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GLOBAL BIFURCATION RESULT FOR DISCRETE BOUNDARY VALUE PROBLEM INVOLVING THE MEAN CURVATURE OPERATOR*

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Abstract In this paper, by applying bifurcation technique, we obtain that there are two distinct unbounded continua $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ for a class of discrete Dirichlet problem involving the mean curvature operator which bifurcate from intervals of the line of trivial solutions. Under some suitable conditions on nonlinear term near at the origin, we will show the existence and multiplicity of nontrivial solutions.

Keywords Singular ϕ-Laplacian, interval bifurcation, difference equation.

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1. Introduction

Denote Z and $\mathbb{R}$ the sets of integers and real numbers, respectively. For $m, n \in Z$ with $m < n$, $[m,n]_Z := \{m, m+1, \cdots, n\}$. We will establish the unilateral interval bifurcation results for the singular discrete Dirichlet problem

$$\begin{cases} -\nabla[\phi(\Delta u_t)] + q_t u_t = \lambda m_t u_t + m_t f(t, u_t, \lambda), \quad t \in [1, N-1]_Z, \\ u_0 = u_N = 0, \end{cases} \tag{1.1}$$

where $\Delta u_t = u_{t+1} - u_t$ is the forward difference operator, $\nabla u_t = u_t - u_{t-1}$ is the backward difference operator, $\lambda \in \mathbb{R}$ is a parameter, $\phi : (-1,1) \to \mathbb{R}$ is given by $\phi(x) = \frac{x}{\sqrt{1-x^2}}$. $N \geq 4$ is a integer. $\mathbf{q}= (q_1, q_2, \cdots, q_{N-1}) \in \mathbb{R}^{N-1}$, $\mathbf{m}= (m_1, m_2, \cdots, m_{N-1}) \in \mathbb{R}^{N-1}$ and $f \in C([1, N-1]_Z \times \mathbb{R}^2, \mathbb{R})$ satisfy the following assumption:

($\mathbf{C_1}$) $q_t \in C([1, N-1]_Z, [0, +\infty))$ and $q_{t_0} > 0$ for some $t_0 \in [1, N-1]_Z$, $m_t \in C([1, N-1]_Z, [0, +\infty))$ and $m_{t_0} > 0$ for some $t_0 \in [1, N-1]_Z$;

($\mathbf{C_2}$) There exist $f_0, f^0 \in \mathbb{R}$ with $f_0 \neq f^0$, where

$$f_0 = \liminf_{|s| \to 0^+} \frac{f(t,s,\lambda)}{s}, \qquad f^0 = \limsup_{|s| \to 0^+} \frac{f(t,s,\lambda)}{s}$$

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uniformly for $t \in [1, N-1]_Z, 0 < |s| < 1$ and for all $\lambda \in \mathbb{R}$.

Singular ϕ-Laplacian problems in the continuous case have been extensively studied in the literature, refer to [2, 7–9], etc. Recently, Dai [9] established a unilateral global result for the following singular Dirichlet problem

$$\begin{cases} -\mathrm{div}(\frac{\mathrm{grad}\ u}{\sqrt{1-|\mathrm{grad}\ u|^2}}) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, and λ is a real parameter. When the nonlinear term f is superlinear, asymptotically linear, sublinear and jumping near zero, the author proved the existence and multiplicity of one-sign solutions.

Problem (1.1) may be regarded as a more general discrete analog of the problem (1.2) with one dimension. There are many authors have discussed the existence and multiplicity of solutions for discrete boundary value problems involving the mean curvature operator, we refer to [3, 5, 6, 18] and references therein. These results were usually obtained by the applicability of the topological methods such as the upper and lower solutions technique, critical theory, Leray-Schauder degree, etc. In this paper, under the condition that f is not differentiable at the origin, we shall show that there are two distinct unbounded continua $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ of problem (1.1), which emanate from the interval of the line of trivial solutions.

For $\mathbf{u} \in \mathbb{R}^p$ set $\|\mathbf{u}\| = \max\limits_{1\le t\le p} |u_t|$. Let $\mathbf{u}= (u_0, u_1, \cdots, u_N) \in \mathbb{R}^{N+1}$, define

$$\Delta\mathbf{u} = (\Delta u_0, \cdots, \Delta u_{N-1}) \in \mathbb{R}^N,$$

and if $\|\Delta\mathbf{u}\| < 1$, define

$$\nabla[\phi(\Delta\mathbf{u})] = (\nabla[\phi(\Delta u_1)], \cdots, \nabla[\phi(\Delta u_{N-1})]) \in \mathbb{R}^{N-1}.$$

Let E^{N-1} is defined by

$$E^{N-1} = \{\mathbf{u} \in \mathbb{R}^{N+1} : u_0 = u_N = 0, \|\Delta\mathbf{u}\| < 1\}$$

equipped with the norm $\|\mathbf{u}\| = \max\limits_{t\in[1,N-1]_Z} |u_t|$, then $(E^{N-1}, \|\cdot\|)$ is a Banach space. A solution of the problem (1.1) is a vector $\mathbf{u} = (u_0, u_1, \cdots, u_N) \in \mathbb{R}^{N+1}$ satisfying (1.1) and $\|\Delta\mathbf{u}\| < 1$. A nontrivial solution of problem (1.1) is a solution of problem (1.1) such that $\mathbf{u} \neq \mathbf{0}$. Note that if $\mathbf{u}$ is a solution of (1.1), from the fact that $\max\limits_{t\in[0,N-1]_Z} |\Delta u_t| < 1$, it is easy to varify that $\|\mathbf{u}\| < \frac{N}{2}$.

It is well known (see [12]) that the eigenvalue problem

$$\begin{cases} -\nabla(\Delta u_t) + q_t u_t = \lambda m_t u_t, & t \in [1, N-1]_Z, \\ u_0 = u_N = 0, \end{cases} \tag{1.3}$$

possesses $N-1$ real eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N-1}$, all of which are simple. The eigenfunction $\varphi_{\mathbf{k}}$ corresponding to λ_k has exactly $k-1$ simple generalized zeros in $[1, N-1]_Z$.

We give a definition of simple generalized zeros which will play a very important role in our discussions.

Definition 1.1. Let $\mathbf{u} = (u_0, u_1, \cdots, u_N) \in E^{N-1}$. If $u_{t_0} = 0$, then t_0 is a zero of $\mathbf{u}$. If $u_{t_0} = 0$ and $u_{t_0-1}u_{t_0+1} < 0$ for some $t_0 \in [2, N-2]_Z$, then t_0 is a simple zero of $\mathbf{u}$. If $u_{t_0}u_{t_0+1} < 0$ for some $t_0 \in [1, N-2]_Z$, then we say that $\mathbf{u}$ has a node at the points $t_* = \frac{t_0 u_{t_0+1} - (t_0+1)u_{t_0}}{u_{t_0}u_{t_0+1}} \in (t_0, t_0+1)$. The simple zeros and nodes of $\mathbf{u}$ are called the simple generalized zeros of $\mathbf{u}$.

Let us denote $S_k^+ = \{\mathbf{u} \in E^{N-1} : \mathbf{u}$ has exactly $k-1$ simple generalized zeros in $[1, N-1]_Z$ and $u_t > 0$ near $t = 0\}$ and let $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E^{N-1}. Furthermore, we use C to denote the closure in $\mathbb{R} \times E^{N-1}$ of the set of nontrivial solutions of (1.1) and C_k^σ to denote the closure in $\mathbb{R} \times E^{N-1}$ of the set of solutions of (1.1) with $\mathbf{u} \in S_k^\sigma$. In addition, we use the terminology of Rynne [17]. For any $\lambda \in \mathbb{R}$, we say that a subset $C' \subset C$ meets $(\lambda, \mathbf{0})$ if there is a sequence $(\lambda_n, \mathbf{u_n}) \in C'(\mathbf{u_n} = (u_0^n, u_1^n, \cdots, u_N^n), n = 1, 2, \cdots)$ such that $\lambda_n \to \lambda, \|\mathbf{u_n}\| \to \mathbf{0}$ as $n \to +\infty$. Furthermore, we will say that $C' \subset C$ meets $(\lambda, \mathbf{0})$ through $\mathbb{R} \times S_k^\sigma$ if the sequence $(\lambda_n, \mathbf{u_n}) \in C'(n = 1, 2, \cdots)$ can be chosen such that $\mathbf{u_n} \in S_k^\sigma$ for all n. If $I \subset \mathbb{R}$ is a bounded interval we say that $C' \subset C$ meets $I \times \{\mathbf{0}\}$ if C' meets $(\lambda, \mathbf{0})$ for some $\lambda \in I$. Similarly, we can define C' meets $I \times \{\mathbf{0}\}$ through $\mathbb{R} \times S_k^\sigma$, where $\sigma = +$ or $-$. The main result of this paper is the following theorem.

Theorem 1.1. *Assume that* $(\mathbf{C_1})$-$(\mathbf{C_2})$ *hold. Let* $I_k = [\lambda_k - f^0, \lambda_k - f_0]$. *For every* $k \in [1, N-1]_Z$ *and* $\sigma \in \{+, -\}$, *the connected component* $\mathcal{C}_k^\sigma$, *containing* $I_k \times \{\mathbf{0}\}$ *is unbounded and* $\mathcal{C}_k^\sigma \subset (\mathbb{R} \times S_k^\sigma) \bigcup (I_k \times \{\mathbf{0}\})$.

From Theorem 1.1, we can easily conclude the existence and multiplicity of nontrivial solutions.

Corollary 1.1. *Assume that* $(\mathbf{C_1})$-$(\mathbf{C_2})$ *hold, then for all* $\lambda \in (\lambda_k - f_0, +\infty)$, *problem (1.1) has at least two solutions* $\mathbf{u}_k^+$ *and* $\mathbf{u}_k^-$ *such that* $\mathbf{u}_k^+$ *has exactly* $k-1$ *simple generalized zeros in* $[1, N-1]_Z$ *and* $u_t > 0$ *near* $t = 0$, $\mathbf{u}_k^-$ *has exactly* $k-1$ *simple generalized zeros in* $[1, N-1]_Z$ *and* $u_t < 0$ *near* $t = 0$.

This paper is organized as follows: Section 2 contains several lemmas needed to prove the main results. Section 3 contains the proof of the Theorem 1.1.

2. Preliminary results

In this section, we introduce some lemmas and well-known results which will be used in the subsequent section.

Lemma 2.1. $\nabla[\phi(\Delta u_t)] = \nabla(\Delta u_t)\Big[\frac{\sqrt{1-(\Delta u_{t-1})^2}\sqrt{1-(\Delta u_t)^2}+\Delta u_{t-1}\Delta u_t+1}{\sqrt{1-(\Delta u_{t-1})^2}\sqrt{1-(\Delta u_t)^2}[\sqrt{1-(\Delta u_{t-1})^2}+\sqrt{1-(\Delta u_t)^2}]}\Big]$.

Proof. The conclusion can be obtained by calculation. □

Lemma 2.2. *Let* $g(\Delta u_{t-1}, \Delta u_t) = \frac{\sqrt{1-(\Delta u_{t-1})^2}\sqrt{1-(\Delta u_t)^2}[\sqrt{1-(\Delta u_{t-1})^2}+\sqrt{1-(\Delta u_t)^2}]}{\sqrt{1-(\Delta u_{t-1})^2}\sqrt{1-(\Delta u_t)^2}+\Delta u_{t-1}\Delta u_t+1}$ *for* $\Delta u_{t-1} < 1, \Delta u_t < 1$, *then* $g(\Delta u_{t-1}, \Delta u_t) < 2$ *and*

$$\lim_{(\Delta u_{t-1}, \Delta u_t) \to (0,0)} \frac{g(\Delta u_{t-1}, \Delta u_t) - 1}{\max\{|\Delta u_{t-1}|, |\Delta u_t|\}} = 0.$$

Proof. Obviously,

$$\begin{aligned}
&g(\Delta u_{t-1}, \Delta u_t)\\
=&\frac{\sqrt{1-(\Delta u_{t-1})^2}\sqrt{1-(\Delta u_t)^2}[\sqrt{1-(\Delta u_{t-1})^2}+\sqrt{1-(\Delta u_t)^2}]}{\sqrt{1-(\Delta u_{t-1})^2}\sqrt{1-(\Delta u_t)^2}+\Delta u_{t-1}\Delta u_t+1}\\
\le&\frac{\sqrt{1-(\Delta u_{t-1})^2}\sqrt{1-(\Delta u_t)^2}[\sqrt{1-(\Delta u_{t-1})^2}+\sqrt{1-(\Delta u_t)^2}]}{\sqrt{1-(\Delta u_{t-1})^2}\sqrt{1-(\Delta u_t)^2}}\\
=&\sqrt{1-(\Delta u_{t-1})^2}+\sqrt{1-(\Delta u_t)^2}<2.
\end{aligned}$$

It is easy to see that $\sqrt{1-(\Delta u_{t-1})^2} = 1-(\Delta u_{t-1})^2 + o((\Delta u_{t-1})^2)$ when $\Delta u_{t-1} \to 0$, and $\sqrt{1-(\Delta u_t)^2} = 1-(\Delta u_t)^2 + o((\Delta u_t)^2)$ when $\Delta u_t \to 0$. Furthermore,

$$\begin{aligned}
&g(\Delta u_{t-1}, \Delta u_t)-1\\
=&(1-(\Delta u_{t-1})^2+o((\Delta u_{t-1})^2))(1-(\Delta u_t)^2+o((\Delta u_t)^2))\\
&\times\frac{[2-(\Delta u_{t-1})^2+o((\Delta u_{t-1})^2)-(\Delta u_t)^2+o((\Delta u_t)^2)]}{(1-(\Delta u_{t-1})^2+o((\Delta u_{t-1})^2))(1-(\Delta u_t)^2+o((\Delta u_t)^2))+\Delta u_{t-1}\Delta u_t+1}-1\\
=&\frac{2-3(\Delta u_{t-1})^2-3(\Delta u_t)^2+o((\Delta u_{t-1})^2))+o((\Delta u_t)^2))}{2-(\Delta u_{t-1})^2-(\Delta u_t)^2+\Delta u_{t-1}\Delta u_t+o((\Delta u_{t-1})^2))+o((\Delta u_t)^2))}-1\\
=&\frac{-2(\Delta u_{t-1})^2-2(\Delta u_t)^2-\Delta u_{t-1}\Delta u_t+o((\Delta u_{t-1})^2))+o((\Delta u_t)^2))}{2-(\Delta u_{t-1})^2-(\Delta u_t)^2+\Delta u_{t-1}\Delta u_t+o((\Delta u_{t-1})^2))+o((\Delta u_t)^2))}.
\end{aligned}$$

Thus,

$$\lim_{(\Delta u_{t-1},\Delta u_t)\to(0,0)}\frac{g(\Delta u_{t-1},\Delta u_t)-1}{\max\{|\Delta u_{t-1}|,|\Delta u_t|\}}=0.$$

□

It is clear that the solutions of (1.1) is equivalent to the solutions of the following problem

$$\begin{cases}-\nabla(\Delta u_t)+q_t u_t g(\Delta u_{t-1},\Delta u_t)=\lambda m_t u_t g(\Delta u_{t-1},\Delta u_t)\\ \qquad\qquad +m_t f(t,u_t,\lambda)g(\Delta u_{t-1},\Delta u_t),\quad t\in[1,N-1]_Z,\\ u_0=u_N=0.\end{cases}\tag{2.1}$$

Obviously, we can get the following result:

Lemma 2.3. $\mathbf{u}\in E^{N-1}$ *is a solution of problem (1.1) if and only if* $\mathbf{u}\in E^{N-1}$ *is a solution of the following problem:*

$$\begin{cases}-\nabla(\Delta u_t)+q_t u_t=\lambda m_t u_t+m_t f(t,u_t,\lambda)+[\lambda m_t u_t+m_t f(t,u_t,\lambda)-q_t u_t]\\ \qquad\qquad \times(g(\Delta u_{t-1},\Delta u_t)-1),\qquad t\in[1,N-1]_Z,\\ u_0=u_N=0.\end{cases}\tag{2.2}$$

By Lemma 2.2, we can see that

$$\lim_{\|\Delta\mathbf{u}\|\to 0}\frac{g(\Delta u_{t-1},\Delta u_t)-1}{\|\Delta\mathbf{u}\|}=0.$$

Let $h(t,u_t,\lambda)=[\lambda m_t u_t+m_t f(t,u_t,\lambda)-q_t u_t](g(\Delta u_{t-1},\Delta u_t)-1)$, since $\|\Delta u_t\|=\max\limits_{1\le t\le N-1}|\Delta u_t|\le\max\limits_{1\le t\le N-1}(|u_t|+|u_{t+1}|)$. Combining these facts with our hypothesis, we obtain $\lim\limits_{\|\mathbf{u}\|\to 0}\frac{h(t,u_t,\lambda)}{\|\mathbf{u}\|}=0$.

Lemma 2.4. *Let $(\lambda,\mathbf{u})$ be a solution of (2.2) under the assumptions $(\mathbf{C_1})$ and $(\mathbf{C_2})$, if there exists $t_0\in[1,N-1]_Z$ such that either $u_{t_0}=0,\Delta u_{t_0}=0$, or $u_{t_0}=0,u_{t_0-1}u_{t_0+1}\ge 0$. Then $\mathbf{u}\equiv 0$.*

Proof. If $u_{t_0}=0,\Delta u_{t_0}=0$. By (2.2), we have

$$\Delta u_{t_0-1}-\Delta u_{t_0}+q_{t_0}u_{t_0}=\lambda m_{t_0}u_{t_0}+m_{t_0}f(t,u_{t_0},\lambda)+h(t_0,u_{t_0},\lambda).$$

Combining $u_{t_0}=0$ with the hypothesis of f, we obtain that $\Delta u_{t_0-1}-\Delta u_{t_0}=0$. Thus, $u_{t_0-1}=0$. Similarly, from $\Delta u_{t_0}-\Delta u_{t_0+1}=0$, we conclude that $u_{t_0+2}=0$. Therefore, $\Delta u_{t_0-1}=\Delta u_{t_0+1}=0$. Step by step, it follows that $\mathbf{u}\equiv 0$.

If $u_{t_0}=0,u_{t_0-1}u_{t_0+1}\ge 0$. Similarly, we find that $-u_{t_0-1}-u_{t_0+1}=0$. Since $u_{t_0-1}u_{t_0+1}\ge 0$, this implies that $u_{t_0-1}=u_{t_0+1}=0$ is the only possibility. Combining the above facts, it follows that $\mathbf{u}\equiv\mathbf{0}$. □

To get the main results, we introduce the following approximate problem

$$\begin{cases}-\nabla(\Delta u_t)+q_t u_t=\lambda m_t u_t+m_t f(t,u_t|u_t|^{\varepsilon},\lambda)+h(t,u_t,\lambda), & t\in[1,N-1]_Z,\\ u_0=u_N=0.\end{cases}\tag{2.3}$$

Lemma 2.5. *Let $\varepsilon_n\to 0$, $0<\varepsilon_n<1$. If there exists a sequence $\{(\lambda_n,\mathbf{u}_n)\}\subset\mathbb{R}\times S_k^{\sigma}$ such that $(\lambda_n,\mathbf{u}_n)$ a nontrivial solution of (2.3) corresponding to $\varepsilon=\varepsilon_n$, and $(\lambda_n,\mathbf{u}_n)\to(\lambda,\mathbf{0})$ in $\mathbb{R}\times E^{N-1}$. Then $\lambda\in I_k$, where $\sigma\in\{+,-\}$, $\mathbf{u}_n=(u_0^n,u_1^n,\cdots,u_N^n)$.*

Proof. Let $\mathbf{v}_n=\frac{\mathbf{u}_n}{\|\mathbf{u}_n\|},\mathbf{v}_n=(v_0^n,v_1^n,\cdots,v_N^n)$, thus $\mathbf{v}_n$ satisfies

$$\begin{cases}-\nabla(\Delta v_t^n)+q_t v_t^n=\lambda m_t v_t^n+m_t f_n(t)+h_n(t), & t\in[1,N-1]_Z,\\ v_0^n=v_N^n=0,\end{cases}\tag{2.4}$$

where $f_n(t)=\frac{f(t,u_t^n|u_t^n|^{\varepsilon_n},\lambda_n)}{\|\mathbf{u}_n\|},h_n(t)=\frac{h(t,u_t^n,\lambda_n)}{\|\mathbf{u}_n\|}$.

In view of the assumption of f and $\mathbf{v}_n$ is bounded, then

$$|f_n(t)|=|\frac{f(t,u_t^n|u_t^n|^{\varepsilon_n},\lambda_n)}{\|\mathbf{u}_n\|}|\le|\frac{f^0u_t^n|u_t^n|^{\varepsilon_n}}{\|\mathbf{u}_n\|}|\le f^0|v_t^n|$$

for any $t\in[1,N-1]_Z$. Furthermore, $h_n\to 0$ as $\mathbf{u}\to\mathbf{0}$. Therefore, by the Arzela-Ascoli theorem, we may assume that $\mathbf{v}_n\to\mathbf{v}$ and $\|\mathbf{v}\|=1$. Therefore, $\mathbf{v}\in\overline{S}_k^{\sigma}$.

Now, let us prove that $\mathbf{v}\in S_k^{\sigma}$. If $\mathbf{v}\notin S_k^{\sigma}$, then $\mathbf{v}\in\partial S_k^{\sigma}$. Hence $\mathbf{v}$ has at least one double zero in $[1,N-1]_Z$. We assume that there exists $t_0\in[1,N-1]_Z$ such that either $v^n_{t_0} \to 0, \Delta v^n_{t_0} \to 0$ or $v^n_{t_0} \to 0, v^n_{t_0-1}v^n_{t_0+1} \geq 0$ as $n \to +\infty$. By Lemma 2.4, we can see that $\mathbf{v}_n \to \mathbf{0}$, which contradicts $\|\mathbf{v}\| = 1$. Hence $\mathbf{v} \in S^\sigma_k$.

We know that $\varphi_{\mathbf{k}} = (\varphi^k_0, \varphi^k_1, \cdots, \varphi^k_N)$ satisfies

$$-\nabla(\Delta\varphi^k_t) + q_t\varphi^k_t = \lambda_k m_t \varphi^k_t.$$

Let $[\xi, \eta]_Z \subset [1, N-1]_Z$, one has

$$\sum_{t=\xi}^{\eta}\Big(-\nabla(\Delta v^n_t) + q_t v^n_t\Big)\varphi^k_t - \sum_{t=\xi}^{\eta}\Big(-\nabla(\Delta\varphi^k_t) + q_t\varphi^k_t\Big)v^n_t$$
$$= \sum_{t=\xi}^{\eta}(\lambda - \lambda_k)m_t\varphi^k_t v^n_t + \sum_{t=\xi}^{\eta} f_n(t)m_t\varphi^k_t.$$

Taking the limit as $n \to +\infty$, there is

$$\sum_{t=\xi}^{\eta}\Big(-\nabla(\Delta v_t) + q_t v_t\Big)\varphi^k_t - \sum_{t=\xi}^{\eta}\Big(-\nabla(\Delta\varphi^k_t) + q_t\varphi^k_t\Big)v_t$$
$$= \sum_{t=\xi}^{\eta}(\lambda - \lambda_k)m_t\varphi^k_t v_t + \lim_{n\to+\infty}\sum_{t=\xi}^{\eta} f_n(t)m_t\varphi^k_t.$$

Since $\mathbf{v} \in S^\sigma_k, \varphi_{\mathbf{k}} \in S^\sigma_k$. In view of Lemma 2 in [4], we obtain that there exist two intervals $[\xi_1 + 1, \eta_1 - 1]_Z$ and $[\xi_2 + 1, \eta_2 - 1]_Z$ in $[1, N-1]_Z$ where $\mathbf{v}$ and φ_k do not vanish and have the same sign and such that $v_{\xi_1} = v_{\eta_1} = 0$, and the same for $[\xi_2 + 1, \eta_2 - 1]_Z$ with $\mathbf{v}$ replaced by $\varphi_{\mathbf{k}}$. Furthermore, we can show that

$$\sum_{t=\xi_1+1}^{\eta_1-1}(\lambda - \lambda_k)m_t\varphi^k_t v_t + \limsup_{n\to+\infty}\sum_{t=\xi_1+1}^{\eta_1-1} f_n(t)m_t\varphi^k_t \geq 0,$$

$$\sum_{t=\xi_2+1}^{\eta_2-1}(\lambda - \lambda_k)m_t\varphi^k_t v_t + \liminf_{n\to+\infty}\sum_{t=\xi_2+1}^{\eta_2-1} f_n(t)m_t\varphi^k_t \leq 0.$$

Since

$$f_0\sum_{t=\xi}^{\eta} m_t v_t\varphi^k_t \leq \lim_{n\to+\infty}\sum_{t=\xi}^{\eta} f_n(t)m_t\varphi^k_t \leq f^0\sum_{t=\xi}^{\eta} m_t v_t\varphi^k_t$$

for n large enough. Therefore,

$$\sum_{t=\xi_1+1}^{\eta_1-1}(\lambda - \lambda_k + f^0)m_t\varphi^k_t v_t \geq 0, \qquad \sum_{t=\xi_2+1}^{\eta_2-1}(\lambda - \lambda_k + f_0)m_t\varphi^k_t v_t \leq 0,$$

it follows that $\lambda_k - f^0 \leq \lambda \leq \lambda_k - f_0$. Hence $\lambda \in I_k$. □

3. Proof of Theorem 1.1

Proof. Without loss of generality, we only prove the case of $\mathcal{C}^-_k$. Let $\mathcal{C}^-_k$ be the component of $C^-_k \cup (I_k \times \{\mathbf{0}\})$ containing $I_k \times \{\mathbf{0}\}$. Firstly, we prove that

$\mathcal{C}_k^- \subset (\mathbb{R}\times S_k^-)\cup(I_k\times\{\mathbf{0}\})$. For every $(\lambda,\mathbf{u})\in\mathcal{C}_k^-$, then either $\mathbf{u}\in S_k^-$ or $\mathbf{u}\in\partial S_k^-$. If $\mathbf{u}\in S_k^-$, obviously, $(\lambda,\mathbf{u})\in\mathbb{R}\times S_k^-$. If $\mathbf{u}\in\partial S_k^-$, then $\mathbf{u}\equiv\mathbf{0}$. Hence, there exists a sequence $\{(\lambda_n,\mathbf{u}_n)\}\subset\mathbb{R}\times S_k^-$ such that $(\lambda_n,\mathbf{u}_n)$ is a solution of (2.3) corresponding to $\varepsilon=0$, and $(\lambda_n,\mathbf{u}_n)\to(\lambda,\mathbf{0})$ in $\mathbb{R}\times S_k^-$. Lemma 2.5 implies that $\lambda\in I_k$, thus $\mathcal{C}_k^-\cap(\mathbb{R}\times\{\mathbf{0}\})\subset I_k\times\{\mathbf{0}\}$. Therefore, $\mathcal{C}_k^-\subset(\mathbb{R}\times S_k^-)\cup(I_k\times\{\mathbf{0}\})$. Similarly, $\mathcal{C}_k^+\subset(\mathbb{R}\times S_k^+)\cup(I_k\times\{\mathbf{0}\})$.

Next, we need to prove that $\mathcal{C}_k^-$ is unbounded. Assume for contradiction that $\mathcal{C}_k^-$ is bounded. We know that $\mathcal{C}_k^-$ is compact in $\mathbb{R}\times E^{N-1}$. Following [4], we can find a neighborhood O of $\mathcal{C}_k^-$ such that $\partial O\cap\mathcal{C}_k^-=\emptyset$. We consider the problem (2.3) for $\varepsilon>0$. By the Theorem 1.24 of [16], there exists an unbounded continuum $\mathrm{C}_{k,\varepsilon}$ of solutions of (2.3), which bifurcates from $(\lambda_k,\mathbf{0})$, and $\mathrm{C}_{k,\varepsilon}\subset(\mathbb{R}\times S_k\cup\{(\lambda_k,\mathbf{0})\})$. In addition, there are two continua $\mathrm{C}_{k,\varepsilon}^+$ and $\mathrm{C}_{k,\varepsilon}^-$, consisting of the bifurcation branch $\mathrm{C}_{k,\varepsilon}$. Furthermore, $\mathrm{C}_{k,\varepsilon}^+$ and $\mathrm{C}_{k,\varepsilon}^-$ are both unbounded. Therefore, for any $\varepsilon>0$, there exists $(\lambda_\varepsilon,\mathbf{u}_\varepsilon)\in\mathrm{C}_{k,\varepsilon}^-\cap\partial O$. Combining the fact that O is bounded in $\mathbb{R}\times E^{N-1}$. Thus, we can take a sequence $\varepsilon_n\to0, n\to\infty$ such that $(\lambda_{\varepsilon_n},\mathbf{u}_{\varepsilon_n})\to(\lambda,\mathbf{u})$, where $(\lambda,\mathbf{u})$ is a solution of (1.1). Therefore, $\mathbf{u}$ lies in the closure of S_k^-.

If $\mathbf{u}\in\partial S_k^-$, it is easy to see from Lemma 2.4 that $\mathbf{u}\equiv0$. By Lemma 2.5, we know that $\lambda\in I_k$, which is impossible, since O is a neighborhood of $I_k\times\{\mathbf{0}\}$. If $\mathbf{u}\in S_k^-$, then $(\lambda,\mathbf{u})\in\partial O\cap C_k^-$. Hence $\partial O\cap C_k^-\neq\emptyset$, which contradicts the assumption that $\mathcal{C}_k^-$ is bounded. Consequently, $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are both unbounded in $\mathbb{R}\times E^{N-1}$.

The fact $\|\mathbf{u}\|<\frac{N}{2}$ for any fixed $(\lambda,\mathbf{u})\in\mathcal{C}_k^\sigma$ means the projection of $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ on $\mathbb{R}$ is unbounded. □

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