Journal of Applied Analysis and Computation
Volume 11, Number 4, August 2021, 2090–2101
Website:http://www.jaac-online.com
DOI:10.11948/20200316

EIGENVALUE PROBLEM OF A WEAKLY SINGULAR COMPACT INTEGRAL OPERATOR BY DISCRETE LEGENDRE PROJECTION METHODS

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Abstract In this article, the discrete version of Legendre projection and iterated Legendre projection methods are considered to find the approximate eigenfunctions (eigenvalues and eigenvectors) of a weakly singular compact integral operator. Making use of a sufficiently accurate numerical quadrature rule, we establish the error bounds of the approximated eigenvalues and eigenvectors by discrete Legendre projection and iterated discrete Legendre projection methods in both L^2 and uniform norm. In particular, we obtain the optimal convergence rates $\mathcal{O}(n^{-m})$ for the eigenfunctions in iterated discrete Legendre projection method in L^2 and uniform norms, where n is the highest degree of the Legendre polynomial employed in the approximation and m is the smoothness of the eigenvectors. Numerical examples are presented to illustrate the theoretical results.

Keywords Convergence rates, weakly singular kernels, discrete Projection methods, hyperinterpolation operator, legendre polynomials, spectral projection, eigenvalues.

MSC(2010) 45B05, 45G10, 65R20.

1. Introduction

Considering $\mathbb{X}$ is a Banach space, find the eigenvalue $\lambda \in \mathbb{C} \setminus \{0\}$ and eigenvector $\phi \in \mathbb{X}$ such that

$$\mathcal{F}\phi = \lambda\phi, \ \|\phi\| = 1, \tag{1.1}$$

where $\mathcal{F}$ is a compact linear integral operator with weakly singular kernels of algebraic and logarithmic type. Numerical approximation methods such as degenerate kernel method, Nyström method, Galerkin, collocation and petrov-Galerkin methods (see [1, 3, 10, 12–17]) have invited much awareness, over the years for the approximation of the eigenfunctions (eigenvalues and eigenvectors) for the compact integral operator $\mathcal{F}$. The analysis for the convergence of Galerkin, petrov-Galerkin, collocation, Nyström and degenerate kernel methods are well documented in [1-5].

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In [14], authors discussed the wavelet Galerkin method and obtained the superconvergence results for the eigenfunctions. In the piecewise polynomial based Galerkin and collocation methods, the bounds of error functions for the eigenvalues, iterated eigenvectors of the above defined integral operator with smooth kernels have the convergence rate $\mathcal{O}(h^{2m})$, whereas the error bounds for the spectral subspaces is of the order $\mathcal{O}(h^m)$, where h is the norm of the partition. In [3], authors considered the fast collocation methods for the same integral operator with weakly singular kernels and obtained the convergence rates for the eigenfunctions.

In the standard projection methods, $\mathcal{F}$ is approximated by $\pi_n \mathcal{F} \pi_n$, a finite rank operator, where π_n from $\mathbb{X}$ onto a finite dimensional subspace $\mathbb{X}_n$, is either an orthogonal projection or interpolatory projection for the Galerkin method or collocation method, respectively. Generally, in projection methods, the matrix eigenvalue problem related to $\pi_n \mathcal{F} \pi_n$ cannot be evaluated precisely, due to the various integrals appearing from the integral operator $\mathcal{F}$ and inner products. Hence, it is necessary to replace these integrals by an appropriate numerical quadrature rule. Replacement by numerical quadrature rules for these integrals leads to discrete projection methods. In [8], authors discussed the discrete multi-projection methods using piecewise polynomial bases for the approximation of eigenfunctions of a compact integral operator for smooth kernels. In all these discrete and non-discrete piecewise polynomially based approximated eigenvalue problems, one needs to solve large size of matrix eigenvalue problem, which is numerically expensive to compute. To avoid this complexity, one can use global polynomials as basis functions instead of piecewise polynomials. In particular, one can use Legendre polynomials as basis functions. In [13], B. L. Panigrahi et. al. discussed the eigenvalue problem of a weakly singular compact operator (1.1) by Legendre Galerkin and Legendre multi Galerkin methods and obtained the convergence results for the eigenfunctions. In this paper, our main aim is to discuss the discrete Legendre projection methods of a weakly singular compact integral operator and to find the error bounds for the approximate eigenfunctions. We will show that eigenfunctions in the iterated discrete Legendre Galerkin method have optimal convergence rates in L^2 and uniform norm.

This paper is organized as follows: The discrete version of Legendre projection and iterated Legendre projection methods are introduced in Section 2. The convergence results for the eigenfunctions in L^2 and uniform norm are explored in Section 3. In Section 4, to verify the hypothetical results, numerical illustrations are provided. Assume that c is a generic constant, all over the paper.

2. Discrete Legendre projection method

Let $\mathbb{X} = \mathcal{C}[-1,1]$, a Banach space with the norm $\|\cdot\|_\infty$. Our aim is to find the eigenfunctions $\phi \in \mathbb{X}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ for the following eigenvalue problem defined on $\mathbb{X}$:

$$\mathcal{F}\phi = \lambda\phi, \quad 0 \neq \lambda \in \mathbb{C}, \ \|\phi\| = 1, \tag{2.1}$$

where the integral operator $\mathcal{F}$ is defined by

$$\mathcal{F}\phi(x) = \int_{-1}^{1} k(x,t)\phi(t)dt, \quad x \in [-1,1], \tag{2.2}$$

with the kernel $k(x,t) = m(x,t)a(x,t)$, where

$$m(x,t) = \begin{cases} |x-t|^{\alpha-1}, & \text{if } \frac{1}{2} < \alpha < 1, \\ \log(|x-t|), & \text{if } \alpha = 1, \end{cases} \tag{2.3}$$

and $a(x,t)$ is sufficiently smooth w.r.t. both the variables x and t.

Note that the kernel $k(x,t)$, for $\frac{1}{2} < \alpha \leq 1$ satisfies the following condition

A1. $\sup\limits_{x\in[-1,1]} \int_{-1}^{1} |k(x,s)|^2\,ds = M < \infty.$

A2. $\lim\limits_{x\to x'} \int_{-1}^{1} |k(x,t) - k(x',t)|^2\,dt = 0, \qquad -1 \leq x \leq 1.$

Then $\mathcal{F} : \mathcal{C}[-1,1] \to \mathcal{C}[-1,1]$ is a compact linear operator (see [7] and [9]).

Let the space $BL(\mathcal{X})$ denotes all bounded linear operators on $\mathbb{X}$, and

$$\rho(\mathcal{F}) = \{z \in \mathbb{C} : (\mathcal{F} - z\mathcal{I})^{-1} \in BL(\mathbb{X})\} \tag{2.4}$$

and $\sigma(\mathcal{F}) = \mathbb{C} \setminus \rho(\mathcal{F})$ be the resolvent set and spectrum of $\mathcal{F}$, respectively.

Let the eigenvalue λ of $\mathcal{F}$ have the ascent ℓ and algebraic multiplicity r. Let $\Gamma \subset \rho(\mathcal{F})$ be a rectifiable curve, which is simple closed such that $\sigma(\mathcal{F}) \cap int\Gamma = \{\lambda\},\ 0 \notin int\Gamma$, where $int\ \Gamma$ denotes the interior of Γ. Let

$$\mathcal{P} = -\frac{1}{2\pi i}\int_{\Gamma} (\mathcal{F} - zI)^{-1}\,dz, \tag{2.5}$$

be the spectral projection associated with $\mathcal{F}$ and λ (see [2]).

Let $\mathcal{R}(\mathcal{P})$, the range of $\mathcal{P}$, the spectral subspace associated with $\mathcal{F}$ and λ. Then $\mathcal{R}(\mathcal{P}) = (N(\mathcal{F} - \lambda I)^{\ell})$ and r is the dimension of $\mathcal{R}(\mathcal{P})$.

Let $\mathbb{Y}_1, \mathbb{Y}_2$ be any two non zero closed subspaces of $\mathbb{X}$ and for $p = 2, \infty$, let

$$\delta_p(\mathbb{Y}_1, \mathbb{Y}_2) = \sup\{dist_p(y, \mathbb{Y}_2) : y \in \mathbb{Y}_1,\ \ \|y\|_p = 1\}.$$

Then the gap between the subspaces is

$$\hat{\delta}_p(\mathbb{Y}_1, \mathbb{Y}_2) = \max\{\delta_p(\mathbb{Y}_1, \mathbb{Y}_2), \delta_p(\mathbb{Y}_2, \mathbb{Y}_1)\}.$$

Next, we discuss the discrete Legendre projection methods for eigenvalue problem (2.1). To do this, let $\mathbb{X}_n = \text{span}\{\varpi_0, \varpi_1, \varpi_2, \dots \varpi_n\}$ be the subspace of orthonormal Legendre polynomials of degree $\leq n$ on $[-1,1]$. Here ϖ_i's are given by

$$\varpi_i(x) = \sqrt{\frac{2i+1}{2}}L_i(x), i = 0, 1, \cdots, n,$$

L_i's are Legendre polynomials of degree $\leq i$,, generated from the following relation

$$L_0(x) = 1, \quad L_1(x) = x, \qquad x \in [-1,1],$$

and

$$(i+1)L_{i+1}(x) = (2i+1)xL_i(x) - iL_{i-1}(x), \quad i = 1, 2, 3 \cdots, n-1.$$

Let $\pi_n : \mathbb{X} \to \mathbb{X}_n$ be orthogonal projection. For the eigenvalue problem (2.1), the Galerkin method is to find $\phi_n \in \mathbb{X}_n$ and $\lambda_n \in \mathbb{C}$ such that

$$\pi_n \mathcal{F} \pi_n \phi_n = \lambda_n \phi_n,\ \|\phi_n\| = 1. \tag{2.6}$$

Using $\phi_n = \sum_{j=0}^{n} \alpha_j \varpi_j \in \mathbb{X}_n$, we have

$$\sum_{j=0}^{n} \alpha_j < \mathcal{F}\varpi_j, \varpi_i >= \lambda_n \sum_{j=0}^{n} \alpha_j < \varpi_j, \varpi_i >, \; i = 0, 1, 2, 3, \dots n, \tag{2.7}$$

and iterated eigenfunction is defined as

$$\tilde{\phi_n} = \frac{1}{\lambda_n} \mathcal{F}\phi_n. \tag{2.8}$$

In general, the integrals appeared in the projection methods for solving (2.6)-(2.8) due to the inner products, the integral operator $\mathcal{F}$ can not be evaluated precisely. The replacement of these integrals by numerical quadratures lead to the projection methods in its discrete form.

For doing so, we choose a numerical integration as follows: for $f \in \mathcal{C}[-1, 1]$ let

$$\int_{-1}^{1} f(t)\, dt \simeq \sum_{p=1}^{R(n)} w_p f(t_p), \tag{2.9}$$

where
(i) the weights w_p are s.t.

$$w_p > 0, \quad p = 1, 2, \cdots, R(n), \tag{2.10}$$

(ii) the quadrature rule having the degree of precision is at least $2n$, and for all polynomials of degree $\leq 2n$, we have

$$\int_{-1}^{1} f(t)\, dt = \sum_{p=1}^{R(n)} w_p f(t_p). \tag{2.11}$$

From now on, we set $R(n) = R$, for the notational convenient. We define the discrete inner product by utilizing the above quadrature rule (2.9)-(2.11) (see [6], [18]) as

$$\langle f, g \rangle_R = \sum_{p=1}^{R} w_p f(t_p) g(t_p), \quad f, g \in \mathbb{C}[-1, 1]. \tag{2.12}$$

Since $w_j > 0$, it follows that

$$2 = \int_{-1}^{1} ds = \sum_{i=1}^{R} w_i. \tag{2.13}$$

Hyperinterpolation operator: The hyperinterpolation operator (Discrete orthogonal projection operator) $\mathcal{L}_n : \mathbb{X} \to \mathbb{X}_n$ (see [18]), defined by

$$\mathcal{L}_n y = \sum_{j=0}^{n} \langle y, \varpi_j \rangle_R \varpi_j, \quad y \in \mathbb{X}, \tag{2.14}$$

and $\mathcal{L}_n$ satisfies

$$\langle \mathcal{L}_n y, y_n \rangle_R = \langle y, y_n \rangle_R, \quad y \in \mathbb{X}, \;\; y_n \in \mathbb{X}_n. \tag{2.15}$$

Now we quote some essential properties of $\mathcal{L}_n$ from Sloan [18].

Lemma 2.1. *Let $\mathcal{L}_n : \mathbb{X} \to \mathbb{X}_n$, the hyperinterpolation operator defined by (2.14). Then the following hold*

i) For any $y \in \mathbb{X}$,

$$\langle y - \mathcal{L}_n y, y - \mathcal{L}_n y\rangle_R = \min_{\chi \in \mathbb{X}_n} \langle y - \chi, y - \chi\rangle_R. \tag{2.16}$$

ii) For any $y \in \mathbb{X}$,

$$\|\mathcal{L}_n y\|_{L^2} \leq p\|y\|_\infty, \tag{2.17}$$

where p is a constant and

$$\|\mathcal{L}_n y - y\|_{L^2} \leq 2\sqrt{2} \inf_{\phi \in \mathbb{X}_n} \|y - \phi\|_\infty \to 0, n \to \infty. \tag{2.18}$$

iii) In particular, $y \in \mathcal{C}^m[-1, 1]$;

$$\|\mathcal{L}_n y - y\|_{L^2} = cn^{-m}\|y^{(m)}\|_\infty, \tag{2.19}$$

c is a constant not dependent of n, and $n \geq m$.

From Lemma 2.4 of [5], we have that for any $y \in \mathcal{C}^m[-1, 1]$, there holds

$$\|\mathcal{L}_n y - y\|_\infty \leq cn^{-m+1}\|y^{(m)}\|_\infty, \quad n \geq m, \tag{2.20}$$

c is a constant not dependent of y and n, and $n \geq m$.

From [4] and [19], we have

$$\|\mathcal{L}_n\|_\infty = \mathcal{O}(n), \tag{2.21}$$
$$\|y - \mathcal{L}_n y\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty, \tag{2.22}$$
$$\|y - \mathcal{L}_n y\|_\infty \nrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{2.23}$$

Remark 2.1. If $R = n + 1$, i.e., the number of quadrature points in discrete inner product (2.12) and the dimension of the subspace $\mathbb{X}_n$ are the same, then the discrete orthogonal projection becomes the interpolatory projection (see [18]).

Using the projection $\mathcal{L}_n$, we define the Nyström operator $\mathcal{F}_n : \mathbb{X} \to \mathbb{X}_n$ by

$$\mathcal{F}_n(z)(x) = \int_{-1}^{1} m(x, t)\mathcal{L}_n(a(x, t)z(t))\, dt, \tag{2.24}$$

which is an approximation of the integral operator $\mathcal{F}$. Note that for any $z \in \mathbb{X}$, using

$$\mathcal{L}_n(a(x, t)z(t)) = \sum_{j=0}^{n} \varpi_j(x)a(x, t_j)z(t_j), \tag{2.25}$$

in (2.24), we have

$$\mathcal{F}_n(z)(x) = \sum_{j=0}^{n} w_j(x)a(x, t_j)z(t_j), \tag{2.26}$$

with $w_j(x) = \int_{-1}^{1} \varpi_j(x)m(x, t)\, dt$.

The discrete Lengendre Galerkin method is to find $\psi_n = \sum_{j=0}^{n} \beta_j \varpi_j \in \mathbb{X}_n$ and $\tilde{\lambda}_n \in \mathbb{C} \setminus \{0\}$ such that

$$\sum_{j=0}^{n} \beta_j < \mathcal{F}_n \varpi_j, \varpi_i >_R = \tilde{\lambda}_n \sum_{j=0}^{n} \beta_j < \varpi_j, \varpi_i >_R, \; i = 0, 1, 2, 3, \dots n,$$
$$\sum_{p=1}^{R} \sum_{j=0}^{n} \sum_{i=0}^{n} \beta_j w_p w_j(t_p)(a(t_p, t_i) z(t_i)) = \tilde{\lambda}_n \sum_{p=1}^{R} \sum_{j=0}^{n} \beta_j w_p w_j \varpi_j(t_p) \varpi_i(t_p). \quad (2.27)$$

Using $\mathcal{L}_n$ and $\mathcal{F}_n$, the eigenvalue problem (2.27) can be written as

$$\mathcal{L}_n \mathcal{F}_n \psi_n = \tilde{\lambda}_n \psi_n, \quad \|\psi_n\|_\infty = 1. \quad (2.28)$$

We define the iterated eigenvector as

$$\tilde{\psi}_n = \frac{1}{\tilde{\lambda}_n} \mathcal{F}_n \psi_n. \quad (2.29)$$

Applying $\mathcal{L}_n$ of the equation (2.29), we have

$$\mathcal{L}_n \tilde{\psi}_n = \psi_n. \quad (2.30)$$

and

$$\mathcal{F}_n \mathcal{L}_n \tilde{\psi}_n = \tilde{\lambda}_n \tilde{\psi}_n. \quad (2.31)$$

This is the iterated discrete Legendre Galerkin method for the eigenvalue problem (2.1).

3. Convergence analysis

In this section, we discuss the convergence analysis of the eigenvalues and eigenvectors in the discrete Legendre projection methods.

Lemma 3.1. *Let the operators $\mathcal{F}_n$ and $\mathcal{F}$ defined by* (2.26) *and (2.2), respectively. Then for $a(.,.) \in \mathcal{C}^m([-1,1] \times [-1,1])$ and $\phi \in \mathcal{C}^m[-1,1]$, there hold*

$$\begin{aligned} &\|(\mathcal{F} - \mathcal{F}_n)\phi\|_\infty \leq cn^{-m} \|(a\phi)^{(m)}\|_\infty, \\ &\|(\mathcal{F} - \mathcal{F}_n)\phi\|_{L^2} \leq cn^{-m} \|(a\phi)^{(m)}\|_\infty, \\ &\|\mathcal{F}_n z\|_\infty \leq \sqrt{M_2 p} \|a\|_\infty \|z\|_\infty. \end{aligned} \quad (3.1)$$

Proof. Using Schwartz inequality and estimate (2.19), we have

$$\begin{aligned} \|(\mathcal{F} - \mathcal{F}_n)\phi\|_\infty &= \sup_{x \in [-1,1]} |(\mathcal{F} - \mathcal{F}_n)\phi(x)| \\ &= \sup_{x \in [-1,1]} |\int_{-1}^{1} m(x,t)[(\mathcal{I} - \mathcal{L}_n)(a(x,t))\phi(t)]dt| \\ &\leq \sup_{x \in [-1,1]} \int_{-1}^{1} |m(x,t)[(\mathcal{I} - \mathcal{L}_n)(a(x,t))\phi(t)]|dt \\ &\leq \|m(\cdot,\cdot)\|_{L^2} \|(\mathcal{I} - \mathcal{L}_n)(a(\cdot,\cdot)\phi)\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&\leq M_2 c n^{-m} \|(a(\cdot,\cdot)\phi)^{(m)}\|_\infty \\
&\leq M_2 M_3 c n^{-m} \\
&= \mathcal{O}(n^{-m}).
\end{aligned} \tag{3.2}
$$

Note that $\|(a(\cdot,\cdot)\phi)^{(m)}\|_\infty \leq M_3 < \infty$, for $m = 0, 1, 2, \ldots$.

Now using the estimate (3.2), we obtain

$$
\|(\mathcal{F} - \mathcal{F}_n)\phi\|_{L^2} \leq \sqrt{2}\|(\mathcal{F} - \mathcal{F}_n)\phi\|_\infty = \mathcal{O}(n^{-m}). \tag{3.3}
$$

Next using Holder's inequality, for any $z \in \mathbb{X}$, we have

$$
\begin{aligned}
|\mathcal{F}_n z(t)| = \left| \int_{-1}^{1} m(x,t) \mathcal{L}_n(a(t,x) z(x))\, dx \right| &\leq \|m(\cdot,\cdot)\|_{L^2} \|\mathcal{L}_n(a(x,t) z(t))\|_{L^2} \\
&\leq \sqrt{M_2} p \|a\|_\infty \|z\|_\infty,
\end{aligned}
$$

which implies that

$$
\|\mathcal{F}_n z\|_\infty \leq \sqrt{M_2} p \|a\|_\infty \|z\|_\infty. \tag{3.4}
$$

This completes the proof. □

For the rest of the paper, we assume that $R = n + 1$, the number of quadrature points used in discrete inner product (2.12) and the dimension of the subspace $\mathbb{X}_n$ are the same. In this case, we have

$$
\mathcal{F}_n \mathcal{L}_n(z)(x) = \sum_{j=0}^{n} w_j (a(x,t_j) \mathcal{L}_n z(t_j)) = \sum_{j=0}^{n} w_j (a(x,t_j) z(t_j)) = \mathcal{F}_n(z)(x). \tag{3.5}
$$

Thus $\mathcal{F}_n \mathcal{L}_n = \mathcal{F}_n$.

Lemma 3.2. *The operator $\mathcal{F}_n$ defined in* (2.26) *converges to $\mathcal{F}$ in collectively compact fashion.*

Proof. From (3.2), we see that the operator $\mathcal{F}_n$ converges to $\mathcal{F}$ pointwise and since $\mathcal{F}$ is a compact operator, so to prove the Lemma, it is sufficient to prove that the set $M = \bigcup_{n=1}^{\infty} \{\mathcal{F}_n v : \|v\|_\infty \leq 1, v \in \mathbb{X}\}$ is relatively compact set.

For any $v \in \mathbb{X}$, and $x, x' \in [-1, 1]$, Consider

$$
\begin{aligned}
&|(\mathcal{F}_n v)(x) - (\mathcal{F}_n v)(x')| \\
=& \left| \int_{-1}^{1} m(x,t) \mathcal{L}_n(a(x,t) v(t)) dt - \int_{-1}^{1} m(x',t) \mathcal{L}_n(a(x',t) v(t)) dt \right| \\
=& \left| \int_{-1}^{1} m(x,t) \mathcal{L}_n(a(x,t) v(t)) dt - \int_{-1}^{1} m(x',t) \mathcal{L}_n(a(x,t) v(t)) dt \right. \\
&\left. + \int_{-1}^{1} m(x',t) \mathcal{L}_n(a(x,t) v(t)) dt - \int_{-1}^{1} m(x',t) \mathcal{L}_n(a(x',t) v(t)) dt \right| \\
\leq& \left| \int_{-1}^{1} [m(x,t) - m(x',t)] \mathcal{L}_n(a(x,t) v(t)) dt \right| \\
&+ \left| \int_{-1}^{1} m(x',t) [\mathcal{L}_n(a(x,t) v(t)) - \mathcal{L}_n(a(x',t) v(t))] dt \right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \int_{-1}^{1} |[m(x,t) - m(x',t)]\mathcal{L}_n(a(x,t)v(t))|dt \\
&\quad + \int_{-1}^{1} |m(x',t)[\mathcal{L}_n(a(x,t)v(t)) - \mathcal{L}_n(a(x',t)v(t))]|dt \\
&\leq \|[m(x,\cdot) - m(x',\cdot)]\|_{L^2}\|\mathcal{L}_n(a(x,\cdot)v(\cdot))\|_{L^2} \\
&\quad + \|m(x',\cdot)\|_{L^2}\|[\mathcal{L}_n(a(x,\cdot)v(\cdot)) - \mathcal{L}_n(a(x',\cdot)v(\cdot))]\|_{L^2} \\
&\leq \|[m(x,\cdot) - m(x',\cdot)]\|_{L^2} p_1 \|a\|_\infty \|v\|_\infty \\
&\quad + \|m(x,\cdot)\|_{L^2} p_2 \|v\|_\infty \|[a(x,.) - a(x',.)]\|_\infty. \qquad (3.6)
\end{aligned}$$

Now using A2 and smoothness of $a(\cdot,\cdot)$ in the estimate (3.6), we obtain

$$|(\mathcal{F}_n v)(x) - (\mathcal{F}_n v)(x')| \to 0 \text{ as } x \to x'. \qquad (3.7)$$

Now from (3.4) and (3.7), it follows that the set $M = \bigcup_{n=1}^{\infty}\{\mathcal{F}_n v : \|v\| \leq 1, v \in \mathbb{X}\}$ is uniformly bounded and equicontinuous. Therefore the set

$$M = \bigcup_{n=1}^{\infty}\{\mathcal{F}_n v : \|v\|_\infty \leq 1, v \in \mathbb{X}\} \text{ is relatively compact set.}$$

This complete the proof. □

Since $\mathcal{F}_n$ converges to $\mathcal{F}$ in collectively compact fashion, the results of Chatelin [2] and Osborn [11] are applicable. The spectrum of $\mathcal{F}_n$ inside Γ, i.e., $\Lambda_n = \sigma(\mathcal{F}_n)\bigcap int(\Gamma)$ consist of r eigenvalues say $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,r}$ counted accordingly to their algebraic multiplicities. Let

$$\hat{\hat{\lambda}}_n = \frac{\lambda_{n,1} + \lambda_{n,2} + \cdots + \lambda_{n,r}}{r},$$

denote the arithmetic mean of $\lambda_{n,i}$, for $i = 1, 2, \ldots, r$ and we approximate λ by $\hat{\hat{\lambda}}_n$. Let

$$\mathcal{P}_n = -\frac{1}{2\pi i}\int_\Gamma (\mathcal{F}_n - z\mathcal{I})^{-1} dz,$$

$\mathcal{P}_n$ is the spectral projection associated with $\mathcal{F}_n$ and Λ_n, and let $\mathcal{R}(\mathcal{P}_n)$ denotes the range of $\mathcal{P}_n$. Then by applying the results of [2,11], we have the following theorem.

Theorem 3.1. *For sufficiently large enough n, the following hold*

$$\begin{aligned}
&|\lambda - \lambda_{n,i}|^l \leq c\|(\mathcal{F} - \mathcal{F}_n)|_{\mathcal{R}(\mathcal{P})}\|_{L^2}, \text{ for } i = 1, \ldots, r, \\
&|\lambda - \widetilde{\lambda}_n| \leq c\|(\mathcal{F} - \mathcal{F}_n)|_{\mathcal{R}(\mathcal{P})}\|_{L^2}, \\
&\hat{\delta}_p(\mathcal{R}(\mathcal{P}), \mathcal{R}(\mathcal{P}_n)) \leq c\|(\mathcal{F} - \mathcal{F}_n)|_{\mathcal{R}(\mathcal{P})}\|_{L^p}, \quad p = 2, \infty,
\end{aligned}$$

c is a constant not dependent of n.

Next, we discuss the convergence results for the eigenvalues and eigenvectors.

Theorem 3.2. *Let $\mathcal{F}$ be the integral operator with algebraic kernel $k(x,t) = a(x,t)|x-t|^{\alpha-1}$ for $\frac{1}{2} < \alpha < 1$ or logarithmic kernel $k(x,t) = a(x,t)\log|x-t|$ for $\alpha = 1$, where $a(x,t) \in \mathcal{C}^m([-1,1]\times[-1,1])$. Let the eigenvalue λ of $\mathcal{F}$ has* *ascent ℓ and algebraic multiplicity r, and let $\mathcal{R}(\mathcal{P})$ and $\mathcal{R}(\mathcal{P}_n)$, the ranges of the spectral projections $\mathcal{P}$ and $\mathcal{P}_n$, respectively, with $\mathcal{R}(\mathcal{P}) \subset \mathcal{C}^m([-1,1])$ and $\hat{\tilde{\lambda}}_n$, the arithmetic mean of the eigenvalues $\lambda_{n,j}$, $j = 1, 2, \ldots, r$. Then there hold*

$$|\lambda - \lambda_{n,i}|^l = \mathcal{O}(n^{-m}), \text{ for } i = 1, \ldots, r,$$
$$|\lambda - \hat{\tilde{\lambda}}_n| = \mathcal{O}(n^{-m}),$$
$$\hat{\delta}_p(\mathcal{R}(\mathcal{P}), \mathcal{R}(\mathcal{P}_n)) = \mathcal{O}(n^{-m}), \text{ for } p = 2, \infty.$$

Proof. Note that using Lemma 3.1, we have

$$\|(\mathcal{F} - \mathcal{F}_n)|_{\mathcal{R}(\mathcal{P})}\|_\infty = \sup\{\|(\mathcal{F} - \mathcal{F}_n)\phi\|_\infty; \phi \in \mathcal{R}(\mathcal{P}), \|\phi\|_\infty = 1\} = \mathcal{O}(n^{-m}). \quad (3.8)$$

and

$$\|(\mathcal{F} - \mathcal{F}_n)|_{\mathcal{R}(\mathcal{P})}\|_{L^2} = \sup\{\|(\mathcal{F} - \mathcal{F}_n)\phi\|_{L^2}; \phi \in \mathcal{R}(\mathcal{P}), \|\phi\|_{L^2} = 1\} = \mathcal{O}(n^{-m}). \quad (3.9)$$

Combining this with Theorem 3.1, we obtain the desired results. This completes the proof. ☐

Next let λ, be the simple eigenvalue of $\mathcal{F}$ with the corresponding eigenfunction ϕ_n, i.e., $r = 1$ and $\ell = 1$. Let $\widetilde{\lambda}_n$, be the corresponding simple eigenvalue of $\mathcal{F}_n$ with the corresponding eigenvector ψ_n, i.e.

$$\mathcal{F}_n \psi_n = \tilde{\lambda}_n \psi_n, \qquad \|\psi_n\| = 1. \quad (3.10)$$

Let $\tilde{\psi}_n = \frac{1}{\tilde{\lambda}_n} \mathcal{F} \psi_n$ be the corresponding iterated eigenvector.

Theorem 3.3. *Let the compact integral operator $\mathcal{F}$ with algebraic kernel $k(x,t) = a(x,t)|x-t|^{\alpha-1}$ for $\frac{1}{2} < \alpha < 1$ or logarithmic kernel $k(x,t) = a(x,t) \log|x-t|$ for $\alpha = 1$, where $a(x,t) \in \mathcal{C}^m([-1,1] \times [-1,1])$. Assume that λ, a simple eigenvalue of $\mathcal{F}$, i.e., $r = 1, \ell = 1$. Let $\tilde{\lambda}_n$ be a simple eigenvalue of $\mathcal{F}_n$ with the corresponding eigenfunction ψ_n. Assume that $\mathcal{R}(\mathcal{P}_n), \mathcal{R}(\mathcal{P}) \subset \mathcal{C}^m[-1,1]$, then there hold*

$$|\lambda - \tilde{\lambda}_n| = \mathcal{O}(n^{-m}),$$
$$\|\tilde{\psi}_n - \mathcal{P}\tilde{\psi}_n\|_\infty = \mathcal{O}(n^{-m}), \qquad \|\tilde{\psi}_n - \mathcal{P}\tilde{\psi}_n\|_{L^2} = \mathcal{O}(n^{-m}),$$
$$\|\psi_n - \mathcal{P}\psi_n\|_\infty = \mathcal{O}(n^{-m+1}), \quad \|\psi_n - \mathcal{P}\psi_n\|_{L^2} = \mathcal{O}(n^{-m+1}).$$

Proof. Using Theorem 3.1, we have

$$|\lambda - \tilde{\lambda}_n| \leq c\|(\mathcal{F} - \mathcal{F}_n)|_{\mathcal{R}(\mathcal{P})}\|_\infty \leq c \sup\{\|(\mathcal{F} - \mathcal{F}_n)\phi\|_\infty; \phi \in \mathcal{R}(\mathcal{P}), \|\phi\| = 1\}. \quad (3.11)$$

Now using lemma 3.1 in (3.11),

$$|\lambda - \tilde{\lambda}_n| = \mathcal{O}(n^{-m}).$$

Again from Theorem 3.1, we have

$$\|\tilde{\psi}_n - \mathcal{P}\tilde{\psi}_n\|_{L^2} \leq \hat{\delta}_2(\mathcal{R}(\mathcal{P}), \mathcal{R}(\mathcal{P}_n)) \leq c\|(\mathcal{F} - \mathcal{F}_n)|_{\mathcal{R}(\mathcal{P})}\|_{L^2} = \mathcal{O}(n^{-m}), \quad (3.12)$$

and

$$\|\tilde{\psi}_n - \mathcal{P}\tilde{\psi}_n\|_\infty \leq \hat{\delta}_\infty(\mathcal{R}(\mathcal{P}), \mathcal{R}(\mathcal{P}_n)) \leq c\|(\mathcal{F} - \mathcal{F}_n)|_{\mathcal{R}(\mathcal{P})}\|_\infty = \mathcal{O}(n^{-m}). \quad (3.13)$$

Now using the above estimate and the estimates (2.19) and (2.20), we have

$$\begin{aligned}\|\psi_n - \mathcal{P}\psi_n\|_\infty &= \|\mathcal{L}_n\tilde{\psi}_n - \mathcal{P}\mathcal{L}_n\tilde{\psi}_n\|_\infty \\ &\leq \|\mathcal{P}\tilde{\psi}_n - \tilde{\psi}_n\|_\infty + \|\mathcal{P}(\mathcal{I}-\mathcal{L}_n)\tilde{\psi}_n\|_\infty + \|(\mathcal{I}-\mathcal{L}_n)\tilde{\psi}_n\|_\infty \\ &\leq \mathcal{O}(n^{-m}) + \|\mathcal{P}\|_\infty\|(\mathcal{I}-\mathcal{L}_n)\tilde{\psi}_n\|_\infty + \|(\mathcal{I}-\mathcal{L}_n)\tilde{\psi}_n\|_\infty \\ &\leq \mathcal{O}(n^{-m}) + (1+\|\mathcal{P}\|_\infty)\, n^{-m+1}\|\tilde{\psi}_n^{(m)}\|_\infty \\ &= \mathcal{O}(n^{-m+1}),\end{aligned}$$

and

$$\begin{aligned}\|\psi_n - \mathcal{P}\psi_n\|_{L^2} &= \|\mathcal{L}_n\tilde{\psi}_n - \mathcal{P}\mathcal{L}_n\tilde{\psi}_n\|_{L^2} \\ &\leq \|\mathcal{P}\tilde{\psi}_n - \tilde{\psi}_n\|_{L^2} + \|\mathcal{P}(\mathcal{I}-\mathcal{L}_n)\tilde{\psi}_n\|_{L^2} + \|(\mathcal{I}-\mathcal{L}_n)\tilde{\psi}_n\|_{L^2} \\ &\leq \mathcal{O}(n^{-m}) + (1+\|\mathcal{P}\|_{L^2})\|(\mathcal{I}-\mathcal{L}_n)\tilde{\psi}_n\|_{L^2} \\ &\leq \mathcal{O}(n^{-m}) + cn^{-m}\|\tilde{\psi}_n^{(m)}\|_\infty \\ &= \mathcal{O}(n^{-m}).\end{aligned}$$

This completes the proof. □

Remark 3.1. we observe that if the quadrature rule is minimal, that is when number of quadrature points and the dimension of the approximate space are chosen to be the same, then from Theorems 3.2, 3.3, we conclude that the eigenfunctions in the iterated discrete Legendre Galerkin method improves over discrete Legendre Galerkin method. Also, we obtain the optimal results of convergence for the eigenfunctions in iterated version of discrete Legendre Galerkin method in both uniform and L^2 norms.

4. Numerical results

The numerical results for finding the eigenvalue for the problem (2.1) along with the integral operator $\mathcal{F}$ defined in (2.2)-(2.3) are presented here. We choose the basis functions of $\mathbb{X}_n$ as Legendre polynomials and quadrature rule as defined in section 2. We present the error of the approximated eigenvalues, eigenvectors with exact eigenvalues, eigenvectors by discrete version of Legendre projection and iterated Legendre projection methods in both L^2 and uniform norm. For different kernels, and for distinct n, we compute $\tilde{\lambda}_n$, ψ_n and $\tilde{\psi}_n$ in the discrete version of Legendre projection and iterated Legendre projection methods and compare the results with exact solutions. The numerical tests were performed on a PC Intel(R)Core(TM)i5-3470 CPU@3.20GHz Processor, 4.00GB RAM and 32-bit Operating System on Matlab (R2012b). The computed errors in both L^2 and uniform norm of the approximated eigen solutions to those of the exact eigen solutions are presented in the following Tables [1-2].

Example 4.1. Consider the eigenvalue problem

$$\int_{-1}^{1} \log|x-t|y(t)\,\mathrm{d}t = \lambda y(s).$$

Here $\alpha = 1$.

Table 1. Discrete Legendre Galerkin method

n	$\|\lambda-\tilde{\lambda}_n\|$	$\\|\psi_n-\mathcal{P}\psi_n\\|_{L^2}$	$\\|\tilde{\psi}_n-P\tilde{\psi}_n\\|_{L^2}$	$\\|\psi_n-\mathcal{P}\psi_n\\|_\infty$	$\\|\tilde{\psi}_n-P\tilde{\psi}_n\\|_\infty$
2	1.76932589×10^{-1}	3.32698361×10^{-2}	3.85996321×10^{-3}	2.02158961×10^{-1}	4.54966324×10^{-3}
3	1.25693544×10^{-2}	1.25966369×10^{-2}	2.65982365×10^{-3}	1.32596214×10^{-2}	2.42365414×10^{-3}
4	2.54369144×10^{-3}	5.96325891×10^{-3}	1.45693012×10^{-4}	2.35698744×10^{-3}	1.56932544×10^{-4}
5	7.96358244×10^{-4}	4.68923650×10^{-3}	7.85693214×10^{-5}	2.45632144×10^{-3}	8.52169874×10^{-5}
6	5.54256394×10^{-4}	2.35698105×10^{-4}	5.56932145×10^{-5}	5.36598124×10^{-4}	6.57544289×10^{-5}
7	2.3095442×10^{-5}	9.03480998×10^{-6}	1.30997890×10^{-6}	4.00932102×10^{-5}	3.49867021×10^{-6}
8	1.09983589×10^{-7}	6.02484302×10^{-7}	5.24136952×10^{-7}	4.26536981×10^{-7}	1.25256918×10^{-7}

Example 4.2. Consider the eigenvalue problem

$$\int_{-1}^{1} |x-t|^{-1/4} (xt+1)\, y(t)\mathrm{d}t = \lambda y(s).$$

Here $\alpha = 3/4$.

Table 2. Discrete Legendre Galerkin method

n	$\|\lambda-\tilde{\lambda}_n\|$	$\\|x_n-\mathcal{P}\psi_n\\|_{L^2}$	$\\|\tilde{\psi}_n-\mathcal{P}\tilde{\psi}_n\\|_{L^2}$	$\\|\psi_n-\mathcal{P}\psi_n\\|_\infty$	$\\|\tilde{\phi}_n-\mathcal{P}\tilde{\psi}_n\\|_\infty$
2	3.25693654×10^{-1}	1.45326914×10^{-1}	1.73695845×10^{-2}	1.96321454×10^{-1}	1.85326944×10^{-3}
3	3.22169356×10^{-2}	1.32569801×10^{-2}	4.61258944×10^{-3}	1.24589634×10^{-2}	8.58960325×10^{-4}
4	4.12563012×10^{-3}	4.63201544×10^{-3}	8.75698356×10^{-4}	5.54651325×10^{-3}	1.56923654×10^{-4}
5	3.01236954×10^{-4}	5.15360248×10^{-4}	5.42569354×10^{-5}	1.72569356×10^{-3}	8.25632145×10^{-5}
6	8.74586914×10^{-4}	1.85693244×10^{-4}	1.65893214×10^{-5}	3.56932144×10^{-4}	2.02158694×10^{-5}
7	3.78954331×10^{-5}	7.54430976×10^{-5}	8.65893214×10^{-7}	7.89804356×10^{-5}	3.67765509×10^{-6}
8	9.20390987×10^{-6}	5.56609987×10^{-6}	7.43890965×10^{-8}	7.01877491×10^{-6}	1.89976544×10^{-7}

From Tables 1 and 2 of Examples 1 and 2, we can see that the iterated discrete Legendre projection method improves over the discrete Legendre Galerkin method. Also, we have calculated the CPU times needed for computation of the above numerical results. We see that the CPU times for evaluating the numerical results for the discrete Legendre Galerkin are given in Table 1 and Table 2 are $187.30s$ and $176.21s$, respectively and for iterated discrete Legendre Galerkin methods are given in Tables 1 and 2 are $216.34s$ and $254.87s$, respectively.

Remark 4.1. We use global (Legendre) polynomials of degree n, for solving the eigenvalue problem. We need to solve the matrix eigenvalue problem of size $(n+1)\times(n+1)$. For that we choose $n = 2, 3, 4, 5, 6$, which means that we only need to solve the matrix eigenvalue problem of the size $3\times3, 4\times4, 5\times5, 6\times6$, 7×7 and to obtain the approximate eigenvector.

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