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SHARP BOUNDS ON THE MINIMUM *M*-EIGENVALUE OF ELASTICITY *Z*-TENSORS AND IDENTIFYING STRONG ELLIPTICITY*

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Abstract In this paper, we establish an upper bound and sharp lower bounds on the minimum M-eigenvalue of elasticity Z-tensors without irreducible conditions. Based on the lower bound estimations for the minimum M-eigenvalue, we provide some checkable sufficient conditions for the strong ellipticity. Numerical examples are given to show the efficiency of the proposed results.

Keywords Elasticity Z-tensors, elasticity M-tensors, minimum M-eigenvalue, upper and lower bounds.

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1. Introduction

A fourth-order real tensor $\mathcal{A}$ is called a partially symmetric tensor, denoted by $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$, if

$$a_{ijkl} = a_{jikl} = a_{ijlk},\ i, j, k, l \in N = \{1, 2, \cdots, n\}.$$

The fourth-order partially symmetric tensor is useful in nonlinear elastic material analysis [6–8, 10, 12, 15, 16, 19]. For example, a fourth-order partially symmetric tensor with $n = 2$ or 3, called the elasticity tensor, can be used in the two/three-dimensional field equations for a homogeneous compressible nonlinearly elastic material without body forces [2, 4, 17, 29]. To identify the strong ellipticity in elastic mechanics, Han *et al.* [8] introduced M-eigenvalue of a fourth-order partially symmetric tensor. For $\lambda \in \mathbb{R}, x, y \in \mathbb{R}^n$, if

$$\begin{cases} \mathcal{A}xy^2 = \lambda x, \\ \mathcal{A}x^2y = \lambda y, \\ x^\top x = 1, \\ y^\top y = 1, \end{cases}$$

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where $(\mathcal{A}xy^2)_i = \sum_{j,k,l\in N} a_{ijkl}x_jy_ky_l$, $(\mathcal{A}x^2y)_l = \sum_{i,j,k\in N} a_{ijkl}x_ix_jy_k$, then the scalar λ is called an M-eigenvalue of the tensor $\mathcal{A}$, and x and y are called left and right M-eigenvectors associated with the M-eigenvalue.

Tensors with special structures, such as nonnegative tensors and M-tensors, are becoming the keynote in recent research [1, 3, 5, 6, 11, 25–28]. Particularly, some important properties of M-tensors and nonsingular M-tensors have been established in [6, 26]. Further, some bounds for the minimum H-eigenvalue of nonsingular M-tensors have been proposed in [3, 5, 9, 26, 27]. To characterize the strong ellipticity condition, Ding *et al.* [6] introduced a structured partially symmetric tensor named elasticity Z-tensors and elasticity M-tensors as follows.

Definition 1.1. Tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ is called an elasticity Z-tensor if there exist a nonnegative tensor $\mathcal{B} \in \mathbb{E}_{4,n}$ and a real number s such that

$$\mathcal{A} = s\mathcal{I}_M - \mathcal{B},$$

where $\mathcal{I}_M = (e_{ijkl}) \in \mathbb{E}_{4,n}$ is called elasticity identity tensor with its entries

$$e_{ijkl} = \begin{cases} 1, \text{ if } i = j \text{ and } k = l \\ 0, \text{ otherwise} \end{cases}$$

and $a_{iikk}(i, k \in N)$ is called diagonal entry. Further, if $s \geq \rho_M(\mathcal{B})$, we call $\mathcal{A}$ an elasticity M-tensor, and if $s > \rho_M(\mathcal{B})$, then we call $\mathcal{A}$ a nonsingular elasticity M-tensor.

An interesting problem arises: can the minimum M-eigenvalue of elasticity M-tensors be estimated as the minimum H-eigenvalue of M-tensors? Unfortunately, the following example gives us a negative answer.

Example 1.1. $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,2}$ be an elasticity M-tensor, whose entries are

$$a_{ijkl} = \begin{cases} a_{1111} = 6, a_{1122} = 3, a_{2211} = 4, a_{2222} = 5, \\ a_{1112} = a_{1121} = -2, a_{2212} = a_{2221} = -1, \\ a_{2111} = a_{1211} = -1, a_{1222} = a_{2122} = -1, \\ a_{ijkl} = 0, \text{otherwise.} \end{cases}$$

By computations, we obtain that the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are $(\tau_M(\mathcal{A}), \bar{x}, \bar{y}) = (1.4884, (0.9202, 0.3914), (0.4781, 0.8783))$. It follows from Theorem 2.1 of [9] that

$$[\min_{i\in N} R_i(\mathcal{A}), \max_{i\in N} R_i(\mathcal{A})] = [3, 5].$$

However,

$$\tau_M(\mathcal{A}) = 1.4884 \notin [\min_{i\in N} R_i(\mathcal{A}), \max_{i\in N} R_i(\mathcal{A})].$$

This stimulates researchers to establish new characterizations for the minimum M-eigenvalue of elasticity M-tensors. Based on the minimum diagonal entries, He *et al.* [10] proposed some bounds for the minimum M-eigenvalue of elasticity M-tensors under irreducible conditions. As we know, the strong ellipticity condition holds for an elasticity tensor if and only if it is M-positive, and that it is M-positive if and only if its minimum M-eigenvalue is positive. Certainly, elasticity M-tensors are M-positive and satisfies the strong ellipticity condition [6]. However, elasticity Z-tensors are usually not M-positive definite, and the strong ellipticity condition is not satisfied. Can we provide some checkable sufficient conditions for the strong ellipticity by estimating lower bounds for the minimum M-eigenvalue of elasticity Z-tensors? Meanwhile, many elasticity M-tensors and Z-tensors, such as anisotropic tensors, are not irreducible, which reveals that irreducibility is a relatively strict condition. Inspired by these observations, combining the maximum diagonal entries with accurate eigenvector information, we want to establish sharp bound estimations on the minimum M-eigenvalue of elasticity Z-tensors without irreducible conditions, and identify whether the strong ellipticity condition holds. This constitutes the motivation of the paper.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are recalled. In Section 3, we establish an upper bound and two sharp lower bounds for the minimum M-eigenvalue of elasticity Z-tensors. In Section 4, we propose some sufficient conditions to verify whether an elasticity Z-tensor is a nonsingular elasticity M-tensor and strong ellipticity condition is satisfied. The given numerical experiments show its validity.

2. Preliminaries

In this section, we firstly introduce some definitions and important properties of elasticity M-tensors [6, 10, 16].

Definition 2.1. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$ be an m-th order n dimensional real square tensor, then $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ is called reducible if there exists a nonempty proper index subset $J \subset \{1, 2, \dots, n\}$ such that $a_{i_1 i_2 \dots i_m} = 0, \forall\, i_1 \in J, \forall\, i_2, \dots, i_m \notin J$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ to be irreducible.

Lemma 2.1 (Theorem 1 of [16]). *M-eigenvalues always exist. If x and y are left and right M-eigenvectors of $\mathcal{A}$, associated with an M-eigenvalue λ, then $\lambda = \mathcal{A}x^2y^2$.*

Lemma 2.2 (Theorem 6 of [6]). *The M-spectral radius of any nonnegative tensor in $\mathbb{E}_{4,n}$ is exactly its greatest M-eigenvalue. Furthermore, there is a pair of nonnegative M-eigenvectors corresponding to the M-spectral radius.*

Lemma 2.3 (Lemma 2.4 of [10]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue of $\mathcal{A}$, then*

$$\tau_M(\mathcal{A}) \leq \min_{i,k \in N} \{a_{iikk}\}.$$

In the following, we characterize the M-eigenvector associated with the minimum M-eigenvalue by relaxing the irreducible condition.

Lemma 2.4. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue. Then, there exist nonnegative left and right M-eigenvectors (x, y) corresponding to $\tau_M(\mathcal{A})$ such that*

$$\mathcal{A}xy^2 = \tau_M(\mathcal{A})x, \ \ \mathcal{A}x^2y = \tau_M(\mathcal{A})y.$$

Proof. Since $\mathcal{A}$ is an elasticity Z-tensor, there exist a nonnegative tensor $\mathcal{B} \in \mathbb{E}_{4,n}$ and a real number s such that

$$\mathcal{A} = s\mathcal{I}_M - \mathcal{B}.$$

It follows Lemmas 2.1 and 2.2 that

$$\begin{aligned}\tau_M(\mathcal{A}) &= \min_{x,y}\{f_{\mathcal{A}}(x,y) = \mathcal{A}x^2y^2 : x^\top x = 1 \text{ and } y^\top y = 1\}\\ &= \min_{x,y}\{(s\mathcal{I}_M - \mathcal{B})x^2y^2 : x^\top x = 1 \text{ and } y^\top y = 1\}\\ &= s - \max_{x,y}\{\mathcal{B}x^2y^2 : x^\top x = 1 \text{ and } y^\top y = 1\} = s - \rho_M(\mathcal{B}),\end{aligned}$$

where $\rho_M(\mathcal{B})$ is the greatest M-eigenvalue of $\mathcal{B}$ with a nonnegative eigenvector (x, y). Meanwhile, elasticity identity tensor $\mathcal{I}_M$ has the following property:

$$\begin{cases}\mathcal{I}_M xy^2 = x,\\ \mathcal{I}_M x^2y = y.\end{cases}$$

Further,

$$\mathcal{B}xy^2 = \rho_M(\mathcal{B})x = (s - \tau_M(\mathcal{A}))x = s\mathcal{I}_M xy^2 - \tau_M(\mathcal{A})x,$$

$$\mathcal{B}x^2y = \rho_M(\mathcal{B})y = (s - \tau_M(\mathcal{A}))y = s\mathcal{I}_M x^2y - \tau_M(\mathcal{A})y,$$

which imply

$$\mathcal{A}xy^2 = (s\mathcal{I}_M - \mathcal{B})xy^2 = \tau_M(\mathcal{A})x, \ \mathcal{A}x^2y = (s\mathcal{I}_M - \mathcal{B})x^2y = \tau_M(\mathcal{A})y.$$

□

3. Bounds for the minimum M-eigenvalue of elasticity Z-tensors

In this section, inspired by H-eigenvalue inclusion theorems [13,20,22], Z-eigenvalue intervals and M-eigenvalue intervals [10,11,14,18,21,23], we establish sharp bounds on the minimum M-eigenvalue of the elasticity Z-tensors. To proceed, we give the following lemma.

Lemma 3.1. *For unit vector $x \in \mathbb{R}^n$, it holds that* $\max_{i,j\in N, i\neq j} |x_i||x_j| \le \frac{1}{2}$.

Proof. For all $i \neq j \in N$, it follows from $2|x_i||x_j| \le x_i^2 + x_j^2$ that

$$2|x_i||x_j| \le x_i^2 + x_j^2 \le x_1^2 + x_2^2 + \ldots + x_n^2 = 1,$$

which implies $\max_{i,j\in N, i\neq j} |x_i||x_j| \le \frac{1}{2}$. □

Without irreducible conditions, we propose a sharp upper bound for the minimum M-eigenvalue.

Theorem 3.1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue. Then,*

$$\tau_M(\mathcal{A}) \leq \min\{\min_{i,l\in N} a_{iill}, \frac{\sum\limits_{i\in N} S_i(\mathcal{A})}{n^2}\},$$

where $S_i(\mathcal{A}) = \sum\limits_{j,k,l\in N} a_{ijkl}$.

Proof. Let $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue of $\mathcal{A}$. It follows Lemma 2.1 that

$$\tau_M(\mathcal{A}) = \min_{x,y}\{f_{\mathcal{A}}(x,y) = \mathcal{A}x^2y^2 : x^\top x = 1 \text{ and } y^\top y = 1\}. \tag{3.1}$$

Setting a feasible solution of (3.1)

$$(\bar{x},\bar{y}) = (\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}}),$$

we obtain

$$\tau_M(\mathcal{A}) \leq f_{\mathcal{A}}(\bar{x},\bar{y}) = \sum_{i,j\in N}\sum_{k,l\in N}\frac{a_{ijkl}}{n^2} = \frac{\sum\limits_{i\in[n]} S_i(\mathcal{A})}{n^2}. \tag{3.2}$$

Following the similar arguments to the proof Lemma 2.4 of [10], we obtain

$$\tau_M(\mathcal{A}) \leq \min_{i,l\in N} a_{iill}.$$

Thus,

$$\tau_M(\mathcal{A}) \leq \min\{\min_{i,l\in N} a_{iill}, \frac{\sum\limits_{i\in N} S_i(\mathcal{A})}{n^2}\}.$$

□

Remark 3.1. Without irreducible conditions, we propose an improved upper bound for the minimum M-eigenvalue of elasticity Z-tensors, and extend Lemma 2.4 of [10] from elasticity M-tensors to elasticity Z-tensors.

Next, we propose sharp lower bound estimations for the minimum M-eigenvalue of elasticity Z-tensors based on the maximum diagonal entries.

Theorem 3.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue. Then,*

$$\tau_M(\mathcal{A}) \geq \max\{\min_{i\in N}\{\mu_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\kappa_l - \mathcal{M}_l(\mathcal{A})\}\},$$

where

$$G_i(\mathcal{A}) = \omega_i - \frac{1}{2}r_i(\mathcal{A}),\ \mu_i = \max_{l\in N}\{a_{iill}\},$$
$$\omega_i(\mathcal{A}) = \max_{l\in N}(\mu_i - a_{iill} - \sum_{\substack{j\in N,\\ j\neq i}} a_{ijll}),\ r_i(\mathcal{A}) = \sum_{\substack{j,k,l\in N,\\ k\neq l}} a_{ijkl},$$
$$\mathcal{M}_l(\mathcal{A}) = m_l - \frac{1}{2}c_l(\mathcal{A}),\ \kappa_l = \max_{i\in N}\{a_{iill}\},$$

$$m_l(\mathcal{A}) = \max_{i\in N}(\kappa_l - a_{iill} - \sum_{\substack{k\,\in\,N,\\ k\,\neq\,l}} a_{iikl}),\ \ c_l(\mathcal{A}) = \sum_{\substack{i,\,j,\,k\,\in\,N,\\ i\,\neq\,j}} a_{ijkl}.$$

Proof. Let $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue of $\mathcal{A}$. It follows from Lemma 2.4 that there exist nonnegative left and right M-eigenvectors (x, y) corresponding to $\tau_M(\mathcal{A})$. Setting $x_p = \max_{i\in N}\{x_i\}$, by $x^\top x = 1$, one has $0 < x_p \leq 1$. Recalling the p-th equation of $\tau_M(\mathcal{A})x = \mathcal{A}xy^2$, we obtain

$$((a_{pp11}y_1^2 + \cdots + a_{ppnn}y_n^2) - \tau_M(\mathcal{A}))x_p = -\sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\,\neq\,l}} a_{pjkl}x_jy_ky_l - \sum_{\substack{j,\,l\,\in\,N,\\ j\,\neq\,p}} a_{pjll}x_jy_l^2. \tag{3.3}$$

Setting $\mu_p = \max_{l\in N}\{a_{ppll}\}$, by (3.3) and $a_{pjkl} \leq 0$ for all $j \neq p, j, k, l \in N$, one has

$$\begin{aligned}(\mu_p - \tau_M(\mathcal{A}))x_p &= \sum_{l\in N}(\mu_p - a_{ppll})y_l^2x_p - \sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\,\neq\,l}} a_{pjkl}x_jy_ky_l - \sum_{\substack{j,\,l\,\in\,N,\\ j\,\neq\,p}} a_{pjll}x_jy_l^2\\ &\leq \sum_{l\in N}(\mu_p - a_{ppll})y_l^2x_p - \sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\,\neq\,l}} a_{pjkl}x_py_ky_l - \sum_{\substack{j,\,l\,\in\,N,\\ j\,\neq\,p}} a_{pjll}x_py_l^2\\ &= \sum_{l\in N}(\mu_p - a_{ppll} - \sum_{\substack{j\,\in\,N,\\ j\,\neq\,p}} a_{pjll})y_l^2x_p - \sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\,\neq\,l}} a_{pjkl}x_py_ky_l\\ &\leq \max_{l\in N}(\mu_p - a_{ppll} - \sum_{\substack{j\,\in\,N,\\ j\,\neq\,p}} a_{pjll})x_p - \frac{1}{2}\sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\,\neq\,l}} a_{pjkl}x_p. \end{aligned}\tag{3.4}$$

It follows from (3.4) and the definition of ω_p that

$$(\mu_p - \tau_M(\mathcal{A}))x_p \leq (\omega_p(\mathcal{A}) - \frac{1}{2}r_p(\mathcal{A}))x_p,$$

which implies

$$\tau_M(\mathcal{A}) \geq \mu_p - \omega_p(\mathcal{A}) + \frac{1}{2}r_p(\mathcal{A}) = \mu_p - G_p(\mathcal{A}). \tag{3.5}$$

On the other hand, setting $y_t = \max_{l\in N}\{y_l\}$, from the t-th equation of $\tau_M(\mathcal{A})y = \mathcal{A}x^2y$, we have

$$((a_{11tt}x_1^2 + \cdots + a_{nntt}x_n^2) - \tau_M(\mathcal{A}))y_t = -\sum_{\substack{i,\,j,\,k\,\in\,N,\\ i\,\neq\,j}} a_{ijkt}x_ix_jy_k - \sum_{\substack{i,\,k\,\in\,N,\\ k\,\neq\,t}} a_{iikt}x_i^2y_k. \tag{3.6}$$

Setting $\kappa_t = \max_{i\in N}\{a_{iitt}\}$, by (3.6) and $a_{ijkt} \leq 0$ for all $l \neq t, i, j, k \in N$, we obtain

$$(\kappa_t - \tau_M(\mathcal{A}))y_t = \sum_{i\in N}(\kappa_t - a_{iitt})x_i^2y_t - \sum_{\substack{i,\,j,\,k\,\in\,N,\\ i\,\neq\,j}} a_{ijkt}x_ix_jy_k - \sum_{\substack{i,\,k\,\in\,N,\\ k\,\neq\,t}} a_{iikt}x_i^2y_k$$

$$
\begin{aligned}
&\le \sum_{i\in N}(\kappa_t - a_{iitt})x_i^2 y_t - \sum_{\substack{i,\,k\,\in\,N,\\ k\,\neq\,t}} a_{iikt}y_t x_i^2 - \sum_{\substack{i,\,j,\,k\,\in\,N,\\ i\,\neq\,j}} a_{ijkt}x_i x_j y_t \\
&= \sum_{i\in N}(\kappa_t - a_{iitt} - \sum_{\substack{k\,\in\,N,\\ k\,\neq\,t}} a_{iikt})x_i^2 y_t - \sum_{\substack{i,\,j,\,k\,\in\,N,\\ i\,\neq\,j}} a_{ijkt}x_i x_j y_t \\
&\le \max_{i\in N}(\kappa_t - a_{iitt} - \sum_{\substack{k\,\in\,N,\\ k\,\neq\,t}} a_{iikt})y_t - \frac{1}{2}\sum_{\substack{i,\,j,\,k\,\in\,N,\\ i\,\neq\,j}} a_{ijkt}y_t. \qquad (3.7)
\end{aligned}
$$

It follows from (3.7) and definition of m_t that

$$
(\kappa_t - \tau_M(\mathcal{A}))y_t \le (m_t(\mathcal{A}) - \frac{1}{2}c_t(\mathcal{A}))y_t,
$$

which shows

$$
\tau_M(\mathcal{A}) \ge \kappa_t - m_t(\mathcal{A}) + \frac{1}{2}c_t(\mathcal{A}) = \kappa_t - \mathcal{M}_t(\mathcal{A}). \qquad (3.8)
$$

By (3.5) and (3.8), we obtain the desired results. □

Next, we show that the bound in Theorem 3.2 is tighter than that of Theorem 3.1 of [10].

Lemma 3.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. Then*

$$
\tau_M(\mathcal{A}) \ge \max\{\min_{i\in N}\{\alpha_i - R_i(\mathcal{A}), \min_{l\in N}\{\beta_l - C_l(\mathcal{A})\}\},
$$

where

$$
\alpha_i = \min_{l\in N} a_{iill}, \beta_l = \min_{i\in N} a_{iill}, \gamma_i = \max_{l\in N}\{\sum_{j\in N, j\neq i} |a_{ijll}|\}, \delta_l = \max_{i\in N}\{\sum_{k\in N, k\neq l} |a_{iikl}|\},
$$

$$
r_i(\mathcal{A}) = \sum_{j,k,l\in N, k\neq l} |a_{ijkl}|, c_l(\mathcal{A}) = \sum_{i,j,k\in N, i\neq j} |a_{ijkl},
$$

$$
R_i(\mathcal{A}) = \gamma_i + r_i(\mathcal{A}), C_l(\mathcal{A}) = \delta_l + c_l(\mathcal{A}).
$$

Corollary 3.1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity M-tensor. Then*

$$
\max\{\min_{i\in N}\{\mu_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\kappa_l - M_l(\mathcal{A})\}\} \ge \max\{\min_{i\in N}\{\alpha_i - R_i(\mathcal{A})\}, \min_{l\in N}\{\beta_l - C_l(\mathcal{A})\}\}.
$$

Proof. Since $\mathcal{A}$ an elasticity M-tensor, we obtain $a_{ijkl} \le 0$ except for $i = j$ and $k = l$ and

$$
\begin{aligned}
\alpha_i - R_i(\mathcal{A}) = \alpha_i - \gamma_i(\mathcal{A}) - r_i(\mathcal{A}) &= \alpha_i - \max_{l\in N}\{\sum_{\substack{j\,\in\,N,\\ j\,\neq\,i}} |a_{ijll}|\} - \sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\,\neq\,l}} |a_{ijkl}| \\
&= \alpha_i - \max_{l\in N}\{-\sum_{\substack{j\,\in\,N,\\ j\,\neq\,i}} a_{ijll}\} + \sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\,\neq\,l}} a_{ijkl}.
\end{aligned}
$$

It follows from Theorem 3.2 that

$$
\mu_i - G_i(\mathcal{A}) = \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2}r_i(\mathcal{A}) = \mu_i - \max_{l\in N}(\mu_i - a_{iill} - \sum_{\substack{j\,\in\,N,\\ j\,\neq\,i}} a_{ijll}) + \frac{1}{2}\sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\,\neq\,l}} a_{ijkl}.
$$

Since $\mu_i - a_{iill} \geq 0$ and $\sum\limits_{j\in N, j\neq i} a_{ijll} \leq 0$, we can verify

$$\max_{l\in N}(\mu_i - a_{iill} - \sum_{\substack{j \in N, \\ j \neq i}} a_{ijll}) \leq \max_{l\in N}(\mu_i - a_{iill}) + \max_{l\in N}\{-\sum_{\substack{j \in N, \\ j \neq i}} a_{ijll}\}.$$

From $\mu_i = \max\limits_{l\in N}\{a_{iill}\}$ and $\alpha_i = \min\limits_{l\in N}\{a_{iill}\}$, we obtain $\max\limits_{l\in N}(\mu_i - a_{iill}) = \mu_i - \alpha_i$. Thus,

$$\max_{l\in N}(\mu_i - a_{iill} - \sum_{\substack{j \in N, \\ j \neq i}} a_{ijll}) \leq \mu_i - \alpha_i + \max_{l\in N}\{-\sum_{\substack{j \in N, \\ j \neq i}} a_{ijll}\},$$

which implies

$$\mu_i - \max_{l\in N}(\mu_i - a_{iill} - \sum_{\substack{j \in N, \\ j \neq i}} a_{ijll}) \geq \alpha_i - \max_{l\in N}\{-\sum_{\substack{j \in N, \\ j \neq i}} a_{ijll}\}. \tag{3.9}$$

From $a_{ijkl} \leq 0$ for all $j, k, l \in N, k \neq l$, it holds that

$$\frac{1}{2}\sum_{\substack{j,\, k,\, l \in N, \\ k \neq l}} a_{ijkl} \geq \sum_{\substack{j,\, k,\, l \in N, \\ k \neq l}} a_{ijkl}. \tag{3.10}$$

Summing inequalities (3.9) and (3.10), we deduce

$$\mu_i - G_i(\mathcal{A}) = \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2}r_i(\mathcal{A}) \geq \alpha_i - \gamma_i(\mathcal{A}) - r_i(\mathcal{A}) = \alpha_i - R_i(\mathcal{A}), \forall i \in N. \tag{3.11}$$

Following the similar arguments to the proof of (3.11), we deduce

$$\kappa_l - \mathcal{M}_l(\mathcal{A}) \geq \beta_l - \mathcal{C}_l(\mathcal{A}), \forall\ l \in N.$$

Thus,

$$\max\{\min_{i\in N}\{\mu_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\kappa_l - M_l(\mathcal{A})\}\} \geq \max\{\min_{i\in N}\{\alpha_i - R_i(\mathcal{A})\}, \min_{l\in N}\{\beta_l - C_l(\mathcal{A})\}\}.$$

□

Choosing x_p as a component of x with the largest modulus and x_q as a arbitrary component of left M-eigenvector x, we shall obtain sharp lower bound of the minimum M-eigenvalue.

Theorem 3.3. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor and $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue. Then*

$$\tau_M(\mathcal{A}) \geq \max\{\min_{i\in N}\max_{v\in N, v\neq i}\phi_1(\mathcal{A}), \min_{l\in N}\max_{u\in N, u\neq l}\phi_2(\mathcal{A})\},$$

where

$$\phi_1(\mathcal{A}) = \{\tfrac{1}{2}[\mu_v + \mu_i - \omega_i(\mathcal{A}) + \tfrac{1}{2}(r_i(\mathcal{A}) - r_i^v(\mathcal{A})) - \Delta_{i,v}^{\frac{1}{2}}(\mathcal{A})]\},$$
$$\phi_2(\mathcal{A}) = \{\tfrac{1}{2}[\kappa_u + \kappa_l - m_l(\mathcal{A}) + \tfrac{1}{2}(c_l(\mathcal{A}) - c_l^u(\mathcal{A})) - \theta_{l,u}^{\frac{1}{2}}(\mathcal{A})]\},$$
$$\Delta_{i,v}(\mathcal{A}) = (\mu_v - \mu_i + \omega_i(\mathcal{A}) - \tfrac{1}{2}(r_i(\mathcal{A}) - r_i^v(\mathcal{A})))^2 - 2r_i^v(\mathcal{A})G_v(\mathcal{A}),$$
$$\theta_{l,u}(\mathcal{A}) = (\kappa_u - \kappa_l + m_l(\mathcal{A}) - \tfrac{1}{2}(c_l(\mathcal{A}) - c_l^u(\mathcal{A})))^2 - 2c_l^u(\mathcal{A})\mathcal{M}_u(\mathcal{A}),$$
$$r_i^v(\mathcal{A}) = \sum_{\substack{k,\,l\,\in\,N,\\ k\neq l;\,j=v}} a_{ivkl},\ \ c_l^u(\mathcal{A}) = \sum_{\substack{i,\,j\,\in\,N,\\ i\neq j;\,k=u}} a_{ijul}.$$

Proof. Let $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue of $\mathcal{A}$. It follows from Lemma 2.4 that there exist nonnegative left and right M-eigenvectors (x, y) corresponding to $\tau_M(\mathcal{A})$. Set $x_p = \max_{i\in N}\{x_i\}$. By the p-th equation of $\tau_M(\mathcal{A})x = \mathcal{A}xy^2$, we have

$$((a_{pp11}y_1^2 + \cdots + a_{ppnn}y_n^2) - \tau_M(\mathcal{A}))x_p = -\sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\neq l}} a_{pjkl}x_jy_ky_l - \sum_{\substack{j,\,l\,\in\,N,\\ j\neq p}} a_{pjll}x_jy_l^2. \tag{3.12}$$

Setting $\mu_p = \max_{l\in N}\{a_{ppll}\}$, from (3.12), we obtain

$$\begin{aligned}
&(\mu_p - \tau_M(\mathcal{A}))x_p\\
&= \sum_{\substack{l\,\in\,N,\\ j=p}} (\mu_p - a_{ppll})y_l^2x_p - \sum_{\substack{j,\,l\,\in\,N,\\ j\neq p}} a_{pjll}x_jy_l^2 - \sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\neq l}} a_{pjkl}x_jy_ky_l\\
&\le \sum_{\substack{l\,\in\,N,\\ j=p}} (\mu_p - a_{ppll})y_l^2x_p - \sum_{\substack{j,\,l\,\in\,N,\\ j\neq p}} a_{pjll}x_py_l^2\\
&\quad - \sum_{\substack{k,\,l\,\in\,N,\\ k\neq l;\,j=q}} a_{pqkl}x_qy_ky_l - \sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\neq l;\,j\neq q}} a_{pjkl}x_py_ky_l\\
&= \sum_{l\in N}(\mu_p - a_{ppll} - \sum_{\substack{j\,\in\,N,\\ j\neq p}} a_{pjll})y_l^2x_p - \sum_{\substack{k,\,l\,\in\,N,\\ k\neq l;\,j=q}} a_{pqkl}x_qy_ky_l - \sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\neq l;\,j\neq q}} a_{pjkl}x_py_ky_l\\
&\le \max_{l\in N}(\mu_p - a_{ppll} - \sum_{\substack{j\,\in\,N,\\ j\neq p}} a_{pjll})x_p - \frac{1}{2}\sum_{\substack{k,\,l\,\in\,N,\\ k\neq l;\,j=q}} a_{pqkl}x_q - \frac{1}{2}\sum_{\substack{j,\,k,\,l\,\in\,N,\\ k\neq l;\,j\neq q}} a_{pjkl}x_p,
\end{aligned} \tag{3.13}$$

which shows

$$(\mu_p - \tau_M(\mathcal{A}) - \omega_p(\mathcal{A}) + \frac{1}{2}(r_p(\mathcal{A}) - r_p^q(\mathcal{A})))x_p \le -\frac{1}{2}r_p^q(\mathcal{A})x_q. \tag{3.14}$$

If $x_q = 0$, we can verify that $\tau_M(\mathcal{A}) \ge \mu_p - \omega_p + \frac{1}{2}(r_p(\mathcal{A}) - r_p^q(\mathcal{A})) \ge \phi_1(\mathcal{A})$. Otherwise, for any $q \in N, q \neq p, x_q > 0$. Recalling the q-th equation of $\tau_M(\mathcal{A})x = \mathcal{A}xy^2$ and $\mu_q = \max_{l\in N}\{a_{qqll}\}$, we deduce

$$(\mu_q - \tau_M(\mathcal{A}))x_q$$

$$
\begin{aligned}
&= \sum_{\substack{l \in N,\\ j=q}} (\mu_q - a_{qqll}) y_l^2 x_q - \sum_{\substack{j,l \in N,\\ j \neq q}} a_{qjll} x_j y_l^2 - \sum_{\substack{j,k,l \in N,\\ k \neq l}} a_{qjkl} x_j y_k y_l \\
&\leq \sum_{\substack{l \in N,\\ j=q}} (\mu_q - a_{qqll}) x_p y_l^2 - \sum_{\substack{j,l \in N,\\ j \neq q}} a_{qjll} x_p y_l^2 - \sum_{\substack{j,k,l \in N,\\ k \neq l}} a_{qjkl} x_p y_k y_l \\
&= \sum_{l\in N} (\mu_q - a_{qqll} - \sum_{\substack{j \in N,\\ j \neq q}} a_{qjll}) x_p y_l^2 - \sum_{\substack{j,k,l \in N,\\ k \neq l}} a_{qjkl} x_p y_k y_l \\
&\leq \max_{l\in N} (\mu_q - a_{qqll} - \sum_{\substack{j \in N,\\ j \neq q}} a_{qjll}) x_p - \frac{1}{2} \sum_{\substack{j,k,l \in N,\\ k \neq l}} a_{qjkl} x_p, \qquad (3.15)
\end{aligned}
$$

which implies

$$0 \leq (\mu_q - \tau_M(\mathcal{A})) x_q \leq (\omega_q(\mathcal{A}) - \frac{1}{2} r_q(\mathcal{A})) x_p = G_q(\mathcal{A}) x_p. \qquad (3.16)$$

Multiplying inequalities (3.14) with (3.16) yields

$$[\mu_p - \tau_M(\mathcal{A}) - \omega_p(\mathcal{A}) + \frac{1}{2}(r_p(\mathcal{A}) - r_p^q(\mathcal{A}))](\mu_q - \tau_M(\mathcal{A})) \leq -\frac{1}{2} r_p^q(\mathcal{A}) G_q(\mathcal{A}).$$

Then, solving for $\tau_M(\mathcal{A})$, we obtain

$$\tau_M(\mathcal{A}) \geq \frac{1}{2}[\mu_q + \mu_p - \omega_p(\mathcal{A}) + \frac{1}{2}(r_p(\mathcal{A}) - r_p^q(\mathcal{A})) - \Delta_{p,q}^{\frac{1}{2}}(\mathcal{A})]$$

where $\Delta_{p,q}(\mathcal{A}) = (\mu_q - \mu_p + \omega_p(\mathcal{A}) - \frac{1}{2}(r_p(\mathcal{A}) - r_p^q(\mathcal{A})))^2 - 2 r_p^q(\mathcal{A}) G_q(\mathcal{A})$. From the arbitrariness of q, we have

$$\tau_M(\mathcal{A}) \geq \max_{q\in N, q\neq p} \{\frac{1}{2}[\mu_q + \mu_p - \omega_p(\mathcal{A}) + \frac{1}{2}(r_p(\mathcal{A}) - r_p^q(\mathcal{A})) - \Delta_{p,q}^{\frac{1}{2}}(\mathcal{A})]\}.$$

Further,

$$\tau_M(\mathcal{A}) \geq \min_{i\in N} \max_{v\in N, v\neq i} \{\frac{1}{2}[\mu_v + \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2}(r_i(\mathcal{A}) - r_i^v(\mathcal{A})) - \Delta_{i,v}^{\frac{1}{2}}(\mathcal{A})]\}. \qquad (3.17)$$

On the other hand, setting $y_t = \max_{l\in N}\{y_l\}$, from the t-th equation of $\tau_M(\mathcal{A})y = \mathcal{A}x^2y$, we deduce

$$((a_{11tt}x_1^2 + \cdots + a_{nntt}x_n^2) - \tau_M(\mathcal{A})) y_t = - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt} x_i x_j y_k - \sum_{\substack{i,k=1,\\ k\neq t}}^{n} a_{iikt} x_i^2 y_k. \qquad (3.18)$$

Setting $\kappa_t = \max_{i\in N}\{a_{iitt}\}$, by(3.18), we obtain

$$
\begin{aligned}
&(\kappa_t - \tau_M(\mathcal{A})) y_t \\
&= \sum_{\substack{i \in N,\\ k=t}} (\kappa_t - a_{iitt}) x_i^2 y_t - \sum_{\substack{i,k \in N,\\ k \neq t}} a_{iikt} x_i^2 y_k - \sum_{\substack{i,j,k \in N,\\ i \neq j}} a_{ijkt} x_i x_j y_k
\end{aligned}
$$

$$\begin{aligned}
&\le \sum_{\substack{i\in N,\\ k=t}} (\kappa_t - a_{iitt}) x_i^2 y_t - \sum_{\substack{i,k\in N,\\ k\ne t}} a_{iikt} x_i^2 y_t \\
&\quad - \sum_{\substack{i,j\in N,\\ i\ne j;\, k=s}} a_{ijst} x_i x_j y_s - \sum_{\substack{i,j,k\in N,\\ i\ne j;\, k\ne s}} a_{ijkt} x_i x_j y_t \\
&= \sum_{i\in N} (\kappa_t - a_{iitt} - \sum_{\substack{k\in N,\\ k\ne t}} a_{iikt}) x_i^2 y_t - \sum_{\substack{i,j\in N,\\ i\ne j;\, k=s}} a_{ijst} x_i x_j y_s - \sum_{\substack{i,j,k\in N,\\ i\ne j;\, k\ne s}} a_{ijkt} x_i x_j y_t \\
&\le \max_{i\in N} (\kappa_t - a_{iitt} - \sum_{\substack{k\in N,\\ k\ne t}} a_{iikt}) y_t - \frac{1}{2} \sum_{\substack{i,j\in N,\\ i\ne j;\, k=s}} a_{ijst} y_s - \frac{1}{2} \sum_{\substack{i,j,k\in N,\\ i\ne j;\, k\ne s}} a_{ijkt} y_t. \qquad (3.19)
\end{aligned}$$

Thus,

$$(\kappa_t - \tau_M(\mathcal{A}) - m_t(\mathcal{A}) + \frac{1}{2}(c_t(\mathcal{A}) - c_t^s(\mathcal{A})))y_t \le -\frac{1}{2} c_t^s(\mathcal{A}) y_s. \qquad (3.20)$$

Similarly, for any $s \in N, s \ne t$, in the view of the s-th equation of $\tau_M(\mathcal{A})y = \mathcal{A}x^2 y$ and $\kappa_s = \max_{i\in N}\{a_{iiss}\}$, we have

$$\begin{aligned}
&(\kappa_s - \tau_M(\mathcal{A}))y_s \\
&= \sum_{\substack{i\in N,\\ k=s}} (\kappa_s - a_{iiss}) x_i^2 y_s - \sum_{\substack{i,k\in N,\\ k\ne s}} a_{iiks} x_i^2 y_k - \sum_{\substack{i,j,k\in N,\\ i\ne j}} a_{ijks} x_i x_j y_k \\
&\le \sum_{\substack{i\in N,\\ k=s}} (\kappa_s - a_{iiss}) x_i^2 y_t - \sum_{\substack{i,k\in N,\\ k\ne s}} a_{iiks} x_i^2 y_t - \sum_{\substack{i,j,k\in N,\\ i\ne j}} a_{ijks} x_i x_j y_t \\
&= \sum_{i\in N} (\kappa_s - a_{iiss} - \sum_{\substack{k\in N,\\ k\ne s}} a_{iiks}) x_i^2 y_t - \sum_{\substack{i,j,k\in N,\\ i\ne j}} a_{ijks} x_i x_j y_t \\
&\le \max_{i\in N} (\kappa_s - a_{iiss} - \sum_{\substack{k\in N,\\ k\ne s}} a_{iiks}) y_t - \frac{1}{2} \sum_{\substack{i,j,k\in N,\\ i\ne j}} a_{ijks} y_t. \qquad (3.21)
\end{aligned}$$

Therefore,

$$0 \le (\kappa_s - \tau_M(\mathcal{A}))y_s \le (m_s(\mathcal{A}) - \frac{1}{2} c_s(\mathcal{A}))y_t = \mathcal{M}_s(\mathcal{A}) y_t. \qquad (3.22)$$

Multiplying inequalities (3.20) with (3.22) yields

$$(\kappa_t - \tau_M(\mathcal{A}) - m_t(\mathcal{A}) + \frac{1}{2}(c_t(\mathcal{A}) - c_t^s(\mathcal{A})))(\kappa_s - \tau_M(\mathcal{A})) \le -\frac{1}{2} c_t^s(\mathcal{A}) \mathcal{M}_s(\mathcal{A}).$$

Then, solving for $\tau_M(\mathcal{A})$, we obtain

$$\tau_M(\mathcal{A}) \ge \frac{1}{2}[\kappa_s + \kappa_t - m_t(\mathcal{A}) + \frac{1}{2}(c_t(\mathcal{A}) - c_t^s(\mathcal{A})) - \theta_{t,s}^{\frac{1}{2}}(\mathcal{A})]$$

where $\theta_{t,s}(\mathcal{A}) = (\kappa_s - \kappa_t + m_t(\mathcal{A}) - \frac{1}{2}(c_t(\mathcal{A}) - c_t^s(\mathcal{A})))^2 - 2c_t^s(\mathcal{A})\mathcal{M}_s(\mathcal{A})$. From the arbitrariness of s, we have

$$\tau_M(\mathcal{A}) \geq \max_{s\in N, s\neq t}\{\frac{1}{2}[\kappa_s + \kappa_t - m_t(\mathcal{A}) + \frac{1}{2}(c_t(\mathcal{A}) - c_t^s(\mathcal{A})) - \theta_{t,s}^{\frac{1}{2}}(\mathcal{A})]\}.$$

Further,

$$\tau_M(\mathcal{A}) \geq \min_{l\in N}\max_{u\in N, u\neq l}\{\frac{1}{2}[\kappa_u + \kappa_l - m_l(\mathcal{A}) + \frac{1}{2}(c_l(\mathcal{A}) - c_l^u(\mathcal{A})) - \theta_{l,u}^{\frac{1}{2}}(\mathcal{A})]\}. \quad (3.23)$$

Thus, the desired resluts hold from (3.17) with (3.23). □

Next, we prove that the bound in Theorem 3.3 is always tighter than that of Theorem 3.2.

Corollary 3.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor. Then*

$$\begin{aligned}&\max\{\min_{i\in N}\max_{j\in N, j\neq i}\phi_1(\mathcal{A}), \min_{l\in N}\max_{u\in N, u\neq l}\phi_2(\mathcal{A})\}\\ \geq &\max\{\min_{i\in N}\{\mu_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\kappa_l - \mathcal{M}_l(\mathcal{A})\}\}.\end{aligned}$$

Proof. We now break up the argument into two cases.
Case 1: For any $i, j \in N$, $j \neq i$, if $\mu_i - G_i(\mathcal{A}) \leq \mu_j - G_j(\mathcal{A})$, then

$$\mu_j - \mu_i + G_i(\mathcal{A}) \geq G_j(\mathcal{A}) \geq 0. \quad (3.24)$$

It follows from (3.24), $2r_i^j(\mathcal{A}) \leq 0$ and $G_i(\mathcal{A}) = \omega_i(\mathcal{A}) - \frac{1}{2}r_i(\mathcal{A})$ that

$$\begin{aligned}&(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})))^2 - 2r_i^j(\mathcal{A})G_j(\mathcal{A})\\ \leq&(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})))^2 - 2r_i^j(\mathcal{A})(\mu_j - \mu_i + G_i(\mathcal{A}))\\ =&(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})))^2 - 2r_i^j(\mathcal{A})(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}r_i(\mathcal{A}))\\ =&(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}r_i(\mathcal{A}) - \frac{1}{2}r_i^j(\mathcal{A}))^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\mu_j + \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A}))\\ &- \sqrt{(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})))^2 - 2r_i^j(\mathcal{A})G_j(\mathcal{A})}\\ \geq&\mu_j + \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})) - (\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}r_i(\mathcal{A}) - \frac{1}{2}r_i^j(\mathcal{A}))\\ =&2(\mu_i - \omega_i(\mathcal{A}) + \frac{1}{2}r_i(\mathcal{A})) = 2(\mu_i - G_i(\mathcal{A})),\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{2}\{\mu_j + \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A}))\\ &- \sqrt{(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})))^2 - 2r_i^j(\mathcal{A})G_j(\mathcal{A})}\} \geq \mu_i - G_i(\mathcal{A}).\end{aligned} \quad (3.25)$$

Case 2: For any $i, j \in N$, $j \neq i$, if $\mu_i - G_i(\mathcal{A}) \geq \mu_j - G_j(\mathcal{A})$, then

$$\mu_i - \omega_i(\mathcal{A}) - \mu_j + G_j(\mathcal{A}) \geq -\frac{1}{2} r_i(\mathcal{A}) \geq 0. \tag{3.26}$$

It follows from (3.26) that

$$\begin{aligned}
&(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})))^2 - 2r_i^j(\mathcal{A})G_j(\mathcal{A}) \\
\leq &(\mu_j - \mu_i + \omega_i(\mathcal{A}) + \mu_i - \omega_i(\mathcal{A}) - \mu_j + G_j(\mathcal{A}) + \frac{1}{2} r_i^j(\mathcal{A})))^2 - 2r_i^j(\mathcal{A})G_j(\mathcal{A}) \\
= &(G_j(\mathcal{A}) - \frac{1}{2} r_i^j(\mathcal{A}))^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mu_j + \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})) \\
&- \sqrt{(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})))^2 - 2r_i^j(\mathcal{A})G_j(\mathcal{A})} \\
\geq &\mu_j + \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2} r_i(\mathcal{A}) - \frac{1}{2} r_i^j(\mathcal{A})) - G_j(\mathcal{A}) + \frac{1}{2} r_i^j(\mathcal{A}) \\
= &\mu_j - G_j(\mathcal{A}) + \mu_i - G_i(\mathcal{A}) \geq 2(\mu_j - G_j(\mathcal{A})).
\end{aligned}$$

Further,

$$\begin{aligned}
&\frac{1}{2}\{\mu_j + \mu_i - \omega_i(\mathcal{A}) + \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})) \\
&- \sqrt{(\mu_j - \mu_i + \omega_i(\mathcal{A}) - \frac{1}{2}(r_i(\mathcal{A}) - r_i^j(\mathcal{A})))^2 - 2r_i^j(\mathcal{A})G_j(\mathcal{A})}\} \\
\geq &\mu_j - G_j(\mathcal{A}).
\end{aligned} \tag{3.27}$$

Using (3.25) and (3.27), we deduce

$$\min_{i \in N} \max_{j \in N, j \neq i} \{\phi_1(\mathcal{A})\} \geq \min_{i \in N} \{\mu_i - G_i(\mathcal{A})\}. \tag{3.28}$$

Following the similar arguments to the proof of (3.28), we obtain

$$\min_{l \in N} \max_{u \in N, u \neq l} \{\phi_2(\mathcal{A})\} \geq \min_{l \in N} \{\kappa_l - \mathcal{M}_l(\mathcal{A})\}. \tag{3.29}$$

Thus, the desired result follows (3.28) and (3.29). □

We use Example 3.1 of [10] to show the superiority of our results.

Example 3.1. $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,2}$ be an elasticity M-tensor, whose entries are

$$a_{ijkl} = \begin{cases}
a_{1111} = a_{1122} = 4.1, a_{2211} = a_{2222} = 5, \\
a_{1112} = a_{1121} = -1, a_{2212} = a_{2221} = -1, \\
a_{2111} = a_{1211} = -1, a_{1222} = a_{2122} = -1, \\
a_{ijkl} = 0, \text{otherwise}.
\end{cases}$$

By computations, we obtain that the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are $(\tau_M(\mathcal{A}), \bar{x}, \bar{y}) = (2.4534, (0.8398, 0.5430), (0.7071, 0.7071))$. The bounds via different estimations given in the literature are shown in the following table:

References	bounds
Lemma 2.4 and Theorem 3.1 of [10]	$1.10 \leq \tau_M(\mathcal{A}) \leq 4.10$
Lemma 2.4 and Theorem 3.2 of [10]	$1.29 \leq \tau_M(\mathcal{A}) \leq 4.10$
Theorems 3.1 and 3.2	$2.10 \leq \tau_M(\mathcal{A}) \leq 2.55$
Theorems 3.1 and 3.3	$2.21 \leq \tau_M(\mathcal{A}) \leq 2.55$

4. Identifying strong ellipticity condition and an elasticity M-tensor

As we know, for an elasticity tensor, the strong ellipticity condition holds if and only if it is M-positive, and that it is M-positive if and only if its minimum M-eigenvalue is positive. Meanwhile, an elasticity Z-tensor is an elasticity M-tensor if and only if the minimum M-eigenvalue is positive [6]. In this section, we establish some sufficient conditions for identifying an elasticity M-tensor and strong ellipticity condition based on the results in Theorems 3.2 and 3.3.

Theorem 4.1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor. If*

$$\max\{\min_{i\in N}\{\mu_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\kappa_l - \mathcal{M}_l(\mathcal{A})\}\} > 0, \tag{4.1}$$

then strong ellipticity condition of $\mathcal{A}$ is satisfied, and $\mathcal{A}$ is an elasticity M-tensor.

Proof. It follows from Theorem 3.2 and (4.1) that

$$\tau_M(\mathcal{A}) \geq \max\{\min_{i\in N}\{\mu_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\kappa_l - \mathcal{M}_l(\mathcal{A})\}\} > 0,$$

which implies that strong ellipticity condition is satisfied. Further, $\mathcal{A}$ is an elasticity M-tensor. □

Theorem 4.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an elasticity Z-tensor. If*

$$\max\{\min_{i\in N}\max_{v\in N, v\neq i}\phi_1(\mathcal{A}), \min_{l\in N}\max_{u\in N, u\neq l}\phi_2(\mathcal{A})\} > 0,$$

then strong ellipticity condition of $\mathcal{A}$ is satisfied, and $\mathcal{A}$ is an elasticity M-tensor.

Proof. Following the similar arguments to the proof of Theorem 4.1, we obtain the desired results. □

The following example shows that the results given in Theorems 4.1 and 4.2 can check whether an elasticity Z-tensor is an elasticity M-tensor and verify the strong ellipticity condition of an elasticity Z-tensor.

Example 4.1. Consider an elasticity Z-tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,3}$ defined by the conditions

$$a_{ijkl} = \begin{cases} a_{1111} = a_{2222} = a_{3333} = 5, a_{1122} = a_{1133} = a_{2233} = 6, \\ a_{2211} = a_{3311} = a_{3322} = 7, a_{2123} = a_{1223} = a_{2132} = a_{1232} = -0.2, \\ a_{1112} = a_{1121} = -1, a_{2212} = a_{2221} = -0.5, a_{1222} = a_{2122} = -2, \\ a_{3313} = a_{3331} = -0.5, a_{1333} = a_{3133} = -2, a_{1311} = a_{3111} = -1, \\ a_{2223} = a_{2232} = -0.5, a_{2322} = a_{3222} = -1, a_{2333} = a_{3233} = -2, \\ a_{1213} = a_{1231} = a_{2113} = a_{2131} = -0.2, \\ a_{3132} = a_{3123} = a_{1332} = a_{1323} = -0.2, a_{ijkl} = 0, \text{otherwise}. \end{cases}$$

By computations, we obtain that the the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are

$$(\tau_M(\mathcal{A}), \bar{x}, \bar{y}) = (2.5000, (0.7071, 0, 0.7071), (0.7071, 0.7071, 0)).$$

Hence, $\mathcal{A}$ is an elasticity M-tensor and strong ellipticity condition holds. The bounds given in the different literatures are shown in the following table:

References	bounds
Lemma 2.4 and Theorem 3.1 of [10]	$-0.80 \leq \tau_M(\mathcal{A}) \leq 5.00$
Lemma 2.4 and Theorem 3.2 of [10]	$-0.66 \leq \tau_M(\mathcal{A}) \leq 5.00$
Theorems 3.1 and 3.2	$0.30 \leq \tau_M(\mathcal{A}) \leq 3.40$
Theorems 3.1 and 3.3	$0.39 \leq \tau_M(\mathcal{A}) \leq 3.40$

5. Conclusions

In this paper, we characterized the eigenvector associated with the minimum M-eigenvalue of elasticity Z-tensors. Further, we established new bounds on the minimum M-eigenvalue of elasticity Z-tensors without irreducible conditions, which extended bound estimations on the minimum M-eigenvalue from elasticity M-tensors to elasticity Z-tensors. Finally, our approach to estimate the minimum M-eigenvalue was based on maximum diagonal entries, which is different from that of the literature [10].

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