)(c(s)u(s-\sigma))'-f(s,u(s-\sigma))))ds\\=&(\tilde{\mathcal{B}}u)(t),\end{aligned}$$

which show that $(\tilde{\mathcal{A}}u)(t)$ and $(\tilde{\mathcal{B}}u)(t)$ are ω-periodic. For any u_1, $u_2\mathcal{K}\bigcap\mathfrak{B}$, we get

$$|(\tilde{\mathcal{A}}u_1)(t)-(\tilde{\mathcal{A}}u_2)(t)|=\left|\frac{1}{c(t+\sigma)}u_1(t+\sigma)-\frac{1}{c(t+\sigma)}u_2(t+\sigma)\right|$$

$$
\begin{aligned}
&=\left|\frac{1}{c(t+\sigma)}\right| |u_1(t+\sigma)-u_2(t+\sigma)|\\
&\le\frac{1}{c_0}\|u_1-u_2\|.
\end{aligned}
$$

Thus, we have from $c(t)\in(1,+\infty)$ that $\tilde{\mathcal{A}}$ is contractive. It is clear that $\tilde{\mathcal{B}}$ is completely continuous.

Next, we claim that any fixed point u of equation (4.4) for any $\lambda\in(0,1)$ must satisfy $\|u\|\neq r$. Assume, by way of contradiction, that the above claim dues to not hold. Then, there exists a u of fixed point of equation (4.4) for some $\lambda\in(0,1)$ such that $\|u\|=r$. From the (H_4) and (H_5), we have

$$
\begin{aligned}
&u(t)\\
=&\lambda\frac{1}{c(t+\sigma)}\int_0^\omega \tilde{G}(t+\sigma,s)\left(a(s)c(s)u(s-\sigma)+b(s)(c(s)u(s-\sigma))'-f(s,u(s-\sigma))\right)ds\\
&+\lambda\frac{1}{\lambda}\frac{1}{c(t+\sigma)}u(t+\sigma)\\
=&\lambda\frac{1}{c(t+\sigma)}\int_0^\omega \tilde{G}(t+\sigma,s)\Big(a(s)c(s)u(s-\sigma)+b(s)c'(s)u(s-\sigma)\\
&+b(s)c(s)u'(s-\sigma)-f(s,u(s-\sigma))\Big)ds+\lambda\frac{1}{\lambda}\frac{1}{c(t+\sigma)}u(t+\sigma)\\
=&\lambda\frac{1}{c(t+\sigma)}\int_0^\omega \tilde{G}(t+\sigma,s)\Big(a(s)c(s)u(s-\sigma)+b(s)c'(s)u(s-\sigma)+\delta b(s)c(s)u(s-\sigma)\\
&-f(s,u(s-\sigma))\Big)ds+\lambda\frac{1}{\lambda}\frac{1}{c(t+\sigma)}u(t+\sigma)\\
=&\lambda\frac{1}{c(t+\sigma)}\int_0^\omega \tilde{G}(t+\sigma,s)\left(u(s-\sigma)(a(s)c(s)+b(s)c'(s)+\delta b(s)c(s))-f(s,u(s-\sigma))\right)ds\\
&+\lambda\frac{1}{\lambda}\frac{1}{c(t+\sigma)}u(t+\sigma)\\
\le&\frac{1}{c_0}\int_0^\omega \tilde{G}(t+\sigma,s)k(s)h(u)ds+\frac{1}{c_0}u(t+\sigma)\\
\le&\frac{1}{c_0}K^*h(r)+\frac{1}{c_0}r\\
<&r.
\end{aligned}
$$

Thus, $r=\|u\|<r$, this is a contradiction. Applying Lemma 3.1, we see that $u=\tilde{\mathcal{A}}u+\tilde{\mathcal{B}}u$ has a fixed point u in $\mathcal{K}\bigcap\mathfrak{B}$. Therefore, equation (1.1) has at least one positive ω-periodic solution $u(t)$. □

Corollary 4.1. *Assume $c(t)\in(1,+\infty)$ and condition (A) hold. Furthermore, suppose that the nonlinear term f satisfies the following condition:*
(F_2) *There exist a continuous positive ω-periodic function $d(t)$ and positive constants $\rho,\ \mu$ such that*

$$
\begin{aligned}
&f(t,u(t-\sigma))=-\mu d(t)u^\rho(t)+u(t-\sigma)(a(t)c(t)+b(t)c'(t)+\delta b(t)c(t)),\\
&\textit{for all } (t,u)\in[0,\omega]\times\mathbb{R}.
\end{aligned}
$$

(i) If $\rho<1$, then equation (1.1) has at least one positive ω-periodic solution for each $\mu>0$.

(ii) If $\rho \geq 1$, then equation (1.1) *has at least one positive ω-periodic solution for each $0 < \mu < \mu_3 := \sup\limits_{r>0} \frac{(c_0-1)r^{1-\rho}}{\Phi^*}$.*

4.2. Equation (1.1) in the case that $c(t) \in (-\infty, -1)$.

Theorem 4.2. *Suppose $c(t) \in (-\infty, -1)$, conditions (H_4) and (A) hold. Furthermore, assume that the following condition is satisfied:*
(H_6) There exists a constant $r > 0$ such that

$$K^* < \frac{c_0 r}{h(r)}.$$

Then equation (1.1) has at least one positive ω-periodic solution $u(t)$.

Proof. We use similar notations and method as in the proof of Theorem 4.1. We claim that any fixed point u of equation (4.4) for any $\lambda \in (0,1)$ must satisfy $\|u\| \neq r$. Assume, by way of contradiction, that the above claim dues to not hold. Then, there exists a u of fixed point of equation (4.4) for some $\lambda \in (0,1)$ such that $\|u\| = r$. From the conditions (H_4) and (H_6), we get

$$\begin{aligned}
u(t) =& \lambda \frac{1}{c(t+\sigma)} \int_0^\omega G(t+\sigma, s)\Big(a(s)c(s)u(s-\sigma) + b(s)(c(s)u(s-\sigma))' \\
&- f(s, u(s-\sigma))\Big)ds + \lambda \frac{1}{\lambda} \frac{1}{c(t+\sigma)} u(t+\sigma) \\
=& \lambda \frac{1}{c(t+\sigma)} \int_0^\omega G(t+\sigma, s)\Big(a(s)c(s)u(s-\sigma) + b(s)c(s)'u(s-\sigma) \\
&+ b(s)c(s)u'(s-\sigma) - f(s, u(s-\sigma))\Big)ds + \lambda \frac{1}{\lambda} \frac{1}{c(t+\sigma)} u(t+\sigma) \\
=& \lambda \frac{1}{c(t+\sigma)} \int_0^\omega \tilde{G}(t+\sigma, s)\Big(u(s-\sigma)(a(s)c(s) + b(s)c'(s) + \delta b(s)c(s)) \\
&- f(s, u(s-\sigma))\Big)ds + \lambda \frac{1}{\lambda} \frac{1}{c(t+\sigma)} u(t+\sigma) \\
\leq& \frac{1}{c_0} \int_0^\omega G(t+\sigma, s)k(s)h(u)ds - \left|\frac{1}{c(t+\sigma)}\right| u(t+\sigma) \\
\leq& \frac{1}{c_0} K^* h(r) - \frac{1}{c_0} r \\
\leq& \frac{1}{c_0} K^* h(r) \\
<& r.
\end{aligned}$$

Thus, $r = \|u\| < r$, this is a contradiction. Applying Lemma 3.1, we see that $u = \tilde{\mathcal{A}}u + \tilde{\mathcal{B}}u$ has a fixed point u in $\mathcal{K} \bigcap \mathfrak{B}$. Therefore, equation (1.1) has at least one positive ω-periodic solution $u(t)$. □

Corollary 4.2. *Assume that $c(t) \in (-\infty, -1)$, conditions (A) and (F_2) hold, then we have*
(i) If $\rho < 1$, then equation (1.1) *has at least one positive ω-periodic solution for each $\mu > 0$.*

(ii) If $\rho \geq 1$, then equation (1.1) *has at least one positive ω-periodic solution for each* $0 < \mu < \mu_4 := \sup\limits_{r>0} \frac{c_0 r^{1-\rho}}{\Phi^*}$.

References

[1] A. Ardjouni and A. Djoudi, *Existence of positive periodic solutions for two types of second-order nonlinear neutral differential equations with variable delay*, Proyecciones. J. Math., 2013, 32, 377–391.

[2] T. Candan, *Existence of positive periodic solutions of first order neutral differential equations with variable coefficients*, Appl. Math. Lett., 2016, 52, 142–148.

[3] Z. Cheng and F. Li, *Positive periodic solutions for a kind of second-order neutral differential equations with variable coefficient and delay*, Mediterr. J. Math., 2018, 15, 19.

[4] Z. Cheng, F. Li and S. Yao, *Positive periodic solutions for second-order neutral differential equations with time-dependent deviating arguments*, Filomat, 2019, 33, 3627–3638.

[5] Z. Cheng and J. Ren, *Periodic solutions for second order damped differential equations with attractive-repulsive singularities*, Rocky Mountain J. Math., 2018, 48, 753–768.

[6] Z. Cheng and J. Ren, *Existence of periodic solution for fourth-order Liénard type p-Laplacian generalized neutral differential equation with variable parameter*, J. Appl. Anal. Comput., 2015, 5, 704–720.

[7] W. Cheung, J. Ren and W. Han, *Positive periodic solution of second-order neutral functional differential equations*, Nonlinear Anal., 2009, 71, 3948–3955.

[8] J. Cid, G. Infante, M. Tvrdý and M. Zima, *A topological approach to periodic oscillations related to the Liebau phenomenon*, J. Math. Appl., 2015, 423, 1546–1556.

[9] B. Dhage and D. O'Regan, *A fixed point theorem in Banach algebras with applications to functional intergral equations*, Funct. Differ. Equ., 2000, 7, 259–267.

[10] W. Han and J. Ren, *Some results on second-order neutral functional differential equations with infinite distributed delay*, Nonlinear Anal., 2009, 70, 1393–1406.

[11] B. Liu and L. Huang, *Existence and uniqueness of periodic solutions for a kind of second order neutral functional differential equations*, Nonlinear Anal. Real World Appl., 2007, 8, 222–229.

[12] S. Lu and W. Ge, *Periodic solutions for a kind of second order differential equation with multiple deviating arguments*, Appl. Math. Comput., 2003, 146, 195–209.

[13] Y. Luo, W. Wei and J. Shen, *Existence of positive periodic solutions for two kinds of neutral functional differential equations*, Appl. Math. Lett., 2008, 21, 581–587.

[14] L. Lv and Z. Cheng, *Positive periodic solution to superlinear neutral differential equation with time-dependent parameter*, Appl. Math. Lett., 2019, 98, 271–277.

[15] J. Ren, Z. Cheng and S. Siegmund, *Neutral operator and neutral differential equation*, Abstr. Appl. Anal., 2011, ID 969276, 29.

[16] H. Wang, *Positive periodic solutions of functional differential equations*, J. Differential Equations, 2004, 202, 354–366.

[17] J. Wu and Z. Wang, *Two periodic solutions of second-order neutral functional differential equations*, J. Math. Anal. Appl., 2007, 329, 677–689.

[18] M. Zhang, *Periodic solutions of linear and quasilinear neutral functional differential Equations*, J. Math. Anal. Appl., 1995, 189, 378–392.