Journal of Applied Analysis and Computation
Volume 11, Number 4, August 2021, 1749–1767
Website:http://www.jaac-online.com
DOI:10.11948/20200067

WELL-POSEDNESS AND CONVERGENCE FOR TIME-SPACE FRACTIONAL STOCHASTIC SCHRÖDINGER-BBM EQUATION*

Shang Wu[1], Jianhua Huang[1,†] and Yuhong Li[2,†]

Abstract In this paper, the Banach fixed point theorem combined with Mittag-Leffler functions has been used to obtain the existence and uniqueness of global mild solution for a kind of time-space fractional stochastic Schrödinger-BBM equation driven by Gaussian noise. The spatial-temporal regularity of the nonlocal stochastic convolution is established. Furthermore the convergence and simulation is provided by the Galerkin finite element method as well.

Keywords Schrödinger-BBM Equation, Caputo fractional derivative, mild solution, Galerkin finite element method.

MSC(2010) 37L55, 60H15.

1. Introduction

The propagation of unidirectional, one-dimensional, small-amplitude long waves in nonlinear dispersive media is sometimes well approximated by the Benjamin-Bona-Mahony (BBM) equation, which is generally understood as an alternative to the Korteweg-de Vries (KdV) equation [2, 3],

$$n_t(t,x) + n_x(t,x) + nn_x(t,x) - n_{xxt}(t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R},$$

where the real function $n(t,x), t \geq 0, x \in \mathbb{R}$ is an approximation for moderately long waves of small but finite amplitude in particular physical systems.

Zakharov studied the one-dimensional long-wave Langmuir turbulence in a plasma by the following set of coupled equations [1, 6, 17],

$$\begin{cases} i\epsilon_t(t,x) + \epsilon_{xx}(t,x) - n\epsilon(t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ n_{tt}(t,x) - n_{xx}(t,x) - (|\epsilon(t,x)|^2)_{xx} = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \end{cases}$$

[†]The corresponding author.
Email: jhhuang32@nudt.edu.cn(J. Huang), liyuhong@hust.edu.cn(Y. Li)
[1]College of Liberal Arts and Science, National University of Defense Technology, Changsha, China
[2]School of Hydropower and Information Engineering, HuaZhong University of Science and Technology, Wuhan, China
*The authors were supported by National Natural Science Foundation of China (Nos. 11771449, 61841302).

where the complex value function $\epsilon(t,x)$ is the electric field of Langmuir oscillations and the real value function $n(t,x)$ is the low-frequency density perturbation.

By means of the integral estimation method and the fixed point theorem, Guo established the global solvability of the following Cauchy problem of the coupled system of BBM-nonlinear Schrödinger equations [8],

$$\begin{cases} i\epsilon_t(t,x) + \epsilon_{xx}(t,x) - n(t,x)\epsilon(t,x) + \beta q(|\epsilon(t,x)|^2)\epsilon(t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ n_t(t,x) + f(n(t,x))_x - n_{xxt}(t,x) + (|\epsilon(t,x)|^2)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \epsilon(0,x) = \epsilon_0(x), n(0,x) = n_0(x), \quad x \in \mathbb{R}, \end{cases}$$

where $\beta \in \mathbb{R}$ is a constant, both $f(s)$ and $q(s)$ are real valued functions, and $\epsilon(t,x), n(t,x)$ denote complex and real function, respectively.

Guo et al. in [7] used the Strichartz estimate technique and the contraction mapping principle to establish the global existence for the Cauchy problem of the following Schrödinger-BBM equations,

$$\begin{cases} i\epsilon_t(t,x) + \epsilon_{xx}(t,x) = \epsilon(t,x)n(t,x) + a|\epsilon(t,x)|^{p-1}\epsilon(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ n_t(t,x) - n_{xxt}(t,x) = (|\epsilon(t,x)|^2 + n(t,x)^2)_x, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \epsilon(0,x) = \epsilon_0(x), \quad n(0,x) = n_0(x), \quad x \in \mathbb{R}, \end{cases}$$

where $a \in \mathbb{R}$, $1 < p < +\infty$, and $\epsilon(t,x), n(t,x)$ denote complex and real function, respectively. The paper [18] is devoted to the large time behavior and especially to the regularity of the global attractor for the semi-discrete in time Crank-Nicolson scheme to discretize a class of system of nonlinear Schrödinger-BBM equations on $\mathbb{R} \times \mathbb{R}$.

There are many papers focusing on the space-fractional Schrödinger equation, which used the path integral over the Lévy-like quantum mechanical paths, see [9] for details. Recently, the existence and uniqueness of the mild solution and the Hölder continuity for the time fractional and space nonlocal stochastic nonlinear Schrödinger equation driven by multiplicative white noise are established in [11] with the following form,

$$\begin{cases} iD_t^\alpha u = (-\Delta)^{\frac{\beta}{2}} u + \lambda |u|^2 u + g(u)\dot{W}, \\ u(t,x) = 0, \quad x \in \partial D^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where the fixed number $\alpha \in (0,1)$ represents the order of time fractional differential operator, D_t^α is the Caputo time fractional derivative, $\lambda \in \mathbb{R}, \beta \in (1,2)$, $d \in \mathbb{N}^*$, $D^d = [0,1]^d$ and W stands for $L^2(D^d)$-Q Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\mathcal{F}_t = \sigma\{W(s) : s \le t, t \in [0,T]\}$. And the space-fractional operator $(-\triangle)^{\beta/2}$ can be realized through the Fourier transform:

$$\widehat{(-\Delta)^{\beta/2}} u(\xi) = |\xi|^\beta \widehat{u}(\xi),$$

where $\widehat{u}$ represents the Fourier transform of u. Moreover, a full discrete scheme with spectral Galerkin method in space and exponential method in time was provided to obtain the convergence order of the time discretization and the spatial discretization for the equations (1.1) respectively.

Motivated by the idea presented in [7, 9, 11], we consider the following time-space fractional stochastic Schrödinger-BBM equation (FSSBE for short) driven by Gaussian white noise:

$$\begin{cases} iD_t^{\alpha}u + \Delta^{\frac{\beta_1}{2}}u = un + a|u|^{p-1}u + \dot{W}_1(t), \\ D_t^{\alpha}(n - \Delta^{\frac{\beta_2}{2}}n) = (|u|^2 + n^2)_x + \dot{W}_2(t), \\ u(0) = u_0(x), n(0) = n_0(x). \end{cases} \tag{1.2}$$

where D_t^{α} $(0 < \alpha < 1)$ denotes the α-order Caputo time derivative operator, $(-\triangle)^{\frac{\beta_i}{2}}, i = 1, 2$ is the space fractional Laplacian operator with $\beta_i \in (1, 2), i = 1, 2$. $\dot{W}_1(t)$ and $\dot{W}_2(t)$ are Gaussian white noises which are independent of each other.

To understand the effect of the index α for Caputo type time fractional operator, the index β_1, β_2 for the space-fractional operator $(-\Delta)^{\beta_i}$ and the noise on the FSSBE equation (1.2), the time regularity and the space regularity for the nonlocal stochastic convolution are established firstly. The restrictions $\alpha \in (\frac{3}{4}, 1)$ on the order of time fractional derivative and $2\alpha < \beta_i \leq 2$, $i = 1, 2$, $p < 3$ on the order of spatial nonlocal are required. The local existence and uniqueness of mild solutions of the FSSBE (1.2) in space H^{σ} for $\frac{1}{6} \leq \sigma \leq \frac{1}{2}$ are proven by the application of a Banach fixed point Theorem. Due to the nonlinear term $(|u|^2 + n^2)_x$, we restrict $\beta_2 = 2$ to obtain the global existence of the mild solution of the FSSBE equation (1.2). Moreover, in contrast to the result of integer-order-time FSSBE equations (1.2) for $\beta_1 = \beta_2 = 2, \alpha = 1$, we obtain the convergence order of the time discretization with the exponential method and the space discretization with the Galerkin finite element method respectively.

The rest of the paper is organized as follows. The definitions of fractional operators, Mainardi function, Mittag-Leffler function and the mild solutions for stochastic system are provided in section 2. In section 3, the regularity of the nonlocal stochastic convolution is established. The local and global existence and uniqueness of mild solutions for FSSBE equations are also presented. The convergence order of the time and space discretization are provided respectively in section 4.

2. Preliminaries

In this section, we present the definitions of fractional operators, special functions and the mild solutions for the stochastic system (1.1), which are cited from [4].

Definition 2.1. For $\alpha > 0$, the Riemann-Liouville fractional integral operator of order α for function $f \in L^1([0, T], \mathbb{R})$ is defined by

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds, \qquad t \in [0, T]. \tag{2.1}$$

Definition 2.2. For $\alpha \in (0, 1)$, the Caputo fractional derivative of order α for function $f \in C([0, T]; \mathbb{R})$ is defined by

$$D_t^{\alpha} f(t) := \frac{d}{dt}[I_t^{1-\alpha}(f(t) - f(0))] = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} f'(s) ds. \tag{2.2}$$

Definition 2.3. The Mainardi's function is given by

$$M_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}, \ 0 < \alpha < 1, \ z \in \mathbb{C}. \tag{2.3}$$

Moreover, $M_\alpha(z) \geq 0$ for all $t \geq 0$ and satisfies the following equality

$$\int_0^\infty t^r M_\alpha(t)dt = \frac{\Gamma(r+1)}{\Gamma(\alpha r+1)},\ r > -1,\ 0 < \alpha < 1. \tag{2.4}$$

Definition 2.4. The Mittag-Leffler functions is defined by

$$E_{\alpha,\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\eta)},\ \alpha,\eta \in \mathbb{C},\ Re(z) > 0. \tag{2.5}$$

Definition 2.5. The Mittag-Leffler families operators based on the analytic semigroup $S(t)$ generated by the space fractional operator $(-\Delta)^\gamma$ is defined by:

$$T_{\alpha,\gamma}(t) = \int_0^\infty M_\alpha(s)S(st^\alpha)ds = \int_0^\infty M_\alpha(s)e^{-st^\alpha(-\Delta)^\gamma}ds \tag{2.6}$$

and

$$S_{\alpha,\gamma}(t) = \int_0^\infty \alpha s M_\alpha(s)S(st^\alpha)ds = \int_0^\infty \alpha s M_\alpha(s)e^{-st^\alpha(-\Delta)^\gamma}ds. \tag{2.7}$$

Remark 2.1. Just as Section 3 of [4], although Mittag-Leffler operator does not have the same good properties of a classical semigroup, it is important to notice that for each fixed $x \in X$, the function $t \to T_{\alpha,\gamma}(t)x$ is continuous and satisfies

$$cD_t^\alpha T_{\alpha,\gamma}(t)x = -A^\gamma T_{\alpha,\gamma}(t)x.$$

Then, for any $v \in L^2$, there exists a constant $C = C(N,\alpha) > 0$ such that

$$\|T_{\alpha,\gamma}(-t^\alpha A)v\|_{L^2} \leq C(N,\alpha)\|v\|_{L^2}, \quad t > 0.$$

Definition 2.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space,the Gaussian white noise is defined as

$$W_i(t) = \sum_{k=1}^{\infty} \sqrt{a_k}\beta_{k,i}(t)e_k(x), \qquad i = 1,2 \tag{2.8}$$

with $a := \sum_{k=1}^\infty a_k < \infty$. Here the $\beta_{k,i}(t)$ are standard one dimensional Wiener process on $t > 0$, $e^k(x)$ are an orthonormal basis in $L^2(\mathbb{R})$.

Now we shall introduce some notations of functional spaces given as follow:

$$H^\sigma = \left\{ u = \sum u_n e_n \in L^2[\mathbb{R}] \,\Big|\, \|u\|_{H^\sigma} = (\Sigma u_n^2 \lambda_n^\sigma)^{\frac{1}{2}} < \infty \right\},$$

with the norm: $\|u\|_{H^\sigma} = (\Sigma u_n^2 \lambda_n^\sigma)^{\frac{1}{2}}$, where $e_n(x) = \sqrt{2/\pi}\sin(n\pi x)$, $\lambda_n = \pi^2 n^2$. Then $(e_n(x), \lambda_n^\beta)$ are the eigenvectors and eigenvalues of $(-\Delta)^\beta$ with Dirichlet boundary conditions, and the operator $(-\Delta)^{\frac{\sigma}{2}}$ is well defined in the Hilbert space H^σ.

Definition 2.7. A (H^σ, L^2)-valued stochastic process $(u(t), n(t))$ is called a mild solution of (1.2) with initial value u_0 and n_0 if the following equation is satisfied:

$$\begin{cases} u(t,x) = T_{\alpha,\beta_1}u_0 + \displaystyle\int_0^t (t-\tau)^{\alpha-1}S_{\alpha,\beta_1}(t-\tau)(un + a|u|^{p-1}u)d\tau \\ \qquad\qquad + \displaystyle\int_0^t (t-\tau)^{\alpha-1}S_{\alpha,\beta_1}(t-\tau)dW_1(\tau), \\ n(t,x) = n_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1}R(x) * (|u|^2 + n^2)d\tau \\ \qquad\qquad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1}R_0(x) * \dot{W}_2(\tau)d\tau. \end{cases} \tag{2.9}$$

Here $R(x) = \mathcal{F}^{-1}\left(\frac{ik}{1-(ik)^{\beta_2}}\right)$, $R_0(x) = \mathcal{F}^{-1}\left(\frac{1}{1-(ik)^{\beta_2}}\right)$, the $*$ denote the convolution on $\mathbb{R}$ and

$$\int_0^t (t-\tau)^{\alpha-1} R(x) * \dot{W}_2(\tau) d\tau \triangleq \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} R(x-y) dW(\tau, y) dy.$$

Remark 2.2. when $\beta = 2$, the $\Delta^{\beta/2} n = n_{xx}$. And the $R(x) = \frac{1}{2} sgn(x) e^{-|x|}$, more details can be seen in [7] .

3. Well-Posedness of the mild solution

In this section, we will establish the basic properties of the following stochastic integrals:

$$z_1(t,x) = \int_0^t (t-\tau)^{\alpha-1} S_{\alpha,\beta}(t-\tau) dW_1(\tau, y), \tag{3.1}$$

$$z_2(t,x) = \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} R(x-y) dW_2(\tau, y) dy. \tag{3.2}$$

Lemma 3.1. *Let $1 < \beta_2 \leq 2$, then for every $t \in \mathbb{R}^+$, the stochastic convolution $z_2(t)$ belongs to L^2, i.e.,*

$$\mathbb{E}\|z_2(t,x)\|_{L^2}^2 < \infty. \tag{3.3}$$

Proof. It follows from the fact that $\|e_k\|_\infty < 1$, then

$$\begin{aligned}
\mathbb{E}\|z_2(t,x)\|^2 &= \mathbb{E}\left\| \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} R(x-y) dW(\tau,y) dy \right\|_{L^2}^2 \\
&= \mathbb{E}\left\| \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} \sum_{k=1}^{\infty} R(x-y)\sqrt{a_k} e_k(y) d\beta_k(\tau) dy \right\|_{L^2}^2 \\
&= \mathbb{E}\left\| \int_0^t (t-\tau)^{\alpha-1} \int_{\mathbb{R}} R(x-y) \sum_{k=1}^{\infty} \sqrt{a_k} e_k(y) dy d\beta_k(\tau) \right\|_{L^2}^2 \\
&\leq \int_{\mathbb{R}} \mathbb{E}|\sum_{k=1}^{\infty} \int_0^t (t-\tau)^{\alpha-1} R * f_k(x) d\beta_k(\tau)|^2 dx \\
&\leq \int_{\mathbb{R}} \int_0^t (t-\tau)^{2(\alpha-1)} |R * f(x)|^2 d\tau dx \\
&\leq C\|R * f\|_{L^2}^2 < \infty,
\end{aligned}$$

where $f_k(x) := \sqrt{a_k} e_k(x)$, and $f(x) := \sum_{k=1}^{\infty} f_k(x)$. □

Lemma 3.2. *Let $\frac{1}{2} < \alpha \leq 1$, then the stochastic convolution $z_2(t), t \in [0,T]$ has a continuous version.*

Proof. For $s, t \in [0,t], s < t$, we have

$$\begin{aligned}
&\mathbb{E}|z_2(t,x) - z_2(s,x)|^2 \\
&= \mathbb{E}| \int_s^t \int_{\mathbb{R}} (t-\tau)^{\alpha-1} R(x-y) dW(\tau,y) dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^s \int_{\mathbb{R}} \left((t-\tau)^{\alpha-1}-(s-\tau)^{\alpha-1}\right) R(x-y)dW(\tau,y)dy|^2 \\
=& 2\mathbb{E}\int_s^t \left|(t-\tau)^{\alpha-1}\int_{\mathbb{R}} R(x-y)\sum_{k=1}^{\infty}\sqrt{a_k}e_k(y)dy\right|^2 d\tau \\
&+2\mathbb{E}\int_0^s \left|\left[(t-\tau)^{\alpha-1}-(s-\tau)^{\alpha-1}\right]\int_{\mathbb{R}} R(x-y)\sum_{k=1}^{\infty}\sqrt{a_k}e_k(y)dy\right|^2 d\tau \\
=& 2I_1+2I_2.
\end{aligned}
$$

Then we caculate that

$$
\begin{aligned}
I_1 =& \mathbb{E}\int_s^t \left|(t-\tau)^{\alpha-1}\int_{\mathbb{R}} R(x-y)\sum_{k=1}^{\infty}\sqrt{a_k}e_k(y)dy\right|^2 d\tau \\
\le& \int_s^t (t-\tau)^{2\alpha-2}\|R(x)\|^2\|\sum_{k=1}^{\infty}\sqrt{a_k}e_k(y)\|^2 d\tau \\
\le& \|R(x)\|^2\|\sum_{k=1}^{\infty}\sqrt{a_k}e_k(y)\|^2\int_s^t (t-\tau)^{2\alpha-2}d\tau \\
\le& C\int_s^t (t-\tau)^{2\alpha-2}d\tau \le C(t-s)^{2\alpha-1},
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 =& \mathbb{E}\int_0^s \left|\left[(t-\tau)^{\alpha-1}-(s-\tau)^{\alpha-1}\right]\int_{\mathbb{R}} R(x-y)\sum_{k=1}^{\infty}\sqrt{a_k}e_k(y)dy\right|^2 d\tau \\
\le& \|R(x)\|^2\|\sum_{k=1}^{\infty}\sqrt{a_k}e_k(y)\|^2\int_0^s \left[(t-\tau)^{\alpha-1}-(s-\tau)^{\alpha-1}\right]^2 d\tau \\
\le& C\int_0^s \frac{[(t-\tau)^{1-\alpha}-(s-\tau)^{1-\alpha}]^2}{(t-\tau)^{2-2\alpha}(s-\tau)^{2-2\alpha}}d\tau \\
\le& C|t-s|^{2-2\alpha}\int_0^s \frac{1}{(t-\tau)^{2-2\alpha}(s-\tau)^{2-2\alpha}}d\tau \\
\le& CT^{4\alpha-3}|t-s|^{2-2\alpha}.
\end{aligned}
$$

Thus, by the Kolmmogorov's test theorem [5], the Lemma was proven. □

The regularity of $z_1(t)$ can be seen in [13], we state them as following

Lemma 3.3 ([13])**.** *Let* $0<\sigma<\frac{1}{2}$ *, then it permits that*

$$
\mathbb{E}\|A^{\sigma/2}z_1(t,x)\|^2_{H^\sigma}<\infty. \tag{3.4}
$$

Lemma 3.4 ([13])**.** *Let* $\beta_1\in(\frac{3}{4},1), 2\alpha<\beta_1\le 2$*, then the stochastic convolution* $z_1(t), t\in[0,T]$ *has a continuous version.*

In order to give the piro estimation of solution, we need the following Gronwall inequality

Lemma 3.5 ([16])**.** *Suppose that* $\beta>0$*,a(t) is a nonnegative function that is locally integrable on* $0\le t<T$ *and g(t) is a nonnegative, nondecreasing continuous*

function defined on $0 \le t < T$, $g(t) < M$ *(constant), and suppose* $u(t)$ *is nonnegative and locally integrable with*

$$u(t) \le a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) ds.$$

on $0 \le t < T$, *then*

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=0}^{\infty} \frac{[g(t)\Gamma(\beta)]^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds.$$

Let X be the product space of $C([0,t], H^\sigma) \times C([0,t], L^2)$, with the norm

$$\|(u,n)\|_X = \sqrt{\|u\|^2_{H^\sigma} + \|n\|^2_{L^2}}.$$

To establish the local well-posedness of the mild solution, consider the space B_R^T by

$$B_R^T = \{(u,n) \in X | \, \|(u,n)\|_X \le R\}.$$

Denote the $(u', n') = (u - z_1, n - z_2)$,then the (u', n') solves the problem

$$\begin{cases} id[I_t^{1-\alpha}(u' - u_0)] + \Delta^{\frac{\beta_1}{2}} u' dt = \big((u'+z_1)(n'+z_2) + a|u'+z_1|^{p-1}(u'+z_1)\big)\, dt, \\ d[I_t^{1-\alpha}(n' - \Delta^{\frac{\beta_2}{2}} n')] = \big(|u'+z_1|^2 + (n'+z_2)^2)_x\big)\, dt. \end{cases} \tag{3.5}$$

Now we give the well-posedness of the mild solution of (3.5).

Theorem 3.1. *Assume* $(u_0, n_0) \in B_R^T$, *then for* $\alpha \in (\frac{3}{4}, 1)$, $2\alpha < \beta_i \le 2 (i = 1, 2)$, $p < 3$, *and* $\frac{1}{6} \le \sigma \le \frac{1}{2}$, *there exist a random variable* $T > 0$, *such that the equation has a unique local mild solution in* B_R^T.

Proof. Taking any $(u', n') \in B_R^T$, and denote the operator $\mathcal{L}$ by

$$\mathcal{L}(u', n') = (U', N'),$$

where

$$U' = T_{\alpha,\beta_1} u_0 + \int_0^t (t-\tau)^{\alpha-1} S_{\alpha,\beta}(t-\tau) \left[(u'+z_1)(n'+z_2) + a|u'+z_1|^{p-1}(u'+z_1) \right] d\tau,$$

and

$$N' = n_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} R(x) * \left[|u'+z_1|^2 + (n'+z_2)^2 \right] d\tau.$$

Then we have

$$\begin{aligned} \|U'\|_{H^\sigma} \le & \|T_{\alpha,\beta_1} u_0\|_{H^\sigma} + \left\| \int_0^t (t-\tau)^{\alpha-1} S_{\alpha,\beta_1}(t-\tau)(u'+z_1)(n'+z_2) d\tau \right\|_{H^\sigma} \\ & + \left\| \int_0^t (t-\tau)^{\alpha-1} S_{\alpha,\beta_1}(t-\tau) a|u'+z_1|^{p-1}(u'+z_1) d\tau \right\|_{H^\sigma} := I_1 + I_2 + I_3. \end{aligned}$$

We deduce that

$$I_1 = \left\| A^{\frac{\sigma}{2}} \int_0^\infty M_\alpha(s) e^{-st^\alpha A^{\frac{\beta_1}{2}}} u_0 ds \right\| \le \left\| A^{\frac{\sigma}{2}} u_0 \right\| = \|u_0\|_{H^\sigma},$$

and

$$
\begin{aligned}
I_2 &\leq \left\| A^{\frac{\sigma}{2}} \int_0^\infty \alpha s M_\alpha(s) e^{-s(t-\tau)^\alpha A^{\frac{\beta_1}{2}}} (u' + z_1)(n' + z_2) d\tau \right\|^2 \\
&= \sum_{k=1}^\infty Re \left\langle \lambda_k^{\frac{\sigma}{2}} \int_0^\infty \alpha s M_\alpha(s) e^{-s(t-\tau)^\alpha \lambda_k^{\frac{\beta_1}{2}}} (u' + z_1)(n' + z_2), e_k \right\rangle^2 \\
&\leq C \sum_{k=1}^\infty Re \left\langle \lambda_k^{\frac{\sigma}{2} - \frac{\beta_1 \theta}{2}} \int_0^\infty \alpha s^{1-\theta} M_\alpha(s)(t-\tau)^{-\alpha\theta}(u' + z_1)(n' + z_2), e_k \right\rangle^2 \\
&\leq C \sum_{k=1}^\infty k^{2(\sigma - \beta_1\theta)} (t-\tau)^{-2\alpha\theta} Re\langle (u' + z_1)(n' + z_2), e_k \rangle^2 \\
&\leq C \sum_{k=1}^\infty k^{2(\sigma - \beta_1\theta)} (t-\tau)^{-2\alpha\theta} \|u' + z_1\|_{L^2} \|n' + z_2\|_{L^2} \\
&\leq C \sum_{k=1}^\infty k^{2(\sigma - \beta_1\theta)} (t-\tau)^{-2\alpha\theta} \left(\|u'\|_{H^\sigma} + \|A^{\frac{\sigma}{2}} z_1\| \right) \|n' + z_2\|_{L^2} < \infty.
\end{aligned}
$$

For I_3, we have

$$
\begin{aligned}
I_3 &\leq \left\| A^{\frac{\sigma}{2}} \int_0^\infty \alpha s M_\alpha(s) e^{-s(t-\tau)^\alpha A^{\frac{\beta_1}{2}}} a|u' + z_1|^{p-1}(u' + z_1) d\tau \right\|^2 \\
&= \sum_{k=1}^\infty Re \left\langle \lambda_k^{\frac{\sigma}{2}} \int_0^\infty \alpha s M_\alpha(s) e^{-s(t-\tau)^\alpha \lambda_k^{\frac{\beta_1}{2}}} a|u' + z_1|^{p-1}(u' + z_1), e_k \right\rangle^2 \\
&\leq C \sum_{k=1}^\infty Re \left\langle \lambda_k^{\frac{\sigma}{2} - \frac{\beta_1 \theta}{2}} \int_0^\infty \alpha s^{1-\theta} M_\alpha(s)(t-\tau)^{-\alpha\theta} |u' + z_1|^{p-1}(u' + z_1), e_k \right\rangle^2 \\
&\leq C \sum_{k=1}^\infty k^{2(\sigma - \beta_1\theta)} (t-\tau)^{-2\alpha\theta} Re\langle |u' + z_1|^{p-1}(u' + z_1), e_k \rangle^2 \\
&\leq C \sum_{k=1}^\infty k^{2(\sigma - \beta_1\theta)} (t-\tau)^{-2\alpha\theta} \|u'\|_{L^p}^p + \|z_1\|_{L^p}^p \\
&\leq C \sum_{k=1}^\infty k^{2(\sigma - \beta_1\theta)} (t-\tau)^{-2\alpha\theta} \|u'\|_{H^\sigma}^p + \|A^{\frac{\sigma}{2}} z_1\|^p < \infty,
\end{aligned}
$$

where $p \leq 3$. Choosing $\frac{1}{6} < \theta < \frac{1}{2\alpha}$, $\alpha > \frac{3}{4}$ ensures that $I_1 < \infty$. Then for the N',we have

$$
\begin{aligned}
\|N'\|_{L^2} &= \|n_0\|_{L^2} + \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t-\tau)^{\alpha-1} R(x) * \left[|u' + z_1|^2 + (n' + z_2)^2 \right] d\tau \right\|_{L^2} \\
&:= J_1 + J_2.
\end{aligned}
$$

Recalling that $\|n_0\|_{L^2} < \infty$, then we deduce that

$$
\begin{aligned}
J_2 &\leq \int_0^t (t-\tau)^{\alpha-1} \left\| R(x) * \left[|u' + z_1|^2 + (n' + z_2)^2 \right] \right\|_{L^2} d\tau \\
&\leq CT^\alpha \left\| R(x) * \left[|u' + z_1|^2 + (n' + z_2)^2 \right] \right\|_{L^2}
\end{aligned}
$$

$$\le CT^{\alpha}\left\|R(x)\right\|_{L^2}\left\|\left|u'+z_1\right|^2+(n'+z_2)^2\right\|_{L^1}$$
$$\le CT^{\alpha}\|R(x)\|_{L^2}\left(\|u'+z_1\|_{L^2}^2+\|n'+z_2\|_{L^2}^2\right)$$
$$\le CT^{\alpha}\left\|R(x)\right\|_{L^2}\left(\|u'\|_{H^\sigma}^2+\|A^{\frac{\sigma}{2}}z_1\|^2+\|n'\|^2+\|z_2\|^2\right)<\infty.$$

Thus, we obtained that $\mathcal{L}: B_R^T \to B_R^T$. Next we will prove $\mathcal{L}$ is compressed for any $(u_1, n_1), (u_2, n_2) \in B_R^T$. To the end, it suffices to show that

$$\mathcal{L}[(u_1,n_1)-(u_2,n_2)]=(U'_{1,2},N'_{1,2}).$$

In fact, direct computation implies that

$$\begin{aligned}\|U'_{1,2}\|_{H^\sigma} \le& \left\|\int_0^t (t-\tau)^{\alpha-1}S_{\alpha,\beta_1}(t-\tau)\left[(u'_1+z_1)(n'_1+z_2)-(u'_2+z_1)(n'_2+z_2)\right]d\tau\right\|_{H^\sigma}\\ &+\left\|\int_0^t (t-\tau)^{\alpha-1}S_{\alpha,\beta}(t-\tau)(u'_1+z_1)a\left[|u'_1+z_1|^{p-1}(u'_1+z_1)\right.\right.\\ &-\left.\left.|u'_2+z_1|^{p-1}(u'_2+z_1)\right]d\tau\right\|_{H^\sigma}:=K_1+K_2,\end{aligned}$$

and

$$\begin{aligned}\|N'_{1,2}\|_{L^2}=&\left\|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}R(x)*\left[|u'_1+z_1|^2-|u'_2+z_1|^2\right.\right.\\ &+\left.\left.(n'_1+z_2)^2-(n'_2+z_2)^2\right]d\tau\right\|_{L^2}.\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}K_1 \le& \int_0^t (t-\tau)^{\alpha-1}S_{\alpha,\beta}(t-\tau)\left[(u'_1+z_1)(n'_1+z_2)-(u'_2+z_1)(n'_2+z_2)\right]d\tau\\ \le& C\sum_{k=1}^{\infty}k^{2(\sigma-\beta\theta)}(t-\tau)^{-2\alpha\theta}Re\langle(u'_1+z_1)(n'_1+z_2)-(u'_2+z_1)(n'_2+z_2),e_k\rangle^2\\ \le& C\sum_{k=1}^{\infty}k^{2(\sigma-\beta\theta)}(t-\tau)^{-2\alpha\theta}\|(u'_1-u'_2)(n'_2+z_2)+(u'_1+z_1)(n'_1-n'_2)\|_{L^1}^2\\ \le& CR^2\sum_{k=1}^{\infty}k^{2(\sigma-\beta\theta)}(t-\tau)^{-2\alpha\theta}\left(\|(u'_1-u'_2)\|_{H^\sigma}^2+\|(n'_1-n'_2)\|_{L^2}^2\right),\end{aligned}$$

and

$$\begin{aligned}K_2 \le& C\sum_{k=1}^{\infty}k^{2(\sigma-\beta\theta)}(t-\tau)^{-2\alpha\theta}Re\langle\left[|u'_1+z_1|^{p-1}+|u'_2+z_1|^{p-1}\right](u'_1-u'_2),e_k\rangle^2\\ \le& C\sum_{k=1}^{\infty}k^{2(\sigma-\beta\theta)}(t-\tau)^{-2\alpha\theta}\|u'_1-u'_2\|_{L^3}(\|u'_1\|_{L^3}^{2p}+\|u'_2\|_{L^3}^{2p}+\|z_1\|_{L^3}^{2p})\\ \le& CR^{2p}\sum_{k=1}^{\infty}k^{2(\sigma-\beta\theta)}(t-\tau)^{-2\alpha\theta}\|u'_1-u'_2\|_{H^\sigma}.\end{aligned}$$

As for $N'_{1,2}$, we deduce that

$$\|N'_{1,2}\|_{L^2}\le CT^{\alpha}\left\|R(x)\right\|_{L^2}\left\|\left[|u'_1+z_1|^2-|u'_2+z_1|^2+(n'_1+z_2)^2-(n'_2+z_2)^2\right]\right\|_{L^1}$$

$$
\begin{aligned}
&\leq CT^{\alpha}\left\|R(x)\right\|_{L^2} \left\| \left[|u_1'|^2 - |u_2'|^2 + 2Re\left(\bar{z_1}(n_1' - n_2')\right)\right.\right.\\
&\quad \left.\left. +(n_1')^2 - (n_2')^2 + 2z_2(n_1' - n_2')\right] \right\|_{L^1}\\
&\leq CT^{\alpha}\||u_1'|^2 - |u_2'|^2\|_{L^1} + \|2Re\left(\bar{z_1}(n_1' - n_2')\right)\|_{L^1}\\
&\quad + \|(n_1')^2 - (n_2')^2\|_{L^1} + 2\|z_2(n_1' - n_2')\|_{L^1}\\
&\leq CR^2T^{\alpha}\left[\|u_1' - u_2'\|_{L^2} + 2\|(n_1' - n_2')\|_{L^2} + \|n_1' - n_2'\|_{L^2} + 2\|(n_1' - n_2')\|_{L^2}\right]\\
&\leq CR^2T^{\alpha}\left[\|u_1' - u_2'\|_{H^\sigma} + \|(n_1' - n_2')\|_{L^2}\right].
\end{aligned}
$$

Choosing T be small enough such that $\mathcal{L}$ is compressed. Thus, $\mathcal{L}$ has a unique point in B_R^T. The proof is complete. □

Next,we will prove the global well-poseness of (1.2). Some prior estimates are required.

Lemma 3.6. *Assume that* (u', n') *is the solution over* $[0, T]$ *and* $\beta_2 = 2$*, then it holds that*

$$
\begin{aligned}
&\|u'\| + \|n'\| + \|n_x'\| + \|(-\Delta)^{\sigma/2}u'\|^2\\
\leq& C(T,\epsilon,\|u_0\|,\|n_0\|,\|A^{\sigma/2}z_1\|,\|A^{\sigma/2}z_2\|)\\
&+ \int_0^t \sum_{k=1}^{\infty} \frac{c^k}{\Gamma(\alpha k)}(t-s)^{\alpha k-1} I_t^{\alpha}\left(\|A^{\sigma/2}z_1\| + \|A^{\sigma/2}z_2\|\right) ds.
\end{aligned}
\tag{3.6}
$$

Proof. Mulitiply the first equation of (3.5) by $\bar{u}'$, and take the real part :

$$
\begin{aligned}
&\|\Delta^{\beta_1/4}u'\|^2 + \langle |u' + z_1|^2, (n' + z_2)\rangle + \|u' + z_1\|_{L^p}^p\\
=& Re\langle (u' + z_1)(n' + z_2), z_1\rangle + Re\langle |u' + z_1|^{p-1}(u' + z_1), z_1\rangle.
\end{aligned}
$$

Using the Hölder inequality, we have

$$
\begin{aligned}
&Re\langle (u' + z_1)^2(n' + z_2), z_1\rangle + Re\langle |u' + z_1|^{p-1}(u' + z_1), z_1\rangle\\
\leq& \epsilon\langle |u' + z_1|^2, (n' + z_2)\rangle + C(\epsilon)\langle |n' + z_2|, z_1\rangle + \epsilon\|u' + z_1\|_{L^p}^p + C(\epsilon)\|z_1\|_{L^p}\\
\leq& \epsilon\langle |u' + z_1|^2, (n' + z_2)\rangle + \epsilon\|u' + z_1\|_{L^p}^p + C(\epsilon)\|n + z_2\| + C(\epsilon)\|z_1^2\|_{L^p}\\
\leq& \epsilon\langle |u' + z_1|^2, (n' + z_2)\rangle + \epsilon\|u' + z_1\|_{L^p}^p + C(\epsilon)\|n + z_2\| + C(\epsilon)\|A^{\sigma/2}z_1\|.
\end{aligned}
$$

Take $\epsilon < 1$ to get

$$
\|\Delta^{\beta/4}u'\|^2 + \langle |u' + z_1|^2, (n' + z_2)\rangle + \|u' + z_1\|_{L^p}^p \leq C(\epsilon)\|n' + z_2\| + C(\epsilon)\|A^{\sigma/2}z_1\|. \tag{3.7}
$$

Taking the imaginary part, and using (3.7), we have

$$
\begin{aligned}
D_t^{\alpha}\|u'\|^2 &\leq Im\langle (u' + z_1)(n' + z_2), z_1\rangle + Im\langle |u' + z_1|^{p-1}(u' + z_1), z_1\rangle\\
&\leq C(\epsilon)\|n' + z_2\| + C(\epsilon)\|A^{\sigma/2}z_1\|,
\end{aligned}
$$

then with the Gronwall inequality, we obtain

$$
\|u'\|^2 \leq C(\epsilon)I_t^{\alpha}\left(\|n' + z_2\| + \|A^{\sigma/2}z_1\|\right) + C(T, \epsilon, \|u_0\|, \|n_0\|). \tag{3.8}
$$

Multiplying the second equation of (3.5) by n'. By (3.7) and Gagliardo-Nirenberg inequality, we have

$$
D_t^{\alpha}(\|n'^2\| + \|n_x'\|^2)
$$

$$
\begin{aligned}
&\leq \int_\Omega (|u+z_1|^2+(n+z_2)^2)|n'_x|dx \\
&\leq \epsilon\|u'+z_1\|_{L^4}+\epsilon\|n'+z_2\|_{L^4}+C(\epsilon)\|n_x\| \\
&\leq \epsilon\|u'\|_{L^4}+\epsilon\|n'\|_{L^4}+C(\epsilon)\|A^{\sigma/2}z_2\|+C(\epsilon)\|A^{\sigma/2}z_1\|+C(\epsilon)\|n'_x\| \\
&\leq \epsilon\|A^{\beta/4}u'\|\|u'\|+\epsilon\|n'_x\|\|n'\|+C(\epsilon)\|A^{\sigma/2}z_2\|+C(\epsilon)\|A^{\sigma/2}z_1\|+C(\epsilon)\|n'_x\| \\
&\leq C(\epsilon)\left(\|n'\|+\|n'_x\|+\|A^{\beta/4}u'\|+\|A^{\sigma/2}z_2\|+\|A^{\sigma/2}z_1\|\right) \\
&\leq C(\epsilon)\left(\|n'\|+\|n'_x\|+\|z_2\|+\|A^{\sigma/2}z_2\|+\|A^{\sigma/2}z_1\|\right).
\end{aligned}
$$

It is easily to show that $\|A^\sigma z_2\|<\infty$ when $\beta=2$ in the second equation. Then together with the (3.7), (3.8) and Gronwall inequality, the following conculsion can be present,

$$
\begin{aligned}
&\|u'\|+\|n'\|+\|n'_x\| \\
\leq& C(t,\epsilon,\|u_0\|,\|n_0\|,\|A^{\sigma/2}z_1\|,\|A^{\sigma/2}z_2\|) \\
&+\int_0^t\sum_{k=1}^\infty \frac{c^k}{\Gamma(\alpha k)}(t-s)^{\alpha k-1}I_t^\alpha\left(\|A^{\sigma/2}z_1\|+\|A^{\sigma/2}z_2\|\right)ds.
\end{aligned}
\tag{3.9}
$$

Multiply the first equation of (3.5) by $A^\sigma\bar{u'}$, and take the real part :

$$
\begin{aligned}
\|A^{\beta/4+\sigma/2}u'\|^2 =& Re\langle(u'+z_1)(n'+z_2),A^{\sigma/2}u'\rangle \\
&+Re\langle|u'+z_1|^{p-1}(u'+z_1),A^{\sigma/2}u'\rangle.
\end{aligned}
$$

Using the Hölder inequality, we have

$$
\begin{aligned}
&Re\langle(u'+z_1)^2(n'+z_2),A^{\sigma/2}u'\rangle+Re\langle|u'+z_1|^{p-1}(u'+z_1),A^{\sigma/2}u'\rangle \\
\leq& \epsilon\langle|u'+z_1|^2,(n'+z_2)\rangle+C(\epsilon)\langle|n+z_2|,A^{\beta/4+\sigma/2}u'\rangle \\
&+\epsilon\|u'+z_1\|_{L^p}^p+C(\epsilon)\|A^{\beta/4+\sigma/2}u'\|_{L^4} \\
\leq& \epsilon\|A^{\beta/4+\sigma/2}u'\|^2+\epsilon\|u'+z_1\|_{L^4}^2+C(\epsilon)\|n+z_2\|+C(\epsilon)\|A^{\beta/4+\sigma/2}z_1\|.
\end{aligned}
$$

Take $\epsilon<1$ to get

$$
\|A^{\beta/4+\sigma/2}u'\|^2\leq C(\epsilon)\|n+z_2\|+C(\epsilon)\|A^{\sigma/2}z_1\|. \tag{3.10}
$$

Taking the imaginary part,

$$
\begin{aligned}
D_t^\alpha\|A^{\sigma/2}u'\|^2 &\leq Im\langle(u'+z_1)^2(n'+z_2),A^{\sigma/2}u'\rangle+Im\langle|u'+z_1|^{p-1}(u'+z_1),A^{\sigma/2}u'\rangle \\
&\leq C(\epsilon)\|n'+z_2\|+C(\epsilon)\|A^{\beta/4+\sigma/2}u'\|^2.
\end{aligned}
$$

Using (3.10) and applying the generalized Gronwall inequality, we can obtain that the conclusion. Thus, the proof is completed. □

If we set R be the bounded gotten in Lemma 3.6, the solution exists in the interval $[0,T^*]$. Then we can repeat the proof of Theorem 3.1 to get the existence and uniqueness in $[T^*,2T^*],[2T^*,3T^*],...$, hence we can get the following theorem

Theorem 3.2. *Assume that* $(u_0,n_0)\in B_R^T$, *then for* $\alpha\in(\frac{3}{4},1)$, $2\alpha<\beta_1\leq 2$, $p<3$, $\beta_2=2$, *and* $\frac{1}{6}\leq\sigma\leq\frac{1}{2}$, *the equations (1.1) posses a global mild solution* $\{(u,n),t\in[0,T]\}$ *in* $C([0,T],H^\sigma)\times C([0,T],L^2)$ *for all* $u_0\in H^\sigma,n_0\in L^2$.

4. Numerical simulation

In this section, we set $\beta_1 = \beta_2 = 2$, and use the Galerkin finite element method to approximate the solution of time fractional equation (4.1),

$$\begin{cases} iD_t^\alpha u - (-\Delta)u = un + a|u|^{p-1}u + \dot{W}_1(t), \\ D_t^\alpha(n - \Delta n) = (|u|^2 + n^2)_x + \dot{W}_2(t), \\ u(0) = u_0(x), n(0) = n_0(x). \end{cases} \tag{4.1}$$

This method is developed by Li et al. [10] for stochastic space-time fractional wave equations. Zou [19] developed Galerkin finite element method for time-fractional stochastic heat equation. Liang et al. [11] gave the analysis of time fractional and space nonlocal stochastic nonlinear Schrödinger equation driven by multiplicative white noise.

Let V_J denoted the Galerkin subspace by

$$V_J := span\{e_i\}_{i=1}^J,$$

and the orthogonal projection operator P_J is defined as

$$u_h = P_J u = \sum_{j=1}^J \langle u, e_j\rangle e_j, u = \sum_{j=1}^\infty \langle u, e_j\rangle e_j.$$

Now we give the semi-discrete finite element approximation. For convenience, we set $\beta_1 = \beta_2 = 2$, denote (u_h, n_h) be the numerical approximation

$$\begin{cases} iD_t^\alpha u_h + \Delta u_h = P_J(u_h n_h) + aP_J(|u_h|^{p-1}u_h) + P_J dW_1(t), \\ D_t^\alpha(n_h - \Delta n_h) = P_J(|u_h|^2 + n_h^2)_x + dW_2(t), \\ u(0) = P_J u_0(x), n(0) = P_J n_0(x). \end{cases} \tag{4.2}$$

Similar to the proof of Theorem 3.1, we have the following conclusion

Theorem 4.1. *Assume $(u_0, n_0) \in B_R^T$, for $\alpha \in (\frac{3}{4}, 1)$, and $\frac{1}{6} \le \sigma \le \frac{1}{2}$, $p = 3$,there exist a random variable $T > 0$, such that the equation has a unique mild solution with the form*

$$\begin{cases} u_h(t,x) = T_{\alpha,\beta_1} u_h(0) + \int_0^t (t-\tau)^{\alpha-1} S_{\alpha,\beta_1}(t-\tau) P_J(u_h n_h + a|u_h|^{p-1}u_h) d\tau \\ \qquad + \int_0^t (t-\tau)^{\alpha-1} S_{\alpha,\beta_1}(t-\tau) P_J dW_1(\tau), \\ n_h(t,x) = n_h(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} R(x) * P_J(|u_h|^2 + n_h^2) d\tau \\ \qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} R_0(x) * P_J \dot{W}_2(\tau_h) d\tau. \end{cases} \tag{4.3}$$

Firstly, we give the following two lemmas

Lemma 4.1 ([11]). *If P_J defined as above, and I denote the identify operator. then we have the following estimate:*

$$\|I - P_J\|_{\mathcal{L}(H^\sigma, L^2)} \le CJ^{-\sigma}. \tag{4.4}$$

Lemma 4.2 ([14]). *For complex function U, V, u, n we have*

$$\left||U|^2V-|u|^2v\right|\le(\max\{|U|,|V|,|u|,|n|\})*(2|U-u|+|V-v|). \tag{4.5}$$

In the sequel, denote

$$\epsilon_1(t)=T_{\alpha,2}(t),\quad \epsilon_2(t)=t^{\alpha-1}S_{\alpha,2}(t).$$

The regularity of $\epsilon_1(t),\epsilon_2(t)$ can be found from [11, 19]. Then we can deduce the spatial convergence order of the semi-discrete form(4.2).

Theorem 4.2. *Assume* $(u_0,n_0)\in B_R^T$ *and* $\alpha\in(\frac{3}{4},1)$, $\frac{1}{6}\le\sigma\le\frac{1}{2}$. *Then for all* $t\in(0,T)$, *it holds that*

$$\mathbb{E}\|u(t)-u_h(t)\|^2+\mathbb{E}\|n(t)-n_h(t)\|^2\le CJ^{-2\sigma}. \tag{4.6}$$

Proof. It suffices to estimate the norms between (4.1) and (4.3) . In fact, we have

$$\begin{aligned}u(t)-u_h(t) &= \epsilon_1(t)\,[u(0)-u_h(0)]+\int_0^t\epsilon_2(t-\tau)\,[un-P_Ju_hn_h]\,d\tau\\ &\quad+\int_0^t\epsilon_2(t-\tau)\left[|u|^{p-1}u-|u_h|^{p-1}u_h\right]d\tau+\int_0^t\epsilon_2(t-\tau)\,[dW_1(\tau)-P_JdW_1(\tau)]\\ &:= I_1+I_2+I_3+I_4,\end{aligned}$$

and

$$\begin{aligned}n(t)-n_h(t,x) &= n(0)-n_h(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}R(x)*\left[u^2-P_J|u_h|^2\right]d\tau\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}R(x)*\left[n^2-P_J|n_h|^2\right]d\tau\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}R_0(x)*\left[\dot{W}_2(\tau_h)-P_J\dot{W}_2(\tau_h)\right]d\tau\\ &:= J_1+J_2+J_3+J_4.\end{aligned}$$

It follows from Lemma 4.1 and Theorem 3.2 that

$$\mathbb{E}\|I_1\|^2\le C\mathbb{E}\|u_0\|_{H^\sigma}J^{-2\sigma}.$$

For I_2, we can derive that

$$\begin{aligned}\mathbb{E}\|I_2\|^2 &= \int_0^t\epsilon_2(t-\tau)\,[un-P_Jun]\,d\tau+\int_0^t\epsilon_2(t-\tau)\,[P_Jun-P_Ju_hn_h]\,d\tau\\ &:= I_{21}+I_{22}.\end{aligned}$$

We deduce from Lemma 4.1 and Lemma 4.2 that

$$\begin{aligned}\mathbb{E}\|I_{21}\|^2 &\le \mathbb{E}\int_0^t\|\epsilon_2(t-\tau)\,[un-P_Jun]\,\|^2d\tau\\ &\le \mathbb{E}\int_0^t(t-\tau)^{2\alpha-2}\|(I-P_J)(un)\|^2d\tau\le Ct^{2\alpha-1}J^{-2\sigma},\\ \mathbb{E}\|I_{22}\|^2 &\le \mathbb{E}\int_0^t\|\epsilon_2(t-\tau)P_J\,[un-u_hn_h]\,\|^2d\tau\end{aligned}$$

$$\leq \int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|P_J(un-u_hn_h)\|^2 d\tau$$
$$\leq C\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|u-u_h\|^2+\mathbb{E}\|n-n_h\|^2 d\tau.$$

Similar to I_3, we can get

$$\begin{aligned}\mathbb{E}\|I_3\|^2 &= \int_0^t \epsilon_2(t-\tau)\left[|u|^2u-P_J|u|^2\right]d\tau+\int_0^t \epsilon_2(t-\tau)\left[P_J|u|^2u-P_J|u_h|^2u_h\right]d\tau\\ &= I_{31}+I_{32}.\end{aligned}$$

Then use Lemma 4.1 and 4.2, we have

$$\mathbb{E}\|I_{31}\|^2 \leq \mathbb{E}\int_0^t (t-\tau)^{2\alpha-2}\|(I-P_J)(|u|^2u)\|^2 d\tau \leq Ct^{2\alpha-1}J^{-2\sigma}$$

and

$$\begin{aligned}\mathbb{E}\|I_{32}\|^2 &\leq \int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|P_J(|u|^2u-|u_h|^2u_h)\|^2 d\tau\\ &\leq C\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|u-u_h\|^2 d\tau.\end{aligned}$$

For I_4, we have the following estimate

$$\begin{aligned}\mathbb{E}\|I_4\|^2 &= \int_0^t \epsilon_2(t-\tau)\left[dW_1(\tau)-P_JdW_1(\tau)\right] \leq \int_0^t \|\epsilon_2(t-\tau)(I-P_J)\Phi\|^2 d\tau\\ &\leq \int_0^t (t-\tau)^{2\alpha-2}\|(I-P_J)\Phi\|^2 d\tau \leq Ct^{2\alpha-1}J^{-2\sigma}.\end{aligned}$$

Then for J_1 and J_2, according to the regularity of $R(x)$, we have

$$\mathbb{E}\|J_1\|^2 \leq CJ^{-\sigma}$$

and

$$\begin{aligned}\mathbb{E}\|J_2\|^2 &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}R(x)*\left[|u|^2-P_J|u|^2\right]d\tau\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}R(x)*\left[P_J|u|^2-P_J|u_h|^2\right]d\tau\\ &\leq \int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|u-u_h\|^2 d\tau+Ct^{2\alpha-1}J^{-\sigma}.\end{aligned}$$

Then we estimate the J_3

$$\mathbb{E}\|J_3\|^2 \leq \int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|n-n_h\|^2 d\tau+Ct^{2\alpha-1}J^{-\sigma}.$$

For J_4, we have

$$\mathbb{E}\|J_4\|^2 \leq \int_{\mathbb{R}}\int_0^t (t-\tau)^{\alpha-1}R(x-y)dW(\tau,y)dy$$

$$
\begin{aligned}
&- \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} R(x-y) P_J dW(\tau, y) dy \\
&\leq \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} R(x-y)(I-P_J) dW(\tau, y) dy \\
&\leq C t^{2\alpha-1} J^{-2\sigma}.
\end{aligned}
$$

Applying the Gronwall inequality, the proof of the conclusion can be drawn. □

Next, we will provide a fully discrete scheme. Let δt be the time mesh size, u_h^n be the approximation of $u_h(t_n)$. Then the fully discretized scheme at time t_n is defined as follows

$$
\begin{cases}
u_h^n = \epsilon_1(t_n) u_h(0) + \delta t \sum_{j=0}^{n-1} (t_n - t_j)^{\alpha-1} \epsilon_2(t_n)(t_n - t_j) P_J(u_h n_h + a|u_h|^2 u_h) \\
\qquad + \delta t \sum_{j=0}^{n-1} (t_n - t_j)^{\alpha-1} \epsilon_2(t_n)(t_n - t_j) P_J \delta W_j, \\
n_h^n = n_h(0) + \dfrac{\delta t}{\Gamma(\alpha)} \sum_{j=1}^{n} (t_n - t_j)^{\alpha-1} R(x) * P_J(|u_h|^2 + n_h^2) \\
\qquad + \dfrac{\delta t}{\Gamma(\alpha)} \sum_{j=1}^{n} (t_n - t_j)^{\alpha-1} R(x) * P_J(\delta W_2(t_j)).
\end{cases}
\tag{4.7}
$$

The temporal convergence order of the fully discrete scheme is given as following theorem:

Theorem 4.3. *Assume $(u_0, n_0) \in B_R^T$, and $\alpha \in (\frac{3}{4}, 1)$, $\frac{1}{6} \leq \sigma \leq \frac{1}{2}$. Then for all $t \in (0, T)$, it holds that*

$$
\mathbb{E}\|u_h(t_j) - u_h^j\|^2 + \mathbb{E}\|n_h(t_j) - n_h^j\|^2 \leq C \delta t^{2\alpha-1}. \tag{4.8}
$$

Proof. Firstly, direct calculation gives

$$
\begin{aligned}
u_h(t_n) - u_h^n = &\int_0^{t_n} \epsilon_2(t_n - \tau) P_J(u_h n_h) d\tau - \delta t \sum_{j=1}^{n-1} \epsilon_2(t_n - t_j) P_J(u_h^j n_h^j) \\
&+ \int_0^{t_n} \epsilon_2 P_J(|u_h|^2 u_h) d\tau - i\delta t \sum_{j=1}^{n-1} \epsilon_2(t_n - t_j) P_J(|u_h^j|^2 u_h^j) \\
&+ \int_0^{t_n} \epsilon_2 dW(\tau) - \delta t \sum_{j=1}^{n-1} \epsilon_2(t_n - t_j) P_J \delta W^j := I_1 + I_2 + I_3.
\end{aligned}
$$

For I_1, we have

$$
\begin{aligned}
I_1 &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \epsilon_2(t_n - \tau) P_J(u_h n_h) - \epsilon_2(t_n - t_j) P_J(u_h^j n_h^j) d\tau \\
&= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \epsilon_2(t_n - \tau) P_J(u_h n_h) - \epsilon_2(t_n - \tau) P_J(u_h(t_j) n_h(t_j)) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\epsilon_2(t_n-\tau)P_J(u_h(t_j)-\epsilon_2(t_n-t_j)P_J(u_h(t_j)n_h(t_j))d\tau\\
&+\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\epsilon_2(t_n-t_j)P_J(u_h(t_j)n_h(t_j))-\epsilon_2(t_n-t_j)P_J(u_hj)n_h^j)d\tau\\
=\,& I_{11}+I_{12}+I_{13}.
\end{aligned}
$$

It follows from Lemma 4.1 that

$$
\begin{aligned}
\mathbb{E}\|I_{11}\|^2 &\le \mathbb{E}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_n-\tau)^{2\alpha-2}\|u_hn_h-u_h(t_j)n_h(t_j)\|^2d\tau\\
&\le C\mathbb{E}\int_{t_j}^{t_{j+1}}(t_n-\tau)^{2\alpha-2}d\tau \le \frac{t^{2n-1}}{2\alpha-1}\delta t^{2\alpha-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\|I_{12}\|^2 &\le \mathbb{E}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|\epsilon_2(t_n-\tau)-\epsilon_2(t_n-t_j)P_J(\|u_h(t_j)n_h(t_j)\|^2d\tau\\
&\le \mathbb{E}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(\tau-t_j)^2\|P_J(\|u_h(t_j)n_h(t_j)\|^2d\tau \le \delta t^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}\|I_{13}\|^2 &\le \mathbb{E}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_n-\tau)^{2\alpha-2}\|u_hn_h-u_h(t_j)n_h(t_j)\|^2d\tau\\
&\le C\mathbb{E}\int_{t_j}^{t_{j+1}}(t_n-\tau)^{2\alpha-2}\mathbb{E}\|u_h(t_j)-u_h^j\|^2+\mathbb{E}\|n_h(t_j)-n_h^j\|^2d\tau\\
&\le \delta t^{2\alpha-1}\sum_{j=1}^{n-1}\mathbb{E}\|u_h(t_j)-u_h^j\|^2+\mathbb{E}\|n_h(t_j)-n_h^j\|^2.
\end{aligned}
$$

Direct computation gives that

$$
\mathbb{E}\|I_2\|^2 \le C\delta t^2 + C\delta t^{2\alpha-1}\mathbb{E}\sum_{j=1}^{n-1}\|u_h(t_j)-u_h^j\|^2, \mathbb{E}\|I_3\|^2 \le C\delta t^{2\alpha-1}.
$$

Then we talk about n_h

$$
\begin{aligned}
n_h(t_n)-n_h^n =& \frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}R(x)*P_J(|u_h|^2)d\tau\\
&-\frac{\delta t}{\Gamma(\alpha)}\sum_{j=1}^{n}(t_n-t_j)^{\alpha-1}R(x)*P_J(|u_h|^2)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}R(x)*P_J(n_h^2)d\tau\\
&-\frac{\delta t}{\Gamma(\alpha)}\sum_{j=1}^{n}(t_n-t_j)^{\alpha-1}R(x)*P_J(n_h^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}R_0(x)*P_J\dot{W}_2(\tau_h)d\tau \\
&-\frac{\delta t}{\Gamma(\alpha)}\sum_{j=1}^{n}(t_n-t_j)^{\alpha-1}R(x)*P_J(\delta W_2(t_j)) \\
:=&K_1+K_2+K_3.
\end{aligned}
$$

For K_1 and K_2, we have

$$
\begin{aligned}
\mathbb{E}\|K_1\|^2 &\le C\delta t^{2\alpha-1}\sum_{j=1}^{n-1}\mathbb{E}\|u_h(t_j)-u_h^j\|^2, \\
\mathbb{E}\|K_2\|^2 &\le C\delta t^{2\alpha-1}\sum_{j=1}^{n-1}\mathbb{E}\|n_h(t_j)-n_h^j\|^2.
\end{aligned}
$$

For K_3, we derive that

$$
\begin{aligned}
K_3 \le& \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}R_0(x)*P_J\dot{W}_2(\tau)d\tau \\
&-\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n}(t_n-t_j)^{\alpha-1}R(x)*P_J(\delta W_2(t_j))\delta t \\
\le& \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t-\tau)^{\alpha-1}R_0(x)*P_J\dot{W}_2(\tau)d\tau \\
&-\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t-t_j)^{\alpha-1}R_0(x)*P_J\delta W_2(\tau_j)d\tau \\
\le& \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left[(t-\tau)^{\alpha-1}-(t-t_j)^{\alpha-1}\right]R_0(x)*P_J\dot{W}_2(\tau)d\tau \\
&+\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t-t_j)^{\alpha-1}R_0(x)*P_J[\dot{W}(\tau)-\delta W_2(\tau_j)]d\tau.
\end{aligned}
$$

Finally, The Gronwall inequality guarantees the conclusion (4.8) holds. □

As we can see , when we take $\alpha = 0.5, 0.8$, respectively. Figure 1 shows the simulation results (u, n) in coupled time fractional Schrödinger-BBM equations driven by Gaussian noises obtained by numerical scheme. When we set $n = 0$, the equation (4.1) reduces to a time fractional Schrödinger equation driven by Gaussian noises. The simulation results obtained by the same method are shown as Figure 2.

We can see from those figures that the smaller the time derivative is, the faster the solution of the equation decays. At the same time, n decays faster than u. Under the influence of n, u will have more fluctuation than when there is no influence.

Acknowledgements

The authors would like to express their sincere gratitude to all editors and reviewers for their careful reading of our manuscript and valuable suggestions that helped to improve the article.

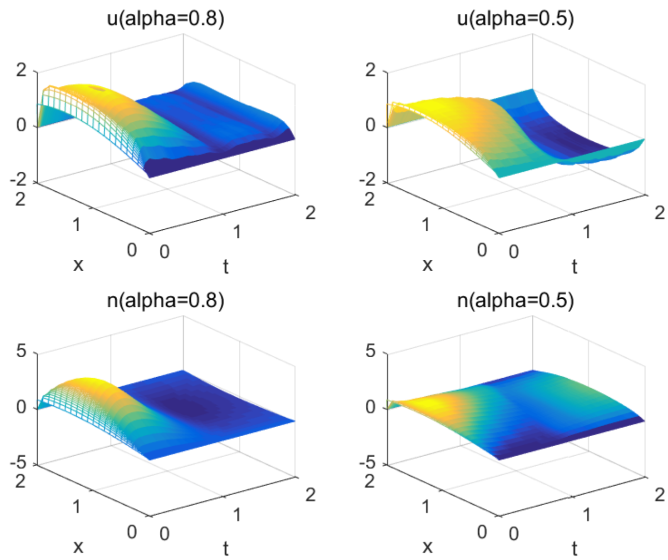

Figure 1. Coupled time fractional Schrödinger-BBM equations driven by Gaussian noises.

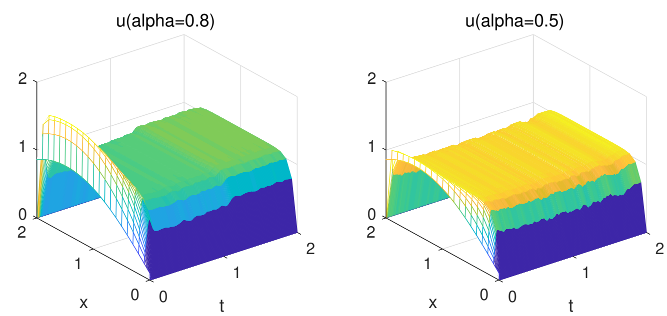

Figure 2. Fractional Schrödinger equations driven by Gaussian noises.

References

[1] K. Appert and J. Vaclavik, *Dynamics of coupled solitons*, Phys. Fluids, 1977, 20(11), 1845–1849.

[2] T. Benjamin, J. Bona and J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Math. Phys. Sci., 1972, 272(1220), 47–78.

[3] J. Bona and N. Tzvetkov, *Sharp well-posedness results for the BBM equation*, Discrete Contin. Dyn. Syst, 2009, 23(4), 1241–1252.

[4] P. Carvalho and G. Planas, *Mild solutions to the time fractional Navier-Stokes equations in* R^n, J. Diff. Eqs., 2015, 259(7), 2948–2980.

[5] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge university press, Cambridge, 2014.

[6] J. Gibbons, S. Thornhill and M. Wardrop, *On the theory of Langmuir solitons*. J. Plasma Phys., 1977, 17(2), 153–170.

[7] B. Guo, C. Miao and H. Huang, *Global flow generated by coupled system of Schrödinger-BBM equations*, Sci. China Math, 1998, 41(2), 131–138.

[8] B. Guo, *The Global solution for the coupled system of BBM-Schrödinger equations*, J. Eng. Math (in Chinese), 1987, 4(3), 1–12

[9] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A, 2000, 268(4-6), 298–305.

[10] Y. Li, Y. Wang and W. Deng, *Galerkin finite element approximations for stochastic space-time fractional wave equations*, SIAM J. Numer. Anal., 2017, 55(6), 3173–3202.

[11] J. Liang, X. Qian, T. Shen and S. Song, *Analysis of time fractional and space nonlocal stochastic nonlinear Schrödinger equation driven by multiplicative white noise*, J. Math. Anal. Appl., 2018, 466(2), 1525–1544.

[12] V. G. Makhankov, *Dynamics of classical solitons (in non-integrable systems)*, Phys. Rep., 1978, 35(1), 1–128.

[13] T. Shen, J. Xin and J. Huang, *Time-space fractional stochastic Ginzburg-Landau equation driven by Gaussian white noise*, Stoch. Ana. Appl., 2018, 36(1), 103–113.

[14] Z. Sun and D. Zhao, *On the L^∞ convergence of a difference scheme for coupled nonlinear Schrödinger equations*, Compu. Math. Appl., 2010, 59(10), 3286–3300.

[15] G. Whitham, *Linear and nonlinear waves*, John Wiley & Sons, New Jersey, 2011.

[16] H. Ye, J. Gao and Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*. J. Math. Anal. Appl., 2007, 328(2), 1075–1081.

[17] V. E. Zakharov, *Collapse of Langmuir waves*. Sov. Phys. Jetp, 1972, 35(5), 908–914.

[18] M. Zhao, C. Zhu and Y. Li, *Global attractor for a class of semi-discrete system of nonlinear Schrödinger-BBM equations*, Acta Math. Sin. (in Chinese), 2015, 58(2), 227–242.

[19] G. Zou, *A Galerkin finite element method for time-fractional stochastic heat equation*, Computers. Math. Appl., 2018, 75(11), 4135–4150.