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DETERIORATED HSS-LIKE METHODS FOR THE WEIGHTED TOEPLITZ LEAST SQUARES PROBLEM FROM IMAGE RESTORATION*

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Abstract In this paper, we construct a deteriorated HSS-like (DHSS-like) iteration method for a class of large and sparse block two-by-two linear systems from image restoration. The detailed spectral properties and the quasi-optimal iteration parameters are investigated in detail. Because the DHSS-like iteration method naturally leads to a DHSS-like preconditioner, then we can use the circulant matrix to replace the Toeplitz matrix in the DHSS-like preconditioner approximately to obtain a circulant matrix-based DHSS-like (CDHSS-like) preconditioner. It is pointed out that the workload of the new preconditioned method is about $O(n \log n)$. Theoretically analysis shows that the eigenvalues of the CDHSS-like preconditioned matrix are clustered around 1. Implementations in linear systems from the image restoration problems are made to verify the correctness of the theoretical results and the efficiency of both the CDHSS-like iteration method and the CDHSS-like preconditioned method.

Keywords Deteriorated HSS-like method, matrix splitting iteration method, convergence, preconditioner, Krylov subspace method.

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1. Introduction

Consider the Tikhonov minimization problems of the following form:

$$\min_{x \in \mathbb{R}^n} \|Bx - \tilde{b}\|_2,$$

where

$$B = \begin{pmatrix} \Xi K \\ \sqrt{\nu} I \end{pmatrix} \quad \text{and} \quad \tilde{b} = \begin{pmatrix} \Xi f \\ 0 \end{pmatrix},$$

where $K \in \mathbb{R}^{m \times n} (m \geq n)$ is a full-rank Toeplitz matrix, $\Xi \in \mathbb{R}^{m \times m}$ is a carefully selected regularization matrix and usually is a symmetric positive definite (SPD)

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matrix (as a weighting matrix) and $\nu > 0$ is called the regularization parameter (generally small, i.e., $0 < \nu < 1$). $I \in \mathbb{R}^{n\times n}$ is the identity matrix and $f \in \mathbb{R}^m$ is a given vector.

Denote by $y = \Xi^T\Xi(f - Kx)$, then according to [23], the above Tikhonov minimization problems can be expressed as an augmented linear system of the following form

$$\mathcal{A}u := \begin{pmatrix} W & K \\ -K^T & \nu I \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} := b, \tag{1.1}$$

where $W = (\Xi^T\Xi)^{-1} \in \mathbb{R}^{m\times m}$ is an SPD matrix and $\mathcal{A}$ is nonsingular.

As is shown in [19] that the displacement rank is very large because of the spatially variant property of weighted Toeplitz matrix ΞK. Hence, it is of great task to develop fast iterative methods for solving the weighted Toeplitz regularized least squares system (1.1).

It can be easy found that the coefficient matrix of the linear system (1.1) possesses a special generalized saddle point matrix of block two-by-two structure. Recent years, lots of efficient methods are developed by researchers in existing references. Some of these methods are constructed based on the classical successive overrelaxation method (SOR) method [26], e.g., the preconditioned GSOR (PGSOR) method [14], the accelerated GSOR (AGSOR) method [13], the generalized SOR method [6], the parameterized inexact Uzawa method [2], the preconditioned AGSOR (PAGSOR) method [16], the new relaxed splitting preconditioned method [21], and so on. Besides, by making use of the Hermitian and skew-Hermitian (HS) splitting of the coefficient matrix A, Bai and the co-authors proposed a preconditioned Hermitian and skew-Hermitian splitting (HSS) iteration method [3] for solving the block two-by-two linear system (1.3) from distributed control problems. For the block two-by-two linear systems of skew-Hamiltonian coefficient matrices, Bai, Chen and Wang [4] introduced an additive block diagonal preconditioning technique. Most recent iterative methods based on the HS splitting of the coefficient matrix can be found in [16].

Many researchers try to use the existing methods for generalized saddle point problem to solve the linear system (1.1). One of the most classical methods is the HSS iteration method [7]. Based on the HS splitting [5]

$$\begin{pmatrix} W & K \\ -K^T & \nu I \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & \nu I \end{pmatrix} + \begin{pmatrix} 0 & K \\ -K^T & 0 \end{pmatrix} := H + S, \tag{1.2}$$

the HSS iteration method for solving the linear system (1.1) from the image restoration problem can be written as,

$$\begin{cases} (\alpha I + H)u^{(k+\frac{1}{2})} = (\alpha I - S)u^{(k)} + b, \\ (\alpha I + S)u^{(k+1)} = (\alpha I - H)u^{(k+\frac{1}{2})} + b, \end{cases}$$

for $k = 0, 1, 2, \cdots$, where $u^{(0)}$ is an arbitrary initial guess, I is the identity matrix with the proper dimension (here and in the sequence, we will omit the subscript) and $\alpha > 0$ is a given parameter. The HSS iteration method naturally induces the

HSS preconditioner as

$$P_1 = \frac{1}{2\alpha}(H+\alpha I)(S+\alpha I) = \frac{1}{2\alpha}\begin{pmatrix}\alpha I + W & 0 \\ 0 & (\alpha+\nu)I\end{pmatrix}\begin{pmatrix}\alpha I & K \\ -K^T & \alpha I\end{pmatrix}. \tag{1.3}$$

In [7], the authors gave detailed theoretically analysis for the case $\alpha = \nu$. However, it was also mentioned that taking $\alpha = \nu$ does not lead to a very good performance. To obtain a better performance, based on the preconditioning idea in [25], Ng and Pan further construct the modified HSS (MHSS) preconditioner in [24] as

$$P_2 = \frac{1}{2}\Sigma^{-1}(\Sigma + H)(\Sigma + S), \tag{1.4}$$

where

$$\Sigma = \begin{pmatrix}\alpha I & 0 \\ 0 & \nu I\end{pmatrix}$$

with $\alpha > 0$ being a given constant. From their numerical results we see that the MHSS iteration method is efficient for some examples when the parameters are chosen appropriately. However, we can further improve the efficiency for other examples. Moreover, they are also some efficient methods for the case $m = n$. For example, based on the HSS iteration method, Aghazadeh constructed a generalized HSS iteration method [1]. Cui constructed a modified special HSS iteration method [11, 12]. Liao and Zhang proposed a new variant of HSS iteration method [22]. Besides, Zak and Toutounian proposed a shifted nested splitting iteration method [28]. Those methods have improved the efficiency of the HSS iteration method for solving linear systems of the form (1.1).

Although most of the existing methods can be used to solve the Toeplitz-like linear systems, but they are convergent slowly . Besides, for those Toeplitz-like linear systems, the circulant matrix approximation of the Toeplitz matrix cannot be diagonalized by the fast Fourier transforms (FFTs). Therefore, exploiting more efficient methods and preconditioners are very important for solving the linear system (1.1) from image restoration.

In this work, we focus on the case of $m = n$. Following the idea of alternative direction iteration, we will design a deteriorated HSS-like iteration method based on the HSS method for the saddle point problem in [2]. The eigenvalues and the corresponding eigenvectors will be proposed. The quasi-optimal parameter will be analyzed theoretically. Furthermore, we will construct a circulant-based DHSS-like (CDHSS-like) preconditioner to accelerate the Krylov subspace method by using a circulant matrix to approximate the Toeplitz matrix. Moreover, the conclusion about the CDHSS-like preconditioner superiority to the HSS preconditioner and the MHSS preconditioner will be verified through the numerical experiments.

The organization of the paper is as follows. In Section 2, we construct an HSS-like iteration method and then propose a deteriorated HSS-like preconditioner for the large and sparse block structured linear system (1.1). We also investigate the spectral properties and derive the choice of the quasi-optimal iterative parameter for the DHSS-like preconditioner. After replacing the Toeplitz matrix K by a circulant matrix approximately, we obtain a CDHSS-like preconditioner in Section 3. We also analyze the clustering properties of the CDHSS-like preconditioned matrix. Section

4 devotes to the numerical implementations both from the one-dimensional example of linear system and the image restoration problem to illustrate the efficiency of the CDHSS-like preconditioned method. Finally in Section 5, a brief concluding remark will be drawn to end this work.

2. The DHSS-like preconditioner

To begin with, we will construct an HSS-like iteration method for the linear equations (1.1). Based on the HS splitting (1.2), we rewrite the linear system (1.1) as

$$(\alpha J + H)u = (\alpha J - S)u + b$$

and

$$(\alpha J + S)u = (\alpha J - H)u + b,$$

where J is a given matrix and $\alpha > 0$ is a given parameter. Using the above identities and the special structure of the matrices H and S, we can construct the HSS-like iteration method as the following algorithm.

Algorithm 2.1. (The HSS-like iteration method)
Given an initial guess $u^{(0)}$, for $k = 0, 1, 2, \ldots$, until $\{u^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha J + H)u^{(k+\frac{1}{2})} = (\alpha J - S)u^{(k)} + b, \\ (\alpha J + S)u^{(k+1)} = (\alpha J - H)u^{(k+\frac{1}{2})} + b, \end{cases}$$

where $\alpha > 0$ is a given positive constant and

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

with I being an identity matrix with the proper dimension.

After a few simple algebra computations, we can rewrite the HSS-like iteration method as a fixed-point iterative method as

$$u^{(k+1)} = u^{(k)} + P_3^{-1}(b - \mathcal{A}u^{(k)}), \quad k = 0, 1, 2, \ldots,$$

where

$$P_3 = \frac{1}{2\alpha}(\alpha J + H)J(\alpha J + S) = \frac{1}{2\alpha} \begin{pmatrix} \alpha I & W \\ \nu I & \alpha I \end{pmatrix} \begin{pmatrix} 0 & \alpha I + K \\ \alpha I - K^T & 0 \end{pmatrix}.$$

Inspired by the relaxed preconditioning idea proposed in [8,9,29], we modify the preconditioner P_3 as the following deteriorated HSS-like (DHSS-like) preconditioner:

$$P = \frac{1}{\alpha} \begin{pmatrix} \alpha I & W \\ \nu I & -K^T \end{pmatrix} \begin{pmatrix} 0 & \alpha I + K \\ \alpha I & 0 \end{pmatrix} = \begin{pmatrix} W & \alpha I + K \\ -K^T & \nu I + \frac{\nu}{\alpha} K \end{pmatrix}. \tag{2.1}$$

Note that the DHSS-like preconditioner can be constructed from the splitting of the coefficient matrix $\mathcal{A}$

$$\mathcal{A} = P - R = \begin{pmatrix} W & \alpha I + K \\ -K^T & \nu I + \frac{\nu}{\alpha} K \end{pmatrix} - \begin{pmatrix} 0 & \alpha I \\ 0 & \frac{\nu}{\alpha} K \end{pmatrix}.$$

Hence, when we use the preconditioned Krylov subspace method with the preconditioner P, we need to solve a generalized residual equation $Pz = r$ at each iterative step, with $z = (z_1^T, z_2^T)^T$ and $r = (r_1^T, r_2^T)^T$ being the current and generalized residual vectors, respectively. By simple computations, we obtain

$$\begin{aligned} z = P^{-1} r &= \begin{pmatrix} W & \alpha I + K \\ -K^T & \nu I + \frac{\nu}{\alpha} K \end{pmatrix}^{-1} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \\ &= \alpha \begin{pmatrix} 0 & \frac{1}{\alpha} I \\ (\alpha I + K)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \alpha I & W \\ \nu I & -K^T \end{pmatrix}^{-1} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \\ &= \alpha \begin{pmatrix} 0 & \frac{1}{\alpha} I \\ (\alpha I + K)^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & -\frac{1}{\alpha} W \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} I & 0 \\ 0 & -(\frac{\nu}{\alpha} W + K^T)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\frac{\nu}{\alpha} I & I \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \end{aligned}$$

According to the above results, we can describe the DHSS-like preconditioned method in the following practical procedure.

Algorithm 2.2. (The DHSS-like iteration method)

The implementing process of the DHSS-like preconditioner:

Given a residual vector $r = (r_1^T, r_2^T)^T \in \mathbb{R}^{2n}$. Compute z according to the following steps:

Step 1. solve $(\nu I + \alpha K^T W^{-1})\tilde{s} = \nu r_1 - \alpha r_2$ to obtain $\tilde{s}$.

Step 2. solve $(\alpha I + K) z_2 = r_1 - \tilde{s}$ to obtain z_2.

Step 3. set $z_1 = W^{-1}\tilde{s}$ to obtain the generalized residual vector $z = (z_1^T, z_2^T)^T$.

Remark 2.1. We see that the main workload of Algorithm 2.2 is to solve two linear subsystems at Step 1 and Step 2. Because the Toeplitz matrix K and the positive diagonal matrix W may be very ill-conditioned [24], then we can use the Krylov subspace method, e.g., the GMRES method [26], with a proper preconditioner to solve the corresponding linear system. More implementation details will be shown in the next section.

In the following of this section, we will investigate the spectral properties of the preconditioned matrix $P^{-1}\mathcal{A}$. Firstly, we give the results about the eigenvalues distribution of the matrix $P^{-1}\mathcal{A}$.

Theorem 2.1. *Assume that the coefficient matrix $\mathcal{A} \in \mathbb{R}^{2n\times 2n}$ of the linear system (1.1) is nonsingular. $\nu > 0$ is the regularization parameter, generally small enough. $W \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix and $K \in \mathbb{R}^{n\times n}$ is a full-rank Toeplitz matrix. Let α be a real positive constant such that $\alpha I + K$ and $\nu W + \alpha K^T$ are both nonsingular matrices. Then for the preconditioned matrix $P^{-1}\mathcal{A}$, the following results hold.*

(1) $P^{-1}\mathcal{A}$ has an eigenvalue 1 with multiplicity at least n. The corresponding eigenvectors are

$$\begin{pmatrix} u_l \\ 0 \end{pmatrix}, \quad (l = 1, 2, \ldots, n),$$

where $u_l (l = 1, 2, \ldots, n)$ are arbitrary linearly independent vectors.
(2) The remaining nonunit eigenvalues λ of $P^{-1}\mathcal{A}$ are the eigenvalues of Θ_2, i.e., $\Theta_2 v_l = \lambda v_l$. The corresponding eigenvectors are

$$\begin{pmatrix} \frac{1}{1-\lambda}\Theta_1 v_l \\ v_l \end{pmatrix}, \quad (l = 1, 2, \ldots, n).$$

Here $v_l \neq 0$ is the eigenvector of the matrix Θ_2 corresponding to the eigenvalue λ with $\Theta_1 = \nu(\nu W + \alpha K^T)^{-1}(\alpha I - K)$ and $\Theta_2 = (\alpha I + K)^{-1}(K + W\Theta_1)$.

Proof. After simple computations, we have

$$\begin{aligned}
P^{-1}\mathcal{A} &= P^{-1}(P - R) = I - P^{-1}R \\
&= I - \alpha \begin{pmatrix} 0 & \frac{1}{\alpha}I \\ (\alpha I + K)^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & -\frac{1}{\alpha}W \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha}I & 0 \\ 0 & -(\frac{\nu}{\alpha}W + K^T)^{-1} \end{pmatrix} \times \ldots \\
&\quad \times \begin{pmatrix} I & 0 \\ -\frac{\nu}{\alpha}I & I \end{pmatrix} \begin{pmatrix} W & K \\ -K^T & \nu I \end{pmatrix} \\
&= \begin{pmatrix} I & -\nu(\nu W + \alpha K^T)^{-1}(\alpha I - K) \\ 0 & I - \alpha(\alpha I + K)^{-1} + \nu(\alpha I + K)^{-1}W(\nu W + \alpha K^T)^{-1}(\alpha I - K) \end{pmatrix} \\
&= \begin{pmatrix} I & -\Theta_1 \\ 0 & \Theta_2 \end{pmatrix}.
\end{aligned}$$

The above last equation can be induced from

$$\begin{aligned}
&(\alpha I + K)^{-1}(K + W\Theta_1) \\
=&I - \alpha(\alpha I + K)^{-1} + \nu(\alpha I + K)^{-1}W(\nu W + \alpha K^T)^{-1}(\alpha I - K) \\
=&\Theta_2.
\end{aligned}$$

Let $\lambda \in sp(P^{-1}\mathcal{A})$, i.e., λ be an eigenvalue of the preconditioned matrix $P^{-1}\mathcal{A}$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ be the corresponding eigenvector. Then we have

$$P^{-1}\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} I & -\Theta_1 \\ 0 & \Theta_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix},$$

i.e.,

$$\begin{cases} u - \Theta_1 v = \lambda u, \\ \Theta_2 v = \lambda v. \end{cases}$$

If $\lambda = 1$, from the first equation we have $v = 0$. Hence, the eigenvectors corresponding to the eigenvalue 1 are $\begin{pmatrix} u_l \\ 0 \end{pmatrix}$ with $\{u_l | l = 1, 2, \ldots, n\}$ being any linear independent vectors of dimension n.

If $\lambda \neq 1$, from the first equation, we have

$$u = \frac{1}{1-\lambda} \Theta_1 v,$$

where (λ, v) is an eigenvalue pair of the matrix Θ_2. Hence, the eigenvectors corresponding to these eigenvalues can be described as

$$\begin{pmatrix} \frac{1}{1-\lambda} \Theta_1 v_l \\ v_l \end{pmatrix}, \quad (l = 1, 2, \ldots, n),$$

with (λ, v_l) being the eigenvalue pair of the matrix Θ_2. □

Theorem 2.2. *Let the DHSS-like preconditioner P be defined as in (2.1). Then the degree of the minimal polynomial of the preconditioned matrix $P^{-1}\mathcal{A}$ is at most $n + 1$.*

Proof. Suppose that μ_l $(l = 1, 2, \ldots, n)$ are the eigenvalues of the matrix Θ_2, then according to the proof of Theorem 2.1, we know the characteristic polynomial of the matrix $P^{-1}\mathcal{A}$ can be expressed as

$$\Phi_{P^{-1}\mathcal{A}}(\lambda) = \det(P^{-1}\mathcal{A} - \lambda I) = (\lambda - 1)^n \prod_{l=1}^{n} (\lambda - \mu_l).$$

Let

$$\Upsilon(\lambda) = (\lambda - 1) \prod_{l=1}^{n} (\lambda - \mu_l),$$

then we have

$$\begin{aligned} \Upsilon(P^{-1}\mathcal{A}) &= (P^{-1}\mathcal{A} - I) \prod_{l=1}^{n} (P^{-1}\mathcal{A} - \mu_l I) \\ &= \begin{pmatrix} 0 & -\Theta_1 \prod_{l=1}^{n} (\Theta_2 - \mu_l I) \\ 0 & (\Theta_2 - I) \prod_{l=1}^{n} (\Theta_2 - \mu_l I) \end{pmatrix}. \end{aligned}$$

Using the Hamilton-Cayley theorem, we have $\prod_{l=1}^{n} (\Theta_2 - \mu_l I) = 0$. Hence, the degree of the minimal polynomial of the preconditioned matrix $P^{-1}\mathcal{A}$ is at most $n + 1$. □

Theorem 2.3. *The quasi-optimal parameter for the DHSS-like preconditioner is $\alpha_{opt} = \frac{\sqrt{\nu}}{\sqrt[4]{n}} \cdot \sqrt[4]{tr(K^T K)}$.*

Proof. Using the same strategy in [15], we determine the quasi-optimal parameter α by minimizing the following functional:

$$f(\alpha) = \|P - \mathcal{A}\|_F = \|R\|_F = \mathrm{tr}(R^T R) = n\alpha^2 + \frac{\nu^2}{\alpha^2}\mathrm{tr}(K^T K).$$

Letting the first derivative of $f(\alpha)$ be equal to 0 will lead to

$$f'(\alpha) = 2n\alpha - \frac{2\nu^2}{\alpha^3}\mathrm{tr}(K^T K) = 0.$$

Therefore, the quasi-optimal parameter is

$$\alpha = \alpha_{opt} = \frac{\sqrt{\nu}}{\sqrt[4]{n}} \cdot \sqrt[4]{\mathrm{tr}(K^T K)}.$$

□

3. The circulant matrix based DHSS-like precondo-tioner

From Section 2, we see that when we use the DHSS-like preconditioner, we have to solve two linear subsystems with coefficient matrices being $\alpha I + K$ and $\nu I + \alpha K^T W^{-1}$. Due to the ill-conditioned property of the matrix K, we will use the circulant matrix C (e.g., see [27]) to replace K approximately. Besides, because W is a diagonal matrix, then a scaled matrix ωI will be used instead, where ω is the average value of the elements in the matrix W, i.e.,

$$\omega = \frac{1}{n}\sum_{i=1}^{n} w_i,$$

where $W = \mathrm{diag}(w_1, w_2, \ldots, w_n)$. Hence we can obtain the CDHSS-like preconditioner P_C with

$$P_C^{-1} = \alpha \begin{pmatrix} 0 & \frac{1}{\alpha}I \\ (\alpha I + C)^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & -\frac{1}{\alpha}W \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha}I & 0 \\ 0 & -(\frac{\nu}{\alpha}\omega I + C^T)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\frac{\nu}{\alpha}I & I \end{pmatrix}.$$

Or equivalently,

$$P_C = \begin{pmatrix} W & \alpha I + C \\ \frac{\nu}{\alpha}(W - \omega I) - C^T & \nu I + \frac{\nu}{\alpha}C \end{pmatrix}. \tag{3.1}$$

Therefore, the implementation details of the CDHSS-like preconditioned method can be formulated in the following algorithm.

Algorithm 3.1. (The CDHSS-like preconditioned method)

The implementing process of the CDHSS-like preconditioner:

Given a residual vector $r = (r_1^T, r_2^T)^T \in \mathbb{R}^{2n}$. Compute z according to the following steps:

Step 1. solve $(\nu\omega I + \alpha C^T)W^{-1}\tilde{s} = \nu r_1 - \alpha r_2$ to obtain $\tilde{s}$.

Step 2. solve $(\alpha I + C)z_2 = r_1 - \tilde{s}$ to obtain z_2.

Step 3. set $z_1 = W^{-1}\tilde{s}$ to obtain the generalized residual vector $z = (z_1^T, z_2^T)^T$.

It can be seen from the implementing steps of the CDHSS-like preconditioned method that the main workloads of Step 1 and Step 2 are carrying out the FFTs. The task of Step 3 is to calculate the product of a diagonal matrix and a vector. Hence, the total computation workloads are about $O(n \log n)$.

Obviously, the convergence property of the CDHSS-like preconditioned Krylov subspace iteration method is dependent on the approximation degree of the matrix P_C to the original coefficient matrix $\mathcal{A}$, i.e., the eigenvalues clustering of the matrix $P_C^{-1}\mathcal{A}$. Because the matrix $P_C^{-1}\mathcal{A}$ can be rewritten as $P_C^{-1}PP^{-1}\mathcal{A}$, then to obtain the properties of the matrix $P_C^{-1}\mathcal{A}$ needs one to discuss the accuracy about P approximated by P_C and the clustering condition about the eigenvalues of $P^{-1}\mathcal{A}$.

What follows will be some lemmas about the properties of the Toeplitz matrix K and its Strang's approximate circulant matrix C [27].

Lemma 3.1 ([10]). *If the generating function of K is in the Wiener class, then for any $r_0 > 0$, there exist $N_0 > 0$ and $K_0 > 0$, such that for all $n > N_0$ (n is the dimension of the Toeplitz matrix K), it holds*

$$K - C = E_0 + F_0,$$

where the matrix E_0 is of low rank, satisfying rank$(E_0) < r_0$, and the matrix F_0 is of small norm, satisfying $\|F_0\| \le K_0$.

Lemma 3.2 ([18]). *If the generating function of K is in the Wiener class, then the circulant matrix C and its inverse matrix are bounded, i.e., there exists a constant $X > 0$, such that $\|C\| \le X$ (or $\|C^{-1}\| \le X$).*

Using the above two lemmas, we will describe the eigenvalues distribution of the CDHSS-like preconditioned matrix $P_C^{-1}\mathcal{A}$ in the following theorem.

Theorem 3.1. *Suppose that the generating function of the Toeplitz matrix K from the image restoration is in the Wiener class. Let the matrix P and ω be defined previously and the conditions of Theorem 2.1 be satisfied. Assume that there exists a positive constant $\bar{w}$, such that, for any $1 \le i \le n$, it holds $|w_i - \omega| \le \bar{w}$. Then the following results hold.*

(1) Given a positive parameter α, then for any $r_C > 0$, there exist $N_C > 0$ and $K_C > 0$, such that for all $n > N_C$, it holds

$$P - P_C = E_C + F_C,$$

where the matrices E_C and F_C satisfy rank$(E_C) \le r_C$ and $\|F_C\| \le K_C$.

(2) Given a positive parameter α, then for any $r > 0$, there exist $N > 0$ and $M > 0$, such that for all $n > N$, it holds

$$P_C^{-1}\mathcal{A} = P^{-1}\mathcal{A} + E + F,$$

where the matrices E and F satisfy rank$(E) \le r$ and $\|F\| \le M$.

Proof. (1) According to Lemma 3.1, we have

$$P - P_C = \begin{pmatrix} 0 & K - C \\ \frac{\nu}{\alpha}(W - \omega I) - (K - C)^T & \frac{\nu}{\alpha}(K - C) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E_0 \\ -E_0^T & \frac{\nu}{\alpha}E_0 \end{pmatrix} + \begin{pmatrix} 0 & F_0 \\ \frac{\nu}{\alpha}(W-\omega I) - F_0^T & \frac{\nu}{\alpha}F_0 \end{pmatrix}$$
$$= E_C + F_C.$$

It can be easily verified that, there exists $N_C = N_0$, such that when $n > N_C$, it holds

$$\text{rank}(E_C) = \text{rank}(E_0) + \text{rank}(E_0^T) \leq 2r_0 := r_C.$$

Hence, for any given parameter α, we have

$$\begin{aligned} \|F_C\| &\leq \|F_0\| + \|F_0^T\| + \frac{\nu}{\alpha}\|F_0\| + \frac{\nu}{\alpha}(\|W-\omega I\|) \\ &\leq (2+\frac{\nu}{\alpha})K_0 + \frac{\nu}{\alpha}(\|\bar{w}I\|) \\ &= (2+\frac{\nu}{\alpha})K_0 + \frac{\nu\bar{w}}{\alpha} := K_C. \end{aligned}$$

(2) We rewrite $P_C^{-1}\mathcal{A}$ as

$$\begin{aligned} P_C^{-1}\mathcal{A} &= P^{-1}\mathcal{A} + P_C^{-1}(P-P_C)P^{-1}\mathcal{A} \\ &= P^{-1}\mathcal{A} + P_C^{-1}(E_C+F_C)P^{-1}\mathcal{A} \\ &= P^{-1}\mathcal{A} + E + F, \end{aligned}$$

where

$$\begin{cases} E = P_C^{-1}E_C P^{-1}\mathcal{A}, \\ F = P_C^{-1}F_C P^{-1}\mathcal{A}. \end{cases}$$

Obviously, we have

$$\text{rank}(E) \leq \text{rank}(E_C) \leq r_C = r.$$

If K is in the Wiener class, then $\alpha I + K$ and $\alpha I + K^T$ are also in the Wiener class. Then according to Lemma 3.2, $\alpha I + C$ and $\frac{\nu}{\alpha}\omega I + C^T$ are bounded, i.e., $\exists X$, such that $\|(\alpha I + C)^{-1}\| \leq X$ and $\|(\frac{\nu}{\alpha}\omega I + C^T)^{-1}\| \leq X$. Therefore, it follows,

$$\begin{aligned} \|P_C^{-1}\| &= \left\|\alpha \begin{pmatrix} 0 & \frac{1}{\alpha}I \\ (\alpha I + C)^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & -\frac{1}{\alpha}W \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha}I & 0 \\ 0 & -(\frac{\nu}{\alpha}\omega I + C^T)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\frac{\nu}{\alpha}I & I \end{pmatrix}\right\| \\ &= \left\|\begin{pmatrix} 0 & \frac{1}{\alpha}I \\ (\alpha I + C)^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & -\frac{1}{\alpha}W \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha}I & 0 \\ 0 & -(\frac{\nu}{\alpha}\omega I + C^T)^{-1} \end{pmatrix} \begin{pmatrix} \alpha I & 0 \\ -\nu I & \alpha I \end{pmatrix}\right\| \\ &\leq (\frac{1}{\alpha} + X)(\frac{\omega_{\max}}{\alpha} + 1)(\frac{1}{\alpha} + X)(\alpha + \nu) \\ &= (\frac{1}{\alpha} + X)^2(\frac{\omega_{\max}}{\alpha} + 1)(\alpha + \nu), \end{aligned}$$

where

$$\omega_{\max} = \max_{1 \leq i \leq n}\{w_i | W = \text{diag}(w_1, w_2, \ldots, w_n)\}.$$

If the conditions of Theorem 2.1 are satisfied, let $\Lambda = \max_{\lambda \in sp(P^{-1}\mathcal{A})}\{|\lambda|\}$, then we have

$$\|F\| \leq \|P_C^{-1}\| \cdot \|F_C\| \cdot \|P^{-1}\mathcal{A}\| \leq (\frac{1}{\alpha} + X)^2(\frac{\omega_{\max}}{\alpha} + 1)(\alpha + \nu) \cdot K_C \cdot \Lambda := M.$$

Hence, it follows the result that, for any $r > 0$, there exist $N = N_C > 0$ and $M > 0$, such that for all $n > N$, it holds

$$P_C^{-1}\mathcal{A} = P^{-1}\mathcal{A} + E + F,$$

where E and F are matrices satisfying $\operatorname{rank}(E) \le r$ and $\|F\| \le M$. □

Remark 3.1. Theorem 3.1 indicates that the CDHSS-like preconditioned matrix $P_C^{-1}\mathcal{A}$ is a good approximation to the DHSS-like preconditioned matrix $P^{-1}\mathcal{A}$ in terms of both rank and norm. Hence, if the eigenvalues of $P^{-1}\mathcal{A}$ are tightly clustered and its eigenvectors are well conditioned, then the Krylov subspace iteration methods, when incorporated with the CDHSS-like preconditioner, are expected to converge to the exact solution of the linear system (1.1) accurately and stably. In fact, from the results obtained in Section 2, we see that the eigenvalues of $P^{-1}\mathcal{A}$ are tightly clustered and the corresponding eigenvectors are well conditioned.

4. Numerical results

In this section, we are going to test the feasibility and the efficiency of the CDHSS-like preconditioned method for solving the linear system (1.1). We use the CDHSS-like preconditioner to improve the convergence property of the generalized minimum residual (GMRES) method [26]. Comparisons between the conjugate gradient (CG) method and the modified HSS (MHSS) preconditioner [24] are given from the point of view of the number of iteration counts (denoted as "IT") and CPU time (denoted as "CPU"). Our experiments are carried out in MATLAB R2017a on Intel(R) Core(TM) CPU 3.4Ghz and 8.00 GB of RAM, with machine precision 10^{-16}.

In our implementations, the initial guess $u^{(0)} = (y^{(0)}, x^{(0)})^T$ is chosen to be zero vector and the stopping criteria for all the methods are

$$\frac{\|f - Wy^{(k)} - Kx^{(k)}\|_2 + \| - K^T y^{(k)} + \nu x^{(k)}\|_2}{\|f\|_2} \le 10^{-6},$$

where $u^{(k)} = (y^{(k)}, x^{(k)})^T$ is the current approximation. The maximum iteration count 1000 is set in all the experiments.

Example 4.1 ([7]). Consider the one-dimensional example of linear system (1.1) with K being a square Toeplitz matrix defined by

(i) Case 1: $K = (t_{ij}) \in \mathbb{R}^{n\times n}$ with $t_{ij} = \frac{1}{\sqrt{|i-j|+1}}$;

(ii) Case 2: $K = (t_{ij}) \in \mathbb{R}^{n\times n}$ with $t_{ij} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{|i-j|^2}{2\sigma^2}}$.

In the tests, Ξ is set to be a positive diagonal random matrix generated by MATLAB and its diagonal entries are scaled so that the condition number of the diagonal matrix is around 10^3. The regularization parameter ν is set to be 0.001.

The theoretical optimal parameters (marked by '$*$') and the experimental optimal parameters (marked by '$\star$') of the CDHSS-like methods are listed in Table 1. By using the parameters in Table 1, we obtain the IT and CPU correspondingly, for the tested methods, i.e., the CG method, the unpreconditioned GMRES method, the MHSS preconditioned GMRES method (denoted as 'MHSSPre'), the

CDHSS-like iteration method (denoted as ‘CDHSS-like’) and the CDHSS-like preconditioned GMRES method (denoted as ‘CDHSS-likePre’). All the results are reported in Table 2 for both Case 1 and Case 2.

From Table 2, we find that when the mesh grid increases, the iteration counts of the CG method and the unpreconditioned GMRES method grow rapidly. However, for both the MHSS preconditioned method and the CDHSS-like methods, the iteration counts keep steady. But the CDHSS-like preconditioned GMRES method uses the least iteration counts. Besides, from the results, we also find that the CDHSS-like preconditioned method outperforms other methods with respect to both IT and CPU. Therefore, we can draw a conclusion that, for the linear systems of the form (1.1), the CDHSS-like preconditioned method would be a good choice for solving this class of block two-by-two linear systems.

To further illustrate the efficiency of the CDHSS-like methods, we plot the eigenvalues distribution for the normal matrix (i.e., $K^T\Xi^T\Xi K+\nu I$), the original coefficient matrix (i.e., the coefficient matrix of (1.1)), the MHSS preconditioned matrix (i.e., $P_2^{-1}\mathcal{A}$) and the CDHSS-like preconditioned matrix (i.e., $P_C^{-1}\mathcal{A}$) in Fig. 1 – Fig. 4 for $n = 1024$. From these figures, we see that the original coefficient matrices for both case 1 and case 2 are ill-conditioned. However, by using the MHSS preconditioner and the CDHSS-like preconditioner, the coefficient matrices become better conditioned (see, e.g., Fig. 3 and Fig. 4). Comparing Fig. 3 with Fig. 4, we find that the eigenvalues of the CDHSS-like preconditioned matrices are more clustered than the eigenvalues of the MHSS preconditioned matrices. Hence, the CDHSS-like preconditioned matrix is much better conditioned.

Table 1. The optimal parameters of Example 4.1.

h : Method	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}
(Case 1)					
MHSS	32.6	47.7	69.3	100	144
CDHSS-like*	0.05449	0.05634	0.05807	0.05968	0.0612
CDHSS-like⋆	[0.02,0.22]	[0.03,0.21]	[0.02,0.22]	[0.02,0.22]	[0.02,0.22]
(Case 2)					
MHSS	0.0898	0.0898	0.0898	0.0898	0.0898
CDHSS-like*	0.020712	0.020716	0.020718	0.020719	0.02072
CDHSS-like⋆	[0.012,0.026]	[0.018,0.022]	[0.018,0.022]	[0.018,0.022]	[0.018,0.022]

To further illustrate the efficiency of the CDHSS-like iteration method, we will consider an image restoration problem with Gaussian noise in the next example.

Example 4.2. Consider the image restoration problem

$$f = Kx + G\eta,$$

where f and x represent the observed image and the original image, respectively. Here K is the blurring matrix generated by the discrete point spread function

$$k(x,y) = \exp(-\frac{x^2+y^2}{2})$$

Table 2. The numerical results of Example 4.1.

h	Method:	CG	GMRES	MHSSPre	CDHSS-like	CDHSS-likePre
(Case 1)						
2^{10}	IT	64	132	8	6	6
	CPU	0.018	0.148	0.01	0.006	0.006
2^{11}	IT	40	172	8	6	6
	CPU	0.026	0.753	0.014	0.009	0.009
2^{12}	IT	51	213	8	6	6
	CPU	0.047	1.612	0.018	0.014	0.014
2^{13}	IT	70	269	8	6	6
	CPU	0.109	4.062	0.044	0.036	0.036
2^{14}	IT	83	317	9	6	6
	CPU	0.245	8.48	0.093	0.082	0.082
(Case 2)						
2^{10}	IT	262	488	24	13	11
	CPU	0.066	1.118	0.025	0.013	0.01
2^{11}	IT	338	641	25	13	11
	CPU	0.132	13.068	0.042	0.021	0.016
2^{12}	IT	476	732	25	13	11
	CPU	0.46	24.455	0.084	0.042	0.038
2^{13}	IT	375	640	25	13	11
	CPU	0.647	35.922	0.175	0.086	0.076
2^{14}	IT	633	649	25	13	11
	CPU	1.989	40.437	0.255	0.127	0.102

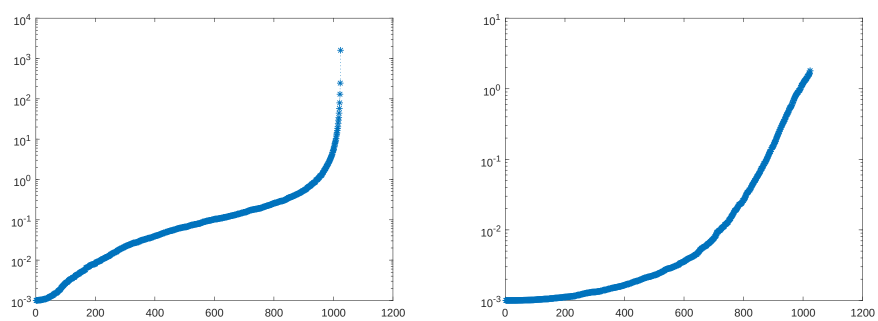

Figure 1. Eigenvalues distribution for the normal equation (Case 1: left; Case 2: right).

and η is set to be the Gaussian white noise. We use the Gaussian filter G to generate the colored noise $G\eta$. Then the image restoration problem can be reformulated into the weighted Toeplitz regularized least squares problem [17, 20]

$$\min_x \|Kx - f\|^2_{(G^TG)^{-1}} + \nu\|x\|_2^2.$$

Or equivalently, it follows the linear system (1.1), where $W = G^TG$.

This example is a modification from the second example in [24]. We test two images: `House` and `Lena`. The noise η is set to be the Gaussian white noise with

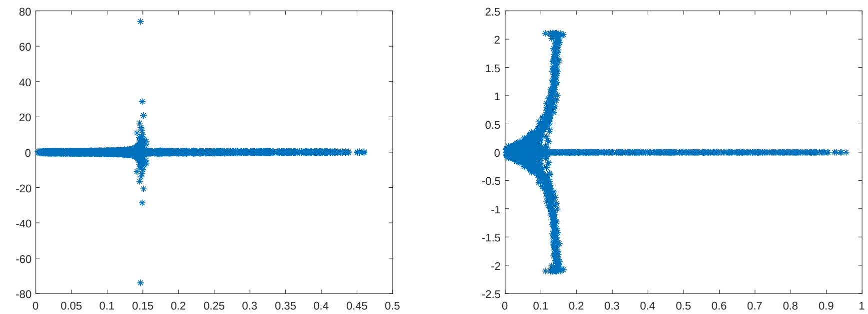

Figure 2. Eigenvalues distribution for the original coefficient matrix (Case 1: left; Case 2: right).

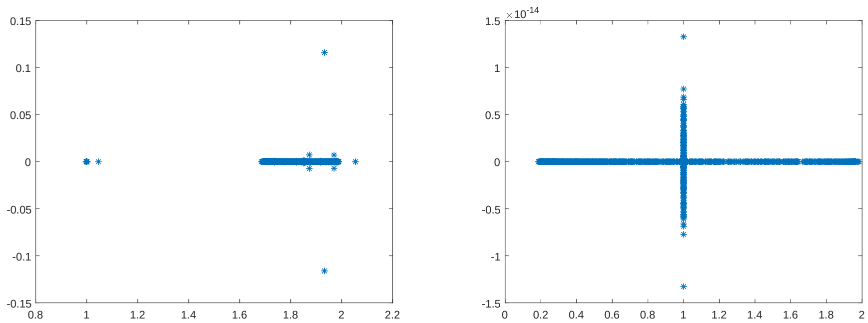

Figure 3. Eigenvalues distribution for the MHSS preconditioned matrix (Case 1: left; Case 2: right).

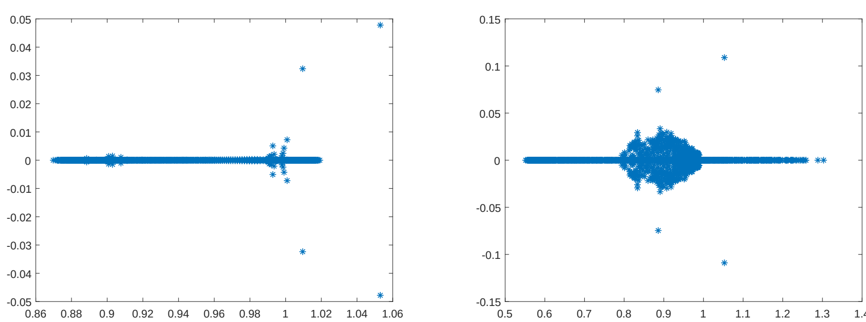

Figure 4. Eigenvalues distribution for the CDHSS-like preconditioned matrix (Case 1: left; Case 2: right).

signal-to-noise ratios (SNR) of 20dB, 30dB and 40 dB, respectively. The regularization parameters used in this example are $\{1.1 \times 10^{-4}, 3 \times 10^{-5}, 1.2 \times 10^{-5}\}$ (`House`) and $\{3 \times 10^{-4}, 5.5 \times 10^{-5}, 1.1 \times 10^{-6}\}$ (`Lena`), for SNR being 20dB, 30dB and 40 dB, respectively. The Gaussian filter G is artificial by a positive diagonal random matrix generated by MATLAB and its diagonal entries are scaled. We use the quasi-optimal parameters according to the theoretical analysis previously. The results of IT, CPU and relative error (RES) are reported in Table 3, in which the

RES of the restored images is used to measure image quality, defined as

$$\mathrm{RES} = \frac{\|x^{(k)} - x^*\|_2}{\|x^*\|_2},$$

where x^* is the original image and $x^{(k)}$ is the restored image. In this example, we compare the CDHSS-like preconditioner with the HSS preconditioner P_1 and the MHSS preconditioner P_2. The theoretical optimal parameters α according to [7,24] and Theorem 2.3 are used. The corresponding experimental results are shown in Table 3.

From Table 3, we find that the relative errors of the three preconditioned methods are almost the same. However, the CDHSS-like preconditioned method needs only a few steps to achieve the high quality restored image. Furthermore, to show the quality of the restored images, we plot the original images in Fig. 5. The noisy images and restored images with respect to different images and different SNR are shown in Fig. 6–Fig. 11. From Fig. 6–Fig. 11, we see that the CDHSS-like preconditioner shows good performance and the corresponding images are of high quality.

Therefore, we can draw a conclusion that the CDHSS-like preconditioner would be a good choice for solving the linear system of the form (1.1).

Table 3. The numerical results of Example 4.2.

SNR		20 dB			30 dB			40 dB		
	Pre.	IT	CPU	RES	IT	CPU	RES	IT	CPU	RES
House	HSS	58	12.48	0.243	57	12.41	0.243	59	12.51	0.242
	MHSS	17	4.77	0.242	17	4.77	0.242	17	4.76	0.242
	CDHSS-like	8	0.027	0.242	8	0.027	0.242	8	0.027	0.241
Lena	HSS	56	12.37	0.304	57	12.41	0.298	58	12.48	0.296
	MHSS	20	5.15	0.295	21	5.56	0.297	21	5.56	0.295
	CDHSS-like	8	0.027	0.291	8	0.027	0.295	8	0.027	0.293

Figure 5. Original images.

5. Conclusions

In this work, we construct an HSS-like iteration method and a DHSS-like preconditioner for weighted Toeplitz least squares computation from image restoration.

Figure 6. The noisy image and the restored images for SNR=20.

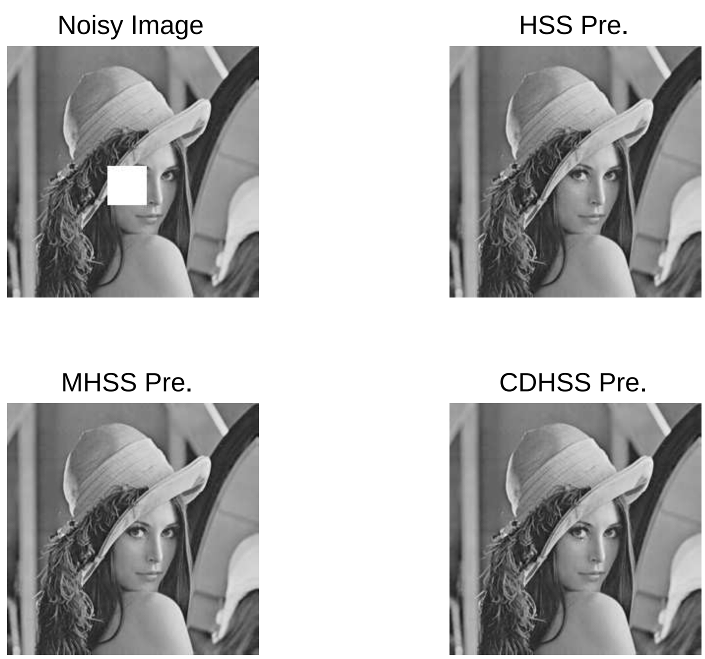

Figure 7. The noisy image and the restored images for SNR=30.

Figure 8. The noisy image and the restored images for SNR=40.

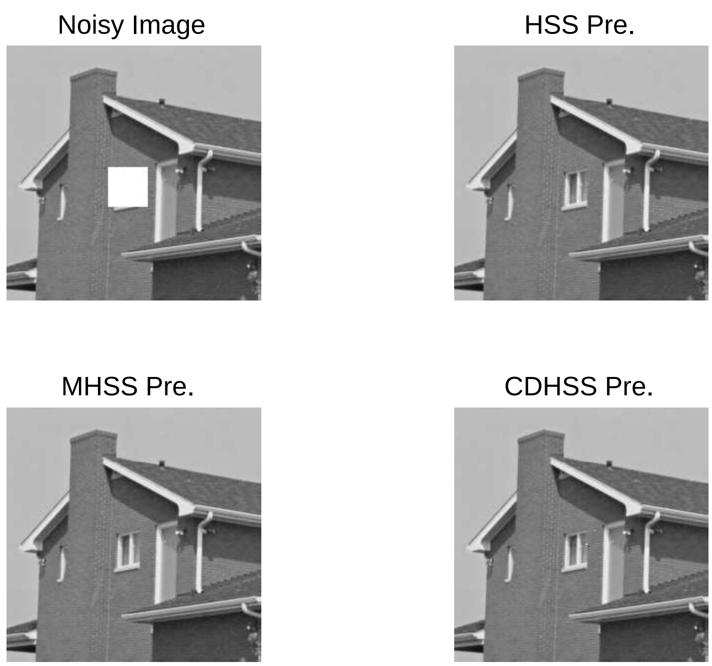

Figure 9. The noisy image and the restored images for SNR=20.

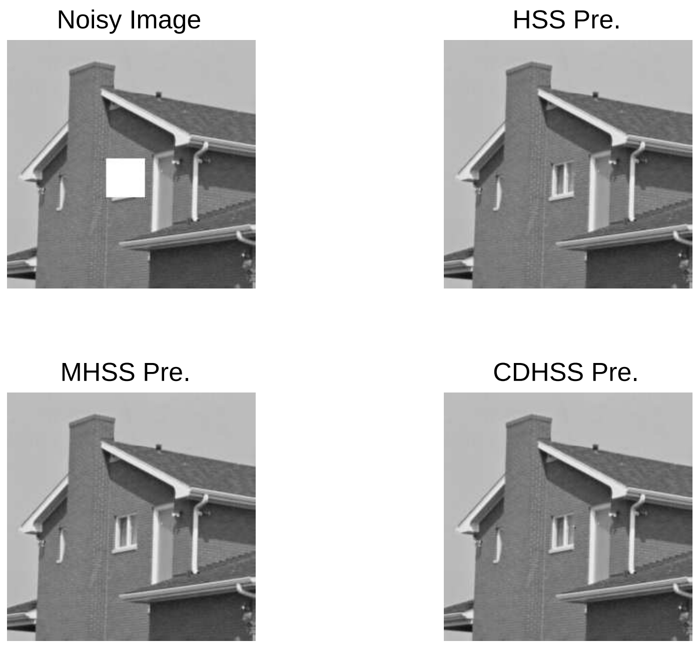

Figure 10. The noisy image and the restored images for SNR=30.

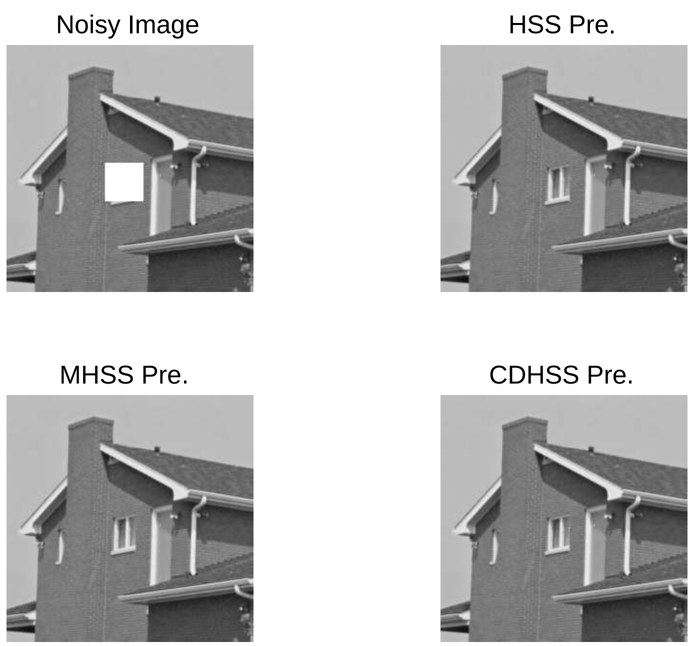

Figure 11. The noisy image and the restored images for SNR=40.

The spectral properties and the quasi-optimal parameters of the DHSS-like preconditioned matrix are investigated in detail. As the implementing of the DHSS-like preconditioned method needs one to solve two linear subsystems, which are ill-conditioned. Hence, the circulant matrix approximate is used. We then naturally obtain a circulant matrix-based DHSS-like preconditioner. Theoretically analysis shows that the eigenvalues of the CDHSS-like preconditioned matrix are clustered around 1. Implementations for the image restoration problems are made to verify the correctness of the theoretical results and the efficiency of the CDHSS-like iteration method and the CDHSS-like preconditioner.

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