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LINEAR 2-ARBORICITY OF PLANAR GRAPHS WITH MAXIMUM DEGREE NINE*

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Abstract The linear 2-arboricity $\mathrm{la}_2(G)$ of a graph G is the least integer k such that G can be partitioned into k edge-disjoint forests, whose component trees are paths of length at most 2. In this paper, we show that every planar graph G with maximum degree $\Delta = 9$ has $\mathrm{la}_2(G) \leq 8$, which extends a known result that every planar graph G with $\Delta \geq 10$ has $\mathrm{la}_2(G) \leq \Delta - 1$.

Keywords Plane graph, linear 2-arboricity, maximum degree.

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1. Introduction

All graphs considered in this paper are finite and simple. A graph G is *planar* if it can be embedded in the plane such that any two edges intersect only at their ends. Given a planar graph G, we use $V(G), E(G), \Delta(G)$, and $\delta(G)$ to denote its vertex set, edge set, maximum degree, and minimum degree in G, respectively. If no confusion arises, we abbreviate $\Delta(G)$ to Δ.

An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \ldots, G_m$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. A *linear k-forest* is a graph whose components are paths of length at most k. The *linear k-arboricity* of G, denoted by $\mathrm{la}_k(G)$, is the least integer m such that G can be edge-partitioned into m linear k-forests.

It is obvious that $\mathrm{la}_k(G) \geq \mathrm{la}_{k+1}(G)$ for any $k \geq 1$. Furthermore, $\mathrm{la}_1(G)$ is the edge chromatic number $\chi'(G)$ of G; $\mathrm{la}_\infty(G)$ corresponds to the linear arboricity $\mathrm{la}(G)$ of G.

In 1982, Habib and Peroche [6] introduced the concept of linear k-arboricity and put forward to the following conjecture:

Conjecture 1.1.

$$\mathrm{la}_k(G) \leq \begin{cases} \left\lceil \frac{n\Delta+1}{2\lfloor \frac{kn}{k+1} \rfloor} \right\rceil & \text{if } \Delta \neq n-1, \\ \left\lceil \frac{n\Delta}{2\lfloor \frac{kn}{k+1} \rfloor} \right\rceil & \text{if } \Delta = n-1. \end{cases}$$

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The linear k-arboricity of graphs has been extensively investigated in past decades, see Aldred and Wormald [1], Bermond et al. [2], Chang et al. [3], Chen and Huang [4], Jackson and Wormald [7], and Thomassen [12].

When $k = 2$, Conjecture 1.1 can be expressed as follows:

Conjecture 1.2.

$$\mathrm{la}_2(G) \leq \begin{cases} \left\lceil \frac{n\Delta+1}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil & \text{if } \Delta \neq n-1, \\ \left\lceil \frac{n\Delta}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil & \text{if } \Delta = n-1. \end{cases}$$

Suppose that G is a planar graph. Let $g(G)$ denote its girth, i.e., the length of a shortest cycle in G. In 2004, Lih, Tong and Wang [8] proved that (i) $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 12$; (ii) $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 6$ if $g(G) \geq 4$; (iii) $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 2$ if $g(G) \geq 5$; and (iv) $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 1$ if $g(G) \geq 7$. In 2009. Ma, Wu and Hu [11] proved that $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 6$ if G contains no 5-cycles or 6-cycles. When G is outerplanar, Lih, Tong and Wang [9] showed that $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 1$ and this upper bound is tight. A big breakthrough about the linear 2-arboricity of planar graphs is that $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 6$ for any planar graph G, see [15] and [16].

A graph G is called *toroidal* if it can be embedded in the torus such that any two edges intersect only at their ends. Wang et al. [14] showed that if G is a toroidal graph, then $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 7$. A graph is called *1-planar* if it can be drawn in the plane so that each edge is crossed by at most one other edge. Recently, Liu et al. [10] proved that every 1-planar graph G satisfies $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 14$.

To obtain a better upper bound of linear 2-arboricity of planar graphs, Wang et al. [16] proved a key and interesting result: every planar graph G with $\Delta \geq 10$ has $\mathrm{la}_2(G) \leq \Delta - 1$. The purpose of this paper is to extend this result by showing the following:

Theorem 1.1. *If G is a planar graph with $\Delta \geq 9$, then $\mathrm{la}_2(G) \leq \Delta - 1$.*

Corollary 1.1. *Every planar graph G with $\Delta = 9$ has $\mathrm{la}_2(G) \leq 8$.*

2. Preliminary

Now we give some basic notions that needed in the sequel. A *plane* graph is a particular drawing in the Euclidean plane of a planar graph. For a plane graph G, let $F(G)$ denote the face set of G. For a face $f \in F(G)$, let $b(f)$ denote the boundary walk of f and write $f = [v_1 v_2 \cdots v_k]$ if $v_1, v_2, \cdots, v_k$ are the vertices of $b(f)$ in clockwise order, where repeated occurrences of a vertex are allowed. The *degree* of a face is the number of edge-steps in its boundary walk. Note that each cut-edge is counted twice. Let $d_G(x)$ denote the degree of x in G for $x \in V(G) \cup F(G)$. If a vertex v is degree k (at most k, at least k, respectively), then it is called a *k-vertex* (*k^--vertex, k^+-vertex,* respectively). Similarly, we can define *k-face, k^--face* and *k^+-face.* For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of neighbors of v in G. Since G is assumed to be simple, we derive that $d_G(v) = |N_G(v)|$. For an element $x \in V(G) \cup F(G)$ and an integer $i \geq 1$, let $n_i(x)$ ($m_i(x)$, respectively) denote the number of i-vertices (i-faces, respectively) adjacent or incident to x. For a vertex $v \in V(G)$, let $t(v)$ denote the number of 3-faces that are incident to v.

To show Theorem 1.1, we consider the linear edge-coloring of a graph G, i.e., a mapping ϕ from $E(G)$ to the color set C such that every color class induces a subgraph whose components are paths of length at most 2. We call such coloring *linear-k-coloring* of G if C contains k colors. Clearly, a graph G has linear 2-arboricity at most k if and only if G is linear-k-colorable.

A function L is called an *list assignment* for the graph G if each edge e of G is assigned by a list $L(e)$ of possible colors. If G has a linear edge-coloring ϕ such that $\phi(e) \in L(e)$ for all edges e, then we say that ϕ is an L-linear edge-coloring of G.

Let S_m denote a star consisting of m edges $e_1, e_2, \ldots, e_m$, where $m \geq 2$. The following Lemma 2.1 appeared in [15].

Lemma 2.1 (Wang, [15]). *Let S_m be a star defined as above and L be a list assignment for the edges in S_m satisfying the following conditions, then S_m admits an L-linear edge-coloring.*

(1) $m = 2$, *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$;
(2) $m = 3$, *and* $|L(e_3)| \geq 2$ *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$;
(3) $m = 4$, *and* $|L(e_i)| \geq 2$ *for* $i = 3, 4$, *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$;
(4) $m = 5$, *and* $|L(e_5)| \geq 3$, $|L(e_i)| \geq 2$ *for* $i = 3, 4$, *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$;
(5) $m = 6$, *and* $|L(e_i)| \geq 3$ *for* $i = 5, 6$, $|L(e_i)| \geq 2$ *for* $i = 3, 4$, *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$.

Lemma 2.2 (Bermond et al., [2]). *For any graph G,* $\mathrm{la}_2(G) \leq \Delta$.

In addition, the following relation is an easy observation:

Lemma 2.3. *If a graph G can be edge-partitioned into two subgraphs G_1 and G_2, then* $\mathrm{la}_2(G) \leq \mathrm{la}_2(G_1) + \mathrm{la}_2(G_2)$.

3. Proof of Theorem 1.1

In fact, it suffices to show the following theorem:

Theorem 3.1. *If G is a planar graph with $\Delta \leq 9$, then* $\mathrm{la}_2(G) \leq 8$.

Proof. If $\Delta \leq 8$, then the result holds by Lemma 2.2. So suppose that $\Delta = 9$. Assume to the contrary that the result is not true. Let G be a minimum counterexample such that $\sigma(G) = |V(G)| + |E(G)|$ is the least possible. So G is connected and $\delta(G) \geq 1$. For any subgraph G' of G with $\sigma(G') < \sigma(G)$, H has a linear-8-coloring ϕ using the color set $C = \{1, 2, \ldots, 8\}$.

For a vertex $v \in V(G')$, we use $C(v)$ to denote the set of colors used in edges incident to v in G'. For an edge $xy \in E(G) \setminus E(G')$, let $C(xy)$ denote the set of colors used in the edges incident to x or y in G'. That is, $C(xy) = C(x) \cup C(y)$. For a vertex $v \in V(G')$, we use $S(v)$ to denote the sequence of colors assigned to the edges incident to v in G'. For example, $S(v) = (1, 1, 2, 3, 4, 5)$ means that the color 1 appears twice, and each of the colors $2, 3, 4, 5$ appears exactly once on the edges incident to v.

Lemma 3.1. *G contains no an edge xy such that $d_G(x) + d_G(y) \leq 9$.*

Proof. Suppose that G contains such an edge xy. Let $G' = G - xy$. Then H has a linear-8-coloring using the color set C. Since $|C(xy)| \leq |C(x)| + |C(y)| \leq$

$d_{G'}(x) + d_{G'}(y) = d_G(x) - 1 + d_G(y) - 1 \leq 9 - 2 = 7 < 8 = |C|$, there is a color $a \in C \setminus C(xy)$, which can be properly assigned to the edge xy. Thus, ϕ is extended to a linear-8-coloring of G, contradicting the minimality of G. □

Lemma 3.2. *Let $v \in V(G)$ be a k-vertex with $5 \leq k \leq 9$ and $v_1, v_2, \ldots, v_k$ be the neighbors of v with $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_k)$. Then the following hold.*

(1) $n_{10-k}(v) \leq 1$.

(2) *If $n_{10-k}(v) = 1$, then $n_{11-k}(v) \leq 1$; If $n_{11-k}(v) = 1$, then $n_{12-k}(v) \leq 1$.*

(3) $n_{10-k}(v) + n_{11-k}(v) \leq 3$; *And if $n_{10-k}(v) + n_{11-k}(v) = 3$, then $n_{12-k}(v) = 0$.*

(4) *If $k = 9$, then $n_1(v) + n_2(v) + n_3(v) \leq 5$; And if $n_1(v) + n_2(v) \geq 3$, then $n_3(v) = 0$.*

Proof. By Lemma 3.1, $d_G(v_i) \geq 10 - k$ for all $1 \leq i \leq k$.

(1) Assume to the contrary that $n_{10-k}(v) \geq 2$, say $d_G(v_1) = d_G(v_2) = 10-k$. Let $G' = G - \{vv_1, vv_2\}$. By the minimality of G, G' has a linear-8-coloring ϕ using the color set C. Since $|C \setminus C(vv_i)| \geq 8-(d_{G'}(v)+d_{G'}(v_i)) = 8-(k-2)-(10-k-1) = 1$ for $i = 1, 2$, by Lemma 2.1(1), ϕ can be extended to a linear-8- coloring of G, a contradiction.

(2) First suppose that $n_{10-k}(v) = 1$ and $n_{11-k}(v) \geq 2$, say $d_G(v_1) = 10 - k$ and $d_G(v_2) = d_G(v_3) = 11 - k$. Let $G' = G - \{vv_1, vv_2, vv_3\}$, which has a linear-8-coloring ϕ. Since $|C \setminus C(vv_1)| \geq 8-(d_{G'}(v)+d_{G'}(v_1)) = 8-(k-3)-(10-k-1) = 2$, and $|C \setminus C(vv_i)| \geq 8-(d_{G'}(v)+d_{G'}(v_i)) = 8-(k-3)-(11-k-1) = 1$ for $i = 2, 3$, we can extend ϕ to a linear-8-coloring of G by Lemma 2.1(2), a contradiction.

Next suppose that $n_{10-k}(v) = n_{11-k}(v) = 1$, and $n_{12-k}(v) \geq 2$, say $d_G(v_1) = 10 - k$, $d_G(v_2) = 11 - k$, and $d_G(v_3) = d_G(v_4) = 12 - k$. Let $G' = G - \{vv_1, vv_2, vv_3, vv_4\}$, which has a linear-8-coloring ϕ. Because $|C \setminus C(vv_1)| \geq 8 - (d_{G'}(v) + d_{G'}(v_1)) = 3$, $|C \setminus C(vv_2)| \geq 2$, and $|C \setminus C(vv_i)| \geq 1$ for $i = 3, 4$, Lemma 2.1(2) asserts ϕ can be extended to G, a contradiction.

(3) Suppose that $n_{10-k}(v)+n_{11-k}(v) \geq 3$ and $n_{10-k}(v)+n_{11-k}(v)+n_{12-k}(v) \geq 4$, say $10 - k \leq d_G(v_i) \leq 11 - k$ for $i = 1, 2, 3$, and $10 - k \leq d_G(v_4) \leq 12 - k$. Let $G' = G - \{vv_1, vv_2, vv_3, vv_4\}$, which has a linear- 8-coloring ϕ. Since $|C \setminus C(vv_i)| \geq 8-(d_{G'}(v)+d_{G'}(v_i)) = 8-(k-4)-(11-k-1) = 2$ for $i = 1, 2, 3$, and $|C \setminus C(vv_4)| \geq 8 - (d_{G'}(v) + d_{G'}(v_4)) = 8 - (k - 4) - (12 - k - 1) = 1$, ϕ can be extended to a linear-8-coloring of G by Lemma 2.1(3), a contradiction.

(4) Suppose that $n_1(v) + n_2(v) + n_3(v) \geq 6$, say $1 \leq d_G(v_i) \leq 3$ for $i = 1, 2, 3, 4, 5, 6$. Let $G' = G - \{vv_1, vv_2, vv_3, vv_4, vv_5, vv_6\}$, which has a linear- 8-coloring ϕ. Since $|C \setminus C(vv_i)| \geq 8 - (d_{G'}(v) + d_{G'}(v_i)) = 8 - 3 - 2 = 3$ for $i = 1, 2, 3, 4, 5, 6$, by Lemma 2.1(5), we can extend ϕ to a linear-8-coloring of G, a contradiction.

Moreover, if $n_1(v) + n_2(v) \leq 3$ and $n_3(v) \geq 1$, say $d_G(v_i) \leq 2$ for $i = 1, 2, 3$ and $d_G(v_4) \leq 3$, then we can also get a linear-8-coloring of G, which is a contradiction. Otherwise, let $G' = G - \{vv_1, vv_2, vv_3, vv_4\}$, which has a linear-8-coloring ϕ. Since $|C \setminus C(vv_i)| \geq 2$ for $i = 1, 2, 3$, and $|C \setminus C(vv_4)| \geq 1$, ϕ can be extended to G by Lemma 2.1(3). □

Let H be a largest component of the graph which is obtained by removing all 1-vertices and 2-vertices of G. Then H is a connected plane graph with $\Delta(H) \leq 9$. For an edge $uv \in E(H)$, we call it a $(d_H(u), d_H(v))$-*edge*. For example, if $d_H(u) = 5$ and $d_H(v) = 5$, then uv is a $(5, 5)$-edge.

Lemma 3.3. $\delta(H) \geq 3$.

Proof. For each vertex $v \in V(H)$, we have $v \in V(G)$ with $d_G(v) \geq 3$ and $d_H(v) = d_G(v) - n_1(v) - n_2(v)$. If $d_G(v) \leq 7$, then $n_1(v) = n_2(v) = 0$ by Lemma 3.1, and hence $d_H(v) = d_G(v) \geq 3$. If $d_G(v) = 8$, then $n_1(v) = 0$ by Lemma 3.1 and $n_2(v) \leq 1$ by Lemma 3.2(1). Thus, $d_H(v) = d_G(v) - n_1(v) - n_2(v) \geq 8 - 1 = 7$. If $d_G(v) = 9$, then $n_1(v) + n_2(v) \leq 3$ by Lemma 3.2(3), henceforth $d_H(v) = d_G(v) - n_1(v) - n_2(v) \geq 9 - 3 = 6$. □

Lemma 3.4. *If $d_H(v) \leq 5$, then $d_H(v) = d_G(v)$.*

Proof. Suppose that $d_G(v) > d_H(v)$. Since $d_H(v) = d_G(v) - n_1(v) - n_2(v)$, we derive that $n_1(v) + n_2(v) > 0$. If $n_1(v) > 0$, then Lemmas 3.1 and 3.2 imply that $d_G(v) = 9$, $n_1(v) = 1$ and $n_2(v) \leq 1$. Hence $d_H(v) = d_G(v) - n_1(v) - n_2(v) \geq 9 - 1 - 1 = 7$, a contradiction. Otherwise, suppose that $n_1(v) = 0$ and $n_2(v) > 0$. By Lemma 3.1, $8 \leq d_G(v) \leq 9$. If $d_G(v) = 8$, then $n_2(v) \leq 1$ by Lemma 3.2(1), and hence $d_H(v) = d_G(v) - n_2(v) \geq 8 - 1 = 7$, a contradiction. If $d_G(v) = 9$, then $n_2(v) \leq 3$ by Lemma 3.2(3). Thus, $d_H(v) = d_G(v) - n_2(v) \geq 9 - 3 = 6$, also a contradiction. □

The following useful Remark follows easily from Lemmas 3.1 and 3.2:

Remark 3.1. Let $v \in V(H)$. The following statements hold.

(a) If $d_H(v) = 6$, then $d_G(v) = 6$; or $d_G(v) = 9$ with $n_2(v) = 3$.

(b) If $d_H(v) = 7$, then $d_G(v) = 7$; or $d_G(v) = 8$ with $n_2(v) = 1$; or $d_G(v) = 9$ with $n_1(v) + n_2(v) = 2$.

(c) If $d_H(v) = 8$, then $d_G(v) = 8$; or $d_G(v) = 9$ with $n_1(v) = 1$ or $n_2(v) = 1$.

For a vertex $v \in V(H)$ and an integer $i \geq 3$, let $n'_i(v)$ denote the number of i-vertices adjacent to v in H, and $m'_i(v)$ denote the number of i-faces incident to v in H. By Lemma 3.4, $n'_i(v) = n_i(v)$ for $3 \leq i \leq 5$.

Lemma 3.5. *If $v \in V(H)$ with $d_H(v) = 6$, then $n'_3(v) = 0$.*

Proof. By Remark 3.1(a), if $d_G(v) = 6$, then $n_1(v) = n_2(v) = n_3(v) = 0$ by Lemma 3.1, and hence $n'_3(v) = 0$. If $d_G(v) = 9$ and $n_2(v) = 3$, then by Lemma 3.2(4), $n_3(v) = 0$. This shows that $n'_3(v) = 0$. □

Lemma 3.6. *If $v \in V(H)$ with $d_H(v) = 7$, then the following statements hold.*

(a) $n'_3(v) \leq 1$.

(b) *If $n'_3(v) = 1$, then $n'_4(v) \leq 1$.*

(c) *If $n'_3(v) = 1$ and $n'_4(v) = 1$, then $n'_5(v) \leq 1$.*

Proof. Let $v_1, v_2, \ldots, v_k$ be the neighbors of v with $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_k)$. If $d_G(v) = 7$, then by Lemma 3.2, (a), (b) and (c) can be established. If $d_G(v) = 8$ and $n_2(v) = 1$, then by Lemma 3.2(2), (a) and (b) hold obviously. To show (c), assume that $n'_5(v) \geq 2$, say $d_G(v_1) = 2$, $d_G(v_2) = 3$, $d_G(v_3) = 4$ and $d_G(v_i) = 5$ for $i = 4, 5$. Let $G' = G - \{vv_1, vv_2, vv_3, vv_4, vv_5\}$, which has a linear-8-coloring ϕ. It is easy to check that $|L(vv_1)| \geq 8 - 3 - (d_G(v_1) - 1) = 4$, $|L(vv_2)| \geq 8 - 3 - (d_G(v_2) - 1) = 3$, $|L(vv_3)| \geq 8 - 3 - (d_G(v_3) - 1) = 2$ and $|L(vv_i)| \geq 8 - 3 - (d_G(v_i) - 1) = 1$ for $i = 4, 5$. By Lemma 2.1(4), ϕ can be extended to G, a contradiction.

So assume that $d_G(v) = 9$ and $n_1(v) + n_2(v) = 2$.

To show (a), suppose that $n'_3(v) \geq 2$, say $d_G(v_i) \leq 2$ for $i = 1, 2$ and $d_G(v_j) = 3$ for $j = 3, 4$. Let $G' = G - \{vv_1, vv_2, vv_3, vv_4\}$, which has a linear-8-coloring ϕ. Since $|L(vv_i)| \geq 8-5-(d_G(v_i)-1) = 2$ for $i = 1, 2$, and $|L(vv_j)| \geq 8-5-(d_G(v_j)-1) = 1$ for $j = 3, 4$, ϕ can be extended to a linear-8-coloring of G by Lemma 2.1(3), a contradiction.

To show (b), suppose that $n'_4(v) \geq 2$, say $d_G(v_i) \leq 2$ for $i = 1, 2$, $d_G(v_3) = 3$ and $d_G(v_j) = 4$ for $j = 4, 5$. Let $G' = G - \{vv_1, vv_2, vv_3, vv_4, vv_5\}$. Similar to the proof of (a), it yields that $|L(vv_i)| \geq 3$ for $i = 1, 2$, $|L(vv_3)| \geq 2$ and $|L(vv_j)| \geq 1$ for $j = 4, 5$. By Lemma 3.1(4), ϕ can be extended to G, a contradiction.

To show (c), suppose that $n'_5(v) \geq 2$, say $d_G(v_i) \leq 2$ for $i = 1, 2$, $d_G(v_3) = 3$, $d_G(v_4) = 4$ and $d_G(v_j) = 5$ for $j = 5, 6$. Let $G' = G - \{vv_1, vv_2, vv_3, vv_4, vv_5, vv_6\}$, which has a linear-8-coloring ϕ. Since $|L(vv_i)| \geq 4$ for $i = 1, 2$, $|L(vv_3)| \geq 3$, $|L(vv_4)| \geq 2$ and $|L(vv_j)| \geq 1$ for $j = 5, 6$, by Lemma 3.1(5), we can get a contradiction. □

Lemma 3.7. *If $v \in V(H)$ with $d_H(v) = 8$, then $n'_3(v) \leq 3$.*

Proof. Let $v_1, v_2, \ldots, v_k$ be the neighbors of v with $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_k)$. If $d_G(v) = 8$, then $n'_3(v) \leq 3$ by Lemma 3.2(3). Otherwise, $d_G(v) = 9$ and $n_1(v) + n_2(v) = 1$. Suppose that $n'_3(v) \geq 4$, say $d_G(v_1) \leq 2$ and $d_G(v_i) = 3$ for $i = 2, 3, 4, 5$. By the minimality of G, $G - \{vv_1, vv_2, vv_3, vv_4, vv_5\}$ has a linear-8-coloring ϕ. Since $|L(vv_1)| \geq 8 - 4 - (d_G(v_1) - 1) = 3$ and $|L(vv_i)| \geq 8 - 4 - (d_G(v_i) - 1) = 2$ for $i = 2, 3, 4, 5$, ϕ can be extended to a linear-8-coloring of G by Lemma 2.1(4). □

Lemma 3.8. *If uv is a $(4, 6)$-edge of H, then uv is not incident to any 3-face.*

Proof. Suppose that uv is incident to a 3-face $f = [uvw]$, say $d_H(u) = 6$ and $d_H(v) = 4$. By Lemma 3.4, $d_G(v) = d_H(v) = 4$. By Remark 3.1, $d_G(u) = 6$ or $d_G(u) = 9$ with $n_2(u) = 3$. This implies that $d_G(u_i) = 2$, and u and x_i are the neighbors of u_i for $i = 1, 2, 3$. Let $G' = G - uv$, which has a linear- 8-coloring ϕ. We first remove the colors of uu_1, uu_2 and uu_3.

Claim 1. *If uv can be colored with some color α which appears only once in both $S(v)$ and $S(u)$, then uu_1, uu_2, uu_3 can be colored properly.*

Proof. Note that $|L(uu_i)| \geq 1$ for $i = 1, 2, 3$. If there is an edge u_ix_i such that $\phi(u_ix_i) \neq \alpha$ for $i = 1, 2, 3$, then coloring this edge with α would led to other two edges satisfying $|L(uu_i)| \geq 1$. By Lemma 2.1(1), they can be colored properly. Otherwise, $\phi(u_ix_i) = \alpha$ for $i = 1, 2, 3$. Then $|L(uu_i)| \geq 2$ for $i = 1, 2, 3$. By Lemma 2.1(2), uu_1, uu_2, uu_3 can be colored properly. This proves Claim 1. □

By Claim 1, suppose that uv cannot be colored, so $C = C(u) \cup C(v)$. W.l.o.g, assume that $C(v) = \{1, 2, 3\}$ with $\phi(vw) = 1$, and $C(u) = \{4, 5, 6, 7, 8\}$ with $\phi(uw) = 8$. We claim that $\{4, 5, 6, 7\} \subseteq C(w)$, otherwise, we can recolor vw and color uv with the color 1. Similarly, $\{2, 3\} \subseteq C(w)$. Since $1, 8 \in C(w)$ and $d_G(w) \leq 9$, then there is at least one color $c \in \{1, 8\}$ which only appears once in $S(w)$. Thus, we can first color uv with c, then color uu_1, uu_2, and uu_3 in this order. If $c = 8$, then $|L(uu_i)| \geq 2$ for $i = 1, 2, 3$. By Lemma 2.1(2), uu_1, uu_2 and uu_3 can be colored. Otherwise, $c = 1$, then we have $S(w) = (1, 2, 3, 4, 5, 6, 7, 8, 8)$. Recolor uw with 1 and uv with 8, so that 8 appears only once in $S(v)$ and $S(u)$. By Claim 1, uu_1, uu_2 and uu_3 can be colored and ϕ is extended to G, a contradiction. □

Lemma 3.9. *If uv is a $(5, 5)$-edge of H, then uv is not incident to any 3-face.*

Proof. Suppose that uv is incident to a 3-face $f = [uvw]$. So $d_H(u) = d_H(v) = 5$. By Lemma 3.4, $d_G(u) = d_G(v) = 5$. Let $G' = G - uv$, which has a linear-8-coloring ϕ. If uv can be colored properly, we are done. Otherwise, we may assume that $C(v) = \{1, 2, 3, 4\}$ with $\phi(vw) = 1$, and $C(u) = \{5, 6, 7, 8\}$ with $\phi(uw) = 8$. If there is a color $b \in \{5, 6, 7\} \setminus C(w)$, then we can recolor vw with the color b and color uv with the color 1. Otherwise, $\{5, 6, 7\} \subseteq C(w)$. Similarly, we have $\{2, 3, 4\} \subseteq C(w)$. Since $1 \in C(w)$, $8 \in C(w)$, and $d_G(w) \leq 9$, then there is at least one color $c \in \{1, 8\}$ appearing only once in $S(w)$. Consequently, we can color uv with the color c, a contradiction. □

Lemma 3.10. *If uv is a $(3, 7)$-edge of H, then uv is incident to at most one 3-face.*

Proof. Suppose that uv is incident to two 3-faces $f_1 = [uvw]$ and $f_2 = [uvx]$. Let $d_H(u) = 7$ and $d_H(v) = 3$. By Lemma 3.4, $d_G(v) = d_H(v) = 3$. By Remark 3.1, $d_G(u) = 7$; or $d_G(u) = 8$ with $n_2(u) = 1$; or $d_G(u) = 9$ with $n_1(u) + n_2(u) = 2$. It suffices to discuss the case $d_G(u) = 9$ with $n_1(u) + n_2(u) = 2$, since other cases can be verified similarly.

Assume that $d_G(u_1) \leq 2$, $d_G(u_2) = 2$, and u and y_i are the neighbors of u_i for $i = 1, 2$. Let $G' = G - uv$, which has a linear-8-coloring ϕ. Remove the colors of uu_1 and uu_2.

Claim 2. *If uv can be colored with some color α which appears only once in both $S(v)$ and $S(u)$, then uu_1, uu_2 can be colored properly.*

Proof. If $|C(u)| \leq 6$, then it is obvious that $|L(uu_i)| \geq 1$ for $i = 1, 2$. By Lemma 2.1(1), uu_1 and uu_2 can be colored properly. Otherwise, $|C(u)| = 7$, say $C(u) = \{2, 3, 4, 5, 6, 7, 8\}$. If $\phi(u_iy_i) \neq 1$, without loss of generality, say $i = 1$, then we can color uu_1 with 1 and uu_2 with a color in $\{1, \alpha\} \setminus \{\phi(u_2y_2\}$. Otherwise, $\phi(u_1y_1) = 1$ and $\phi(u_2y_2) = 1$.

If $1 \notin C(v)$, then we recolor uv with 1, and then color uu_1 and uu_2 with α. Otherwise, $1 \in C(v)$, say $\phi(vw) = 1$, and $\phi(uw) = 2$. If $\phi(vx) \neq 2$, then we recolor vw with 2 and uw with 1, and color uu_1 and uu_2 with 2. Otherwise, $\phi(vx) = 2$. It follows that vw can not be recolored, since we can recolor uv with 1 and color uu_1, uu_2 with α. Hence $S(w) = \{1, 1, 2, 3, 4, 5, 6, 7, 8\}$, we may color uu_1 with 2 and uu_2 with α. This proves Claim 2. □

If there is a color $\beta \in C \setminus (C(u) \cup C(v))$, then we can color uv with some color β that appears only once in $S(v)$ and $S(u)$. By Claim 2, ϕ can be extended to G, a contradiction. Otherwise, $C = C(u) \cup C(v)$, say $C(v) = \{1, 2\}$ with $\phi(vw) = 1$ and $\phi(vx) = 2$, and $C(u) = \{3, 4, 5, 6, 7, 8\}$ with $\phi(uw) = 8$ and $\phi(ux) = 3$. If vw can be recolored, then we can color uv with 1 and uu_1, uu_2 can be colored by Claim 2. Otherwise, we may assume that $\{1, 2, 3, 4, 5, 6, 7, 8\} \subseteq C(w)$. Since $d_G(w) \leq 9$, there exists a color $c \in \{1, 8\}$ appearing only once in $S(w)$. If $c = 1$, then we recolor uw with 1 and color uv with 8. Otherwise, we recolor vw with 8 and color uv with 1. By Claim 2, uu_1, uu_2 can be colored properly. □

In Figure 1, we use black point to denote a vertex that has no edges incident to it other than those shown in graph, while point to denote a vertex that may have edges connected to other vertices that are not in the graph, and triangle to denote a 3-face.

Given a vertex $v \in V(H)$, let $v_0, v_1, \ldots, v_{k-1}$ denote the neighbors of v in clockwise order where $k = d_G(v)$. Let $f_0, f_1, \ldots, f_{k-1}$ be the faces of G which

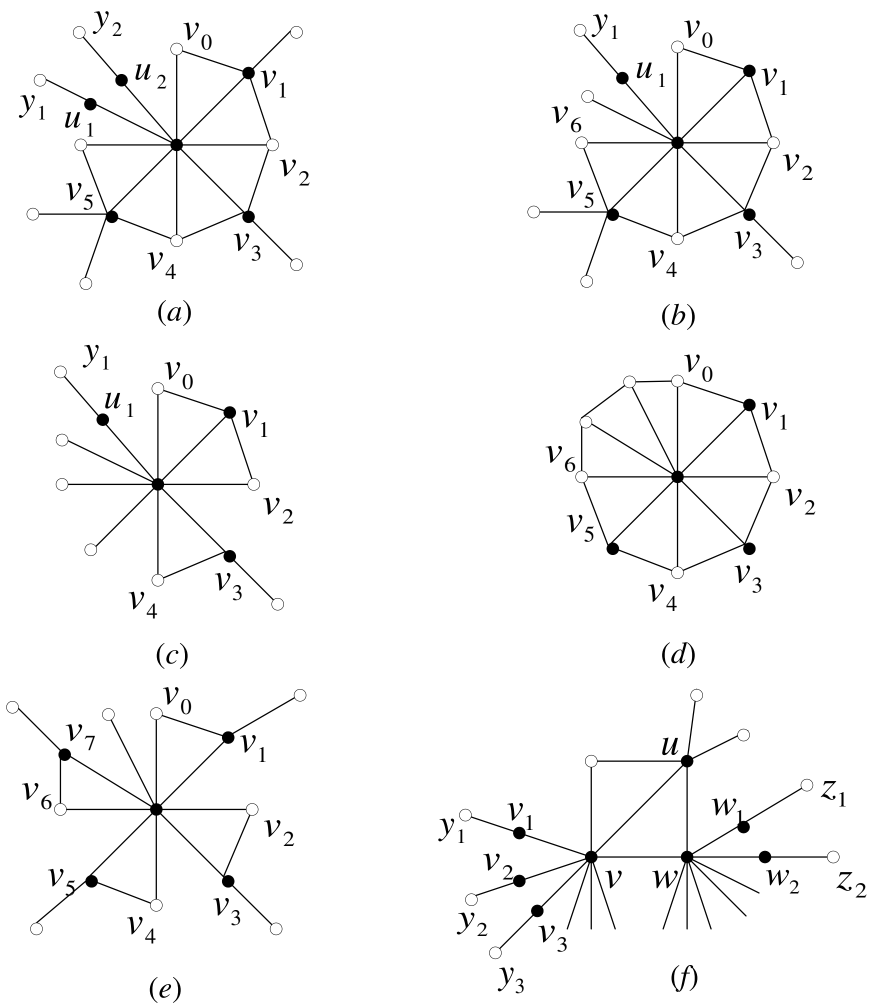

Figure 1. The configurations in Lemmas 3.11–3.16.

are incident to v and $vv_i, vv_{i+1} \in b(f_i)$ for $i = 0, 1, \ldots, k-1$, where indices are taken modulo k.

Lemma 3.11. *If $v \in V(H)$ with $d_H(v) = 7$, $n'_4(v) = 2$ and all the faces incident to v are 3-faces, then other vertices incident to v are 6^+-vertices.*

Proof. By Lemmas 3.1 and 3.9, if v_i is a 5^--vertex, then v_{i-1} and v_{i+1} are 6^+-vertices. Suppose that v_1, v_3 are 4-vertices, and $d_H(v_5) \leq 5$. By Lemma 3.4, $d_G(v_i) = d_H(v_i) = 4$ for $i = 1, 3$ and $d_G(v_5) = d_H(v_5) \leq 5$.

By Remark 3.1, $d_G(v) = 7$, or $d_G(v) = 8$ with $n_2(v) = 1$, or $d_G(v) = 9$ with $n_1(v) + n_2(v) = 2$. In the following, it suffices to discuss the case $d_G(v) = 9$ with $n_1(v) + n_2(v) = 2$, since other cases can be similarly proved.

Assume that $d_G(u_i) \leq 2$, v and y_i are the neighbors of u_i for $i = 1, 2$, as depicted in Fig. 1(a). Let $G' = G - vv_1$, which has a linear-8-coloring ϕ by the minimality of G. First remove the colors of vv_3, vv_5, vu_1 and vu_2. Then color the edges vv_1, vv_3, vv_5, vu_1 and vu_2 as follows.

Claim 3. *Let $\phi(vv_1) = \alpha$, $\phi(vv_3) = \beta$, and $\phi(vv_5) = \gamma$. If α, β, γ only appears once in $S(v)$ and $S(v_1)$, $S(v)$ and $S(v_3)$, $S(v)$ and $S(v_5)$, respectively; or there is a color which appears twice in $S(v)$, then vu_1 and vu_2 can be colored properly.*

Proof. If α, β, γ only appears once in $S(v)$ and $S(v_1)$, $S(v)$ and $S(v_3)$, $S(v)$ and $S(v_5)$, respectively, then vu_1 can be colored with a color in $\{\alpha, \beta, \gamma\} \setminus \{\phi(u_1y_1)\}$ and vu_2 can be colored with a color in $\{\alpha, \beta, \gamma\} \setminus \{\phi(u_2y_2), \phi(vu_1)\}$. Meanwhile, if there is a color appearing twice in $S(v)$, then $|L(vu_i)| \geq 1$ for $i = 1, 2$. By Lemma 2.1(1), vu_1 and vu_2 can be colored properly. □

Now we consider the following two cases.

Case 1. $|L(vv_5)| \geq 1$.

First, if there exists $i \in \{1, 3\}$ such that $|L(vv_i)| \geq 2$, say $|L(vv_1)| \geq 2$, let $\{a, b\} \in L(vv_1)$, $c \in L(vv_3)$ and $d \in L(vv_5)$, then color vv_3 with c, vv_5 with d and vv_1 with a color in $\{a, b\} \setminus \{c\}$. Thus, vu_1 and vu_2 can be colored by Claim 3.

Next suppose that $|L(vv_i)| = 1$ for $i = 1, 3$. Let $a \in L(vv_1)$, $b \in L(vv_3)$ and $c \in L(vv_5)$. If a, b, c are not same, w.l.o.g., assume that $a \neq b$, then color vv_1 with a, vv_3 with b and vv_5 with c. Thus, vu_1 and vu_2 can be colored by Claim 3. Otherwise, $a = b = c$. Let $C(v) = \{1, 2, 3, 4\}$ with $\phi(vv_0) = 1$ and $C(v_1) = \{5, 6, 7\}$ with $\phi(v_0v_1) = 7$, i.e., $a = b = c = 8$. We claim that v_0v_1 and vv_0 cannot be recolored. Otherwise we can color vv_1 with 1 or 7 and then vu_1 and vu_2 can be colored by Claim 3. Therefore, we derive that $S(v_0) = (1, 1, 2, 3, 4, 5, 6, 7, 7)$. In this case, we recolor v_0v_1 with 8, color vv_1 with 7, color vv_3 and vv_5 with 8, and then vu_1 and vu_2 can be colored by Claim 3.

Case 2. $|L(vv_5)| = 0$.

Assume that $C(v) = \{1, 2, 3, 4\}$ with $\phi(vv_4) = 3$, $C(v_5) = \{5, 6, 7, 8\}$. Let $8 \in L(vv_1)$ and $a \in L(vv_3)$, i.e., $a \in \{5, 6, 7, 8\}$.

Since $\phi(v_4v_5) \neq \phi(v_5v_6)$, we can suppose $\phi(v_4v_5) = 7$. First color vv_1 with 8 and vv_3 with the color a, then we color vv_5, vu_1 and vu_2. Claim that vv_4 and v_4v_5 cannot be recolored, otherwise, color vv_5 with 3 or 7, and then vu_1 and vu_2 can be colored by Claim 3. Therefore, we hold that $S(v_4) = (1, 2, 3, 3, 4, 5, 6, 7, \mu)$.

• $a = 8$. We derive that $\mu = 7$, otherwise recolor vv_4 with 7 and color vv_5 with 3. If $3 \in C(v_3)$, then there is a color $e \in \{5, 6, 7\} \setminus C(v_3)$ and hence recolor vv_3 with e, vv_4 with 8 and color vv_5 with 3. Otherwise, $3 \notin C(v_3)$. Recolor vv_3 with 3, vv_4 with 8 and color vv_5 with 3, and then vu_1 and vu_2 can be colored by Claim 3.

• $a \neq 8$. Claim that $\mu = 8$ for otherwise recolor vv_4 with 8 and color vv_5 with 3, and then vu_1 and vu_2 can be colored by Claim 3. Now we only color vv_5, vu_1 and vu_2 can be colored by Claim 3. If $a \neq 7$, then recolor vv_4 with 7 and color vv_5 with 3. Otherwise, $a = 7$. Recolor vv_3 with 3, vv_4 with 7 and color vv_5 with 3. □

Lemma 3.12. *Let $d_H(v) = 8$, and all the faces incident to v be 3-faces. If $n'_3(v) = 1$, $n'_4(v) = 1$ and $n'_5(v) = 1$; or $n'_3(v) = 1$ and $n'_4(v) = 2$, then the other vertices incident to v are 6^+-vertices.*

Proof. By Lemmas 3.1 and 3.9, if v_i is a 5^--vertex, then v_{i-1} and v_{i+1} are 6^+-vertices. Suppose that v_1, v_3 and v_5 are 3-vertex, 4-vertex and 5^--vertex, respectively. Assume to the contrary that v is adjacent to a 5^--vertex except v_1, v_3, v_5. Assume that $d_H(v_7) \leq 5$. By Lemma 3.4, $d_G(v_i) = d_H(v_i)$ for $i = 1, 3, 5, 7$. By Remark 3.1, $d_G(v) = 8$, or $d_G(v) = 9$ with $n_1(v) + n_2(v) = 1$. In the following, it suffices to discuss the case $d_G(v) = 9$ with $n_1(v) + n_2(v) = 1$, since other case can be similarly proved.

Assume that $d_G(u_1) \leq 2$, v and y_1 are the neighbors of u_1, as depicted in Fig. 1(b). Let $G' = G - vv_1$, which has a linear-8-coloring ϕ by the minimality of G. First, we remove the colors of vv_3, vv_5, vv_7, vu_1.

Claim 4. *Let $\phi(vv_i) = \alpha$ and $\phi(vv_j) = \beta$ for $i \neq j$. If the coloring ϕ satisfies one of the following conditions, then vu_1 can be colored properly:*

(i) *α and β appear only once in $S(v)$ and $S(v_i)$, $S(v)$ and $S(v_j)$, respectively,*

(ii) *α appears only once in $S(v)$ and $S(v_i)$ and there is another color appears twice in $S(v)$.*

Proof. If the condition (i) holds, then we color vu_1 with a color in $\{\alpha, \beta\} \setminus \{\phi(u_1y_1)\}$. Assume that the condition (ii) holds. If $\phi(u_1y_1) \neq \alpha$, then we color vu_1 with α. Otherwise, $\phi(u_1y_1) = \alpha$, and vu_1 can be colored properly since $|L(vu_1)| \geq 1$. □

By Lemma 2.1(3), if $|L(vv_1)| \geq 2$, $|L(vv_3)| \geq 2$, $|L(vv_5)| \geq 1$ and $|L(vv_7)| \geq 1$, then $vv_1, vv_3, vv_5, vv_7, vu_1$ can be colored properly. To complete the proof, we have to consider two cases.

Case 1. $|L(vv_5)| \geq 1$ and $|L(vv_7)| \geq 1$.

Due to the above proof, $|L(vv_3)| = 1$. Assume that $C(v) = \{1, 2, 3, 4\}$ with $\phi(vv_0) = 1$, $\phi(vv_2) = 2$, $\phi(vv_4) = 3$ and $\phi(vv_6) = 4$, $C(v_3) = \{5, 6, 7\}$, $\{a, b\} \subseteq L(vv_1)$, $8 \in L(vv_3)$, $c \in L(vv_5)$ and $d \in L(vv_7)$. Obviously, $a, b, c, d \in \{5, 6, 7, 8\}$.

If $c = d = 8$, then color vv_5 and vv_7 with 8 and vv_1 with a color in $\{a, b\} \setminus \{8\}$. Next, we color vv_3 and vu_1. Without loss of generality, we may suppose that $a = 7$. Obviously, we can assume that $\phi(v_3v_4) \neq 7$. Suppose that $\phi(v_3v_4) = 6$. We claim that v_3v_4 and vv_4 cannot be recolored, for otherwise we color vv_3 with 3 or 6. It follows that $S(v_4) = (1, 2, 3, 3, 4, 5, 6, 7, 8)$, and hence we can color vv_3 with 6 and vu_1 can be colored properly by Claim 4.

Otherwise, we color vv_3 with 8, vv_5 with c, vv_7 with d and vv_1 with a color in $\{a, b\}$ which does not appear twice in $S(v)$. Hence vu_1 can be colored by Claim 4.

Case 2. $|L(vv_5)| = 0$ or $|L(vv_7)| = 0$.

Suppose that $|L(vv_7)| = 0$. Furthermore, $C(v) = \{1, 2, 3, 4\}$ with $\phi(vv_0) = 1$, $\phi(vv_2) = 2$, $\phi(vv_4) = 3$, $\phi(vv_6) = 4$, $C(v_7) = \{5, 6, 7, 8\}$, $\{a, b\} \subseteq L(vv_1)$ and $8 \in L(vv_3)$. Obviously, $a, b \in \{5, 6, 7, 8\}$.

- $|L(vv_5)| = 0$. Then $C(v_5) = \{5, 6, 7, 8\}$. Assume that $\phi(v_4v_5) \neq 8$, say $\phi(v_4v_5) = 7$. We first color vv_3 with 8, and color vv_1 with a color in $\{a, b\} \setminus \{7\}$, say, $\phi(vv_1) = a$. Then we color vv_5, vv_7 and vu_1. We claim that vv_4 and v_4v_5 cannot be recolored, for otherwise we can color vv_5 and vv_7 with 3 or color vv_7 with 3 and vv_5 with 5, and then vu_1 can be colored by Claim 4. Thus, we derive that $S(v_4) = (1, 2, 3, 4, 5, 6, 7, 7, 8)$, and $8 \in C(v_4)$, for otherwise we can recolor vv_4 with 8, vv_1 with a color in $\{a, b\} \setminus \{8\}$ and color vv_5 and vv_7 with 3. Suppose that $\phi(v_5v_6) = e$, where $e \in \{5, 6, 8\}$. We first recolor vv_1 with $\{a, b\} \setminus \{8\}$. As observed above, we have that $S(v_6) = (1, 2, 3, 4, 4, 5, 6, 7, 8)$. Therefore, recolor v_5v_6 with 4, vv_6 with e, color vv_5 with 3 and vv_7 with 4. Finally, vu_1 can be colored properly by Claim 4.
- $|L(vv_5)| \geq 1$. Suppose that $c \in L(vv_5)$. Obviously, $c \in \{5, 6, 7, 8\}$. We first color vv_3 with 8, vv_5 with c and vv_1 with a color in $\{a, b\} \setminus \{8\}$. Assume that $\phi(v_6v_7) \neq 8$, say $\phi(v_6v_7) = 7$. Since vv_6 and v_6v_7 cannot be recolored, we conclude that $S(v_6) = (1, 2, 3, 4, 4, 5, 6, 7, 8)$.

Suppose that $c = 8$. If $a \neq 7$, then vv_6 and vv_7 are recolored with 7 and 4, respectively. Suppose that $a = 7$. We claim that $\{5, 6\} \subseteq C(v_1)$, for otherwise we recolor vv_1 so that this case can be reduced to the previous case of $a \neq 7$. Thus, $C(v_1) = \{5, 6, 7\}$. We recolor vv_1 with 4, vv_6 with 7 and color vv_7 with 4. Finally, we color the edge vu_1 by Claim 4.

Suppose that $c \neq 8$. If $c \neq 7$, then recolor vv_6, vv_1 and vv_7 with 7, 4 and 4, respectively. So we assume that $c = 7$. Then $\phi(v_7v_0) \neq 7$, say $\phi(v_6v_7) = d$. Obviously, $d \in \{5, 6, 8\}$. We derive that $S(v_0) = (1, 1, 2, 3, 4, 5, 6, 7, 8)$, as described above. Recolor vv_0 with d, vv_1 with a color in $\{a, b\} \setminus \{d\}$, vv_7 with 1 if $d \in \{5, 6\}$; otherwise, $d = 8$. If $1 \notin C(v_3)$, then recolor vv_3 with 1, vv_0 with 8 and vv_7 with 1. Hence we may assume that $1 \in C(v_3)$. Recolor vv_3 with g, vv_0 with 8 and vv_7 with 1 as there exists a color $g \in \{5, 6, 7\} \setminus C(v_3)$. □

Lemma 3.13. *H does not contain a vertex $v \in V(H)$ with $d_H(v) = 8$, $d_H(v_i) = 3$ and $d_H(v_j) = 3$ with $t(vv_i) = 2$ and $t(vv_j) \geq 1$, where $t(e)$ denotes the number of 3-faces that incident to e.*

Proof. Assume to the contrary that H contains such a vertex v. Let $d_H(v_1) = 3$ with $t(vv_1) = 2$ and $d_H(v_3) = 3$ with $t(vv_3) \geq 1$. W.l.o.g., we may suppose that $[vv_3v_4]$ is a 3-face. Let us consider the case of $d_G(v) = 9$ with $n_1(v) + n_2(v) = 1$.

Recall that $d_G(u_1) \leq 2$, v and y_1 are the neighbors of u_1, shown in Fig. 1(c). Let $G' = G - vv_1$, which has a linear-8-coloring ϕ. Erasing the colors of vv_3 and vu_1, we color vv_1, vv_3, vu_1 in the following ways. If $|L(vv_1)| \geq 2$, $|L(vv_3)| \geq 1$ and $|L(vu_1)| \geq 1$, then by Lemma 2.1(2), they can be colored properly. Otherwise, we need to deal with the following two cases.

Case 1. $|L(vv_1)| \geq 1$ and $|L(vv_3)| \geq 1$.

If $|L(vu_1)| \geq 2$, then vv_1, vv_2 and vv_3 can be colored by Lemma 2.1(2). Otherwise, $|L(vu_1)| = 1$, say $C(v) = \{1, 2, 3, 4, 5, 6\}$ with $\phi(vv_0) = 1$, $\phi(vv_2) = 2$ and $\phi(vv_4) = 3$, $\phi(u_1y_1) = 7$, $a \in L(vv_1)$ and $b \in L(vv_3)$. Obviously, $a, b \in \{7, 8\}$. We first color vv_1 with a and vv_3 with b, then color vu_1. If $a \neq b$ or $a = b = 7$, then we color vu_1 with 8.

Next, we assume that $a = b = 8$. If $7 \notin C(v_1)$, then we recolor vv_1 with 7 and color vu_1 with 8. Otherwise, $7 \in C(v_1)$, w.l.o.g., suppose that $\phi(v_0v_1) = 7$. If $\phi(v_1v_2) \neq 1$, then recolor vv_0 with 7 and v_0v_1 with 1, then color vu_1 with 1. Otherwise, $\phi(v_1v_2) = 1$. It is easy to see that $S(v_2) = (1, 1, 2, 2, 3, 4, 5, 6, 7)$. Recolor vv_2 with 8, vv_1 with 2 and color vu_1 with 2.

Case 2. $|L(vv_1)| = 0$ or $|L(vv_3)| = 0$.

W.l.o.g., we suppose that $|L(vv_3)| = 0$. Let $C(v) = \{1, 2, 3, 4, 5, 6\}$ with $\phi(vv_0) = 1$, $\phi(vv_2) = 2$, $\phi(vv_4) = 3$, and $C(v_3) = \{7, 8\}$.

• $|L(vv_1)| \geq 1$, say $8 \in L(vv_1)$. We first color vv_1 with 8, then color the edges vv_3 and vu_1. Let $\phi(v_3v_4) = a$, that is, $a \in \{7, 8\}$. If $a = 7$, then it follows that $S(v_4) = (1, 2, 3, 3, 4, 5, 6, 7, 7)$ because vv_4 and v_3v_4 can not be recolored. Recolor vv_4 with 8, color vv_3 with 3 and vu_1 with a color in $\{3, 7\} \setminus \{\phi(u_1y_1)\}$.

If $a = 8$, then $S(v_4) = (1, 2, 3, 3, 4, 5, 6, 7, 8)$. We claim that $3 \in C(v_1)$, for otherwise we recolor vv_4 with 8, v_3v_4 with 3 and vv_1 with 3, and color vv_3 with 8 and vu_1 with a color in $\{3, 7\} \setminus \{\phi(u_1y_1)\}$. Thus, $C(v_1) = \{3, 7, 8\}$. We may assume $\phi(v_0v_1) = 7$. We claim that v_0v_1 cannot be recolored. If not, we recolor vv_1 with 7 and then vv_3 and vu_1 can be colored as the former case of $a = 7$. Thus, it imply that $\{2, 4, 5, 6\} \subseteq C(v_0)$ and 1 appears twice in $S(v_0)$. Thus, we have $S(v_0) = (1, 1, 2, 4, 5, 6, 7, 7, 8)$. Recolor vv_0 and v_3v_4 with 3, vv_4 with 8, and color vv_1 and vv_3 with 1, vu_1 with a color in $\{7, 8\} \setminus \{\phi(u_1y_1)\}$.

• $|L(vv_1)| = 0$. Let $\phi(v_0v_1) = 7$ and $\phi(v_1v_2) = 8$. We first consider $S(v_2)$ if $\phi(u_1y_1) = 8$; and $S(v_0)$ otherwise. By symmetry, assume that $\phi(u_1y_1) \neq 8$. We have $S(v_0) = (1, 2, 3, 4, 5, 6, 7, 7, 8)$ for v_0v_1 and vv_0 cannot be recolored. Similarly, we can obtain that $S(v_2) = (1, 2, 3, 4, 5, 6, 7, 8, 8)$. Therefore, we color vv_1 with 1, vv_3 with 2 and vu_1 with 8. □

Lemma 3.14. *Let $v \in V(H)$ with $d_H(v) = 9$. If all the faces incident to v are 3-faces, then $n_3'(v) \leq 2$.*

Proof. Suppose that $n_3'(v) \geq 3$, say, v_1, v_3, v_5 are 3-vertices. For $i = 1, 3, 5$, $d_G(v_i) = d_H(v_i)$ by Lemma 3.4, depicted in Fig. 1(d). Let $G' = G - vv_1$, which has a linear-8-coloring ϕ. First remove the colors of vv_3 and vv_5. Then color the edges vv_1, vv_3 and vv_5. If $|L(vv_1)| \geq 2$, $|L(vv_3)| \geq 1$ and $|L(vv_5)| \geq 1$, then the coloring is available by Lemma 2.1(2). Otherwise, there are three cases to be be considered.

Case 1. $|L(vv_1)| = 1$, $|L(vv_3)| \geq 1$ and $|L(vv_5)| \geq 1$.

Suppose that $8 \in L(vv_1)$, $a \in L(vv_3)$ and $b \in L(vv_5)$. If $a \neq 8$ or $b \neq 8$, then we color vv_1 with 8, vv_3 with a and vv_5 with b. Otherwise, $a = b = 8$. If there is a color in C, say 1, appears twice in $S(v)$, then assume that $S(v) = \{1, 1, 2, 3, 4, 5\}$ with $\phi(vv_6) = 4$, $C(v_i) = \{6, 7\}$ for $i = 1, 3, 5$. Suppose that $\phi(v_4v_5) = 6$ and $\phi(v_5v_6) = 7$. Coloring vv_1, vv_3 with 8, we color vv_5. It follows that $S(v_6) = (1, 2, 3, 4, 4, 5, 6, 7, 7)$. In this case, recolor v_5v_6 with 8 and color vv_5 with 7.

Otherwise, $C(v) = \{1, 2, 3, 4, 5, 6\}$ with $\phi(vv_0) = 1$, $\phi(vv_2) = 2$, $\phi(vv_4) = 3$ and $\phi(vv_6) = 4$. Obviously, $7 \in C(v_i)$ for $i = 1, 3, 5$. We first color vv_1 and vv_3 with 8, then color vv_5, w.l.o.g. say $\phi(v_4v_5) = 7$. So, $S(v_4) = (1, 2, 3, 4, 5, 6, 7, 7, 8)$ and $S(v_6) = (1, 2, 3, 4, 4, 5, 6, 7, 8)$ if $\phi(v_5v_6) = 3$. Suppose $3 \notin C(v_1)$, then recolor vv_1 with 3 and color vv_5 with 8. It follows that $S(v_1) = \{3, 7, 8\}$. Hence recolor v_5v_6 with 4, vv_6 with 3, vv_1 with 4, and color vv_5 with 8.

Suppose $\phi(v_5v_6) \neq 3$. If $3 \notin C(v_1)$, then recolor vv_4 with 7, v_4v_5 with 3, vv_1 with 3 and color vv_5 with 8. Otherwise, we have $S(v_1) = \{3, 7, 8\}$. It turns out that $S(v_3) = \{3, 7, 8\}$ similarly. Assume that $\phi(v_0v_1) = 3$ and $\phi(v_1v_2) = 7$. Then $S(v_2) = (1, 2, 2, 4, 5, 6, 7, 7, 8)$. Recolor v_4v_5 with 3, vv_4 with 7, vv_2 with 3, vv_1 with 2, and color vv_5 with 8.

Case 2. $|L(vv_1)| \geq 1$, and $|L(vv_3)| = 0$ or $|L(vv_5)| = 0$.

Suppose that $|L(vv_5)| = 0$. We may assume that $C(v) = \{1, 2, 3, 4, 5, 6\}$ with $\phi(vv_0) = 1$, $\phi(vv_2) = 2$, $\phi(vv_4) = 3$, $\phi(vv_6) = 4$ and $8 \in L(vv_1)$, $C(v_5) = \{7, 8\}$. Assume that $\phi(v_4v_5) = 7$ and $\phi(v_5v_6) = 8$. If $8 \notin C(v_3)$, then we color vv_1 and vv_3 with 8, and then similar to the above argument, we can verify that $S(v_4) = (1, 2, 3, 3, 4, 5, 6, 7, 7)$ and $S(v_6) = (1, 2, 3, 4, 4, 5, 6, 7, 8)$. Thus, we can recolor v_4v_5 with 8 and color vv_5 with 7. Otherwise, $8 \in C(v_3)$. If $7 \in C(v_3)$, then derive that $S(v_4) = (1, 2, 3, 4, 5, 6, 7, 7, 8)$ and $S(v_6) = (1, 2, 3, 4, 4, 5, 6, 7, 8)$. We recolor vv_6 with 8, v_5v_6 with 4, and color vv_5 with 3 and vv_3 with 4. If $7 \notin C(v_3)$, then $S(v_4) = (1, 2, 3, 3, 4, 5, 6, 7, 8)$ and $S(v_6) = (1, 2, 3, 4, 4, 5, 6, 7, 8)$. If $3 \notin C(v_3)$, then recolor vv_4 with 7 and color vv_3 and vv_5 with 3. Otherwise, recolor vv_6 with 8, v_5v_6 with 4, and color vv_5 with 7 and vv_3 with 4.

Case 3. $|L(vv_1)| = 0$, $|L(vv_3)| = 0$ and $|L(vv_5)| = 0$.

Suppose that $\phi(v_0v_1) = 7$. It follows that $S(v_0) = (1, 2, 3, 4, 5, 6, 7, 7, 8)$. We claim that $\{2, 3, 4, 5, 6\} \subseteq C(v_0)$, for otherwise we recolor v_0v_1, and recolor vv_0 with 7, color vv_1 and vv_3 with 1. Now we color vv_5, w.l.o.g., let $\phi(v_4v_5) = 8$. We obtain that $S(v_4) = (1, 2, 3, 3, 4, 5, 6, 8, 8)$. Thus, recolor v_4v_5, vv_4, vv_3 and vv_5

with 3, 8, 3 and 1, respectively. Furthermore, we conclude that $8 \in S(v_0)$, since, otherwise, we can color vv_0 with 8, v_0v_1 with 1, a vv_1 with 7, vv_3 and vv_5 with 1. Finally, 7 appears twice in $S(v_0)$, if not, we may recolor vv_0 with 7, v_0v_1 with 1, and color vv_1 with 7, vv_3 and vv_5 with 1.

Suppose that $\phi(v_5v_6) = 7$. It is easy to see that $S(v_6) = (1, 2, 3, 4, 5, 6, 7, 7, 8)$. Therefore, we can recolor vv_0 with 7, v_0v_1 with 1, and color vv_1 with 4, vv_3 and vv_5 with 1. □

Lemma 3.15. *H does not contain a vertex v such that $d_H(v) = 9$, $n'_3(v) = 4$ and every 3-vertex v_i satisfies $t(vv_i) \geq 1$.*

Proof. Suppose that $d_H(v_i) = 3$ for $i = 1, 3, 5, 7$. Furthermore, let $[vv_0v_1]$, $[vv_2v_3]$, $[vv_4v_5]$ and $[vv_6v_7]$ be 3-faces. By Lemma 3.4, $d_G(v_i) = d_H(v_i)$ for $i = 1, 3, 5, 7$, depicted in Fig. 1(e). Let $G' = G - vv_1$, which has a linear-8-coloring ϕ. First remove the colors of vv_3, vv_5 and vv_7. If $|L(vv_1)| \geq 2$, $|L(vv_3)| \geq 2$, $|L(vv_5)| \geq 1$ and $|L(vv_7)| \geq 1$, then by Lemma 2.1(3), vv_1, vv_3, vv_5, vv_7 can be colored properly. Otherwise, we assume that $|L(vv_3)| = 1$, $|L(vv_5)| = 1$ and $|L(vv_7)| = 1$. Suppose that $C(v) = \{1, 2, 3, 4, 5\}$ with $\phi(vv_0) = 1$, $\phi(vv_2) = 2$, $\phi(vv_4) = 3$ and $\phi(vv_6) = 4$, $a \in L(vv_3)$, $b \in L(vv_5)$ and $c \in L(vv_7)$. It is easy to observe that $a, b, c \in \{6, 7, 8\}$ are available. Now we consider the following two cases.

- $|L(vv_1)| = 1$. Suppose that $C(v_1) = \{6, 7\}$ and $8 \in L(vv_1)$. If no three colors in $\{a, b, c, 8\}$ are the same, then we can color vv_1 with 8, vv_3 with a, vv_5 with b and vv_7 with c. Otherwise, assume that $a = b = 8$. Thus, $C(v_i) = \{6, 7\}$ for $i = 1, 3, 5$.

If $c \neq 8$, say $c = 7$, then $C(v_7) = \{6, 8\}$. We color vv_1 and vv_3 with 8, vv_7 with 7, and then color vv_5. Assume that $\phi(v_4v_5) = \alpha$, since $C(v_5) = \{6, 7\}$, we can recolor v_4v_5 with 3, vv_4 with α, vv_3 with 3 and color vv_5 with 8. Assume that $c = 8$, then $C(v_7) = \{6, 7\}$. We first color vv_1 and vv_3 with 8, and then color vv_5 and vv_7. W.l.o.g., we may suppose that $\phi(v_0v_1) = 7$. Since v_0v_1 and vv_0 cannot be recolored, we deduce that $S(v_6) = (1, 2, 3, 4, 5, 6, 7, 7, 8)$. Therefore, recolor v_6v_7 with 4, vv_6 with 7 and vv_1 with 4, and color vv_7 with 8 and vv_5 with 4.

- $|L(vv_1)| \geq 2$. The foregoing discussion implies that $L(vv_1) \subseteq \{6, 7, 8\}$. Suppose that $a = 8$, which implies that $C(v_3) = \{6, 7\}$. If $8, b, c$ are mutually distinct, then color vv_3 with 8, vv_5 with b and vv_7 with c. Moreover, vv_1 can be colored properly by $|L(vv_1)| \geq 2$. Otherwise, $b = c = 8$, then $C(v_i) = \{6, 7\}$ for $i = 3, 5, 7$. We first color vv_3, vv_5 with 8, then color vv_1 with some color $\beta \in L(vv_1) \setminus \{8\}$. Since $L(vv_1) \subseteq \{6, 7, 8\}$, we may choose $\beta \in \{6, 7\}$. Finally, we color vv_7, say $\phi(v_6v_7) = 7$. We conclude that $S(v_6) = (1, 2, 3, 4, 4, 5, 6, 7, 8)$. Thus, we may recolor v_6v_7 with 4, vv_6 with 7, vv_5 with 4, and color vv_7 with 8. □

Lemma 3.16. *If $f_1 = [uvw]$ is a $(5, 6, 7^-)$-face in H, then the other face f_2 incident to the $(5, 6)$-edge is not a 3-face.*

Proof. By contradiction. Let f_2 be a 3-face. Set $d_H(u) = 5$, $d_H(v) = 6$, $d_H(w) \leq 7$, and $f_2 = [uvx]$. Note that it suffices to consider the case that $d_G(v) = 9$ with $n_2(v) = 3$ and $d_G(w) = 9$ with $n_1(w) + n_2(w) = 2$. Suppose that $d_G(v_i) = 2$, v and y_i are the neighbors of v_i for $i = 1, 2, 3$, $d_G(w_j) \leq 2$ with w and z_j being the neighbors of w_j for $j = 1, 2$ (see Fig. 1(f)). Let $G' = G - uv$, which has a linear-8-coloring ϕ. Remove the colors of vv_1, vv_2 and vv_3.

Claim 5. *Let $\phi(uv) = \alpha$ which only appears once in $S(v)$ and $S(u)$. Then the edges vv_1, vv_2, and vv_3 can be colored properly.*

Proof. It can readily be checked that $|L(vv_i)| \geq 1$ for $i = 1, 2, 3$. If there exists an edge v_iy_i with $\phi(v_iy_i) \neq \alpha$ for $i = 1, 2, 3$, say $\phi(v_1y_1) \neq \alpha$, then color the edge vv_1 with α. Hence vv_2 and vv_3 can be colored properly, by Lemma 2.1(1). Otherwise, $\phi(v_iy_i) = \alpha$, and then we have $|L(vv_i)| \geq 2$ for $i = 1, 2, 3$. Therefore, again by Lemma 2.1(2), vv_1, vv_2 and vv_3 can be colored properly. □

If $C \setminus (C(u) \cup C(v)) \neq \varnothing$, then ϕ can be extend to G by Claim 5, a contradiction. It follows that $C = C(u) \cup C(v)$, without loss of generality, we may suppose that $\phi(uw) = 1$. Next, we color the edges uv, vv_1, vv_2 and vv_3.

Case 1. $\phi(vw) = 1$.

Assume that $C(u) = \{1, 2, 3, 4\}$ with $\phi(ux) = 4$, $C(v) = \{1, 5, 6, 7, 8\}$ with $\phi(vx) = 8$. Because ux and vx cannot be recolored, $S(x) = (2, 3, 4, 4, 5, 6, 7, 8, 8)$. Furthermore, because uw and vw cannot be recolored, $S(w) = (1, 1, 2, 3, 4, 5, 6, 7, 8)$. Hence, we may assume that $\phi(ww_1) \in \{2, 3, 4\}$ (the case of $\phi(ww_1) \in \{5, 6, 7, 8\}$ is proved similarly). Assume, without loss of generality, $\phi(ww_1) = 2$. We first color the edge uv. Recolor vw with 2, xv with 1 and color uv with 8 if $\phi(w_1z_1) \neq 2$; otherwise, $\phi(w_1z_1) = 2$. Then recolor ww_1 with 1, vw with 2, xv with 1 and color uv with 8. Therefore, again by Claim 4, vv_1, vv_2 and vv_3 can be colored.

Case 2. $\phi(vw) \neq 1$

Without loss of generality, we may assume that $\phi(vw) = 8$.

Case 2.1. Each of the colors 1 and 8 only appears once in $S(u)$ and $S(v)$, and $1 \notin C(v)$ and $8 \notin C(u)$.

Suppose that the color 2 appears twice in $S(u)$ or $S(v)$, or $2 \in C(u) \cap C(v)$, then it implies that each of the other colors appears once in $S(u)$ and $S(v)$. Because uw and vw cannot be recolored, $S(w) = (1, 1, 3, 4, 5, 6, 7, 8, 8)$.

• $\phi(ww_1) \in \{3, 4, 5, 6, 7\}$. Without loss of generality, assume that $\phi(ww_1) = 3$. We first color the edge uv, then recolor ww_1 with the color 2, vw or uw with 3, uv with 8 or 1 if $\phi(w_1z_1) \neq 2$; otherwise, $\phi(w_1z_1) = 2$. Then recolor vw or uw with 3, and color uv with 8 or 1. Finally, by Claim 5, we may color the edges vv_1, vv_2 and vv_3. The case of $\phi(ww_2) \in \{3, 4, 5, 6, 7\}$ can be disposed of similarly

• $\phi(ww_1) \in \{1, 8\}$. Without loss of generality, we may assume that $\phi(ww_1) = 1$ and $\phi(ww_2) = 8$. First, color the edge uv, then recolor ww_1 with the color 2, vw with 1 and color uv with 8 if $\phi(w_1z_1) \neq 2$; otherwise, $\phi(w_1z_1) = 2$. $\phi(w_2z_2) = 2$ can be shown by the same approach. Thus, we can recolor ww_1 with 8, vw with 1 and color uv with 8, and color the edges vv_1, vv_2 and vv_3 by Claim 5. The case of $\phi(ww_2) \in \{1, 8\}$ is proved similarly.

Case 2.2. The color 1 appears twice in $S(u)$ or $1 \in C(v)$.

• If $1 \in C(v)$, say $\phi(vx) = 1$, then without loss of generality, we assume that $C(u) = \{1, 2, 3, 4\}$ with $\phi(ux) = 4$, $C(v) = \{1, 5, 6, 7, 8\}$ with $\phi(vw) = 8$. Now claim that vx cannot be recolored. Suppose not, it can be reduced to the case of Case 2.1. Because vx and ux cannot be recolored, $S(x) = (1, 2, 3, 4, 4, 5, 6, 7, 8)$. Based on this evidence, we immediately deduce that $S(w) = (1, 2, 3, 4, 5, 6, 7, 8, 8)$. In conclusion, $\phi(ww_1)$ or $\phi(ww_2)$ is contained in $\{2, 3, 4, 5, 6, 7\}$. Without loss of generality, we may assume that $\phi(ww_1) = 2$. If $\phi(w_1z_1) \neq 2$, then recolor vw with 2 and color uv with 8; otherwise, $\phi(w_1z_1) = 2$. Then recolor ww_1 with 1, wv with 2 and color uv with 8. Finally, we color the edges vv_1, vv_2 and vv_3 by the Claim 5.

Suppose now that $\phi(vx) \neq 1$. Without loss of generality, assume that $\phi(vx) = 7$. We can confirm that $S(x) = (2, 3, 4, 4, 5, 6, 7, 7, 8)$ and $S(w) = (1, 2, 3, 4, 5, 6, 7, 8, 8)$

similar to the foregoing discussion. Therefore, we can color uv, vv_1, vv_2 and vv_3 similarly.

• The color 1 appears twice in $S(u)$. Without loss of generality, we may assume that $S(u) = (1,1,2,3)$ and $S(v) = (4,5,6,7,8)$ with $\phi(vx) = 7$. We obtain that $S(w) = (1,2,3,4,5,6,7,8,8)$. Hence we have $\phi(ww_1) \in (2,3,4,5,6,7)$. If $\phi(ww_1) \in \{2,3\}$, w.l.o.g., say $\phi(ww_1) = 2$. First, we color the edge uv, then recolor vw with 2 and color uv with 8 if $\phi(w_1z_1) \neq 2$; otherwise $\phi(w_1z_1) = 2$. Then recolor ww_1 with 8, vw with 2 and color uv with 8. By Claim 5, the edges vv_1, vv_2 and vv_3 can be colored. Otherwise, $\phi(ww_1) \in \{5,6,7\}$, which can be solved by the same approach.

Case 2.3. The color 8 appears twice in $S(v)$ or $8 \in C(u)$.

The proof of this situation is similar to the proof of Case 2.2. □

To obtain a contradiction, we use the discharging method. By Euler's Formula, $|V(H)| - |E(H)| + |F(H)| = 2$ and the following fundamental identities

$$\sum_{v\in V(H)} d_H(v) = \sum_{f\in F(H)} d_H(f) = 2|E(H)|,$$

we have

$$\sum_{v\in V(H)} (d_H(v) - 4) + \sum_{f\in F(H)} (d_H(f) - 4) = -8. \tag{3.1}$$

Let ω denote a weight function defined by $\omega(x) = d_H(x) - 4$ for each $x \in V(H) \cup F(H)$. We are going to redistribute the weight between vertices and faces in H so that the resultant weight $\omega'(x) \geq 0$ for every $x \in V(H) \cup F(H)$, while keeping the sum of all weights fixed. Therefore

$$0 \leq \sum_{x\in V(H)\cup F(H)} \omega'(x) = \sum_{x\in V(H)\cup F(H)} \omega(x) = -8 < 0, \tag{3.2}$$

a contradiction.

The following are the discharging rules. For a face $f = [v_1v_2v_3]$, we use $(d_H(v_1), d_H(v_2), d_H(v_3)) \to (\alpha_1, \alpha_2, \alpha_3)$ to denote the amount of weight α_i transferred from v_i to f for $i = 1,2,3$.

(R1) Every 7^+-vertex sends $\frac{1}{3}$ to each adjacent 3-vertex.
(R2) Let $f = [v_1v_2v_3]$ be a 3-face of H with $d_H(v_1) \leq d_H(v_2) \leq d_H(v_3)$. Then
(R2.1) $(3, 7^+, 7^+) \to (0, \frac{1}{2}, \frac{1}{2})$.
(R2.2) $(4, 7^+, 7^+) \to (0, \frac{1}{2}, \frac{1}{2})$.
(R2.3) $(5, 6, 7^-) \to (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$; $(5, 6, 8^+) \to (\frac{1}{5}, \frac{1}{3}, \frac{7}{15})$; $(5, 7^+, 7^+) \to (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$.
(R2.4) $(6^+, 6^+, 6^+) \to (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Let ω' denote the final weight function after **(R1)** and **(R2)** are carried out in H.

Let $f \in F(H)$. If $d_H(f) \geq 4$, then $\omega'(f) = \omega(f) = d_H(f) - 4 = 0$. Assume that $d_H(f) = 3$, say, $f = [xyz]$ satisfies $d_H(x) \leq d_H(y) \leq d_H(z)$. If $d_H(x) = 3$, then by Lemmas 3.1, 3.4 and 3.5, we derive that $d_H(y), d_H(z) \geq 7$. It follows that $\omega'(f) = -1 + 2 \times \frac{1}{2} = 0$ by **(R2.1)**. If $d_H(x) = 4$, by Lemmas 3.1, 3.4 and 3.8, $d_H(y), d_H(z) \geq 7$. By **(R2.2)**, $\omega'(f) = -1 + 2 \times \frac{1}{2} = 0$. By **(R2.4)**, if $d_H(x) \geq 6$, then $\omega'(f) = -1 + 3 \times \frac{1}{3} = 0$. Assume that $d_H(x) = 5$. Applying Lemmas 3.1, 3.4 and 3.9, we obtain that $d_H(y), d_H(z) \geq 6$. So, it is easy to check that $\omega'(f) = 0$ by **(R2.3)**.

Let $v \in V(H)$. Then $d_H(v) \geq 3$ by Lemma 3.3. We have to consider some possibilities, depending on the size $d_H(v)$.

(1) If $d_H(v) = 3$, then v is adjacent to three 7^+-vertices in H by Lemmas 3.1, 3.4 and 3.5. Hence, $\omega'(v) \geq -1 + 3 \times \frac{1}{3} = 0$ by **(R1)**.

(2) If $d_H(v) = 4$, then $\omega'(v) = \omega(v) = d_H(v) - 4 = 0$.

(3) If $d_H(v) = 5$, then v sends at most $\frac{1}{5}$ to each incident face by **(R2.3)**. Thus, $\omega'(v) \geq 1 - 5 \times \frac{1}{5} = 0$.

(4) Assume that $d_H(v) = 6$. By Lemma 3.16, if v is adjacent to a 5-vertex, say v_1, then at least one of f_0 and f_1 is not a 3-face. Thus, v is incident to at most four $(5, 6, 7^-)$-faces. If v is incident to three or four $(5, 6, 7^-)$-faces, then v is incident to at least two 4^+-faces. Thus, $\omega'(v) \geq 2 - 4 \times \frac{2}{5} > 0$. Otherwise, v is incident to at most one $(5, 6, 7^-)$-face, so that $\omega'(v) \geq 2 - 5 \times \frac{2}{5} = 0$ or $\omega'(v) \geq 2 - 6 \times \frac{1}{3} = 0$ by **(R2.3)** and **(R2.4)**.

(5) Assume that $d_H(v) = 7$. By Lemma 3.6(a), we see that $n'_3(v) \leq 1$. If $n'_3(v) = 1$, say $d_H(v_1) = 3$, then $n'_4(v) \leq 1$ by Lemma 3.6(b). If $n'_4(v) = 1$, then $n'_5(v) \leq 1$ by Lemma 3.6(c). Moreover, if f_0 is a 3-face, then f_1 is a 4^+-face by Lemma 3.10. Thus, $\omega'(v) \geq 3 - 3 \times \frac{1}{2} - 2 \times \frac{2}{5} - \frac{1}{3} - \frac{1}{3} = \frac{1}{30} > 0$ by **(R2.1)-(R2.4)**. Otherwise, $n'_4(v) = 0$, then $\omega'(v) \geq 3 - \frac{1}{2} - 5 \times \frac{2}{5} - \frac{1}{3} = \frac{1}{6} > 0$. We may therefore assume that $n'_3(v) = 0$.

By **(R2.1)-(R2.4)**, the following Observation can be easily confirmed.

Observation. Every 5-vertex, 6-vertex and 7^+-vertex sends at most $\frac{1}{5}$, $\frac{2}{5}$ and $\frac{1}{2}$ to an incident 3-face, respectively.

Recall that $t(v)$ denotes the number of 3-faces incident to v. If $t(v) \leq 6$, then $\omega'(v) \geq 3 - 6 \times \frac{1}{2} = 0$. So suppose that $t(v) = 7$. By Lemma 3.12, $n'_4(v) \leq 2$. Moreover, $n'_5(v) = 0$ if $n'_4(v) = 2$, then $\omega'(v) \geq 3 - 4 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$ by **(R2.4)**. If $n'_4(v) = 1$, then $\omega'(v) \geq 3 - 2 \times \frac{1}{2} - 4 \times \frac{2}{5} - \frac{1}{3} = \frac{1}{15} > 0$ by **(R2.3)** and **(R2.4)**. If $n'_4(v) = 0$, then $\omega'(v) \geq 3 - 6 \times \frac{2}{5} - \frac{1}{3} = \frac{4}{15} > 0$ by **(R2.3)** and **(R2.4)**.

(6) Assume that $d_H(v) = 8$. By Lemma 3.7, $n'_3(v) \leq 3$. If $n'_3(v) = 0$, then $\omega'(v) \geq 4 - 8 \times \frac{1}{2} = 0$ by Observation. Otherwise, $1 \leq n'_3(v) \leq 3$.

- $n'_3(v) = 1$. If $t(v) \leq 7$, then $\omega'(v) \geq 4 - 7 \times \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$ by Observation and **(R1)**. Otherwise, $t(v) = 8$. Lemmas 3.1, 3.4, 3.5 and 3.8 yield that $d_H(v_{i-1}), d_H(v_{i+1}) \geq 7$ if v_i is a 4^--vertex in H. By Lemma 3.9 and 3.12, $n'_4(v) \leq 2$ and $n'_5(v) \leq 3$. If $n'_4(v) = 2$, then $n'_5(v) = 0$ by Lemma 3.12. By Observation, **(R1)** and **(R2.4)**, we deduce that $\omega'(v) \geq 4 - 6 \times \frac{1}{2} - 2 \times \frac{1}{3} - \frac{1}{3} = 0$. By Lemma 3.12, $n'_5(v) \leq 1$ if $n'_4(v) = 1$. Hence, Observation, **(R1)**, **(R2.3)** and **(R2.4)** imply that $\omega'(v) \geq 4 - 4 \times \frac{1}{2} - 2 \times \frac{7}{15} - 2 \times \frac{1}{3} - \frac{1}{3} = \frac{1}{15} > 0$. Otherwise, $n'_4(v) = 0$, then $n'_5(v) \leq 3$, we have $\omega'(v) \geq 4 - 2 \times \frac{1}{2} - 4 \times \frac{7}{15} - 2 \times \frac{2}{5} - \frac{1}{3} = 0$.
- $n'_3(v) = 2$. We may assume that $d_H(v_i) = 3$ and $d_H(v_j) = 3$, where $i \neq j$. If $t(v) \leq 6$, then $\omega'(v) \geq 4 - 6 \times \frac{1}{2} - 2 \times \frac{1}{3} = \frac{1}{3} > 0$ by **(R1)** and Observation. Otherwise, assume that $t(v) \geq 7$. It is easy to check that each of the edges vv_i and vv_j is incident to at least one 3-face. If vv_i or vv_j is incident to two 3-faces, then by Lemma 3.13, H does not contain the configuration. Thus, each of the edges vv_i and vv_j is incident to one 3-face, which yields that $t(v) = 7$. Therefore, by Observation, **(R1)** and **(R2.4)**, $\omega'(v) \geq 4 - 6 \times \frac{1}{2} - \frac{1}{3} - 2 \times \frac{1}{3} = 0$.
- $n'_3(v) = 3$. Let $d_H(v_i) = d_H(v_j) = d_H(v_k) = 3$, where $i \neq j \neq k$. If $t(v) \geq 7$, then there exists a 3-vertex, say v_i, such that vv_i incident to two 3-faces. Moreover, each of the edges vv_j and vv_k is incident to at least one 3-face. This contradicts with Lemma 3.13. Hence, $t(v) \leq 6$, then $\omega'(v) \geq 4 - 6 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$ by **(R1)**

and Observation.

(7) Assume that $d_H(v) = 9$. By Lemma 3.2(4), $n'_3(v) \le 5$. If $n'_3(v) \le 2$, then by Observation and **(R1)**, $\omega'(v) \ge 5-8\times\frac{1}{2}-\frac{1}{3}-2\times\frac{1}{3} = 0$. If $n'_3(v) = 3$, then Lemma 3.14 asserts that $t(v) \le 8$, and hence $\omega'(v) \ge 5-8\times\frac{1}{2}-3\times\frac{1}{3} = 0$ by Observation and **(R1)**. If $n'_3(v) = 4$, then we may assume that $d_H(v_i) = d_H(v_j) = d_H(v_k) = d_H(v_l) = 3$. If $t(v) \le 7$, then $\omega'(v) \ge 5-7\times\frac{1}{2}-4\times\frac{1}{3} > 0$ by Observation and **(R1)**. Otherwise, each of the edges vv_i, vv_j, vv_k, vv_l is incident to at least one 3-face. By Lemma 3.15, such configuration is not available in H. Finally, we suppose that $n'_3(v) = 5$. If $t(v) \le 6$, then $\omega'(v) \ge 5-6\times\frac{1}{2}-5\times\frac{1}{3} > 0$ by Observation and **(R1)**. Otherwise, v is adjacent to at least four 3-vertices, and every $(3,9)$-edge is incident with at least one 3-face. By Lemma 3.15, such configuration does not exist in H. □

Proof of Theorem 1.1. If $\Delta(G) = 9$, then Theorem 3.1 asserts that $\mathrm{la}_2(G) \le 8 = \Delta - 1$. So assume that $\Delta(G) \ge 10$. By [13], G is a $\Delta(G)$-edge-coloring ϕ. Let E_i denote the set of edges colored with the color i for $i = 1, 2, \ldots, \Delta(G)$. Set $H = E_1 \cup E_2 \cup \cdots \cup E_9$ and $H' = E_{10} \cup E_{11} \cup \cdots \cup E_{\Delta(G)}$. Then H and H' are planar graphs with $\Delta(H) = 9$ and $\Delta(H') = \Delta(G) - 9$, respectively. By Lemmas 2.2 and 2.3 and Theorem 3.1, we deduce that $\mathrm{la}_2(G) \le \mathrm{la}_2(H) + \mathrm{la}_2(H') \le 8 + \Delta(G) - 9 \le \Delta(G) - 1$. □

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