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EXISTENCE RESULTS FOR ANISOTROPIC FRACTIONAL $(p_1(x,.), p_2(x,.))$-KIRCHHOFF TYPE PROBLEMS

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Abstract In this paper, we investigate the existence and multiplicity of solutions for a class of fractional $(p_1(x,.), p_2(x,.))$-Kirchhoff type problems with Dirichlet boundary data of the following form

$$(\mathcal{P}^s_{M_i})\quad \begin{cases} \displaystyle\sum_{i=1}^{2} M_i\left(\int_Q \frac{1}{p_i(x,y)}\frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\,dxdy\right)(-\Delta)^s_{p_i(x,.)}u(x) \\ \displaystyle +\sum_{i=1}^{2}|u|^{\bar{p}_i(x)-2}u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \qquad \text{in } \mathbb{R}^N\setminus\Omega. \end{cases}$$

More precisely, by means of mountain pass theorem with Cerami condition, we show that the above problem has at least one nontrivial solution. Moreover, using Fountain theorem, we prove that $(\mathcal{P}^s_{M_i})$ possesses infinitely many (pairs) of solutions with unbounded energy.

Keywords Kirchhoff type problems, fractional $p(x,.)$-Laplacian operator, mountain pass theorem, fountain theorem, cerami condition.

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1. Introduction and statement of the main results

The study of differential equations involving $p(x)$-Laplacian operators have been a very interesting and exciting topic in recent years (see in particular the fascinating monograph [24] and the references therein for further details). This type of problems are extremely attractive because they can be used to model dynamical phenomena which arise from the study of electrorheological fluids or elastic mechanics. Problems with variable exponent growth conditions also appear in the modelling of stationary thermorheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes of filtration of an ideal barotropic gas through a porous medium. The detailed application backgrounds of the $p(x)$-Laplacian can be found in [4, 19, 27, 36, 40] and the references therein.

For problems involving different growth rates depending on the underlying domains, they also involve equations with $(p(x), q(x))$-growth conditions where several

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$p(x)$-Laplacian operators involved, interacting with one another. This $(p(x), q(x))$-growth condition is a natural generalization of the anisotropic (p, q)-growth condition. In that context, the systems involving the $(p(x), q(x))$-Laplacian (or $(p_1(x), p_2(x), \cdots, p_n(x))$-Laplacian) can be good candidates for modeling phenomena which ask for distinct behavior of partial differential derivatives in various directions, for related problems we just mention [3, 21, 33, 37].

On the other hand, in the recent years increasing attention has been paid to the study of pseudo-differential and nonlocal fractional operators (as $(-\Delta)^s, (-\Delta)^s_p$ and their generalizations) and related fractional differential equations. This type of operators arises in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastic stabilization of Lévy processes, see for instance [5, 18, 31] and the references therein. For a self-contained overview of the basic properties of fractional Sobolev spaces and fractional Laplacian (or fractional p-Laplacian operator), we refer the reader to [16, 35] and to the references included.

It is therefore, a natural question is to see which results "survive" when the $p(x)$-Laplacian is replaced by the fractional $p(x, .)$-Laplacian. In a few last years, to our best knowledge, there have been some mathematicians extending the study of classical exponent variable case to include fractional case (see for instance [2, 6–11, 13, 14, 17, 23, 29, 32]), the authors established some definitions and basic properties about new fractional Sobolev spaces with variable exponents and obtained some existence results for nonlocal fractional problems.

Motivated by the papers mentioned above and the results introduced in [1, 22] and the references therein, we aim to discuss the existence of a nontrivial solutions for a fractional $(p_1(x, .), p_1(x, .))$-Kirchhoff type problem with homogeneous Dirichlet boundary data of the following form

$$(\mathcal{P}^s_{M_i}) \quad \begin{cases} \displaystyle\sum_{i=1}^{2} M_i\left(\int_Q \frac{1}{p_i(x,y)} \frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\, dxdy\right) (-\Delta)^s_{p_i(x,.)}\, u(x) \\ \displaystyle+ \sum_{i=1}^{2} |u|^{\bar{p}_i(x)-2}u = f(x,u) \quad \text{in} \quad \Omega, \\ u = \quad 0 \qquad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

where:
- $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded open domain and $Q := \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$ with $\Omega^c = \mathbb{R}^N \setminus \Omega$, $N \geqslant 3$.
- $p_i : \overline{Q} \longrightarrow (1, +\infty)$ is a bounded continuous function, $i = 1, 2$, $\bar{p}(x) = p(x, x)$ for any $x \in \overline{\Omega}$, and $s \in (0, 1)$.
- Here, for $i = 1, 2$, the operator $(-\Delta)^s_{p_i(x,.)}$ is the fractional $p_i(x, .)$-Laplacian defined as follows

$$(-\Delta)^s_{p_i(x,.)} u(x) = \text{p.v.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp_i(x,y)}}\, dy \qquad \text{for all } x \in \mathbb{R}^N$$

with p.v. is a commonly used abbreviation in the principal value sense.
- $M_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $i = 1, 2$, is a Kirchhoff function with the following assumptions

(K_0) : $M_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a differentiable function and there exists $m_i > 0$ such that

$$M_i(t) \geqslant m_i \quad \text{for all } t \geqslant 0 \quad i = 1, 2.$$

(K_1) : For $i=1,2$, there exists $\alpha_i \in (0,1)$ such that

$$\alpha_i M_i(t) \geqslant (1-\alpha_i) t M_i'(t) \text{ for all } t \geqslant 0.$$

Note that (K_1) implies that

$(\widehat{K_1})$: $\quad \widehat{M_i}(t) \geqslant (1-\alpha_i)M_i(t)t, \quad$ for all $t \geqslant 0$, where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$.

• The nonlinearity $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that

$$(f_0): \qquad |f(x,t)| \leqslant c_1(1+|t|^{q(x)-1}) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

where c_1 is a positive constant and $q \in C_+(\overline{\Omega})$ such that $1 < q^- \leqslant q(x) < (\bar{p}_{\max})_s^*(x)$ for any $x \in \overline{\Omega}$ (see Notation 2.1 and Section 2).

(f_1) : $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{\frac{p^+_{\max}}{1-\alpha}}} = 0$, uniformly for a.e. $x \in \Omega$, where $\alpha = \min\{\alpha_1, \alpha_2\}$ and $F(x,t) = \int_0^t f(x,\tau)d\tau$.

(f_2) : There exists $\theta \geqslant 1$ such that

$$\theta H(x,t) \geqslant H(x,\beta t) \quad \text{for any } (x,t) \in \Omega \times \mathbb{R} \text{ and } \beta \in [0,1],$$

where $H(x,t) = f(x,t)t - \frac{p^+_{\max}}{1-\alpha}F(x,t)$ (see Notation 2.1).

(f_3) : $\lim_{t\to 0} \frac{F(x,t)}{|t|^{p^+_{\max}}} = 0$, uniformly for a.e. $x \in \Omega$.

Condition (f_2) originates in the study of L. Jeanjean [28] in the case $p(x,y) \equiv 2$ for the Laplacian equation. This condition is crucial to obtain the compactness condition of the Palais-Smale or Cerami type for an elliptic equation in the whole space $\mathbb{R}^N$. In that context, these results for superlinear problems in bounded domains have been initially investigated by Miyagaki and Souto [34], Motivated by this work, many authors studied the existence of nontrivial solutions for nonlinear elliptic problems under the following condition:

(f^*): There is constant $C_* > 0$ such that

$$tf(x,t) - pF(x,t) \leq sf(x,s) - pF(x,s) + C_*$$

for any $x \in \Omega, 0 < t < s$ or $s < t < 0$.

In our study, we suppose that the nonlinearity f satisfies the condition (f_2) instead of the well-known Ambrosetti-Rabinowitz (AR) type condition:

(AR): $\exists \gamma > p^+_{\max}$, $L > 0$ such that

$$0 \leq \gamma F(x,t) \leq f(x,t)t \quad \text{ for all } x \in \Omega, |t| \geq L.$$

If $(x,t) \longmapsto f(x,t)$ is increasing in t, then (AR) implies (f_2) when t is large enough. In fact, we can take $\theta = \frac{1}{1 - \frac{p^+_{\max}}{\gamma(1-\alpha)}} > 1$, then

$$\theta H(x,t) - H(x,\beta t) \geq f(x,t)t - f(x,\beta t)\beta t \geq 0.$$

But, in general, (AR) does not imply (f_2), (see [38, Example 3.4]).

Now, it is worth mentioning that $(-\Delta)^s_{p(x,.)}$ is a nonlocal pseudo-differential operator of elliptic type which can be seen as a generalization of the fractional p-Laplacian operator $(-\Delta)^s_p$ in the constant exponent case (i.e., when $p(x,y)=p=$ constant). On the other hand, we remark that the above expression is the fractional version of the well-known $p(x)$-Laplacian operator $\Delta_{p(x)}u(x)=\operatorname{div}\big(|\nabla u(x)|^{p(x)-2}u(x)\big)$ (where $p(x)=p(x,x)$) which is associated with the variable exponent Sobolev space.

For the Kirchhoff function M, a typical prototype is due to Kirchhoff in 1883, and it is given by

$$M(t)=a+bt^{\alpha-1},\quad a,b\geqslant 0,\quad a+b>0,\quad t\geqslant 0, \tag{1.1}$$

and

$$\begin{cases}\alpha\in(1,+\infty) & \text{if } b>0,\\ \alpha=1 & \text{if } b=0,\end{cases}$$

when $M(t)>0$ for all $t\geqslant 0$, Kirchhoff problems are said to be nondegenerate and this happens for example if $a>0$ and $b\geqslant 0$ in the model case (1.1). Otherwise, if $M(0)=0$ and $M(t)>0$ for all $t>0$, the Kirchhoff problems are called degenerate and this occurs in the model case (1.1) when $a=0$ and $b>0$.

One typical feature of problem $(\mathcal{P}^s_{M_i})$ is the nonlocality, in the sense that the value of $(-\Delta)^s_{p(x,.)}u(x)$ at any point $x\in\Omega$ depends not only on the values of u on Ω, but actually on the entire space $\mathbb{R}^N$. Moreover, the presence of the functions M_i, $i=1,2$, which implies that the first equation in $(\mathcal{P}^s_{M_i})$ is no longer a pointwise equation, it is no longer a pointwise identity, Therefore, the Dirichlet datum is given in $\mathbb{R}^N\setminus\Omega$ (which is different from the classical case of the $p(x)$-Laplacian) and not simply on $\partial\Omega$. Hence, it is often called nonlocal problem. This causes some mathematical difficulties which make the study of such a problem particularly interesting.

As far as we know, there is no work that deals with a nonlocal problem involving fractional $(p_1(x,.),p_2(x,.))$-Laplacian operator except [23] in which the authors considered problem $(\mathcal{P}^s_{M_i})$ for the case $M_1=M_2\equiv 1$ and they established some existence results for the problem with indefinite weights in an appropriate space of functions by means of variational techniques and Ekeland's variational principle. Moreover, in [6], using mountain pass theorem, the authors studied the existence of weak solutions for a quasilinear elliptic system involving the fractional $(p(x,.),q(x,.))$-Laplacian operators. Very recently, the authors in [12] studied the equation $(-\Delta)^s_{p(x,.)}u(x)=f(x,u(x))$ without assuming the (AR) type condition. Therefore, without this condition it becomes a very difficult task to get the compactness condition. That is why, to our best knowledge, the present studied anisotropic Kirchhoff type problem is the first contribution in this direction. The purpose of this work is to improve the results of the above-mentioned papers. So, using the weaker assumption (f_2) instead of (AR)-condition and some variant min-max theorem, we overcome these difficulties and we prove the existence and multiplicity of weak solutions for problem $(\mathcal{P}^s_{M_i})$. Hence, our main results can be stated as follows.

Theorem 1.1. *Assume that the assumptions (K_0), (K_1) and (f_0)-(f_3) hold. If $p^+_{\max}<q^-$, then problem $(\mathcal{P}^s_{M_i})$ has at least one nontrivial solution.*

Theorem 1.2. *Assume that (K_0), (K_1), and (f_0)-(f_3) are satisfied. Moreover, we* *suppose that*

$$(f_4): \qquad f(x,-t) = -f(x,t) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

If $q^- > p^+_{max}$, *then problem* $(\mathcal{P}^s_{M_i})$ *has a sequence of weak solutions* $\{\pm u_k\}_{k=1}^{\infty}$ *such that*

$$\mathcal{J}(\pm u_k) \longrightarrow +\infty \quad \text{as } k \to +\infty.$$

The rest of this paper is organized as follows: In section 2, we give some definitions and fundamental properties of generalized Lebesgue spaces and fractional Sobolev spaces with variable exponent. In section 3, we discuss the existence of nontrivial weak solutions of problem $(\mathcal{P}^s_{M_i})$ by means of mountain pass theorem with Cerami condition. Furthermore, using Fountain theorem, we show that problem $(\mathcal{P}^s_{M_i})$ has infinitely many (pairs) of solutions with unbounded energy. As a conclusion, we extend all our results directly to the fractional multi $p(x,.)$-Laplacian case. Moreover, in order to illustrate our results, we consider a particular example of the Kirchhoff functions M_i and the nonlinearity f.

2. Variatoinal setting and preliminary results

For the reader's convenience, we briefly review the definitions and list some useful properties of the generalized Lebesgue spaces. Furthermore, we recall some qualitative properties of the fractional Sobolev spaces with variable exponent and several important properties of fractional $p(x,.)$-Laplacian operator.

2.1. Variable exponent Lebesgue spaces

In this subsection, we give some basic results of variable exponent Lebesgue spaces $L^{q(\cdot)}(\Omega)$. For more details, we refer the reader to [25,30] and the references therein. Consider the set

$$C_+(\overline{\Omega}) = \left\{ q \in C(\overline{\Omega}) : q(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}.$$

For all $q \in C_+(\overline{\Omega})$, we define

$$q^+ = \sup_{x \in \overline{\Omega}} q(x) \quad \text{and} \quad q^- = \inf_{x \in \overline{\Omega}} q(x),$$

such that

$$1 < q^- \leqslant q(x) \leqslant q^+ < +\infty. \tag{2.1}$$

For any $q \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{q(x)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \;\; \text{measurable} : \int_\Omega |u(x)|^{q(x)} dx < +\infty \right\}.$$

This vector space endowed with the *Luxemburg norm*, which is defined by

$$\|u\|_{L^{q(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leqslant 1 \right\}$$

is a separable reflexive Banach space.

Let $\hat{q} \in C_+(\overline{\Omega})$ be the conjugate exponent of q, that is, $\dfrac{1}{q(x)} + \dfrac{1}{\hat{q}(x)} = 1$. Then we have the following Hölder-type inequality.

Lemma 2.1. *If $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{\hat{q}(\cdot)}(\Omega)$, then*

$$\left| \int_\Omega uv dx \right| \leqslant \left(\frac{1}{q^-} + \frac{1}{\hat{q}^-} \right) \|u\|_{L^{q(\cdot)}(\Omega)} \|v\|_{L^{\hat{q}(\cdot)}(\Omega)} \leqslant 2 \|u\|_{L^{q(\cdot)}(\Omega)} \|v\|_{L^{\hat{q}(\cdot)}(\Omega)}.$$

A very important role in manipulating the generalized Lebesgue spaces with variable exponent is played by the modular of the $L^{q(\cdot)}(\Omega)$ space, which is defined by

$$\begin{aligned} \rho_{q(.)} : L^{q(\cdot)}(\Omega) &\longrightarrow \mathbb{R} \\ u \longmapsto \rho_{q(.)}(u) &= \int_\Omega |u(x)|^{q(x)} dx. \end{aligned}$$

Proposition 2.1. *Let $u \in L^{q(\cdot)}(\Omega)$, $\{u_k\} \subset L^{q(\cdot)}(\Omega)$, $k \in \mathbb{N}$, then we have*

(*i*) $\|u\|_{L^{q(\cdot)}(\Omega)} < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho_{q(.)}(u) < 1$ (*resp.* $= 1, > 1$),

(*ii*) $\|u\|_{L^{q(\cdot)}(\Omega)} < 1 \Rightarrow \|u\|^{q+}_{L^{q(\cdot)}(\Omega)} \leqslant \rho_{q(.)}(u) \leqslant \|u\|^{q-}_{L^{q(\cdot)}(\Omega)}$,

(*iii*) $\|u\|_{L^{q(\cdot)}(\Omega)} > 1 \Rightarrow \|u\|^{q-}_{L^{q(\cdot)}(\Omega)} \leqslant \rho_{q(.)}(u) \leqslant \|u\|^{q+}_{L^{q(\cdot)}(\Omega)}$,

(*iv*) $\lim\limits_{k \to +\infty} \|u_k - u\|_{L^{q(\cdot)}(\Omega)} = 0 \iff \lim\limits_{k \to +\infty} \rho_{q(.)}(u_k - u) = 0.$

2.2. Fractional Sobolev spaces with variable exponent

In this subsection, we present some preliminary results and basic properties of fractional Sobolev spaces with variable exponent that were introduced in [10]. For a deeper treatment on these spaces, we refer the reader to [9, 14, 29].

Let Ω be a Lipschitz bounded open set in $\mathbb{R}^N$. We denote by Q the set

$$Q := \mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^c \times \Omega^c), \quad \text{where} \quad \Omega^c = \mathbb{R}^N \setminus \Omega.$$

Let $p : \overline{Q} \longrightarrow (1, +\infty)$ be a continuous bounded function, we assume that

$$1 < p^- = \min_{(x,y) \in \overline{Q}} p(x,y) \leqslant p(x,y) \leqslant p^+ = \max_{(x,y) \in \overline{Q}} p(x,y) < +\infty \tag{2.2}$$

and

$$p \quad \text{is symmetric, that is,} \quad p(x,y) = p(y,x) \quad \text{for all } (x,y) \in \overline{Q}. \tag{2.3}$$

We set

$$\bar{p}(x) = p(x,x) \quad \text{for any } x \in \overline{\Omega}.$$

Throughout this paper, s is a fixed real number such that $0 < s < 1$.

Due to the non-locality of the operator $(-\Delta)^s_{p(x,.)}$, we introduce the general fractional Sobolev space with variable exponent as in [10] as follows

$$X = W^{s,p(x,y)}(Q) = \left\{ \begin{array}{l} u : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ measurable such that } \ u_{|\Omega} \in L^{\bar{p}(x)}(\Omega) \text{ with} \\ \displaystyle\int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N + sp(x,y)}} \, dx dy < +\infty, \quad \text{for some} \quad \lambda > 0 \end{array} \right\},$$

with the norm

$$\|u\|_X = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_X,$$

where $[.]_X$ is a Gagliardo seminorm with variable exponent, defined by

$$[u]_X = [u]_{s,p(x,y)}(Q) = \inf\left\{\lambda > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dxdy \leqslant 1\right\}.$$

The space $(X, \|.\|_X)$ is a separable reflexive Banach space.

Definition 2.1. Let $p : \overline{Q} \longrightarrow (1,+\infty)$ be a continuous variable exponent and let $s \in (0,1)$, we define the modular $\rho_{p(x,y)} : X \longrightarrow \mathbb{R}$, by

$$\rho_{p(x,y)}(u) = \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy + \int_\Omega |u(x)|^{\bar{p}(x)} dx.$$

Consequently, $\|u\|_{\rho_{p(x,y)}} = \inf\left\{\lambda > 0 : \rho_{p(x,y)}\left(\frac{u}{\lambda}\right) \leqslant 1\right\}$.

Now, let us denote by X_0 the following linear subspace of X

$$X_0 = \left\{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\},$$

with the norm

$$\|u\|_{X_0} = [u]_X = \inf\left\{\lambda > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dxdy \leqslant 1\right\}.$$

The space $(X_0, \|.\|_{X_0})$ is a separable reflexive Banach space (see [10, Lemma 2.3]). We define the modular $\rho^0_{p(x,y)} : X_0 \longrightarrow \mathbb{R}$, by

$$\rho^0_{p(x,y)}(u) = \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy.$$

Consequently, $\|u\|_{\rho^0_{p(x,y)}} = \inf\left\{\lambda > 0 : \rho^0_{p(x,y)}\left(\frac{u}{\lambda}\right) \leqslant 1\right\} = [u]_X$. Similar to Proposition 2.1, $\rho^0_{p(x,y)}$ satisfies the following assertions.

Lemma 2.2. *For any $u \in X_0$ and $\{u_k\} \subset X_0$, we have*

(i) $\|u\|_{X_0} < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho^0_{p(x,y)}(u) < 1$ (*resp.* $= 1, > 1$),

(ii) *for* $u \in X_0 \setminus \{0\}$, $\|u\|_{X_0} = \lambda \Leftrightarrow \rho^0_{p(x,y)}\left(\frac{u}{\lambda}\right) = 1$,

(iii) $1 \leqslant \|u\|_{X_0} \Rightarrow \|u\|^{p^-}_{X_0} \leqslant \rho^0_{p(x,y)}(u) \leqslant \|u\|^{p^+}_{X_0}$,

(iv) $\|u\|_{X_0} \leqslant 1 \Rightarrow \|u\|^{p^+}_{X_0} \leqslant \rho^0_{p(x,y)}(u) \leqslant \|u\|^{p^-}_{X_0}$.

(v) $\|u_k\|_{X_0} \to 0$ (*resp* $\to \infty$) $\Leftrightarrow \rho^0_{p(x,y)}(u_k) \to 0$ (*resp* $\to \infty$).

In [29], the authors introduced the variable exponent Sobolev fractional space as follows

$$\begin{aligned} F &= W^{s,q(x),p(x,y)}(\Omega) \\ &= \left\{u \in L^{q(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dxdy < +\infty, \quad \text{for some} \quad \lambda > 0\right\}, \end{aligned}$$

where $q : \overline{\Omega} \longrightarrow (1, +\infty)$ is a continuous function satisfies (2.1).

We would like to mention that a continuous and compact embedding theorem is proved in [29] under the assumption $q(x) > \bar{p}(x) = p(x, x)$. The authors in [9] give a slightly different version of continuous compact embedding theorem assuming that $q(x) = \bar{p}(x) = p(x, x)$, in this case the space E becomes

$$\begin{aligned} W &= W^{s,p(x,y)}(\Omega) \\ &= \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dxdy < +\infty, \quad \text{for some} \quad \lambda > 0 \right\}. \end{aligned}$$

Theorem 2.1 ([9]). *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and let $s \in (0, 1)$. Let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1, +\infty)$ be a continuous function satisfying (2.2) and (2.3) with $sp^+ < N$. Let $r : \overline{\Omega} \longrightarrow (1, +\infty)$ be a continuous variable exponent such that*

$$1 < r^- = \min_{x\in\overline{\Omega}} r(x) \leqslant r(x) < p_s^*(x) = \frac{N\bar{p}(x)}{N - s\bar{p}(x)} \quad \text{for all } x \in \overline{\Omega}.$$

Then, there exists a constant $C = C(N, s, p, r, \Omega) > 0$ such that, for any $u \in W$,

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_W.$$

That is, the space W is continuously embedded in $L^{r(x)}(\Omega)$. Moreover, this embedding is compact.

In [10], we compared the spaces W and X, and we established the compact and continuous embedding of X into Lebesgue spaces with variable exponent.

Theorem 2.2. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and let $s \in (0, 1)$. Let $p : \overline{Q} \longrightarrow (1, +\infty)$ be a continuous function satisfying (2.2) and (2.3) on $\overline{Q}$ with $sp^+ < N$. Then the following assertions hold:*

(*i*) *If $u \in X$, then $u \in W$. Moreover,*

$$\|u\|_W \leqslant \|u\|_X;$$

(*ii*) *if $u \in X_0$, then $u \in W^{s,p(x,y)}(\mathbb{R}^N)$. Moreover,*

$$\|u\|_W \leqslant \|u\|_{W^{s,p(x,y)}(\mathbb{R}^N)} = \|u\|_X;$$

(*iii*) *if $r : \overline{\Omega} \longrightarrow (1, +\infty)$ be a continuous variable exponent such that*

$$1 < r^- \leqslant r(x) < p_s^*(x) = \frac{N\bar{p}(x)}{N - s\bar{p}(x)} \quad \text{for all } x \in \overline{\Omega}.$$

Then, there exists a constant $C = C(N, s, p, r, \Omega) > 0$ such that, for any $u \in X$,

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_X.$$

That is, the space X is continuously embedded in $L^{r(x)}(\Omega)$. Moreover, this embedding is compact.

Remark 2.1.

(i) The assertion (iii) in Theorem 2.2 remains true if we replace X by X_0.

(ii) Since $1 < p^- \leqslant \bar{p}(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, then by Theorem 2.2-(iii) we have that, $\|.\|_{X_0} = [.]_X$ and $\|.\|_X$ are equivalent on X_0.

Let denote by $\mathcal{L}$ the operator associated to the $(-\Delta_{p(x,.)})^s$ defined as

$$\begin{aligned} \mathcal{L} : X_0 &\longrightarrow X_0^* \\ u &\longmapsto \mathcal{L}(u) : X_0 \longrightarrow \mathbb{R} \\ &\qquad\qquad \varphi \longmapsto \langle \mathcal{L}(u), \varphi \rangle \end{aligned}$$

such that

$$\langle \mathcal{L}(u), \varphi \rangle = \int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}\, dxdy,$$

where $\langle ., . \rangle$ denotes the usual duality between X_0 and its dual space X_0^*.

Lemma 2.3 ([14]). *Assume that assumptions* (2.2) *and* (2.3) *are satisfied. Then, the following assertions hold:*

(i) *$\mathcal{L}$ is a bounded and strictly monotone operator.*

(ii) *$\mathcal{L}$ is a mapping of type (S_+), that is,*
if $u_k \rightharpoonup u$ in X_0 and $\limsup_{k \longrightarrow +\infty} \langle \mathcal{L}(u_k) - \mathcal{L}(u), u_k - u \rangle \leqslant 0$, then $u_k \longrightarrow u$ in X_0.

(iii) *$\mathcal{L}$ is a homeomorphism.*

In order to facilitate the investigation of problem $(\mathcal{P}^s_{M_i})$, the following notations are required.

Notation 2.1. For all $(x,y) \in \overline{Q}$, let us denote

- $p_{\max}(x,y) = \max\{p_1(x,y), p_2(x,y)\} = \max_{i=1,2} p_i(x,y)$,
- $p_{\min}(x,y) = \min_{i=1,2} p_i(x,y)$,
- $\bar{p}_{\max}(x) = \max_{i=1,2} \bar{p}_i(x) = \max_{i=1,2} p_i(x,x), \quad \bar{p}_{\min}(x) = \min_{i=1,2} \bar{p}_i(x)$,
- $p^+_{\max} = \sup_{(x,y)\in\overline{Q}} p_{\max}(x,y) \quad$ and $\quad p^-_{\min} = \inf_{(x,y)\in\overline{Q}} p_{\min}(x,y)$,
- $(\bar{p}_{\max})^*_s(x) = \dfrac{N\bar{p}_{\max}(x)}{N - s\bar{p}_{\max}(x)} \quad$ for any $x \in \overline{\Omega}$.

It is easy to see that $\bar{p}_{\max}, \bar{p}_{\min} \in C_+(\overline{\Omega})$ and $p_{\max}, p_{\min} \in C_+(\overline{Q})$. For simplicity, we set $E = W^{s,p_{max}(x,y)}(\Omega)$ and

$$E_0 = \left\{ u \in E : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

It is clear that E and E_0 are separable and reflexive Banach spaces under the norms

$$\|u\|_E = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_{s,p_{\max}(x,y)} \quad \text{and} \quad \|u\|_{E_0} = [u]_{s,p_{\max}(x,y)}.$$

For $i = 1,2$, we denote by $\rho^0_{p_i(x,y)}$ the modular on $W_0^{s,p_i(x,y)}(Q)$ defined by

$$\rho^0_{p_i(x,y)}(u) = \int_Q \frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\, dxdy.$$

Remark 2.2. Obviously, from Theorem 2.2, for any $q \in C^{+}(\overline{\Omega})$ such that $q(x) < (\bar{p}_{\max})^{*}_{s}(x)$ for all $x \in \overline{\Omega}$, we have $E \hookrightarrow L^{q(x)}(\Omega)$, and this embedding is continuous and compact. Moreover, by Remark 2.1-(i), this result remains true if we replace E by E_0.

Now, we give the definition of the Cerami condition (C) which is introduced by Cerami in [20].

Definition 2.2. Let X be a Banach space and $J \in C^1(X,\mathbb{R})$. Given $c \in \mathbb{R}$, we say that Φ satisfies the Cerami c condition (we denote condition (C_c)), if

(C_1): any bounded sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ has a convergent subsequence,

(C_2): there exist constants $\delta, R, \beta > 0$ such that

$$\|\Phi'(u)\|\,\|u\| \geq \beta \quad \forall u \in \Phi^{-1}([c-\delta, c+\delta]) \quad \text{with} \quad \|u\| \geq R.$$

If $\Phi \in C^1(X,\mathbb{R})$ satisfies condition (C_c) for every $c \in \mathbb{R}$, we say that Φ satisfies condition (C).

Note that condition (C) is weaker than the Palais-Smale condition. However, it was shown in [15] that from condition (C) it is possible to obtain a deformation lemma, which is fundamental in order to get some min-max theorems. More precisely, let us recall the following version of the mountain pass lemma with Cerami condition which will be used in the sequel.

Proposition 2.2. *Let X a Banach space, $\Phi \in C^1(X,\mathbb{R})$, $u_0 \in X$ and $r > 0$, be such that $\|u_0\|_X > r$ and*

$$b := \inf_{\|u\|_X = r} \Phi(u) > \Phi(0) \geq \Phi(u_0).$$

If Φ satisfies the condition (C_c) with

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)),$$
$$\Gamma := \{\gamma \in C([0,1], X) \,|\gamma(0) = 0, \gamma(1) = u_0\}.$$

Then c is a critical value of Φ.

Now, since X is a separable and reflexive Banach space, from [39, Section 17, Theorems 2-3], there exist $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{e_n^*\}_{n=1}^{\infty} \subset X^*$ such that

$$e_n^*(e_m) = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Hence, $X = \overline{span}\{e_n,\ n = 1, 2, ...\}$ and $X^* = \overline{span}\{e_n^*,\ n = 1, 2, ...\}$. For $k = 1, 2, ...$, denote

$$X_k = span\{e_k\}, \quad Y_k = \oplus_{i=0}^{k} X_i, \quad Z_k = \overline{\oplus_{i=k}^{\infty} X_i}.$$

Next, we introduce the Fountain theorem with the condition (C) as in [41].

Proposition 2.3. *Assume that X is a separable Banach space, $\Phi \in C^1(X,\mathbb{R})$ is an even functional satisfying the Cerami condition (C). Moreover, for each $k = 1, 2, \ldots,$ there exist $R_k > r_k > 0$ such that*

- (Φ_1) $\inf\limits_{\{u\in Z_k:\|u\|=r_k\}} \Phi(u) \to +\infty$ *as* $k \to \infty$,
- (Φ_2) $\max\limits_{|u\in Y_k:\|u\|=R_k\}} \Phi(u) \leq 0$.

Then, Φ has a sequence of critical values which tends to $+\infty$.

3. Proof of Existence and multiplicity results

By a weak solution for $(\mathcal{P}^s_{M_i})$, we mean a function $u \in E_0$ such that

$$\sum_{i=1}^{2} M_i\left(\sigma_{p_i(x,y)}(u)\right)\int_Q \frac{|u(x)-u(y)|^{p_i(x,y)-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+sp_i(x,y)}}\,dxdy$$
$$+\sum_{i=1}^{2}\int_\Omega |u(x)|^{\bar{p}_i(x)-2}u(x)\varphi(x)dx - \int_\Omega f(x,u)\varphi dx = 0, \tag{3.1}$$

for all $\varphi \in E_0$, where $\sigma_{p_i(x,y)}(u) = \displaystyle\int_Q \frac{1}{p_i(x,y)}\frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\,dxdy$. In this case, the weak formulation (3.1) is the Euler-Lagrange equation of the energy functional $\mathcal{J} : E_0 \longrightarrow \mathbb{R}$ defined by

$$\mathcal{J}(u) = \sum_{i=1}^{2}\widehat{M_i}\left(\sigma_{p_i(x,y)}(u)\right) + \sum_{i=1}^{2}\int_\Omega \frac{1}{\bar{p}_i(x)}|u|^{\bar{p}_i(x)}dx - \int_\Omega F(x,u)dx.$$

Standard arguments (see, for instance [9, Lemma 3.1]) and the continuity of M_i, $i = 1, 2$, imply that $\mathcal{J}$ is well defined and $\mathcal{J} \in C^1(E_0,\mathbb{R})$. Moreover, for all $u, \varphi \in E_0$, its Gateaux derivative is given by

$$\langle \mathcal{J}'(u),\varphi\rangle = \sum_{i=1}^{2} M_i\left(\sigma_{p_i(x,y)}(u)\right)\int_Q \frac{|u(x)-u(y)|^{p_i(x,y)-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+sp_i(x,y)}}dxdy$$
$$+\sum_{i=1}^{2}\int_\Omega |u|^{\bar{p}_i(x)-2}u\varphi dx - \int_\Omega f(x,u)\varphi dx.$$

Thus, the weak solutions of $(\mathcal{P}^s_{M_i})$ coincide with the critical points of $\mathcal{J}$.

3.1. Compactness Cerami condition for the functional $\mathcal{J}$

In this subsection, we establish the following compactness result which plays the most important role in this chapter.

Lemma 3.1. *Suppose that the conditions (K_0), (K_1), and (f_0)-(f_2) hold. Then, $\mathcal{J}$ satisfies the Cerami condition $(\mathrm{C_c})$.*

Proof. We first show that $\mathcal{J}$ satisfies the assertion (C_1) of Cerami condition $(\mathrm{C_c})$ (see Definition 2.2). Indeed, for all $c \in \mathbb{R}$, let $\{u_n\} \subset E_0$ be a bounded sequence such that

$$\mathcal{J}(u_n) \underset{n\to+\infty}{\longrightarrow} c \quad \text{and} \quad \mathcal{J}'(u_n) \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.2}$$

Since E_0 is a reflexive space, then without loss of generality, we can assume that $u_n \rightharpoonup u$ in E_0, which implies that

$$\langle \mathcal{J}'(u_n), u_n - u\rangle \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus, we have

$$\langle \mathcal{J}'(u_n), u_n - u\rangle = \sum_{i=1}^{2} M_i\left(\sigma_{p_i(x,y)}(u_n)\right) \times \int_Q \frac{|u_n(x)-u_n(y)|^{p_i(x,y)-2}\,(u_n(x)-u_n(y))\,((u_n(x)-u_n(y))-(u(x)-u(y)))}{|x-y|^{N+sp_i(x,y)}}\,dxdy$$
$$+\sum_{i=1}^{2}\int_\Omega |u_n|^{\bar{p}_i(x)-2}u_n(u_n-u)dx - \int_\Omega f(x,u_n)(u_n-u)dx \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.3}$$

On the other hand, by (f_0) and Hölder's inequality in Lemma 2.1, we have

$$\int_\Omega f(x,u_n)(u_n-u)dx \leqslant 2c_1\|1\|_{L^{\hat{q}(x)}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)} + 2c_1\|u_n\|_{L^{q(x)}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)},$$

where $\dfrac{1}{q(x)} + \dfrac{1}{\hat{q}(x)} = 1$. Hence, as $1 < q^- \leqslant q(x) < (\bar{p}_{\max})^*_s(x)$ for all $x \in \overline{\Omega}$, we have that E_0 is compactly embedded in $L^{q(x)}(\Omega)$. It follows that

$$\int_\Omega f(x,u_n)(u_n-u)dx \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.4}$$

Besides this, since $1 < p_i^- \leqslant \bar{p}_i(x) < (\bar{p}_{\max})^*_s(x)$, $i = 1,2$, for all $x \in \overline{\Omega}$, then E_0 is compactly embedded in $L^{\bar{p}_i(x)}(\Omega)$, for $i = 1,2$. So, again by Hölder's inequality in Lemma 2.1, we get

$$\sum_{i=1}^{2}\int_\Omega |u_n|^{\bar{p}_i(x)-2}u_n(u_n-u)dx \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.5}$$

Combining (3.3)-(3.5), we obtain

$$\lim_{n\to+\infty}\sum_{i=1}^{2} M_i\left(\sigma_{p_i(x,y)}(u_n)\right) \times \int_Q \frac{|u_n(x)-u_n(y)|^{p_i(x,y)-2}\,(u_n(x)-u_n(y))\,((u_n(x)-u_n(y))-(u(x)-u(y)))}{|x-y|^{N+sp_i(x,y)}}\,dxdy = 0.$$

Using $(\widehat{K_1})$, for $i = 1,2$, we can easily obtain that

$$\widehat{M_i}(t) \leqslant \tilde{c}_i t^{\frac{1}{1-\alpha_i}}.$$

Hence, from (K_0), it follows that

$$m_0 \leqslant M_i(t) \leqslant \widehat{M}_i(t) \leqslant \frac{\tilde{c}_i}{(1-\alpha_i)} t^{\frac{\alpha_i}{1-\alpha_i}}.$$

Since $\{u_n\} \subset E_0$ and $u \in E_0$, by Lemma 2.2, we deduce that $\left\{M_i\left(\ \sigma_{p_i(x,y)}(u_n)\right)\right\}$ and $\left\{M_i\left(\ \sigma_{p_i(x,y)}(u)\right)\right\}$ are bounded. Thence, by assumption (K_0), we get

$$\sum_{i=1}^{2}\int_Q \frac{|u_n(x)-u_n(y)|^{p_i(x,y)-2}\,(u_n(x)-u_n(y))\,((u_n(x)-u_n(y))-(u(x)-u(y)))}{|x-y|^{N+sp_i(x,y)}}\,dxdy$$

$$:= \left\langle \widetilde{L}(u_n), u_n - u \right\rangle \underset{n\to+\infty}{\longrightarrow} 0 \tag{3.6}$$

Now, since $u_n \rightharpoonup u$ in E_0, using (3.2), we get

$$\langle \mathcal{J}'(u), u_n - u\rangle \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence, by the same argument as before, we deduce that

$$\sum_{i=1}^{2}\int_Q \frac{|u(x)-u(y)|^{p_i(x,y)-2}\,(u(x)-u(y))\,((u_n(x)-u_n(y))-(u(x)-u(y)))}{|x-y|^{N+sp_i(x,y)}}\,dxdy$$

$$:= \left\langle \widetilde{L}(u), u_n - u \right\rangle \underset{n\to+\infty}{\longrightarrow} 0 \tag{3.7}$$

Combining (3.6) and (3.7), we conclude that

$$\limsup_{n\to+\infty} \left\langle \widetilde{L}(u_n) - \widetilde{L}(u), u_n - u \right\rangle \leqslant 0.$$

From Lemma 2.3-(iii), $\mathcal{L}$ is a mapping of type (S_+), and since $\widetilde{L}$ is a sum of two operators of type (S_+). Then, by [26, Lemma 6.8-(b)], $\widetilde{L}$ is also of type (S_+). Hence

$$\begin{cases} \limsup\limits_{n\longrightarrow+\infty} \left\langle \widetilde{L}(u_n) - \widetilde{L}(u), u_n - u \right\rangle \leqslant 0, \\ u_n \rightharpoonup u \text{ in } E_0, \\ \widetilde{L} \ \text{ is a mapping of type } \ (S_+). \end{cases}$$

Consequently, $u_n \longrightarrow u$ (strongly) in E_0.

Next, we show that $\mathcal{J}$ satisfies the assertion (C_2) of Cerami condition $(\mathrm{C_c})$ (see Definition 2.2), we argue by contradiction. Indeed, we assume that there exists $c \in \mathbb{R}$ and $\{u_n\} \subset E_0$ such that

$$\mathcal{J}(u_n) \underset{n\to+\infty}{\longrightarrow} c, \quad \|u_n\|_{E_0} \underset{n\to+\infty}{\longrightarrow} \infty, \quad \text{and} \quad \|\mathcal{J}'(u_n)\|_{E_0^*}\|u_n\|_{E_0} \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.8}$$

From (3.8), it is easy to see that

$$\mathcal{J}(u_n) - \frac{1-\alpha}{p^+_{max}} \langle \mathcal{J}'(u_n), u_n\rangle \underset{n\to+\infty}{\longrightarrow} c. \tag{3.9}$$

Denote $\varphi_n = \dfrac{u_n}{\|u_n\|_{E_0}}$, then $\|\varphi_n\|_{E_0} = 1$, which implies that $\{\varphi_n\}$ is bounded in E_0. Hence, for a subsequence of $\{\varphi_n\}$, still denoted by $\{\varphi_n\}$, and $\varphi \in E_0$, we get

$$\varphi_n \rightharpoonup \varphi \text{ in } E_0, \tag{3.10}$$

$$\varphi_n \longrightarrow \varphi \text{ in } L^{q(x)}(\Omega), \tag{3.11}$$

$$\varphi_n(x) \longrightarrow \varphi(x) \text{ a.e. in } \Omega, \tag{3.12}$$

where q is given in assumption (f_0).

• If $\varphi = 0$, as in [28], we can define a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$\mathcal{J}(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}(tu_n).$$

If there is a number of t_n satisfying the above equality, one choose one of them. Fix any $A > \dfrac{1}{2p^+_{max}}$, let $v_n = \left(2Ap^+_{\max}\right)^{\frac{1}{p^-_{\max}}} \varphi_n$. By (3.11), we get

$$v_n \longrightarrow 0 \text{ in } L^{q(x)}(\Omega).$$

From (f_0), we have

$$|F(x,t)| \leqslant c_1 \left(1 + |t|^{q(x)}\right).$$

Hence, by the continuity of $t \longmapsto F(.,t)$, we have

$$F(.,v_n) \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{in } L^1(\Omega).$$

Therefore,

$$\lim_{n\to+\infty} \int_\Omega F(x,v_n)dx = 0. \tag{3.13}$$

Then, for n large enough, $\dfrac{\left(2Ap^+_{p_{\max}}\right)^{\frac{1}{p^-_{\max}}}}{\|u_n\|_{E_0}} \in (0,1)$, using (K_0), Lemma 2.2, Proposition 2.1, and Remark 2.2, we obtain

$$\begin{aligned}
\mathcal{J}(t_n u_n) \geqslant& \mathcal{J}(v_n) \\
=& \sum_{i=1}^{2} \widehat{M}_i \left(\sigma_{p_i(x,y)}(v_n)\right) + \sum_{i=1}^{2} \int_\Omega \frac{1}{\bar{p}_i(x)} |v_n|^{\bar{p}_i(x)} dx - \int_\Omega F(x,v_n)dx \\
\geqslant& \sum_{i=1}^{2} \frac{m_i}{p_i^+} \int_Q \frac{|v_n(x) - v_n(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\, dxdy + \sum_{i=1}^{2} \frac{1}{p_i^+} \int_\Omega |v_n|^{\bar{p}_i(x)} dx \\
&- \int_\Omega F(x,v_n)dx \\
\geqslant& \sum_{i=1}^{2} \frac{m_i}{p^+_{\max}} \int_Q \left(2Ap^+_{\max}\right) \frac{|\varphi_n(x) - \varphi_n(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\, dxdy \\
&+ \sum_{i=1}^{2} \frac{1}{p^+_{\max}} \int_\Omega \left(2Ap^+_{\max}\right) |\varphi_n|^{\bar{p}_i(x)} dx - \int_\Omega F(x,v_n)dx \\
\geqslant& 2A \sum_{i=1}^{2} m_i \int_Q \frac{|\varphi_n(x) - \varphi_n(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\, dxdy + 2A \sum_{i=1}^{2} \int_\Omega |\varphi_n|^{\bar{p}_i(x)} dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega F(x,v_n)dx\\
&\geqslant 2A\sum_{i=1}^{2} m_i\|\varphi_n\|_{E_0}^{p_i^-}+2A\sum_{i=1}^{2}\|\varphi_n\|^{p_i^-}_{L^{\bar{p}_i(x)}(\Omega)}-\int_\Omega F(x,v_n)dx\\
&\geqslant 2A\sum_{i=1}^{2} m_i\|\varphi_n\|_{E_0}^{p_i^-}+2A\sum_{i=1}^{2}\bar{c}_i^{p_i^-}\|\varphi_n\|_{E_0}^{p_i^-}-\int_\Omega F(x,v_n)dx\\
&\geqslant 2A\min\{m_1,m_2\}+2A\min\{\bar{c}_1^{p_1^-},\bar{c}_2^{p_2^-}\}-\int_\Omega F(x,v_n)dx,
\end{aligned}
$$

that is,

$$\mathcal{J}(t_nu_n)\longrightarrow +\infty,\quad \text{as } n\to+\infty. \tag{3.14}$$

Since $\mathcal{J}(0)=0$ and $\mathcal{J}(u_n)\underset{n\to+\infty}{\longrightarrow} c$, then

$$t_n\in(0,1)\quad\text{and}\quad \langle\mathcal{J}'(t_nu_n),t_nu_n\rangle = t_n\frac{d}{dt}\Big|_{t=t_n}\mathcal{J}(t_nu_n)=0. \tag{3.15}$$

From (3.9) and (f_2), we have

$$
\begin{aligned}
c&=\lim_{n\to+\infty}\left\{\mathcal{J}(u_n)-\frac{1-\alpha}{p^+_{\max}}\langle\mathcal{J}'(u_n),u_n\rangle\right\}\\
&=\lim_{n\to+\infty}\left\{\sum_{i=1}^{2}\widehat{M}_i\left(\sigma_{p_i(x,y)}(u_n)\right)+\sum_{i=1}^{2}\int_\Omega\frac{1}{\bar{p}_i(x)}|u_n|^{\bar{p}_i(x)}dx-\int_\Omega F(x,u_n)dx\right.\\
&\quad-\frac{1-\alpha}{p^+_{\max}}\left(\sum_{i=1}^{2}M_i\left(\sigma_{p_i(x,y)}(u_n)\right)\rho^0_{p_i(x,y)}(u_n)+\sum_{i=1}^{2}\int_\Omega|u_n|^{\bar{p}_i(x)}dx\right.\\
&\quad\left.\left.-\int_\Omega f(x,u_n)u_ndx\right)\right\},\\
c&=\lim_{n\to+\infty}\left\{\sum_{i=1}^{2}\widehat{M}_i\left(\sigma_{p_i(x,y)}(u_n)\right)+\sum_{i=1}^{2}\int_\Omega\frac{1}{\bar{p}_i(x)}|u_n|^{\bar{p}_i(x)}dx\right.\\
&\quad-\frac{1-\alpha}{p^+_{\max}}\left(\sum_{i=1}^{2}M_i\left(\sigma_{p_i(x,y)}(u_n)\right)\rho^0_{p_i(x,y)}(u_n)+\sum_{i=1}^{2}\int_\Omega|u_n|^{\bar{p}_i(x)}dx\right.\\
&\quad\left.\left.-\int_\Omega H(x,u_n)dx\right)\right\}.
\end{aligned}\tag{3.16}
$$

Now, we consider the following function

$$t\in(0,1)\mapsto\theta_i(t)=\widehat{M}_i\Big(\sigma_{p_i(x,y)}(tu)\Big)-\frac{1-\alpha}{p^+_{\max}}M_i\Big(\sigma_{p_i(x,y)}(tu)\Big)\rho^0_{p_i(x,y)}(tu).$$

For $i=1,2$, using (K_1), we obtain

$$\alpha M_i\Big(\sigma_{p_i(x,y)}(tu)\Big)\geqslant(1-\alpha)\frac{dM_i}{dt}\Big(\sigma_{p_i(x,y)}(tu)\Big)\sigma_{p_i(x,y)}(tu).$$

This fact implies that $\dfrac{d\theta_i(t)}{dt}\geqslant 0$ for all $t\geqslant 0$, $i=1,2$. Hence $t\mapsto\theta_i(t)$ is increasing.

Thus, from (f_2) and (3.16) and since $t_n \in (0,1)$, we obtain

$$\begin{aligned} c &\geqslant \lim_{n\to+\infty} \Bigg\{ \sum_{i=1}^{2} \widehat{M}_i\left(\sigma_{p_i(x,y)}(t_n u_n)\right) + \sum_{i=1}^{2} \int_\Omega \frac{1}{\bar{p}_i(x)} |t_n u_n|^{\bar{p}_i(x)} dx \\ &\quad - \frac{1-\alpha}{p^+_{\max}} \left(\sum_{i=1}^{2} M_i\left(\sigma_{p_i(x,y)}(t_n u_n)\right) \rho^0_{p_i(x,y)}(t_n u_n) + \sum_{i=1}^{2} \int_\Omega |t_n u_n|^{\bar{p}_i(x)} dx \right. \\ &\quad \left. - \int_\Omega \frac{H(x, t_n u_n)}{\theta} dx \right) \Bigg\} \\ &\geqslant \lim_{n\to+\infty} \frac{1}{\theta} \left\{ \mathcal{J}(t_n u_n) - \frac{1-\alpha}{p^+_{\max}} \langle \mathcal{J}'(t_n u_n), t_n u_n \rangle \right\}. \end{aligned}$$

Then, by (3.14) and (3.15), we get a contradiction.

• If $\varphi \neq 0$, then the set $\Omega_0 = \{x \in \Omega : \varphi(x) \neq 0\}$ has a positive Lebesgue measure. For $x \in \Omega_0$, we have $|u_n(x)| \underset{n\to+\infty}{\longrightarrow} +\infty$. Thus, by (f_1), we get

$$\frac{F(x, u_n(x))}{|u_n(x)|^{\frac{p^+_{\max}}{1-\alpha}}} |\varphi_n(x)|^{\frac{p^+_{\max}}{1-\alpha}} \underset{n\to+\infty}{\longrightarrow} +\infty. \tag{3.17}$$

Using $(\widehat{K_1})$, for $i = 1, 2$, we can easily deduce that

$$\widehat{M_i}(t) \leqslant \tilde{c}_i t^{\frac{1}{1-\alpha_i}}, \tag{3.18}$$

where $\tilde{c}_i$ is a positive constant. Since $\mathcal{J}(u_n) \underset{n\to+\infty}{\longrightarrow} c$, using (3.18), Lemma 2.2, Proposition 2.1, and the continuous embedding of E_0 into $L^{\bar{p}_i(x)}(\Omega)$, $i = 1, 2$, we deduce via the Fatou lemma that

$$\begin{aligned} \frac{\tilde{c}_1 + \tilde{c}_2}{(p^-_{\min})^{\frac{1}{1-\alpha}}} + \frac{\bar{c}_1 + \bar{c}_2}{p^-_{\min}} - \frac{c + o(1)}{\|u_n\|_{E_0}^{\frac{p^+_{\max}}{1-\alpha}}} &\geqslant \int_\Omega \frac{F(x, u_n)}{\|u_n\|_{E_0}^{\frac{p^+_{\max}}{1-\alpha}}} dx \\ &= \int_{\varphi_n \neq 0} \frac{F(x, u_n)}{|u_n(x)|^{\frac{p^+_{\max}}{1-\alpha}}} |\varphi(x)|^{\frac{p^+_{\max}}{1-\alpha}} dx \\ &\quad + \int_{\varphi_n = 0} \frac{F(x, u_n)}{|u_n(x)|^{\frac{p^+_{\max}}{1-\alpha}}} |\varphi(x)|^{\frac{p^+_{\max}}{1-\alpha}} dx. \end{aligned}$$

By (3.17), we obtain a contradiction. □

3.2. Existence of weak solution via mountain pass theorem

By means of mountain pas theorem with Cerami condition given in Proposition 2.2, we establish the first main result of this paper which is an existence theorem for problem $(\mathcal{P}^s_{M_i})$ as stated in Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.1, $\mathcal{J}$ satisfies the Cerami condition $(\mathrm{C_c})$ in E_0. To apply Proposition 2.2, we will show that $\mathcal{J}$ possesses the mountain pass geometry.

- Firstly, we claim that there exist $R, a > 0$ such that

$$\mathcal{J}(u) \geqslant a \quad \text{for any } u \in E_0 \ \text{ with } \|u\|_{E_0} = R. \tag{3.19}$$

Indeed, Since $\bar{p}_i(x) < (\bar{p}_{\max})^*_s(x)$ for any $x \in \overline{\Omega}$, from Remark 2.2, we have that E_0 embedded in $L^{\bar{p}_i(x)}(\Omega)$, that is, there exist $\bar{c}_i > 0$, $i = 1, 2$, such that

$$\|u\|_{L^{\bar{p}_1(x)}(\Omega)} \leqslant \bar{c}_1 \|u\|_{E_0} \quad \text{and} \quad \|u\|_{L^{\bar{p}_2(x)}(\Omega)} \leqslant \bar{c}_2 \|u\|_{E_0}. \tag{3.20}$$

Moreover, as $p^+_{\max}, q(x) < (\bar{p}_{\max})^*_s(x)$ for any $x \in \overline{\Omega}$, then there exist c_2, $c_3 > 0$ such that

$$\|u\|_{L^{p^+_{\max}}(\Omega)} \leqslant c_2 \|u\|_{E_0} \quad \text{and} \quad \|u\|_{L^{q(x)}(\Omega)} \leqslant c_3 \|u\|_{E_0}. \tag{3.21}$$

Next, let $\varepsilon > 0$ be such that $\varepsilon c_2^{p^+_{\max}} < \dfrac{m_1 + m_2 + \bar{c}_1^{p_1^+} + \bar{c}_2^{p_2^+}}{2p^+_{\max}}$. Combining (f_0) and (f_3), we obtain

$$|F(x,t)| \leqslant \varepsilon |t|^{p^+_{\max}} + c_\varepsilon |t|^{q(x)} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.22}$$

Using (3.20)-(3.22), (K_0), Lemma 2.2, Proposition 2.1, and Remark 2.2, for all $\|u\|_{E_0}$ sufficiently small, we get

$$\begin{aligned}
\mathcal{J}(u) &= \sum_{i=1}^{2} \widehat{M}_i\left(\sigma_{p_i(x,y)}(u)\right) + \sum_{i=1}^{2} \int_\Omega \frac{1}{\bar{p}_i(x)} |u|^{\bar{p}_i(x)} dx - \int_\Omega F(x,u) dx \\
&\geqslant \sum_{i=1}^{2} m_i \sigma_{p_i(x,y)}(u) + \frac{1}{p^+_{\max}} \sum_{i=1}^{2} \int_\Omega |u|^{\bar{p}_i(x)} dx - \varepsilon \int_\Omega |u|^{p^+_{\max}} dx - c_\varepsilon \int_\Omega |u|^{q(x)} dx \\
&\geqslant \frac{1}{p^+_{\max}} \sum_{i=1}^{2} m_i \rho^0_{p_i(x,y)}(u) + \frac{1}{p^+_{\max}} \sum_{i=1}^{2} \|u\|^{p_i^+}_{L^{\bar{p}_i(x)}(\Omega)} - \varepsilon \|u\|^{p^+_{\max}}_{L^{p^+_{\max}}(\Omega)} - c_\varepsilon \|u\|^{q^-}_{L^{q(x)}(\Omega)} \\
&\geqslant \frac{1}{p^+_{\max}} \sum_{i=1}^{2} m_i \|u\|^{p_i^+}_{E_0} + \frac{1}{p^+_{\max}} \sum_{i=1}^{2} \bar{c}_i^{p_i^+} \|u\|^{p_i^+}_{E_0} - \varepsilon c_2^{p^+_{\max}} \|u\|^{p^+_{\max}}_{E_0} - c_\varepsilon c_3^{q^-} \|u\|^{q^-}_{E_0} \\
&= \|u\|^{p^+_{\max}}_{E_0} \left(\frac{m_1 + m_2 + \bar{c}_1^{p_1^+} + \bar{c}_2^{p_2^+}}{p^+_{\max}} - \varepsilon c_2^{p^+_{\max}} - c_\varepsilon c_3^{q^-} \|u\|^{q^- - p^+_{\max}}_{E_0} \right).
\end{aligned}$$

As $p^+_{\max} < q^-$, then there exist $R \in (0,1)$ and $a > 0$ such that (3.19) hold true.

- Secondly, we affirm that there exists $u_0 \in E_0 \setminus \overline{B_{R_6}(0)}$ such that

$$\mathcal{J}(u_0) < 0. \tag{3.23}$$

In fact, from (f_1), we choose a constant $B > \dfrac{\frac{1}{p^-_{min}} \sum_{i=1}^{2} \tilde{c}_i \left(\rho^0_{p_i(x,y)}(\varphi_0)\right)^{\frac{1}{1-\alpha}}}{\int_\Omega |\varphi_0|^{\frac{1}{1-\alpha}} dx}$, and a constant $c_B > 0$ depending on B such that

$$F(x,t) > B|t|^{\frac{p^+_{\max}}{1-\alpha}}, \quad \text{for all } |t| > c_B \text{ and uniformly in } x \in \Omega.$$

Let $l > 1$ be large enough, by the above inequality and (3.18), we have

$$\begin{aligned}
\mathcal{J}(l\varphi_0) &= \sum_{i=1}^{2} \widehat{M}_i\left(\sigma_{p_i(x,y)}(l\varphi_0)\right) + \sum_{i=1}^{2}\int_\Omega \frac{1}{\bar{p}_i(x)}|l\varphi_0|^{\bar{p}_i(x)}dx - \int_\Omega F(x,l\varphi_0)dx \\
&\leqslant \sum_{i=1}^{2} l^{\frac{p_i^+}{1-\alpha}}\tilde{c}_i\left(\sigma_{p_i(x,y)}(\varphi_0)\right)^{\frac{1}{1-\alpha}} + \sum_{i=1}^{2}\frac{l^{p_i^+}}{p_i^-}\int_\Omega |\varphi_0|^{\bar{p}_i(x)}dx \\
&\quad - \int_{\{|l\varphi_0|>c_B\}} F(x,l\varphi_0)dx - \int_{\{|l\varphi_0|\leqslant c_B\}} F(x,l\varphi_0)dx \\
&\leqslant \frac{1}{p_{\min}^-}\sum_{i=1}^{2} l^{\frac{p_i^+}{1-\alpha}}\tilde{c}_i\left(\rho^0_{p_i(x,y)}(\varphi_0)\right)^{\frac{1}{1-\alpha}} + \frac{l^{p_{\max}^+}}{p_{\min}^-}\sum_{i=1}^{2}\int_\Omega |\varphi_0|^{\bar{p}_i(x)}dx \\
&\quad - \int_{\{|l\varphi_0|\leqslant c_B\}} F(x,l\varphi_0)dx - Bl^{\frac{p_{\max}^+}{1-\alpha}}\int_\Omega |\varphi_0|^{\frac{p_{\max}^+}{1-\alpha}}dx + B\int_{\{|l\varphi_0|\leqslant c_B\}} |l\varphi_0|^{\frac{p_{\max}^+}{1-\alpha}}dx \\
&\leqslant \frac{l^{\frac{p_{\max}^+}{1-\alpha}}}{p_{\min}^-}\sum_{i=1}^{2}\tilde{c}_i\left(\rho^0_{p_i(x,y)}(\varphi_0)\right)^{\frac{1}{1-\alpha}} + \frac{l^{p_{\max}^+}}{p_{\min}^-}\sum_{i=1}^{2}\int_\Omega |\varphi_0|^{\bar{p}_i(x)}dx \\
&\quad - Bl^{\frac{p_{\max}^+}{1-\alpha}}\int_\Omega |\varphi_0|^{\frac{p_{\max}^+}{1-\alpha}}dx + c_4.
\end{aligned}$$

Hence, as $\dfrac{p_{\max}^+}{1-\alpha} > p_{\max}^+$, it follows

$$\mathcal{J}(l\varphi_0) \underset{l\to+\infty}{\longrightarrow} -\infty.$$

Consequently, there exist $l_0 > 1$ and $u_0 = l_0\varphi_0 \in E_0 \setminus \overline{B_R(0)}$ such that (3.23) hold true. Hence, in the light of mountain pass theorem with Cerami condition (Proposition 2.2), we deduce that $\mathcal{J}$ has at least one nontrivial critical value, that is, problem $(\mathcal{P}^s_{M_i})$ has at least one nontrivial solution. This completes the proof. □

3.3. Infinitely many solutions for problem $(\mathcal{P}^s_{M_i})$

In this subsection, we provide a multiplicity result for problem $(\mathcal{P}^s_{M_i})$. The main tools used here is the Fountain theorem with Cerami condition (see, Proposition 2.3).

Since E_0 is a separable and reflexive Banach space, from [39, Section 17, Theorems 2-3], there exist $\{e_n\}_{n=1}^\infty \subset E_0$ and $\{e_n^*\}_{n=1}^\infty \subset E_0^*$ such that

$$e_n^*(e_m) = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Hence, $E_0 = \overline{span}\{e_n,\ n = 1, 2, \cdots\}$ and $E_0^* = \overline{span}\{e_n^*,\ n = 1, 2, \cdots\}$. For $k = 1, 2, \cdots$, denote

$$E_{0_k} = span\{e_k\}, \quad Y_k = \oplus_{i=0}^k E_{0_i}, \quad Z_k = \overline{\oplus_{i=k}^\infty E_{0_i}}.$$

To establish the proof of the above result, we need the following auxiliary lemma.

Lemma 3.2. *Let $r\in C_+(\overline{\Omega})$ such that $r(x)<(\bar{p}_{\max})^*_s(x)$ for any $x\in\overline{\Omega}$, define*

$$\theta_k=\sup\left\{\|u\|_{L^{r(x)}(\Omega)}:\|u\|_{E_0}=1,u\in Z_k\right\}.$$

Then $\lim\limits_{k\to+\infty}\theta_k=0$.

Proof. Clearly, $0<\theta_{k+1}\leqslant\theta_k$, so $\theta_k\longrightarrow\bar{\theta}\geqslant 0$. Let $u_k\in Z_k$ satisfy

$$\|u_k\|_{E_0}=1\quad\text{and}\quad 0\leqslant\theta_k-\|u_k\|_{L^{r(x)}(\Omega)}<\frac{1}{k}.$$

Since E_0 is a reflexive space, so $\{u_k\}$ has a weakly convergent subsequence, which we still denoted by $\{u_k\}$, we suppose that $u_k\rightharpoonup u$. We claim that $u=0$. In fact, for any e_n^*, $n=1,2,\cdots$, we have $\langle e_n^*,u_k\rangle=0$ when $k>n$, hence $\langle e_n^*,u_k\rangle\longrightarrow 0$ as $k\to+\infty$, which implies that for any e_n^*, $n=1,2,\cdots$, $\langle e_n^*,u\rangle=0$. Therefore, $u=0$. That is, $u_k\rightharpoonup 0$ into E_0 as $k\to+\infty$. By the compact embedding of E_0 in $L^{r(x)}(\Omega)$, we have that $u_k\longrightarrow 0$ in $L^{r(x)}(\Omega)$. Thus, $\theta_k\longrightarrow 0$ as $k\to+\infty$. □

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We show that $\mathcal{J}$ verifies the assumptions of Fountain theorem given in Proposition 2.3. Indeed, from condition (f_4), $\mathcal{J}$ is an even function. By Lemma 3.1, $\mathcal{J}$ satisfies conditon $(\mathrm{C_c})$. Next, we verify that (Φ_1) and (Φ_2) in Theorem 2.3 are satisfied.
(Φ_1) : For $u\in Z_k$ such that $\|u\|_{E_0}=r_k>1$ (r_k will be given below), using (K_0), (f_0), Lemma 2.2, Proposition 2.1, and Remark 2.2, we get

$$\begin{aligned}
\mathcal{J}(u)&=\sum_{i=1}^{2}\widehat{M}_i\left(\sigma_{p_i(x,y)}(u)\right)+\sum_{i=1}^{2}\int_\Omega\frac{1}{\bar{p}_i(x)}|u|^{\bar{p}_i(x)}dx-\int_\Omega F(x,u)dx\\
&\geqslant\sum_{i=1}^{2}\frac{m_i}{p^+_{\max}}\|u\|_{E_0}^{p_i^-}+\sum_{i=1}^{2}\frac{1}{p^+_{\max}}\|u\|^{p_i^-}_{L^{\bar{p}_i(x)}(\Omega)}-\int_\Omega c_1(|u|+|u|^{q(x)})dx\\
&\geqslant\frac{\bar{m}}{p^+_{\max}}\|u\|_{E_0}^{p^-_{\min}}+\frac{1}{p^+_{\max}}\sum_{i=1}^{2}\bar{c}_i^{p_i^-}\|u\|_{E_0}^{p_i^-}-c_1\int_\Omega|u|^{q(x)}dx-c_1\|u\|_{E_0}\\
&\geqslant\frac{\bar{m}}{p^+_{\max}}\|u\|_{E_0}^{p^-_{\min}}+\frac{\bar{c}}{p^+_{\max}}\|u\|_{E_0}^{p^-_{\min}}-c_1\int_\Omega|u|^{q(x)}dx-c_1\|u\|_{E_0}\\
&\geqslant\begin{cases}\dfrac{\bar{m}+\bar{c}}{p^+_{\max}}\|u\|_{E_0}^{p^-_{min}}-c_1\|u\|^{q^-}_{L^{q(x)}(\Omega)}-c_1\|u\|_{E_0} & \text{if }\|u\|_{L^{q(x)}(\Omega)}\leqslant 1\\ \dfrac{\bar{m}+\bar{c}}{p^+_{\max}}\|u\|_{E_0}^{p^-_{\min}}-c_1\|u\|^{q^+}_{L^{q(x)}(\Omega)}-c_1\|u\|_{E_0} & \text{if }\|u\|_{L^{q(x)}(\Omega)}>1\end{cases}\\
&\geqslant\begin{cases}\dfrac{\bar{m}+\bar{c}}{p^+_{\max}}\|u\|_{E_0}^{p^-_{\min}}-c_1-c_1\|u\|_{E_0} & \text{if }\|u\|_{L^{q(x)}(\Omega)}\leqslant 1\\ \dfrac{\bar{m}+\bar{c}}{p^+_{\max}}\|u\|_{E_0}^{p^-_{\min}}-c_1(\theta_k\|u\|_{E_0})^{q^+}-c_1\|u\|_{E_0} & \text{if }\|u\|_{L^{q(x)}(\Omega)}>1\end{cases}\\
&\geqslant\frac{\bar{m}+\bar{c}}{p^+_{\max}}\|u\|_{E_0}^{p^-_{\min}}-c_1\theta_k^{q^+}\|u\|_{E_0}^{q^+}-c_1\|u\|_{E_0}-c_1.
\end{aligned}$$

Thus, we obtain

$$\mathcal{J}(u)\geqslant r_k^{p^-_{\min}}\left(\frac{\bar{m}+\bar{c}}{p^+_{\max}}-c_1\theta_k^{q^+}r_k^{q^+-p^-_{\min}}\right)-c_1r_k-c_1,$$

when $\bar{m} = \min\{m_1, m_2\}$ and $\bar{c} = \{\bar{c}_1^{p_1^-}, \bar{c}_2^{p_2^-}\}$. We fix r_k as follows

$$r_k = \left(\frac{c_1 q^+ \theta_k^{q^+}}{\bar{m} + \bar{c}} \right)^{\frac{1}{p_{\min}^- - q^+}}.$$

It follows that

$$\begin{aligned}\mathcal{J}(u) &\geqslant (\bar{m} + \bar{c}) r_k^{p_{\min}^-} \left(\frac{1}{p_{\max}^+} - \frac{1}{q^+} \right) - c_1 r_k - c_1 \\ &= r_k \left((\bar{m} + \bar{c}) r_k^{p_{\min}^- - 1} \left(\frac{1}{p_{\max}^+} - \frac{1}{q^+} \right) - c_1 \right) - c_1.\end{aligned}$$

Combining Lemma 3.2 with the fact that $1 < p_{\min}^- \leqslant p_{\max}^+ < q^+$, we deduce that $r_k \longrightarrow +\infty$ as $k \to +\infty$. Consequently,

$$\mathcal{J}(u) \longrightarrow +\infty \quad \text{as} \quad \|u\|_{E_0} \to +\infty.$$

The assertion (Φ_1) is verified.
(Φ_2) : Since $Y_k = \oplus_{i=0}^k E_{0_i}$, then $\dim Y_k < +\infty$ and as all norms are equivalents in the finite-dimensional space, there exists $b_k > 0$, for all $u \in Y_k$ with $\|u\|_{E_0} \geqslant 1$, by (3.18) and Lemma 2.2, we have

$$\begin{aligned}\sum_{i=1}^2 \widehat{M}_i \left(\sigma_{p_i(x,y)}(u) \right) &\leqslant \sum_{i=1}^2 \tilde{c}_i \left(\sigma_{p_i(x,y)}(u) \right)^{\frac{1}{1-\alpha}} \\ &\leqslant \frac{1}{p_{\min}^-} \sum_{i=1}^2 \tilde{c}_i \|u\|_{E_0}^{\frac{p_i^+}{1-\alpha}} \\ &\leqslant b_k \|u\|_{L^{\frac{p_{max}^+}{1-\alpha}}(\Omega)}^{\frac{p_{max}^+}{1-\alpha}}.\end{aligned} \tag{3.24}$$

Using Proposition 2.1 and since $\bar{p}_i(x) \leqslant p_{max}^+$, for all $x \in \overline{\Omega}$, we obtain

$$\begin{aligned}\sum_{i=1}^2 \int_\Omega \frac{1}{\bar{p}_i(x)} |u|^{\bar{p}_i(x)} dx &\leqslant \frac{1}{p_{min}^-} \sum_{i=1}^2 \int_\Omega |u|^{\bar{p}_i(x)} dx \\ &\leqslant \frac{1}{p_{min}^-} \sum_{i=1}^2 \|u\|_{L^{\bar{p}_i(x)}}^{p_i^+} \\ &\leqslant \frac{1}{p_{min}^-} \sum_{i=1}^2 c_i^{p_i^+} \|u\|_{L^{p_{max}^+}(\Omega)}^{p_{max}^+} \\ &\leqslant c_5 \|u\|_{L^{p_{max}^+}(\Omega)}^{p_{max}^+}.\end{aligned} \tag{3.25}$$

Combining assumptions (f_1) and (f_3), we deduce that

$$F(x,t) \geqslant 2b_k |t|^{\frac{p_{max}^+}{1-\alpha}} - \varepsilon |t|^{p_{max}^+} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.26}$$

For (3.24)-(3.26), for any $u \in Y_k$ such that $\|u\|_{E_0} = R_k > r_k$, we deduce that

$$\mathcal{J}(u) = \sum_{i=1}^2 \widehat{M}_i \left(\sigma_{p_i(x,y)}(u) \right) + \sum_{i=1}^2 \int_\Omega \frac{1}{\bar{p}_i(x)} |u|^{\bar{p}_i(x)} dx - \int_\Omega F(x,u) dx$$

$$\leqslant -b_k \|u\|^{\frac{p^+_{max}}{1-\alpha}}_{L^{\frac{p^+_{max}}{1-\alpha}}(\Omega)} + (\varepsilon + c_5)\|u\|^{p^+_{max}}_{L^{p^+_{max}}(\Omega)}$$
$$\leqslant -c_6 \|u\|^{\frac{p^+_{max}}{1-\alpha}}_{E_0} + c_7(\varepsilon + c_5)\|u\|^{p^+_{max}}_{E_0}.$$

Hence, for R_k large enough ($R_k > r_k$), from the above fact, we conclude that

$$\max_{\left\{u\in Y_k : \|u\|_{E_0} = R_k\right\}} \mathcal{J}(u) \leq 0,$$

which implies that the assertion (Φ_2) is verified. Consequently, by the Fountain theorem, we achieve the proof of Theorem 1.2. □

3.4. A multi fractional $p(x,.)$-Laplacian Kirchhoff type problem

We could extend all our results directly to the fractional multi $p(x,.)$-Laplacian case by considering the fractional $(p_1(x,.), p_2(x,.), \cdots, p_n(x,.))$-Laplacian problem of the following form

$$(\mathcal{P}^n_s) \quad \begin{cases} \sum_{i=1}^{n} M_i\left(\sigma_{p_i(x,y)}(u)\right)(-\Delta)^s_{p_i(x,.)}u(x) + \sum_{i=1}^{n}|u|^{\bar{p}_i(x)-2}u = f(x,u) \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases}$$

✠ As a particular case of Kirchhoff functions M_i, we consider

$$M_i(t) = a_i + b_i t, \quad a_i > 0, \quad b_i \geqslant 0 \quad \text{for all } t \geqslant 0, \quad i = 1, \cdots, n.$$

✠ For the nonlinearity f, we can take the function

$$f(x,t) = \frac{p^+_{\max}}{1-\mu}|t|^{\frac{p^+_{\max}}{1-\alpha}-2}t\ln(|t|),$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$. Note that the above function does not satisfy (AR). But it is easy to see that it is satisfies (f_0)-(f_4). In this case problem $(\mathcal{P}^n_s)$ becomes

$$(\mathcal{P}^{s,n}_K) \quad \begin{cases} \sum_{i=1}^{n}\left(a_i + b_i\sigma_{p_i(x,y)}(u)\right)(-\Delta)^s_{p_i(x,.)}u(x) \\ + \sum_{i=1}^{n}|u|^{\bar{p}_i(x)-2}u = \frac{p^+_{\max}}{1-\mu}|t|^{\frac{p^+_{\max}}{1-\alpha}-2}t\ln(|t|) \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases}$$

It is clear that

- $M_i(t) \geqslant a_i$ for all $t \geqslant 0$, $i = 1, \cdots, n$.
- $M'_i(t) = b_i$ for all $t \geqslant 0$, $i = 1, \cdots, n$.

If we take in (K_1), $\alpha_i = \dfrac{1}{2}$, for all $i = 1, \ldots, n$. It follows that M_i satisfies the assumptions (K_0) and (K_1). Therefore, the results obtained in Theorems 1.1 and 1.2 stay true for problem $(\mathcal{P}^{s,n}_K)$. The problem and results are all new.

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References

[1] G. A. Afrouzi, M. Mirzapour and N. T. Chung, *Existence and multiplicity of solutions for a $p(x)$-Kirchhoff type equation*, Rend. Semin. Mat. Univ. Padova, 2016, 136 , 95–109, https://doi.org/10.4171/RSMUP/136-8.

[2] K. B. Ali, M. Hsini, K. Kefi and N. T. Chung, *On a Nonlocal Fractional $p(.,.)$-Laplacian Problem with Competing Nonlinearities*, Complex Analysis and Operator Theory, 2019, 13(3), 1377–1399, https://doi.org/10.1007/s11785-018-00885-9.

[3] M. Allaoui and O. Darhouche, *Existence results for a class of nonlocal problems involving the$(p_1(x), p_2(x))$-Laplace operator*, Complex Var. Elliptic Equ., 2018, 63(1), 76–89, https://doi.org/10.1080/17476933.2017.1282950.

[4] S. N. Antontsev and J. F. Rodrigues, *On stationary thermorheological viscous flows*, Annali Dell' Università Di Ferrara, 2006, 52(1), 19–36, https://doi.org/10.1007/s11565-006-0002-9.

[5] D. Applebaum, *Lévy processes-from probability to fnance quantum groups,* Not. Am. Math. Soc., 2004, 51, 1336–1347 http://www.ams.org/notices/200411/fea-applebaum.pdf.

[6] E. Azroul and A. Boumazourh, *On a class of fractional systems with nonstandard growth conditions*, J. Pseudo–Differ. Oper. Appl., 2019, https://doi.org/10.1007/s11868-019-00310-5.

[7] E. Azroul, A. Benkirane and M. Shimi, *Existence and Multiplicity of solutions for fractional $p(x,.)$-Kirchhoff type problems in $\mathbb{R}^N$*, Applicable Analysis, 2019, https://doi.org/10.1080/00036811.2019.1673373.

[8] E. Azroul, A. Benkirane and M. Shimi, *General fractional Sobolev space with variable exponent and applications to nonlocal problems*, Advances in Operator Theory, 2020, 5(4), 1512–1540, https://doi.org/10.1007/s43036-020-00062-w.

[9] E. Azroul, A. Benkirane and M. Shimi, Eigenvalue problems involving the fractional $p(x)$-Laplacian operator. *Adv. Oper. Theory*, 2019, 4(2), 539–555, https://doi.org/10.15352/aot.1809-1420.

[10] E. Azroul, A. Benkirane, M. Shimi and M. Srati, *On a class of fractional $p(x)$-Kirchhoff type problems*, J. Applicable Analysis, 2021, 100(2), 383–402, https://doi.org/10.1080/00036811.2019.1603372.

[11] E. Azroul, A. Benkirane, A. Boumazourh and M. Shimi, *Existence results for fractional $p(x,.)$-Laplacian problem via the Nehari manifold approach*, Applied Mathematics and Optimization, 2020, https://doi.org/10.1007/s00245-020-09686-z.

[12] E. Azroul, A. Benkirane and M. Shimi, *On a nonlocal problem involving the fractional $p(x,.)$-Laplacian satisfying Cerami condition*, J. Discrete and Continuous Dynamical Systems Series S, 2020, doi: http://www.aimsciences.org/article/doi/10.3934/dcdss.2020425.

[13] E. Azroul and M. Shimi, *Nonlocal eigenvalue problems with variable exponent*, Moroccan J. of Pure and Appl. Anal., 2018, 4(1), 46–61, https://doi.org/10.1515/mjpaa-2018-0006.

[14] A. Bahrouni and V. Rădulescu, *On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent*, Discrete Contin. Dyn. Syst., 2018, 11, 379–389, https://doi.org/10.3934/dcdss.2018021.

[15] P. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlinear Anal., 1983, 7(9), 981–1012, https://doi.org/10.1016/0362-546X(83)90115-3.

[16] G. M. Bisci, V. Rădulescu and R. Servadi , *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of mathematics and its applications, Cambridge University Press, Cambridge, 2016, http://hdl.handle.net/11576/2664250.

[17] L. M. M. Bonaldo, E. J. Hurtado and O. H. Miyagaki, *A class of elliptic equations involving nonlocal integrodifferential operators with sign-changing weight functions*, J. Math. Phys., 2020, 61, 051503, https://doi.org/10.1063/1.514515.

[18] L. Caffarelli, *Nonlocal diffusions, drifts and games*, Nonlinear Partial Differential Equations, 2012, 7, 37–52, https://doi.org/10.1007/978-3-642-25361-4_3.

[19] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image processing*, SIAM J. Appl. Math., 2006, 66 , 1383–1406, https://doi.org/10.1137/050624522.

[20] G. Cerami, *An existence criterion for the critical points on unbounded manifolds*, Istit. Lombardo Accad. Sci. Lett. Rend. A, 1978, 112, 332–336, https://ci.nii.ac.jp/naid/10006654437/#cit.

[21] N. T. Chung, *Multiple solutions for a class of $(p_1(x), p_2(x))$-Laplacian problems with Neumann boundary conditions*, Acta Universitatis Apulensis, 2018, 56, 125–141, https://doi.org/10.17114/j.aua.2018.56.11.

[22] N. T. Chung, *Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities*, Electronic Journal of Qualitative Theory of Differential Equations, 2012, 42, 1–13 https://doi.org/10.14232/ejqtde.2012.1.42.

[23] N. T. Chung and H. Q. Toan, *On a class of fractional Laplacian problems with variable exponents and indefinite weights*, Collectanea Mathematica, 2019, 1–15, https://doi.org/10.1007/s13348-019-00254-5.

[24] L. Diening, P. Harjulehto, P. Hästö and M. Råužička, *Lebesgue and Sobolev Spaces with Variable Exponents, Exponents*, Lecture notes in Mathematics, 1st Edition, 2017, https://doi.org/10.1007/978-3-642-18363-8.

[25] X. Fan and D. Zhao, *On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., 2001 263, 424–446 https://doi.org/10.1006/jmaa.2000.7617.

[26] J. Francåu, *Monotone operators. A survey directed to applications to differential equations,* Aplikace matematiky, 1990, 35(4), 257–301, http://dml.cz/dmlcz/104411.

[27] T. C. Halsey, *Electrorheological fluids*, Science, 1992, 258(5083), 761–766, https://doi.org/10.1126/science.258.5083.761.

[28] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to Landesman-lazer type problem set on* $\mathbb{R}^{\mathbb{N}}$, Proc. Roy. Soc. Edinburgh Sect. A., 1999, 129, 787–809, https://doi.org/10.1017/S0308210500013147.

[29] U. Kaufmann, J. D. Rossi and R. Vidal, *Fractional Sobolev spaces with variable exponents and fractional* $p(x)$*-Laplacians*, Elec. Jour. of Qual. Th, of Diff. Equa., 2017, 76, 1–10, https://doi.org/10.14232/ejqtde.2017.1.76.

[30] O. Kováčik and J. Rákosník, *On Spaces* $L^{p(x)}(\Omega)$ *and* $W^{m,p(x)}(\Omega)$, Czechoslovak Math. Jour., 1991, 41(4), 592–618, https://dml.cz/handle/10338.dmlcz/102493.

[31] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A., 2000, 268, 298–305, https://doi.org/10.1016/S0375-9601(00)00201-2.

[32] J. I. Lee, J. M. Kim , Y. H. Kim and J. Lee, *Multiplicity of weak solutions to non-local elliptic equations involving the fractional* $p(x)$*-Laplacian*, J. Math. Phys., 2020, 61, 011505, https://doi.org/10.1063/1.5111786.

[33] D. Liu, X. Wang and J. Yao, *On* $(p_1(x), p_2(x))$*-Laplace equations*, 2012, http://arxiv.org/pdf/1205.1854.pdf.

[34] O. H. Miyagaki and M. A. S. Souto, *Superlinear problems without Ambrosetti and Rabinowitz growth condition*, J. Differential Equations, 2008, 245, 3628–3638, https://doi.org/10.1016/j.jde.2008.02.035.

[35] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math, 2012 136(5), 521–573, https://doi.org/10.1016/j.bulsci.2011.12.004.

[36] M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Teory*, Lecture Notes in Mathematics, Springer, Berlin,Germany, 2000, https://doi.org/10.1007/BFb0104029.

[37] Z. Yucedag, *Infinitely many nontrivial solutions for nonlinear problem involving* $(p_1(x), p_2(x))$*-Laplace operator*, Acta Univer. Apulensis, 2014, 40, 315–331, https://doi.org/10.17114/j.aua.2014.40.26.

[38] A. Zang, $p(x)$*-Laplacian equations satisfying Cerami condition*, J. Math. Anal. Appl., 2008, 337, 547–555, https://doi.org/10.1016/j.jmaa.2007.04.007.

[39] J. Zhao, *Structure theory of Banach spaces*, Wuhan Univ. Press, Wuhan, 1991.

[40] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, *Math. USSR. Izv.*, 1987, 9, 33–66, https://doi.org/10.1070/IM1987v029n01ABEH000958.

[41] W. Zou, *Variant Fountain theorems and their applications*, Manuscripta Math., 2001, 104, 343–358, https://doi.org/10.1007/s002290170032.