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WELL-POSEDNESS AND NUMERICAL SIMULATIONS OF AN ANISOTROPIC REACTION-DIFFUSION MODEL IN CASE 2D

Anca Croitoru[1], Costică Moroşanu[1,†] and Gabriela Tănase[1]

Abstract This paper presents a qualitative study of a nonlinear second-order parabolic problem, endowed with a nonlinearity of cubic type as well as non-homogeneous Cauchy-Neumann boundary conditions. Under certain hypotheses on the input data $(f(t,x), w(t,x), v_0(x))$, we prove the well-posedness and a priori estimates of a solution in the Sobolev space $W_p^{1,2}(Q)$, extending the results already proven by other authors. Our mathematical model can be applied in many physical phenomena, such as image processing. Numerical simulations illustrate the effectiveness of the mathematical model in image restoration.

Keywords Nonlinear anisotropic reaction-diffusion, well-posedness of solutions, Leray-Schauder principle, finite difference method, numerical approximation scheme, image restoration.

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1. Introduction

Let $\Omega \subset I\!R^2$ be a bounded domain with a C^2 boundary $\partial\Omega$ and let $T > 0$. We examine the next problem of second-order boundary value:

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} v(t,x) = p_2 K(v_x(t,x))\Delta v(t,x) + p_2 \nabla K(v_x(t,x)) \cdot \nabla v(t,x) \\ \qquad\qquad + p_3 \left[v(t,x) - v^3(t,x)\right] + f(t,x), & \text{in } Q, \\ q(t,x)\dfrac{\partial}{\partial \mathbf{n}} v(t,x) + v(t,x) = w(t,x), & \text{on } \Sigma, \\ v(0,x) = v_0(x), & \text{on } \Omega, \end{cases} \tag{1.1}$$

where:

- $t \in (0,T]$, $x = (x_1, x_2) \in \Omega$, $Q = (0,T] \times \Omega$, $\Sigma = (0,T] \times \partial\Omega$;
- $v(t,x)$ (v **in short**) is the unknown function and denote by $\nabla v(t,x) = v_x(t,x)$ ($\nabla v = v_x$ **in short**) the gradient of $v(t,x)$ in x, that is $\nabla v = \left(\dfrac{\partial}{\partial x_1} v, \dfrac{\partial}{\partial x_2} v\right)$. We set $\dfrac{\partial}{\partial x_i} v = v_{x_i}$, $i = 1, 2$, and so $v_x = \left(v_{x_1}, v_{x_2}\right)$;

[†]The corresponding author. Email address: costica.morosanu@uaic.ro (C. Moroşanu) croitoru@uaic.ro(A. Croitoru), gabriela.tanase@uaic.ro(G. Tănase)
[1]Department of Mathematics, University of Iasi, bd. Carol I, Iasi, 700506, Romania

- $\Delta v(t,x)$ is the Laplace operator - a second-order differential operator, defined as the divergence $(\nabla\cdot)$ of the gradient of $v(t,x)$ in x, i.e.

$$\Delta v(t,x) = div(\nabla v(t,x)) = \nabla \cdot \nabla v(t,x) = \nabla^2 v(t,x);$$

- p_1, p_2, p_3, are positive values;
- $K(v_x(t,x))$ - is the *mobility* (attached to the solution $v(t,x)$ of (1.1));
- $q(t,x)$ - is a positive and bounded real function;
- $f(t,x) \in L^p(Q)$ is the distributed control, where

$$p \geq 2; \tag{1.2}$$

- $w(t,x) \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$ is the boundary control;
- $v_0(x) \in W_\infty^{2-\frac{2}{p}}(\Omega)$ and satisfies

$$q(0,x)\frac{\partial}{\partial \mathbf{n}}v_0(x) + v_0(x) = w(0,x);$$

- $\mathbf{n}$=$\mathbf{n}(x)$ is the outward unit normal vector to Ω at a point $x \in \partial\Omega$; $\frac{\partial}{\partial \mathbf{n}}$ means differentiation along $\mathbf{n}$.

Let's note

$$a_i(t,x,v(t,x),v_x(t,x)) = p_2 K\big(v_x(t,x)\big)v_{x_i}(t,x), \qquad i=1,2. \tag{1.3}$$

Then, it is easy to recognize equation $(1.1)_1$ as being a quasi-linear one of type (2.4) in [11] - p. 3 and p. 11, with

$$\begin{aligned} a_{ij}\big(t,x,v(t,x),v_x(t,x)\big) &= \frac{\partial}{\partial v_{x_j}} a_i\big(t,x,v(t,x),v_x(t,x)\big) \\ &= \frac{\partial}{\partial v_{x_j}} p_2 K\big(v_x(t,x)\big)v_{x_i}(t,x), \qquad i=1,2, \end{aligned}$$

and

$$\begin{aligned} a\big(t,x,v(t,x),v_x(t,x)\big) = &-\frac{\partial}{\partial v}\Big(p_2 K(v_x(t,x))v_{x_i}(t,x)\Big)v_{x_i}(t,x) \\ &- \frac{\partial}{\partial x_i} p_2 K\big(v_x(t,x)\big)v_{x_i}(t,x) \\ &- p_3\left[v(t,x) - v^3(t,x)\right] - f(t,x), \end{aligned}$$

while the boundary conditions $(1.1)_2$ are of second type:

$$[a_{ij}\big(t,x,v(t,x),v_x(t,x)\big)v_{x_j}(t,x)\ cos(\mathbf{n},x_i) + v(t,x) - w(t,x)]\Big|_{\Sigma} = 0$$

(see [11] - p. 475, relation (7.2)).

For reader's convenience, we will write problem (1.1) in the equivalent form

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} v(t,x) - \operatorname{div}\Big(p_2 K\big(v_x(t,x)\big)\ \nabla v(t,x)\Big) & \\ = p_3 \left[v(t,x) - v^3(t,x)\right] + f(t,x), & \text{in } \ Q, \\ q(t,x) \dfrac{\partial}{\partial \mathbf{n}} v(t,x) + v(t,x) = w(t,x), & \text{on } \ \Sigma, \\ v(0,x) = v_0(x), & \text{on } \ \Omega. \end{cases} \tag{1.4}$$

Concerning equation $(1.4)_1$, we recall that this is a quasi-linear one with principal part in *divergence form* (see (2.3) of page 11 in [11]), with a_i, $i = 1, 2$, given by (1.3) and

$$a(t,x,v(t,x),v_x(t,x)) = -p_3 \left[v(t,x) - v^3(t,x)\right] - f(t,x).$$

Furthermore, we suppose that equations $(1.1)_1$ [or $(1.4)_1$] are *uniformly parabolic*, that is

$$\nu_1(|u|)\zeta^2 \le a_{ij}\big(t,x,u,z\big)\zeta_i\zeta_j \le \nu_2(|u|)\zeta^2, \tag{1.5}$$

for arbitrary $u(t,x)$ and $z(t,x)$, $(t,x) \in Q$ and $\zeta = (\zeta_1, \zeta_2)$ an arbitrary real vector, where $\nu_1(s)$ and $\nu_2(s)$ are positive continuous functions of $s \ge 0$, $\nu_1(s)$ is nonincreasing and $\nu_2(s)$ is nondecreasing.

The starting point in the elaboration of this paper is [15], with great interest - in theory and applications equaly - focused on finding concrete cases of functions for the general case $K(t,x,u(t,x),u_x(t,x))$ introduced in [15], different from those of [3]. In this respect, we elaborated a complete qualitative and quantitative study and, by using a single method (PSNR), we gained in efficiency of our proposed model (problem (3.2)), as we have 10 iterations (see Figure 4) compared to the 35^{th} ones of [3]. Moreover, the approach of partial time derivative is done by the help of a second order scheme, while in [3] a first order scheme is used.

Another advantage in our paper concerns the non-homogeneous Cauchy-Neumann boundary conditions $(1.1)_2$ which can be seen as *boundary control* in industry. Thus, the system (1.1) can be applied in problems of `moving interface` such as the nucleation of solids, the mixture of two incompressible fluids, vesicle membranes ([1,14,27,32]), the `phase separation and transition` ([4,13,16–23], [25,26,31]), the `anisotropy effects` ([3, 6, 28]), the `image denoising` and `restoration` ([2,3,5,10,29,33]). Another novelty of our study is that in $(1.1)_1$ we consider a cubic nonlinearity $v - v^3$, unlike the linear reaction (see [2,3]). Therefore, the presence of nonlinear terms at the level of diffusion and reaction, increases the possibility to better capture the complexity of real world phenomena.

Definition 1.1. *The function $v(t,x)$ is called a* `classic solution` *of the problem* (1.1) *if it is continuous in $\bar{Q}$, has continuous derivatives v_t, v_x, v_{xx} in Q, verifies* $(1.1)_1$ *in every $(t,x) \in Q$ and verifies* $(1.1)_2$ *and* $(1.1)_3$ *for $(t,x) \in \Sigma$ and $t = 0$, respectively.*

In short, in our paper we study the solvability of the problem (1.4), characterized by the presence of some new physical parameters $(p_1, p_2, p_3,\ K(v_x))$, the principal part in *divergence form* and by considering the cubic nonlinearity $v - v^3$, satisfying the condition H_0 in [24]:

$H_0: \ (v - v^3)|v|^{3p-4}v \leq 1 + |v|^{3p-1} - |v|^{3p}$.

In Theorem 2.1, we prove the existence, the regularity and the uniqueness of the solution for (1.4) in $W_p^{1,2}(Q)$ (see (2.1) for $k = 1$).

See [3] for a numerical study of equation (1.1) corresponding to a linear term $v - v_0$, with homogeneous Neumann boundary condition.

In the sequel we will denote by C some positive constants.

2. Well-posedness of the solution of (1.4)

Theorem 2.1 of this Section presents the dependence of the solution $v(t, x)$ of (1.4) on $f(t, x)$ and $w(t, x)$. In our study, we rely on the following:

- the Leray-Schauder degree theory ([2–5, 7–9, 19–21]);
- the L^p-theory of linear and quasi-linear parabolic equations ([11]);
- Green's first identity

$$-\int_{\Omega} y \operatorname{div} z \, dx = \int_{\Omega} \nabla y \cdot z \, dx - \int_{\partial\Omega} y \frac{\partial}{\partial \mathbf{n}} z \, d\gamma,$$
$$-\int_{\Omega} y \Delta z \, dx = \int_{\Omega} \nabla y \cdot \nabla z \, dx - \int_{\partial\Omega} y \frac{\partial}{\partial \mathbf{n}} z \, d\gamma,$$

for any scalar-valued function y and z - a continuously differentiable vector field in n dimensional space;

- the Lions and Peetre embedding Theorem (see [12], p. 24) to ensure the existence of a continuous embedding $W_p^{1,2}(Q) \subset L^\mu(Q)$, where the real μ is given by (see (1.2)):

$$\mu = \begin{cases} \text{any positive number } \geq 3p, & \text{if} \quad p \geq 2, \\ \left(\dfrac{1}{p} - \dfrac{1}{2}\right)^{-1}, & \text{if} \quad p < 2, \end{cases}$$

and, for $k \in \{1, 2, \cdots\}$ and $1 \leq p \leq \infty$, $W_p^{k,2k}(Q)$ denotes the Sobolev space on Q:

$$W_p^{k,2k}(Q) = \left\{ y \in L^p(Q) : \ \frac{\partial^r}{\partial t^r} \frac{\partial^q}{\partial x^q} \, y \in L^p(Q), \ \text{for } 2r + q \leq 2k \right\}, \tag{2.1}$$

i.e., the spaces of functions whose t-derivatives and x-derivatives up to the order k and $2k$, respectively, belong to $L^p(Q)$ (see [11], p. 5).

Also, we shall use the set $C^{1,2}(\bar{Q})$ ($C^{1,2}(Q)$) of all continuous functions in $\bar{Q}$ (in Q) having continuous derivatives u_t, u_x, u_{xx} in $\bar{Q}$ (in Q), as well as the Sobolev spaces $W_p^l(\Omega)$, $W_p^{l,l/2}(\Sigma)$ with non integral l for the initial and boundary conditions, respectively (see p. 8, p. 70 and p. 81 of [11]).

The main result for study the existence, a priori estimates, uniqueness and regularity for the solution of (1.4) is the next theorem.

Theorem 2.1. *For any classic solution $v(t,x) \in C^{1,2}(Q)$ of* (1.4)*, suppose there are M, M_1, M_2, M_3, $M_4 \in (0,\infty)$ such that the next hypotheses are satisfied:*

$\mathbf{I}_1$. *$|v(t,x_1,x_2)| < M$, $\forall (t,x_1,x_2) \in Q$ and for any $z(t,x)$, the map $K(z(t,x))$ is continuous, differentiable in x, its x-derivatives are measurable bounded and verifies* (1.5)*, as well as*

$$0 < K_{min} \leq K(v_x(t,x)) < K_{max}, \qquad for \quad (t,x) \in Q, \tag{2.2}$$

$$\begin{aligned} &|K(z)v_{x_i}|(1+|z|) + \left|\frac{\partial}{\partial x_1}(K(z)v_{x_1})\right| + \left|\frac{\partial}{\partial x_2}(K(z)v_{x_1})\right| \\ &+ \left|\frac{\partial}{\partial x_1}(K(z)v_{x_2})\right| + \left|\frac{\partial}{\partial x_2}(K(z)v_{x_2})\right| + |v(t,x_1,x_2)| \leq M_1(1+|z|)^2 \end{aligned} \tag{2.3}$$

(see [11] p. 451).

$\mathbf{I}_2$. *For every $\varepsilon > 0$, the functions v and $K(v_x)$ satisfy the relations*

$$\|v\|_{L^s(Q)} \leq M_2, \qquad \|K(v_x)v_{x_i}\|_{L^r(Q)} < M_3, \qquad i = 1,2,$$

where

$$r = \begin{cases} \max\{p,4\} & p \neq 4 \\ 4+\varepsilon & p = 4, \end{cases} \qquad s = \begin{cases} \max\{p,2\} & p \neq 2 \\ 2+\varepsilon & p = 2. \end{cases}$$

Then, for all $f \in L^p(Q)$ and $v_0 \in W_\infty^{2-\frac{2}{p}}(\Omega)$, so that $p \neq \frac{3}{2}$, the problem (1.4) *has a solution $v \in W_p^{1,2}(Q)$ and the following estimate holds:*

$$\begin{aligned} \|v\|_{W_p^{1,2}(Q)} \leq C \Bigg[1 + \|v_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|v_0\|_{L^{3p-2}(\Omega)}^{3-\frac{2}{p}} \\ + \|f\|_{L^p(Q)} + \|w\|_{L^{3p-2}(\Sigma)}^{3-\frac{2}{p}} + \|w\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} \Bigg], \end{aligned} \tag{2.4}$$

where $C > 0$ is a constant, that does not depend on v, f and w.

If $v^1, v^2 \in W_p^{1,2}(Q)$ are two solutions to (1.4)*, corresponding to $\{f^1, w^1, v_0^1\}$ and $\{f^2, w^2, v_0^2\}$, respectively, such that $\|v^1\|_{W_p^{1,2}(Q)} \leq M_4$, $\|v^2\|_{W_p^{1,2}(Q)} \leq M_4$, and*

$$0 < K_{min} \leq q(t,x) < K_{max}, \qquad for \quad (t,x) \in \Sigma, \tag{2.5}$$

then the following estimate holds

$$\max_{(t,x)\in Q} |v^1 - v^2| \leq C_1 e^{CT} \max\left[\max_{(t,x)\in Q} |f^1 - f^2|, \max_{(t,x)\in\Sigma} |w^1 - w^2|, \max_{(t,x)\in\Omega} |v_0^1 - v_0^2|\right], \tag{2.6}$$

where $C, C_1 > 0$ are constants that do not depend on $\{v^1, f^1, w^1, v_0^1\}$ and $\{v^2, f^2, w^2, v_0^2\}$. Particularly, it results that the solution of (1.4) *is unique.*

Proof. For the proof, we use the Leray-Schauder principle. So, we consider the Banach space

$$B = W_p^{0,1}(Q) \cap L^{3p}(Q),$$

endowed with the norm

$$\|u\|_B = \|u\|_{L^p(Q)} + \|u_x\|_{L^p(Q)},$$

and a nonlinear operator $H : B \times [0,1] \to B$ defined by

$$v = v(u,\lambda) = H(u,\lambda) \qquad \text{for all} \qquad (u,\lambda) \in W_p^{0,1}(Q) \cap L^{3p}(Q) \times [0,1], \tag{2.7}$$

where $v(u,\lambda)$ is the unique solution of the next problem

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} v(t,x) - \left[\lambda \dfrac{\partial}{\partial u_{x_j}} (p_2 K(u_x) u_{x_i}) + (1-\lambda)\delta_i^j\right] v_{x_i x_j} \\ = \lambda \Big\{ A(t,x,u,u_x) + p_3 \left[u(t,x) - u^3(t,x)\right] + f(t,x) \Big\}, & \text{in } Q, \\ q(t,x) \dfrac{\partial}{\partial \nu} v(t,x) + v(t,x) = \lambda w(t,x), & \text{on } \Sigma, \\ v(0,x) = \lambda v_0(x), & \text{on } \Omega, \end{cases} \tag{2.8}$$

where $v_{x_i x_j} = \frac{\partial^2}{\partial x_i \partial x_j} v(t,x),\ i,j = 1,2$ and $A(t,x,u,u_x) = p_2 \nabla K(u_x) \cdot \nabla u(t,x)$, for all $(t,x) \in Q$.

Firstly, we show that the nonlinear operator H satisfies the following properties **A** and **B**.

A. If (2.8) has a unique solution, then H is well-defined.

By the right hand of $(2.8)_1$, using I_1 and I_2, it results that

$$A(t,x,u,u_{x_i}) + p_3(u - u^3) + f(t,x) \in L^p(Q), \qquad \forall u \in W_p^{0,1}(Q) \cap L^{3p}(Q). \tag{2.9}$$

Indeed, since $u \in L^{3p}(Q)$, then $\|u\|_{L^{3p}(Q)} \leq Konst$ and thus $\|u^3\|_{L^p(Q)} \leq (Konst)^3$, i.e., for the nonlinear term u^3 in (2.9) we can deduce that $u^3 \in L^p(Q)$ (see [15] for more details). Next, from the properties of K and since $u_{x_i} \in L^p(Q)$, it results that $A(t,x,u,u_{x_i}) \in L^p(Q)$.

According to Theorem 7.4, Theorem 9.1 (for $p = 3$), Theorem 9.2 in [11], p. 491, p. 341 and p. 343, respectively, and (2.9), for $w(t,x) \in W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(\Sigma)$, the parabolic boundary value problem (2.8) has a unique solution $v \in W_p^{1,2}(Q) \subset L^p(Q)$ (see (2.30) at p. 54 of [11]). Thus, the operator H is well-defined.

B. Let us now show that H is continuous and compact. Let $u^n \to u$ in $W_p^{0,1}(Q) \cap L^{3p}(Q)$ and $\lambda_n \to \lambda$ in $[0,1]$. Making the notation

$$v^{n,\lambda_n} = H(u^n,\lambda_n), \qquad v^{n,\lambda} = H(u^n,\lambda) \qquad \text{and} \qquad v^\lambda = H(u,\lambda)$$

and then considering the difference $H(u^n,\lambda_n) - H(u^n,\lambda)$, we obtain from relations (2.7) and (2.8) that

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} V^{n,\lambda_n,\lambda} - \left[\lambda \dfrac{\partial}{\partial u^n_{x_j}} \left(p_2 K(u^n_x) u^n_{x_i}\right) + (1-\lambda)\delta_i^j\right] V^{n,\lambda_n,\lambda}_{x_i x_j} \\ = (\lambda_n - \lambda) \left\{ \left[\frac{\partial}{\partial u^n_{x_j}} (p_2 K(u^n_x) u^n_{x_i}) - \delta_i^j\right] v^{n,\lambda_n}_{x_i x_j} \right. \\ \qquad \left. + A(t,x,u^n,u^n_{x_i}) + p_3\left[u^n - (u^n)^3\right] + f(t,x) \right\}, & \text{in } Q, \\ q(t,x) \frac{\partial}{\partial \mathbf{n}} V^{n,\lambda_n,\lambda} + V^{n,\lambda_n,\lambda} = (\lambda_n - \lambda) w(t,x), & \text{on } \Sigma, \\ v(0,x) = (\lambda_n - \lambda) v_0(x), & \text{on } \Omega, \end{cases} \tag{2.10}$$

where $V^{n,\lambda_n,\lambda} = v^{n,\lambda_n} - v^{n,\lambda}$.

The right-hand side in (2.10) belongs to $L^p(Q)$, since $v^{n,\lambda_n} \in W_p^{1,2}(Q)$. Therefore, the L^p-theory of PDE gives the estimate

$$\begin{aligned}\|V^{n,\lambda_n,\lambda}\|_{W_p^{1,2}(Q)} \leq C|\lambda_n - \lambda| \times \Bigg\{ & \left\| \left[\tfrac{\partial}{\partial u^n_{x_j}} \left(p_2 K(u^n_x) u^n_{x_i} \right) - \delta_i^j \right] v^{n,\lambda_n}_{x_i x_j} \right\|_{L^p(Q)} \\ & + \|A(t,x,u^n,u^n_{x_i})\|_{L^p(Q)} + \|u^n - (u^n)^3\|_{L^p(Q)} \\ & + \|v_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|f\|_{L^p(Q)} + \|w\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} \Bigg\},\end{aligned}$$

with a constant $C(|\Omega|, p_1, p_2, p_3, M, M_1, M_2, M_3)$.

Having u^n bounded in $W_p^{0,1}(Q) \cap L^{3p}(Q)$, it results that $(u^n)^3$ is bounded in $L^p(Q)$ (see, e.g., [8] or [11], p. 42). In addition, the inequality (2.3), the condition I_2 and the inclusion $u^{n,\lambda_n}_{x_i x_j} \in L^p(Q)$ imply the boundedness in $L^p(Q)$ of the terms $A(t,x,u^n,u^n_{x_i})$, $\left(\frac{\partial}{\partial u^n_{x_j}} (p_2 K(u^n_x) u^n_{x_i}) - \delta_i^j \right) v^{n,\lambda_n}_{x_i x_j}$. Also, since $W_\infty^{2-\frac{2}{p}}(\Omega) \subset L^p(\Omega)$, it results that the other terms on the right-hand side from the above inequality are also bounded in $L^p(Q)$. Thus, making $\lambda_n \to \lambda$, we obtain ($V^{n,\lambda_n,\lambda} = v^{n,\lambda_n} - v^{n,\lambda}$)

$$\|v^{n,\lambda_n} - v^{n,\lambda}\|_{W_p^{1,2}(Q)} \to 0 \quad \text{for } n \to \infty. \tag{2.11}$$

In order to evaluate the difference $H(v^n, \lambda) - H(v, \lambda)$, we use again the relations (2.7), (2.8) and we obtain

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} V^{n,1,\lambda} - \left[\lambda \dfrac{\partial}{\partial u^n_{x_j}} \left(p_2 K(u^n_x) u^n_{x_i} \right) + (1-\lambda) \delta_i^j \right] V^{n,1,\lambda}_{x_i x_j} & \\ \quad = \lambda \Big\{ p_2 \left[\frac{\partial}{\partial u^n_{x_j}} (K(u^n_x) u^n_{x_i}) - \frac{\partial}{\partial u_{x_j}} (K(u_x) u_{x_i}) \right] v^\lambda_{x_i x_j} & \\ \quad + A(t,x,u^n,u^n_{x_i}) - A(t,x,u,u_{x_i}) & \\ \quad + p_3 \left[(u^n - u) - \left((u^n)^3 - u^3 \right) \right] \Big\}, & \text{in } Q, \\ q(t,x) \frac{\partial}{\partial \mathbf{n}} V^{n,1,\lambda} + V^{n,1,\lambda} = 0, & \text{on } \Sigma, \\ V^{n,1,\lambda}(0,x) = 0, & \text{on } \Omega, \end{cases} \tag{2.12}$$

where $V^{n,1,\lambda} = v^{n,\lambda} - v^\lambda$.

The L^p-theory applied to (2.12), give us the estimate

$$\begin{aligned}\|V^{n,1,\lambda}\|_{W_p^{1,2}(Q)} \leq C\lambda \Bigg[& \left\| \left(\tfrac{\partial}{\partial u^n_{x_j}} (K(u^n_x) u^n_{x_i}) - \tfrac{\partial}{\partial u_{x_j}} (K(u_x) u_{x_i}) \right) v^\lambda_{x_i x_j} \right\|_{L^p(Q)} \\ & + \|A(t,x,u^n,u^n_{x_i}) - A(t,x,u,u_{x_i})\|_{L^p(Q)} \\ & + \|(u^n - u) - ((u^n)^3 - u^3)\|_{L^p(Q)} \Bigg],\end{aligned}$$

with a constant C. From the convergence $v_n \to v$ in $W_p^{0,1}(Q) \cap L^{3p}(Q)$, the continuity of the Nemytskij operator ([8]), and the continuity of $\frac{\partial}{\partial u^n_{x_j}} (K(u^n_{x_i}) u^n_{x_i})$ and

$A(t,x,u^n,u^n_{x_i})$, it follows that

$$\|v^{n,\lambda} - v^{\lambda}\|_{W_p^{1,2}(Q)} \to 0 \ \text{ as } \ n \to \infty. \tag{2.13}$$

Using (2.11) and (2.13), we obtain the continuity of H defined by (2.7). Now, we prove that H is compact. Indeed, since $\mu > 3p$, the inclusion $W_p^{1,2}(Q) \hookrightarrow W_p^{0,1}(Q) \cap L^{3p}(Q)$ is compact (see [12], p. 21]). Furthermore, writing H as

$$B \times [0,1] \to W_p^{1,2}(Q) \hookrightarrow W_p^{0,1}(Q) \cap L^{3p}(Q) = B,$$

it results that H is compact.

2.1. The regularity of the solution $v(t,x)$

In this subsection, we prove that there exists $\delta > 0$ such that

$$(v,\lambda) \in B \times [0,1], \qquad v = H(v,\lambda) \implies \|v\|_B < \delta. \tag{2.14}$$

Let $v \in B = W_p^{0,1}(Q) \cap L^{3p}(Q)$. Then, the equality $v = H(v,\lambda)$ in (2.14) is equivalent to

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} v(t,x) - \lambda \, \mathrm{div}\Big(p_2 K(v_x)\nabla v\Big) - (1-\lambda)\Delta v & \\ = \lambda\Big[p_3\Big(v(t,x) - v^3(t,x)\Big) + f(t,x)\Big], & \text{in } Q, \\ q(t,x)\dfrac{\partial}{\partial \mathbf{n}} v(t,x) + v(t,x) = \lambda w(t,x), & \text{on } \Sigma, \\ v(0,x) = \lambda v_0(x), & \text{on } \Omega, \end{cases} \tag{2.15}$$

(see (1.1), (1.4) and (2.8)).

Multiplying the first equation in (2.15) by $|v|^{3p-4}v$, integrating over $Q_t := (0,t)\times\Omega$, $t \in (0,T]$, we get

$$\begin{aligned} & p_1 \int_{Q_t} \frac{\partial}{\partial t}|v(\tau,x)|^{3p-2}\, d\tau dx - \lambda \int_{Q_t} \mathrm{div}\Big(p_2 K(v_x)\nabla v\Big)|v|^{3p-4}v \, d\tau dx \\ & - (1-\lambda)\int_{Q_t} \Delta v \ |v|^{3p-4} v \, d\tau dx \\ =& \lambda p_3 \int_{Q_t} (v - v^3)|v|^{3p-4}v \, d\tau dx + \lambda \int_{Q_t} f|v|^{3p-4}v \, d\tau dx. \end{aligned}$$

Owing to the Green's first identity, left inequality in (2.2) and (2.5), the condition (I_2) and the boundary conditions $(2.15)_2$, the previous equality gives us

$$\begin{aligned} & \frac{p_1}{3p-2}\int_{\Omega} |v(t,x)|^{3p-2}\, dx + \lambda \int_{Q_t} K(v_x)\nabla v \cdot \nabla\Big(p_2|v|^{3p-4}v\Big)\, d\tau dx \\ & + (1-\lambda)(3p-3)\int_{Q_t} |\nabla v|^2 |v|^{3p-4}\, d\tau dx \end{aligned}$$

$$+\lambda p_2 \int_{\Sigma_t} |v|^{3p-2}\, d\tau d\gamma + \frac{(1-\lambda)}{K_{max}} \int_{\Sigma_t} |v|^{3p-2}\, d\tau d\gamma$$

$$\leq \lambda \frac{p_1}{3p-2} \int_{\Omega} |v_0(x)|^{3p-2}\, dx$$

$$+\lambda p_3 \int_{Q_t} (v-v^3)|u|^{3p-4} v\, d\tau dx + \lambda \int_{Q_t} f|v|^{3p-4} v\, d\tau dx$$

$$+\lambda p_2 \int_{\Sigma_t} w|v|^{3p-4} v\, d\tau d\gamma + \frac{(1-\lambda)}{K_{min}} \int_{\Sigma_t} w|v|^{3p-4} v\, d\tau d\gamma, \tag{2.16}$$

for every $t \in (0, T]$. Applying Hölder's and Cauchy's Inequalities to the last terms in (2.16), we get

a. $\lambda \int_{Q_t} f|v|^{3p-4} v\, d\tau dx \leq \frac{p-1}{p} \varepsilon^{\frac{p}{p-1}} \int_{Q_t} |v|^{3p}\, d\tau dx + \lambda \frac{1}{p} \varepsilon^{-p} \|f\|^p_{L^p(Q)}$

b. $\lambda p_2 \int_{\Sigma_t} w|v|^{3p-4} v\, d\tau d\gamma$

$$\leq \lambda p_2 \left(1 - \frac{1}{3p-2}\right) \int_{\Sigma_t} |v|^{3p-2}\, d\tau d\gamma + \frac{1}{p_2} \frac{1}{3p-2} \int_{\Sigma_t} |w|^p\, d\tau d\gamma,$$

c. $\frac{(1-\lambda)}{K_{min}} \int_{\Sigma_t} w|v|^{3p-4} v\, d\tau d\gamma$

$$\leq \left(1 - \frac{1}{3p-2}\right) \frac{(1-\lambda)}{K_{max}} \int_{\Sigma_t} |v|^{3p-2} d\tau d\gamma + \frac{K_{max}}{K_{min}} \frac{1}{3p-2} \int_{\Sigma_t} |w|^{3p-2}\, d\tau d\gamma.$$

By H_0, relation (1.2) and Young's inequality, we obtain

$$\lambda p_3 \int_{Q_t} (v-v^3)|v|^{3p-4} v\, d\tau dx$$

$$\leq \lambda p_2 |\Omega| T + \lambda p_2 |\Omega| T \frac{1}{3p} \varepsilon^{-3p} + \frac{3p-1}{3p} \varepsilon^{\frac{3p}{3p-1}} \int_{Q_t} |v|^{3p} d\tau dx$$

$$-\lambda p_3 \int_{Q_t} |v|^{3p} d\tau dx.$$

From the previous inequality, the continuous embedding $L^{3p-2}(\Sigma_t) \subset L^p(\Sigma_t)$, and (**a.-c.**), by (2.16) we derive the following estimate

$$\frac{p_1}{3p-2} \int_{\Omega} |v(t,x)|^{3p-2}\, dx + \lambda \int_{Q_t} K(v_x) \nabla v \cdot \nabla\left(p_2 |v|^{3p-4} v\right)\, d\tau dx$$

$$+(1-\lambda)(3p-3) \int_{Q_t} |\nabla v|^2 |v|^{3p-4}\, d\tau dx + \lambda p_3 \int_{Q_t} |v|^{3p} d\tau dx$$

$$\frac{1}{3p-2} \left[\lambda p_2 + \frac{(1-\lambda)}{K_{max}}\right] \int_{\Sigma_t} |v|^{3p-2} d\tau d\gamma$$

$$
\begin{aligned}
&\le \lambda \frac{p_1}{3p-2} \int_\Omega |v_0(x)|^{3p-2}\, dx \\
&\quad + \left[\frac{3p-1}{3p} \varepsilon^{\frac{3p}{3p-1}} + \frac{p-1}{p} \varepsilon^{\frac{p}{p-1}} \right] \int_{Q_t} |v|^{3p}\, d\tau dx \\
&\quad + \lambda \left(p_2 |\Omega| T + p_2 |\Omega| T \frac{1}{3p} \varepsilon^{-3p} + \frac{1}{p} \varepsilon^{-p} \|f\|^p_{L^p(Q)} \right) \\
&\quad + \frac{1}{3p-2} \left[p_2 + \frac{K_{max}}{K_{min}} \right] \int_{\Sigma_t} |w|^{3p-2}\, d\tau d\gamma.
\end{aligned}
$$

Considering $\varepsilon > 0$ small, from the above inequality it holds

$$
\lambda \| |v|^3 \|^p_{L^p(Q)} \le C_1 \left(1 + \|v_0\|^{3p-2}_{L^{3p-2}(\Omega)} + \|f\|^p_{L^p(Q)} + \|w\|^{3p-2}_{L^{3p-2}(\Sigma_t)} \right), \tag{2.17}
$$

for a positive constant $C_1 = C(|\Omega|, T, n, p, p_1, p_2, p_3, K_{min}, K_{max}, M_1)$.

Applying L^p-theory to problem (2.8) (see [11], p. 341–342]), we get

$$
\begin{aligned}
\|v\|_{W^{1,2}_p(Q)} \le C_2 \Bigg(& \|v_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + p_3 \|(v - v^3)\|_{L^p(Q)} \\
& + \|f\|_{L^p(Q)} + \|w\|_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)} \Bigg),
\end{aligned}
\tag{2.18}
$$

for a constant $C_2 = C(|\Omega|, T, n, p, p_1, p_2, p_3) > 0$.

Using Lemma 1.1 in [24] and (2.17), we obtain

$$
\|v - v^3\|_{L^p(Q)} \le C_1 \left(1 + \|v_0\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Omega)} + \|f\|_{L^p(Q)} + \|w\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Sigma)} \right),
$$

and then (2.18) becomes

$$
\begin{aligned}
\|v\|_{W^{1,2}_p(Q)} \le C_2 \Bigg(& 1 + \|v_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|v_0\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Omega)} \\
& + \|f\|_{L^p(Q)} + \|w\|_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)} + \|w\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Sigma_t)} \Bigg).
\end{aligned}
\tag{2.19}
$$

The continuous embedding $W^{1,2}_p(Q) \subset B = W^{0,1}_p(Q) \cap L^{3p}(Q)$ ensures that

$$
\|v\|_B \le C \|v\|_{W^{1,2}_p(Q)},
$$

which, due to (2.19), implies that there exists $\delta > 0$ verifying (2.14).

Denoting

$$
B_\delta := \Big\{ v \in B : \ \|v\|_B < \delta \Big\},
$$

(2.14) implies that

$$
H(v, \lambda) \ne v \quad \forall v \in \partial B_\delta, \quad \forall \lambda \in [0, 1],
$$

for a large $\delta > 0$. Moreover, acting as in [4], [7], we obtain that (1.4) has a solution $v \in W^{1,2}_p(Q)$ (see also [24], p. 195). The estimate (2.4) results from (2.19) and the proof of the first part in Theorem 2.1 is finished. □

2.2. The uniqueness of the solution $v(t,x)$

Now, we prove (2.6) which implies the uniqueness of the solution of (1.1) or (1.4).

By hypothesis, $v^1, v^2 \in W_p^{1,2}(Q)$ solve problem (1.1) corresponding to $\{f^1, w^1, v_0^1\}$ and $\{f^2, w^2, v_0^2\}$, respectively. So $v^1 - v^2 \in W_p^{1,2}(Q)$.

Let us recall that

$$a_{ij}(t,x,v^1,v_x^1) = \frac{\partial}{\partial v_{x_j}^1} p_2 K(v_x^1) v_{x_i}^1,$$
$$a_{ij}(t,x,v^2,v_x^2) = \frac{\partial}{\partial v_{x_j}^2} p_2 K(v_x^2) v_{x_i}^2,$$

and (see (5.3) at p. 445 of [11]) we write

$$a_{ij}(t,x,v^1,v_x^1) - a_{ij}(t,x,v^2,v_x^2) = \int_0^1 \frac{d}{d\lambda} a_{ij}\left(t,x,v^\lambda,v_x^\lambda\right) d\lambda,$$

where

$$v^\lambda(t,x) = \lambda v^1(t,x) + (1-\lambda) v^2(t,x), \qquad v_x^\lambda(t,x) = \lambda v_x^1(t,x) + (1-\lambda) v_x^2(t,x).$$

Then

$$\begin{aligned} & a_{ij}(t,x,v^1,v_x^1) v^1_{x_i x_j} - a_{ij}(t,x,v^2,v_x^2) v^2_{x_i x_j} \\ =& a_{ij}(t,x,v^1,v_x^1) V_{x_i x_j} + v^2_{x_i x_j} \left[V_{x_i} \int_0^1 \frac{\partial}{\partial v_{x_j}^\lambda} a_{ij}\left(t,x,v^\lambda,v_x^\lambda\right) d\lambda \right. \\ & \left. + V \int_0^1 \frac{\partial}{\partial v^\lambda} a_{ij}\left(t,x,v^\lambda,v_x^\lambda\right) d\lambda \right], \end{aligned} \tag{2.20}$$

where $V(t,x) = v^1(t,x) - v^2(t,x)$.

Regarding $A(t,x,v,v_x)$, we have

$$\begin{aligned} & A(t,x,v^1,v_x^1) - A(t,x,v^2,v_x^2) \\ =& \int_0^1 \frac{d}{d\lambda} A\left(t,x,v^\lambda,v_x^\lambda\right) d\lambda \\ =& V_{x_i} \int_0^1 \frac{\partial}{\partial v_{x_j}^\lambda} A\left(t,x,v^\lambda,v_x^\lambda\right) d\lambda + V \int_0^1 \frac{\partial}{\partial v^\lambda} A\left(t,x,v^\lambda,v_x^\lambda\right) d\lambda. \end{aligned} \tag{2.21}$$

Now, we subtract the equation $(1.1)_1$ for $v^2(t,x)$ from the equations $(1.1)_1$ for $v^1(t,x)$ and by (2.5), (2.20) and (2.21), we get

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} V - \hat{a}_{ij}(t,x) V_{x_i x_j} + \hat{a}_i(t,x) V_{x_i} + \hat{a}(t,x) V = f^1 - f^2, & \text{in } Q, \\ q(t,x) \dfrac{\partial}{\partial \mathbf{n}} V + V = w^1 - w^2, & \text{on } \Sigma, \\ V(0,x) = v_0^1(x) - v_0^2(x), & \text{on } \Omega, \end{cases} \tag{2.22}$$

where

$$\hat{a}_{ij}(t,x) = a_{ij}(t,x,v^1,v_x^1),$$

$$\hat{a}_i(t,x) = -v^2_{x_i x_j} \int_0^1 \frac{\partial}{\partial v^\lambda_{x_j}} a_{i,j}\left(t,x,v^\lambda,v^\lambda_x\right) d\lambda + \int_0^1 \frac{\partial}{\partial v^\lambda_{x_j}} A\left(t,x,v^\lambda,v^\lambda_x\right) d\lambda,$$

$$\hat{a}(t,x) = -v^2_{x_i x_j} \int_0^1 \frac{\partial}{\partial v^\lambda} a_{i,j}\left(t,x,v^\lambda,v^\lambda_x\right) d\lambda + \int_0^1 \frac{\partial}{\partial v^\lambda} A\left(t,x,v^\lambda,v^\lambda_x\right) d\lambda$$
$$- p_2\left[1-\left((v^1)^2+v^1v^2+(v^2)^2\right)\right].$$

By (2.2) and the assumptions on v^1 and v^2, i.e.:

$$\|v^1\|_{W_p^{1,2}(Q)}, \|v^2\|_{W_p^{1,2}(Q)} \leq M_4,$$

the hypothesis of Theorem 2.3 in [11], p. 16 are fulfilled. So, from (2.22) it results that the estimate (2.6) is valid for V, which finishes the proof of Theorem 2.1.

As a consequence, it results the uniqueness for the solution of (1.1).

Corollary 2.1. *For the same initial conditions, the problem* (1.1) *possesses a unique solution* $v(t,x) \in W_p^{1,2}(Q)$.

Proof. Let $f^1 = f^2 = f$ and $w^1 = w^2 = w$ in Theorem 2.1. Then (2.6) demonstrates the corollary (see Theorem 2.4 - [11], p. 17).

Remark 2.1. The nonlinear operator H in (2.7) depends on $\lambda \in [0,1]$ and its fixed point for $\lambda = 1$ are solutions of (2.8).

3. A nonlinear second-order anisotropic reaction-diffusion model in image analysis

The nonlinear parabolic second-order PDE problem (1.4) can be applied for image denoising, enhancement, restoration, and segmentation. In Section 3 we consider a particularization of this differential model by setting the diffusivity (edge-stopping) function $K\big(v_x(t,x)\big)$ as the following form

$$K : [0,\infty) \to (0,1], \qquad K(s) = \frac{1}{e^{\left(\frac{s}{c}\right)^2}}, \tag{3.1}$$

where the parameter c is the conductance (see [28], p. 633). Also, we may take $f = w = 0$ and $q(t,x) = 1$, $t \in [0,T]$, $x = (x_1,x_2) \in \partial\Omega$. Therefore, the following PDE scheme with homogeneous Cauchy-Neumann boundary conditions is achieved:

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} v(t,x) - \operatorname{div}\Big(p_2 K\big(\|v_x(t,x)\|\big)\ \nabla v(t,x)\Big) & \\ = p_3\left[v(t,x) - v^3(t,x)\right], & \text{in } Q, \\ \dfrac{\partial}{\partial \mathbf{n}} v(t,x) + v(t,x) = 0, & \text{on } \Sigma, \\ v(0,x) = v_0(x), & \text{on } \Omega. \end{cases} \tag{3.2}$$

The edge-stopping (diffusivity) function in (3.1) satisfies the main conditions required by a proper diffusion, being positive monotonically decreasing, and converging to zero [28, 33]. Moreover, it is easy to check that K in (3.1) satisfies the assumption (2.3) in Theorem 2.1 and thus the nonlinear anisotropic reaction-diffusion model (3.2) is well-posed, as we proved in the previous section. So, it admits a unique classic solution $v(t,x) \in W_p^{1,2}(Q)$, that represents the evolving image of the observed image $v(0,x) = v_0(x)$. That solution is determined by solving numerically the nonlinear reaction-diffusion model (3.2), using the finite-differences method.

3.1. Numerical approximation

In this subsection we propose a numerical approximation scheme for the new nonlinear reaction-diffusion model (3.2) based on the finite difference method (see also [2, 3]). By using a grid of space size h, one quantizes the space coordinates $x = (x_1, x_2)$ as:

$$x1_i = ih, \quad x2_j = jh, \qquad \text{for all} \quad i = 1, 2 \dots, I, \quad j = 1, 2, \dots, J,$$

where $[Ih \times Jh]$ represents the dimension of the support image.

Given a positive value T and considering M as the number of equidistant nodes in which is divided the time interval $[0, T]$, we set

$$t_m = (m-1)\varepsilon, \qquad m = 1, 2, \dots, M, \qquad \varepsilon = T/(M-1).$$

Let's denote by $v_{i,j}^m$ the approximate values in the point $(t_m, x1_i, x2_j)$ of the unknown function $v(t,x)$ in (3.2), i.e.

$$v_{i,j}^m = v(t_m, x1_i, x2_j), \qquad m = 1, 2, \dots, M, \qquad i = 1, 2 \dots, I, \qquad j = 1, 2, \dots, J,$$

or, for later use

$$v^m = \left(v_{1,1}^m, v_{2,1}^m, \dots, v_{Ih,Jh}^m\right)^T \qquad m = 1, 2, \dots, M. \tag{3.3}$$

From the initial condition $(3.2)_3$, we have

$$v(0,x) \approx v^1 = v(t_1, x1_i, x2_j) = v_0(x1_i, x2_j), \qquad i = \overline{1, I}, \quad j = \overline{1, J}. \tag{3.4}$$

To approximate the partial derivative with respect to time, $\dfrac{\partial}{\partial t} v(t,x)$, we employed a *second-order scheme* (see [30]):

$$\frac{\partial}{\partial t} v(t_{m+1}, x1_i, x2_j) \approx \frac{3v_{i,j}^{m+1} - 4v_{i,j}^m + v_{i,j}^{m-1}}{2\varepsilon}, \tag{3.5}$$

$m = 2, 3, \dots, M-1, \quad i = \overline{1, I}, \quad j = \overline{1, J}.$

The partial differential equation in $(3.2)_1$ can be written in the form ($v_x(t,x) = \nabla v(t,x)$)

$$\begin{aligned} &p_1 \frac{\partial}{\partial t} v(t,x) + p_3 \left[v^3(t,x) - v(t,x)\right] \\ &= p_2 \left[\frac{\partial}{\partial x_1}\Big(K(\|v_x(t,x)\|) v_{x_1}\Big) + \frac{\partial}{\partial x_2}\Big(K(\|v_x(t,x)\|) v_{x_2}\Big)\right]. \end{aligned} \tag{3.6}$$

Its left term is approximated as

$$p_1 \frac{3v_{i,j}^{m+1} - 4v_{i,j}^{m} + v_{i,j}^{m-1}}{2\varepsilon} + p_3 \left[(v_{i,j}^{m})^3 - v_{i,j}^{m} \right],$$

and the right term is discretized using central differences (see [3] and references therein). Thus, we determine $K_{i,j} = K(\|\nabla v_{i,j}\|)$, where

$$\|\nabla v_{i,j}\| \approx \sqrt{\left(\frac{v_{i+1,j}^{m} - v_{i-1,j}^{m}}{2h}\right)^2 + \left(\frac{v_{i,j+1}^{m} - v_{i,j-1}^{m}}{2h}\right)^2},$$
$$\text{for all} \quad i = 1, 2 \ldots, I, \quad j = 1, 2, \ldots, J.$$

The component $\dfrac{\partial}{\partial x_1}(K(\|v_x(t,x)\|)v_{x_1})$ is discretized as

$$K_{i+\frac{1}{2},j}(v_{i+1,j}^{m} - v_{i,j}^{m}) - K_{i-\frac{1}{2},j}(v_{i,j}^{m} - v_{i-1,j}^{m})$$

while $\dfrac{\partial}{\partial x_2}(K(\|v_x(t,x)\|)v_{x_2})$ is approximated by

$$K_{i,j+\frac{1}{2}}(v_{i,j+1}^{m} - v_{i,j}^{m}) - K_{i,j-\frac{1}{2}}(v_{i,j}^{m} - v_{i,j-1}^{m})$$

where

$$K_{i\pm\frac{1}{2},j} = \frac{K_{i\pm1,j} + K_{i,j}}{2}, \qquad K_{i,j\pm\frac{1}{2}} = \frac{K_{i,j\pm1} + K_{i,j}}{2}.$$

Finally, one obtains the following explicit numerical approximation scheme:

$$\begin{aligned}\frac{3p_1}{2\varepsilon} v_{i,j}^{m+1} = & \left(\frac{2p_1}{\varepsilon} + p_3\right) v_{i,j}^{m} - p_3 (v_{i,j}^{m})^3 - \frac{p_1}{2\varepsilon} v_{i,j}^{m-1} + \frac{1}{p_2} v_{i,j}^{m}\Big|_{\partial\Omega} \\ & + p_2 \Big[K_{i+\frac{1}{2},j}(v_{i+1,j}^{m} - v_{i,j}^{m}) - K_{i-\frac{1}{2},j}(v_{i,j}^{m} - v_{i-1,j}^{m}) \\ & + K_{i,j+\frac{1}{2}}(v_{i,j+1}^{m} - v_{i,j}^{m}) - K_{i,j-\frac{1}{2}}(v_{i,j}^{m} - v_{i,j-1}^{m}) \Big].\end{aligned} \tag{3.7}$$

3.2. Experimental results

The iterative numerical approximation scheme provided by (3.7) has been successfully applied in our restoration experiments, for each $m = 1, 2, \ldots, M-1$, starting with $v^1 = v_0(x)$ (see (3.4)), which represents the initial $[Ih \times Jh]$ noisy image. The numerical tests consider the Allen-Cahn equation (see [1]), i.e., in the nonlinear reaction-diffusion equations (3.2) we take $p_1 = 1$, $p_2 = 200$ and $p_3 = 0.0001$. The developed explicit numerical approximation scheme is consistent to the nonlinear second-order anisotropic reaction-diffusion model given by (3.2).

The performance of the proposed restoration technique has been assessed by using the well-known performance measures Peak Signal to Noise Ratio (PSNR) (see [2, 3] and reference therein).

Some restoration results provided by these new techniques are displayed in figures 1–3. All pictures in Figure 1 were contaminated with same amount of noise. Images that have a higher segmentation, first two rows, will be harder to reconstruct

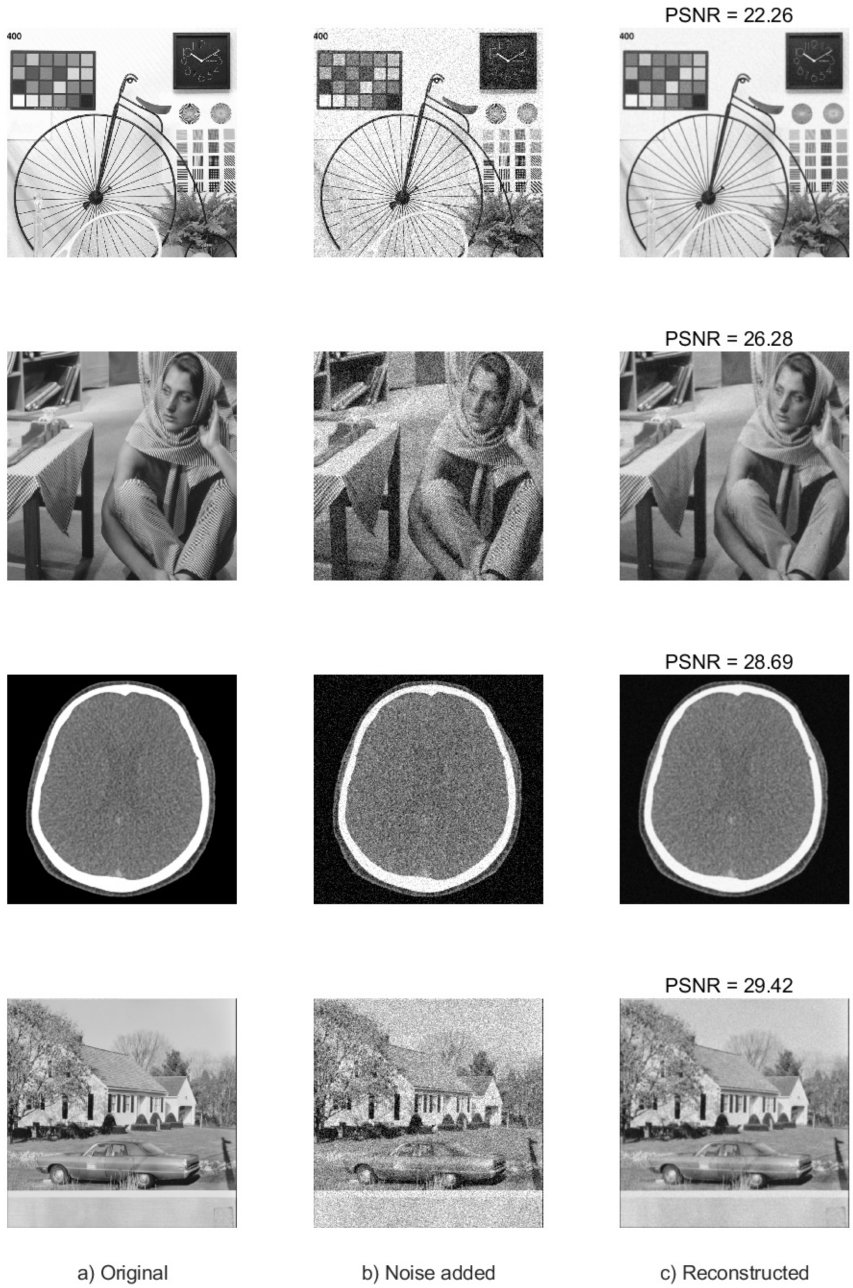

Figure 1. Noise reduction applied to 4 different pictures.

(lower PSNR) while more omogenos images, last two rows, will be reconstructed with a higher affinity grade.

In Figure 2 a) we can see a 100x100 pixel area, taken from original picture in Figure 3 a), represented as a 3D visualization of pixel intensity. In 2 b) the same area but taken from the noise contaminated picture Figure 3 b) second row, is also

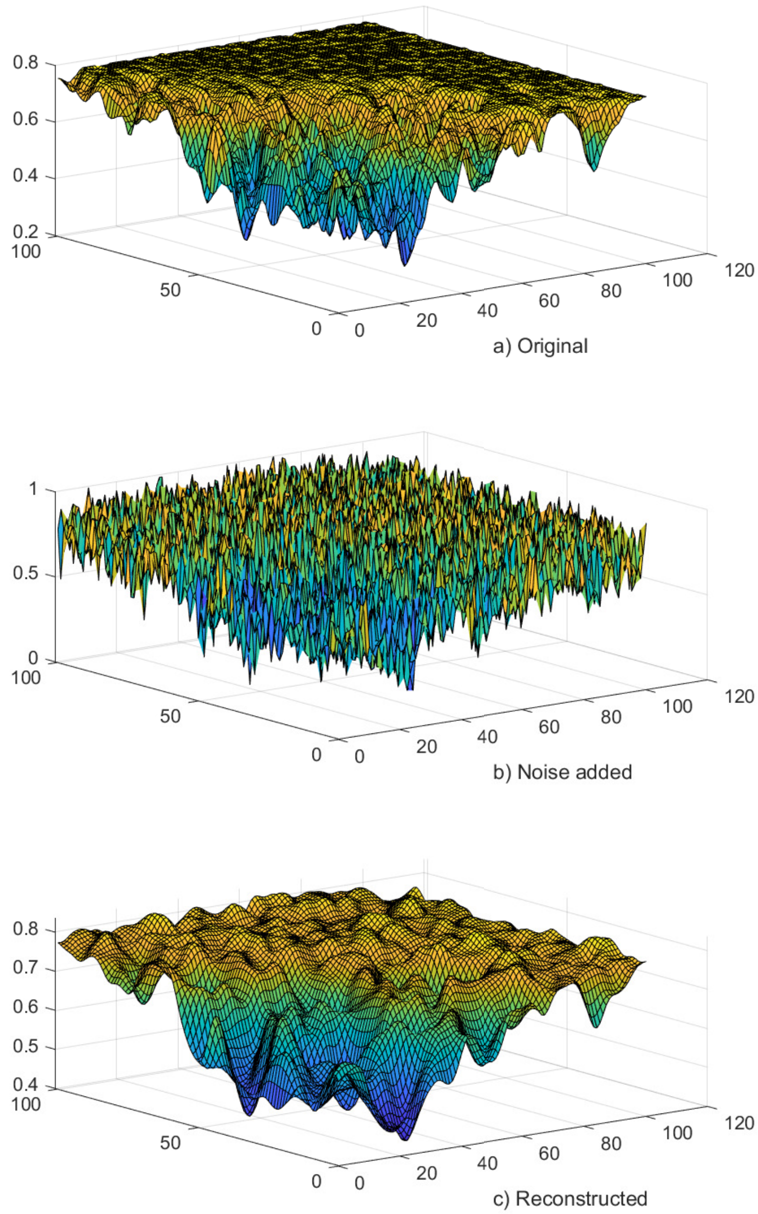

Figure 2. 3D detail of a $100x100$ pixels region for a) original image b) noise contaminated image and c) noise removed image.

represented in same 3D manner. This way we can visualize, comparing 2 a) and b), the amount of noise added to original picture, in b) the edges become almost invisible for the respective $100x100$ pixel area. Applying the noise removal model detailed in (3.2) respectively (3.7) to b) we obtain c). The visualisation in Figure 2 c) shows the noise is removed and the resulting picture is closely following the structure of original picture in a). All edges are well preserved (this we can see comparing a) and c)) while all sharp peaks in b) are removed.

Same original picture contaminated with different amount of noise, is considered in Figure 3. The noise doubles from top to bottom row. As expected, with less amount of noise, the reconstructed image will be closer to original after restoration (top row has the best PSNR).

The performed denoising tests show that our PDE-based scheme reduces considerably the noise, while preserving the image boundaries and other important features. Also, it avoids unintended effects, such as image blurring, blocky effect or speckle noise (see [3, 28], for example).

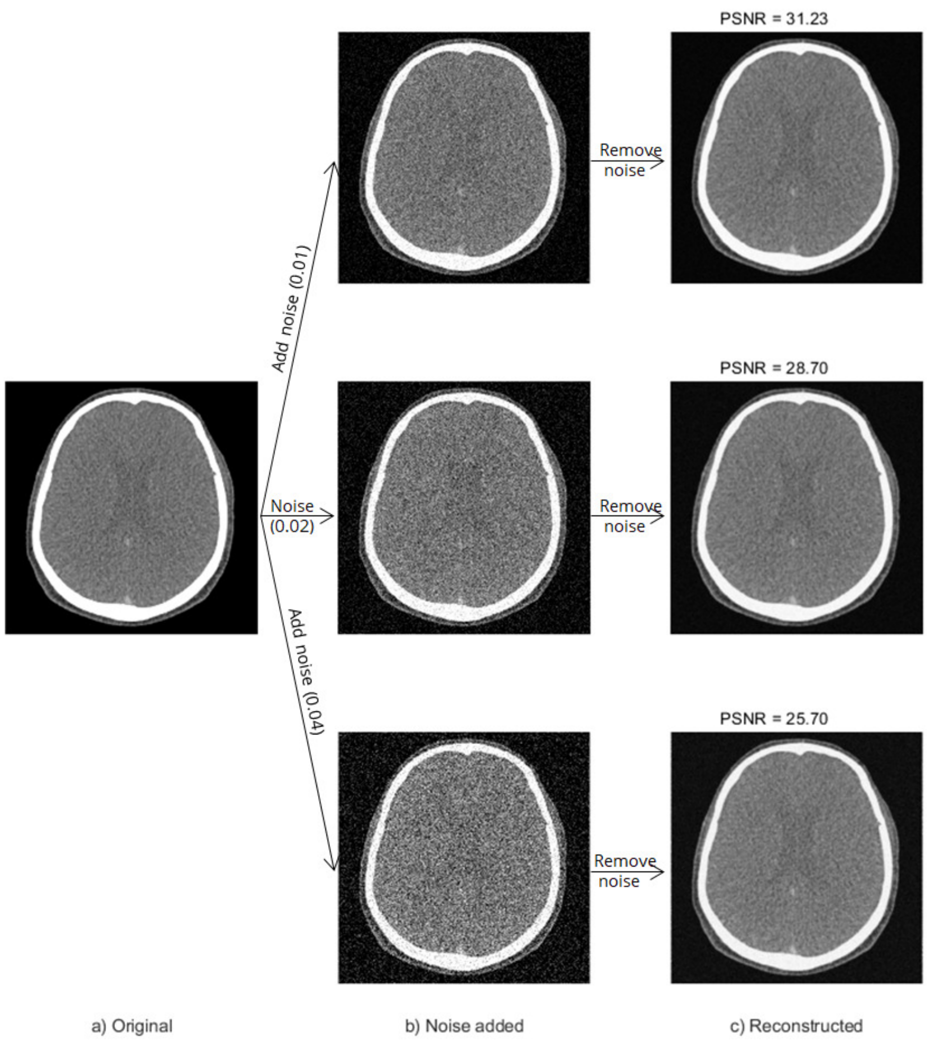

Figure 3. An original picture contaminated with different amount of noise.

In Figure 4 we are showing the evolution of PSNR for up to 20 time iterations. The plots also consider various input noise added to the original picture. We can see that no matter of the input noise the model converges in about 10 iterations. After 10 iterations the PSNR value is stabilized at it's best value for the given input image.

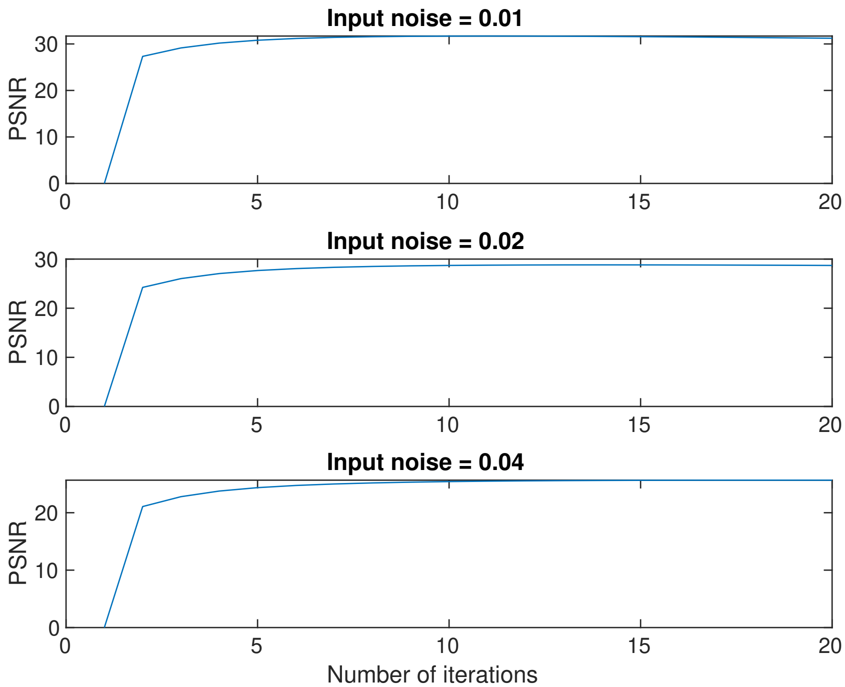

Figure 4. Number of iterations taken to converge to best PSNR, given a picture contaminated with different amount of noise.

4. Conclusions

We study the well-posedness of the solution to a nonlinear second-order anisotropic reaction-diffusion problem (1.4) (or (1.1)) with principal part in *divergence* form and with non-homogeneous Neumann boundary conditions and the *mobility* $K(v_x(t,x))$, $(t,x) \in Q = (0,T] \times \Omega$, $\Omega \subset I\!R^2$. We use the Leray-Schauder principle to prove the existence and uniqueness results and the L^p theory to obtain regularity properties of the solution. Moreover, the *a priori* estimates are done in $L^p(Q)$, implying a better estimates of $v(t,x)$ (see [3,4,7,9,15,20,21,24]).

The rigorous mathematical investigation performed here are mainly used to analyze the well-posedness of the nonlinear anisotropic reaction-diffusion model (3.2), demonstrating the existence of a unique classical solution $v(t,x) \in W_p^{1,2}(Q)$.

Next, using the finite-difference method, an explicit *second-order* approximation scheme is constructed (see (3.7)) for the proposed second-order PDE model (3.2). This scheme converges fast to the approximation of its unique classical solution $v(t,x) \in W_p^{1,2}(Q)$, representing the optimal restoration v^M, since the number of iterations, M, takes low values. Exactly, by using a single method (PSNR), we gained in efficiency of our proposed model (problem (3.2)), as we have 10 iterations (see Figure 4) compared to the 35-th ones of [3, p. 180].

In our future works, we will improve the reaction-diffusion-based restoration scheme (3.2), by modelling *new edge-stopping functions* (see [28]). Also, we will apply this nonlinear PDE model in order to obtain higher-order PDE denoising schemes, such as the fourth-order PDE approaches.

The results of this paper may be applied in the quantitative study of the model (1.4) and in the analysis of distributed and/or boundary nonlinear optimal control problems governed by such a *second-order* boundary value problem.

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