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TIKHONOV REGULARIZATION METHOD OF AN INVERSE SPACE-DEPENDENT SOURCE PROBLEM FOR A TIME-SPACE FRACTIONAL DIFFUSION EQUATION*

Jing Li[1,2,†], Gongsheng Tong[1], Rouzi Duan[1], Shanlin Qin[3]

Abstract The aim of this paper is to identify a space-dependent source term in the time-space fractional diffusion equation with an initial-boundary data and an additional measurement data at the final time point. A series expression for the solution of the direct problem is used to transfer the inverse problem into the first type of Fredholm integral equation. Before solving the inverse problem, the uniqueness of its solution is proved. We then use the Tikhonov regularization method to deal with the integral equation and obtain a series expression for the regularized solution of the inverse problem. Moreover, according to the prior and the posterior regularization parameter selection rules, we prove the convergence rates of the regularization solution. Finally, we provide some numerical experiments to show the effectiveness of our method.

Keywords Inverse space-dependent source problem, time-space fractional diffusion equation, Tikhonov regularization.

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1. Introduction

Fractional derivatives provide a promising tool for the description of the abnormal diffusion in the heterogeneous and fractal media. As an example, the time-space fractional diffusion equation has received much attention among researchers since it can model complex physical phenomena in biology, chemistry and signal processing et al. [3, 4, 9, 20]. There are many theoretical works about the direct problems of the time-space fractional diffusion equations. For example, Guo et al. [5, 13] studied the well-posedness, the attractors, and the asymptotic behavior of the time-space fractional diffusion equation. Different numerical schemes based on the finite

[†]The corresponding author. Email: lijingnew@126.com (J. Li)
[1]School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, 410114, China
[2]Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering
[3]Shenzhen Institutes of Advanced Technology, Chinese Academy of Sciences, Shenzhen, 518055, China
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difference/element methods, the ADI method, the spectral method and the Fourier fast method are proposed to obtain its numerical solution [12, 14, 16, 22, 27, 31].

In many practical applications, the absence of a part of the initial data, the boundary coefficient or the source term will lead to an inverse problem (IP), where these parameters are required to be determined. Recently, the inverse source problems have been extensively considered for the time fractional diffusion equations. For example, Sakamoto and Yamamoto [19] established the uniqueness results of several inverse time-dependent source problems. Sun and Liu [21] employed the conjugate gradient method to identify the unknown time-dependent source in a distributed time fractional diffusion equation. Liu et al. [17] used two regularization methods to recover the time-dependent factors of a time fractional diffusion equation with non-local measured data. Wei and Wang [28] solved an inverse space-dependent source problem for the time fractional diffusion equation by a modified quasi-boundary value method. By using the Tikhonov regularization method, Wei also [29] identified a time-dependent source term in a multi-dimensional time fractional diffusion equation with boundary Cauchy data. This method was also used to investigated an inverse source problem for a time fractional diffusion-wave equation [30]. Nguyen Minh Dien et al. [1] investigated a multi-dimensional space-dependent heat source in a time fractional diffusion equation by using the Tikhonov method. However, the above-mentioned works are predominantly focused on the time fractional diffusion equation, and the inverse problem for the time-space fractional diffusion equation is relatively less studied.

In this paper, we consider the following time-space fractional diffusion equation:

$$\begin{cases} \partial_{0+}^{\alpha} u(x,t) = -(-\Delta)^{\frac{\beta}{2}} u(x,t) + f(x)p(t), & x \in \Omega, \quad t \in (0,T], \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T]. \end{cases} \tag{1.1}$$

with addition data $u(x,T) = g(x)$. In (1.1), the unknown source term $f(x)$ is yet to be determined.

Here, $\Omega \subset \mathbb{R}^d (d = 1, 2 \ or \ 3)$ is a bounded domain with a sufficient smooth boundary $\partial\Omega$; u represents the pollutant concentration at the position x and the time t; and ∂_{0+}^{α} is the Caputo left-sided fractional derivative of order $\alpha \in (0,1]$ defined by

$$\partial_{0+}^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_t(x,\tau)}{(t-\tau)^{\alpha}} d\tau, & 0 < \alpha < 1, \\ u_t(x,t), & \alpha = 1, \end{cases} \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma function. $(-\Delta)^{\frac{\beta}{2}}$ is the space fractional Laplacian operator of order $\beta \in (1,2)$, whose definition will be given in Definition 2.1.

Since the measured noise is unavoidable, we assume that the noisy measurement data $g^{\delta}(x)$ satisfies

$$\| g^{\delta}(x) - g(x) \| \leqslant \delta, \tag{1.3}$$

where $\| \cdot \|$ denotes the $L^2(\Omega)$ norm and $\delta > 0$ is a noise level.

After studying the direct problem of (1.1), Wei and Li [15] combined the boundary element method with a generalized Tikhonov regularization method to identify a time-dependent source term. Tatar et al. [23–25] determined the orders of the time-space fractional derivatives in (1.1). Tuan and Long [26] recovered the source term that depends only on the space by using a truncated Fourier method. Moreover, they gave the convergence estimation and the selection rules of the choices of the regularization parameters, but provided with no numerical calculation. In this paper, we discuss the numerical reconstruction of the source term $f(x)$ in (1.1). We use the Tikhonov regularization method to determine the source $f(x)$ and present some numerical experiments to show the effectiveness of this method. In this work, we focus on:

(**IP**): Assume that the time-dependent source term $p(t)$ is given, we will reconstruct $f(x)$ from the measurement $g^\delta(x)$.

The structure of this paper is organized as follows. In Section 2, we give some preliminary definitions and lemmas. The ill-posedness of the inverse problem, and the conditional stability are studied in Section 3. In Section 4, we present the Tikhonov regularization method and give the convergence rate of the regularized solution. A numerical example is presented to show the effectiveness of our result in Section 5.

2. Preliminaries

Throughout this paper, we use the following definitions and lemmas.

Definition 2.1 ([15]). Let $-\Delta$ be the Laplacian operator in Ω and $\{\overline{\lambda}_n, \varphi_n\}$ be the eigenvalues and the eigenvectors with Dirichlet homogeneous boundary conditions, respectively

$$\begin{cases} -\Delta\varphi_n = \overline{\lambda}_n \varphi_n, & \text{in} \quad \Omega, \\ \varphi_n = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Let

$$\mathcal{H}_0^\beta(\Omega) := \left\{ u = \sum_{n=1}^{\infty} a_n \varphi_n : \| u \|^2_{\mathcal{H}_0^\beta(\Omega)} = \sum_{n=1}^{\infty} a_n^2 \overline{\lambda}_n^\beta < \infty \right\}.$$

For $u \in \mathcal{H}_0^\beta(\Omega)$, the operator $(-\Delta)^{\frac{\beta}{2}}$ can be defined as follows :

$$(-\Delta)^{\frac{\beta}{2}} u = \sum_{n=1}^{\infty} a_n \overline{\lambda}_n^{\beta/2} \varphi_n.$$

This operator maps $\mathcal{H}_0^\beta(\Omega)$ to $L_2(\Omega)$ with

$$\| u \|_{\mathcal{H}_0^\beta(\Omega)} = \| (-\Delta)^{\frac{\beta}{2}} u \|_{L_2(\Omega)} .$$

Definition 2.2. The Mittag-Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ are arbitrary constants.

Lemma 2.1 ([18]). *For $0 < \alpha < 1, \eta > 0$, we have $0 \le E_{\alpha,\alpha}(-\eta) \le \frac{1}{\Gamma(\alpha)}$. Moreover, $E_{\alpha,\alpha}(-\eta)$ is a completely monotone increasing function for $\eta > 0$.*

Lemma 2.2 ([18]). *If $0 < \alpha < 1$, $t > 0$, then we have $0 < E_{\alpha,1}(-t) < 1$. Moreover, $E_{\alpha,1}(-t)$ is a completely monotone increasing function for $t > 0$, and we obtain $0 < E_{\alpha,1}(t) < E_{\alpha,1}(0) = 1,\ \forall t > 0$.*

Lemma 2.3 ([10]). *If $\lambda > 0$, then we have*

$$\frac{d}{dt}E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad t > 0 \text{ and } 0 < \alpha < 1. \tag{2.1}$$

Lemma 2.4 ([10]). *For $\alpha > 0, \beta \in R$, it follows*

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \quad z \in \mathbb{C}.$$

Lemma 2.5 ([28]). *For any λ_n that satisfies $\lambda_n \geq \lambda_1 > 0$, there exists a positive constant $C_1 = 1 - E_{\alpha,1}(-\lambda_1 T^{\alpha})$ depending on α, T, λ_1, such that*

$$\frac{C_1}{T^{\alpha}\lambda_n} \leqslant E_{\alpha,\alpha+1}\left(-\lambda_n T^{\alpha}\right) \leqslant \frac{1}{T^{\alpha}\lambda_n}. \tag{2.2}$$

Lemma 2.6 ([28]). *For constants $q > 0$, $\mu > 0$, $\rho > 0$, $s \geq \lambda_1 > 0$, we have*

$$F(s) = \frac{\mu s^{2-\frac{q}{2}}}{\mu s^2 + \rho} \leqslant \begin{cases} C_2\mu^{\frac{q}{4}}, & 0 < q < 4, \\ C_3\mu, & q \geqslant 4, \end{cases} \tag{2.3}$$

$$G(s) = \frac{\mu s^{1-\frac{q}{2}}}{\mu s^2 + \rho} \leqslant \begin{cases} C_4\mu^{\frac{2+q}{4}}, & 0 < q < 2, \\ C_5\mu, & q \geqslant 2, \end{cases} \tag{2.4}$$

where $C_2 = C_2(q,\rho) > 0$, $C_3 = C_3(q,\rho,\lambda_1) > 0$, $C_4 = C_2(q,\rho) > 0$, $C_5 = C_3(q,\rho,\lambda_1) > 0$ are independent of s.

3. Uniqueness and conditional stability for the inverse source problem

Denote the eigenvalues of the operator $-\Delta$ with the Dirichlet homogeneous boundary condition as $\overline{\lambda}_n$ and the corresponding eigenfunctions as $\varphi_n(x) \in H^2(\Omega) \cap H_0^1(\Omega)$. Then we have $-\Delta\varphi_n = \overline{\lambda}_n\varphi_n$ and $\varphi_n|_{\partial\Omega} = 0$. Additionally,

$$0 < \overline{\lambda}_1 \leq \overline{\lambda}_2 \leq \cdots \leq \overline{\lambda}_n \leq \cdots, \quad \lim_{n\longrightarrow\infty}\overline{\lambda}_n = +\infty,$$

and $\{\varphi_n(x)\}_{n=1}^{\infty}$ is an orthonormal basis in $L_2(\Omega)$.

Theorem 3.1 ([15]). *Let $\phi \in \mathcal{H}_0^{\beta}(\Omega), f \in L^2(\Omega)$, and $p \in AC[0,T]$, where $AC[0,T]$ is the space of functions which are absolutely continuous on $[0,T]$. Then there exists a unique weak solution $u \in C([0,T];L^2(\Omega)) \bigcap L^2(0,T;\mathcal{H}_0^{\beta}(\Omega))$ with $\partial_{0+}^{\alpha}u \in C([0,T];L^2(\Omega)) \bigcap L^2(0,T;L^2(\Omega))$ of (1.1), which is given by*

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty}\left(\phi,\varphi_n\right)E_{\alpha,1}\left(-\lambda_n t^{\alpha}\right)\varphi_n(x) \\ &+ \sum_{n=1}^{\infty}\left(f,\varphi_n\right)\int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^{\alpha}\right)d\tau\varphi_n(x), \end{aligned} \tag{3.1}$$

where $\lambda_n = \overline{\lambda}_n^{\frac{\beta}{2}}$.

Let $t = T$ in (3.1), then we have

$$g(x) = u(x,T) = \sum_{n=1}^{\infty} (\phi, \varphi_n) E_{\alpha,1}(-\lambda_n T^{\alpha}) \varphi_n(x) + \sum_{n=1}^{\infty} (f, \varphi_n) \int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^{\alpha}) d\tau \varphi_n(x).$$

Denote $g_1(x) = g(x) - \sum_{n=1}^{\infty} (\phi, \varphi_n) E_{\alpha,1}(-\lambda_n T^{\alpha}) \varphi_n(x)$, and $Q_n(t) = \int_0^t p(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha}) d\tau$. Setting $f_n = (f, \varphi_n)$, $g_{1n} = (g_1, \varphi_n)$, then we have

$$g_{1n} = f_n Q_n(T). \tag{3.2}$$

To find $f(x)$, we just need to solve the following first kind integral equation

$$(Kf)(x) = \int_{\Omega} k(x,\xi) f(\xi) d\xi = g_1(x), \quad x \in \Omega, \tag{3.3}$$

where

$$k(x,\xi) = \sum_{n=1}^{\infty} Q_n(T) \varphi_n(x) \varphi_n(\xi).$$

Theorem 3.2. *If $p(t) \in C[0,T]$ satisfying $p(t) \geq p_0 > 0$, $\forall t \in [0,T]$, then the solution $f(x)$ of problem (1.1) is unique.*

Proof. By Lemma 2.1, we know $E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha}) \geqslant 0$ when $\tau \leqslant t$. From Lemmas 2.1 - 2.3, we have

$$\begin{aligned} Q_n(T) &\geqslant p_0 \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-\tau)^{\alpha}) d\tau \\ &= \frac{p_0}{\lambda_n} E_{\alpha,1}(-\lambda_n (T-\tau)^{\alpha}) |_0^T \\ &= p_0 T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_n T^{\alpha}). \end{aligned} \tag{3.4}$$

Furthermore, we know $Q_n(T) > 0$ via Lemma 2.5. Thus, from (3.2), it is obvious that if $g_1(x) = 0$, then $f(x) = 0$.

This proves the uniqueness of $f(x)$. □

Remark. K is a compact operator. From Kirsch [11], we conclude that the inverse problem (**IP**) is ill-posed, i.e. the integral equation (3.3) is ill-posed. For example, if we take $\phi(x) = 0$, $g_{1s}(x) = g_s(x) = \frac{\varphi_s(x)}{\sqrt{\lambda_s}}$ in (1.1), then $\| g_s \| = \frac{1}{\sqrt{\lambda_s}} \to 0$ as $s \to \infty$. The corresponding source terms are $f_s(x) = \frac{\varphi_s(x)}{Q_s(T)\sqrt{\lambda_s}}$, and we have $\| f_s \| = \frac{1}{Q_s(T)\sqrt{\lambda_s}}$. By Lemma 2.5, it can be concluded that

$$\begin{aligned} Q_s(T) &\leqslant \| p \|_{C[0,T]} \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_s(T-\tau)^{\alpha}) d\tau \\ &= \| p \|_{C[0,T]} T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_s T^{\alpha}) \leqslant \frac{\| p \|_{C[0,T]}}{\lambda_s}. \end{aligned} \tag{3.5}$$

Hence $\| f_s \| = \frac{1}{Q_s(T)\sqrt{\lambda_s}} \geqslant \frac{\lambda_s}{\|p\|_{C[0,T]}} \to \infty$, as $s \to \infty$.

Next, we'll give a conditional stability about the source term $f(x)$.

Theorem 3.3. *If $p(t) \in C[0,T]$, and $p(t) \geqslant p_0 > 0$, $t \in [0,T]$. Let $f(x) \in \mathcal{H}_0^{\frac{\beta q}{2}}(\Omega)$ satisfy a priori bound condition*

$$\| f \|_{\mathcal{H}_0^{\frac{\beta q}{2}}(\Omega)} \leqslant E, \quad q > 0, \tag{3.6}$$

then we can assert that

$$\| f \| \leqslant C_6 E^{\frac{2}{q+2}} \| g_1 \|^{\frac{q}{q+2}}, \quad q > 0,$$

where $C_6 = (p_0 C_1)^{-\frac{q}{q+2}}$ is a constant depending on $\alpha, T, q, \lambda_1, p_0$.

Proof. From (3.2) and the Hölder inequality, we have

$$\begin{aligned} \| f \|^2 &= \sum_{n=1}^{\infty} f_n^2 = \sum_{n=1}^{\infty} \frac{g_{1n}^2}{Q_n^2(T)} \\ &= \sum_{n=1}^{\infty} \frac{g_{1n}^{\frac{4}{q+2}}}{Q_n^2(T)} g_{1n}^{\frac{2q}{q+2}} \\ &\leqslant \left(\sum_{n=1}^{\infty} \frac{g_{1n}^2}{Q_n^{q+2}(T)} \right)^{\frac{2}{q+2}} \left(\sum_{n=1}^{\infty} g_{1n}^2 \right)^{\frac{q}{q+2}}. \end{aligned} \tag{3.7}$$

Applying Lemma 2.5 and the equation (3.4), it yields

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{g_{1n}^2}{Q_n^{q+2}(T)} &\leqslant \sum_{n=1}^{\infty} \frac{f_n^2}{(p_0 T^\alpha E_{\alpha,\alpha+1}(-\lambda_n T^\alpha))^q} \\ &\leqslant \sum_{n=1}^{\infty} f_n^2 \lambda_n^q (p_0 C_1)^{-q} = \| f \|^2_{\mathcal{H}_0^{\frac{\beta q}{2}}(\Omega)} (p_0 C_1)^{-q}. \end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8), we obtian

$$\| f \|^2 \leqslant C_6^2 \| f \|^{\frac{4}{q+2}}_{\mathcal{H}_0^{\frac{\beta q}{2}}(\Omega)} \| g_1 \|^{\frac{2q}{q+2}},$$

where $C_6 = (p_0 C_1)^{-\frac{q}{q+2}}$. The proof is completed. □

4. Tikhonov regularization method and convergence rate

In this section, we will solve the integral equation (3.3). Due to the ill-posedness of this equation, we utilize the Tikhonov regularization method to obtain the approximation solution. We then prove the convergence rates for the approximation solutions.

The Tikhonov regularization method (see e.g., [6–8]) is used to solve the integral equation (3.3) as follows :

$$\min_{f \in L^2(\Omega)} \| Kf - g_1 \|^2 + \mu \| f \|^2, \tag{4.1}$$

where $\mu > 0$ is a regularization parameter. Let K^* be the adjoint of K. Since $\{\varphi_n\}_{n=1}^{\infty}$ are orthonormal in $L_2(\Omega)$, it is easy to verify

$$K^* K\varphi_n(\xi) = Q_n^2(T)\varphi_n(\xi).$$

Hence, the singular values of K are $\sigma_n = Q_n(T)$. Therefore, we conclude that (see [2])

$$\Psi_n(x) = \frac{K\varphi_n(x)}{\| K\varphi_n(x) \|} = \varphi_n(x).$$

It is clear that $\Psi_n(x)$ are orthonormal in $L_2(\Omega)$ and the following formulas hold

$$\begin{aligned} K\varphi_n(\xi) &= \sigma_n \Psi_n(x) = Q_n(T)\varphi_n(x), \\ K^*\Psi_n(x) &= \sigma_n \varphi_n(\xi) = Q_n(T)\Psi_n(\xi). \end{aligned}$$

Consequently, the singular system of K is $(\sigma_n; \Psi_n(x), \varphi_n)$. The range of the operator K is as follows

$$R(K) = \left\{ g_1 \in L^2(\Omega) \middle| \sum_{n=1}^{\infty} \left(\frac{g_{1n}}{Q_n(t)} \right)^2 < \infty \right\}.$$

By Theorem 5.1 in [2], we know that the minimizer of (4.1) can be denoted by f_μ which satisfies

$$(K^* K + \mu I)\, f_\mu = K^* g_1. \tag{4.2}$$

Hence, for the compact linear operator K with singular systems $(\sigma_n; \Psi_n(x), \varphi_n)$, we obtain

$$f_\mu(x) = \sum_{n=1}^{\infty} \frac{Q_n(T)}{Q_n^2(T) + \mu} \,(g_1, \varphi_n)\, \varphi_n(x). \tag{4.3}$$

Additionally,

$$g_1^\delta = g^\delta - \sum_{n=1}^{\infty} (\phi, \varphi_n) E_{\alpha,1}(-\lambda_n T^\alpha)\varphi_n(x).$$

Thus, the Tikhonov regularized solution is

$$f_\mu^\delta(x) = \sum_{n=1}^{\infty} \frac{Q_n(T)}{Q_n^2(T) + \mu} \left(g_1^\delta, \varphi_n\right) \varphi_n(x). \tag{4.4}$$

Next, we give two convergence rate estimates for $\| f_\mu^\delta - f \|$ by using a priori and a posteriori choice rules for the regularization parameter.

4.1. Convergence rate estimate under an a priori regularization parameter choice rule

Theorem 4.1. *Let $p(t) \in C[0,T]$, $p(t) \geq p_0 > 0$, $\forall t \in [0,T]$. Suppose that the priori condition* (3.6) *and the noise assumption* (1.3) *hold, then*

(i) if $0 < q < 4$ and choose $\mu = (\frac{\delta}{E})^{\frac{4}{q+2}}$, we have

$$\| f_\mu^\delta - f \| \leqslant C_7 E^{\frac{2}{q+2}} \delta^{\frac{q}{q+2}}; \tag{4.5}$$

(ii) if $q \geqslant 4$ and choose $\mu = (\frac{\delta}{E})^{\frac{2}{3}}$, we have

$$\| f_\mu^\delta - f \| \leqslant C_8 E^{\frac{1}{3}} \delta^{\frac{2}{3}}, \tag{4.6}$$

where $C_7 = C_7(\alpha, T, \lambda_1, p_0, q)$ and $C_8 = C_8(\alpha, T, \lambda_1, p_0, q)$ are positive constants.

Proof. From the triangle inequality, we know

$$\| f_\mu^\delta - f \| \leqslant \| f_\mu^\delta - f_\mu \| + \| f_\mu - f \| . \tag{4.7}$$

We firstly give an estimate for the first term on the right side of (4.7),

$$\| f_\mu^\delta - f_\mu \|^2 = \sum_{n=1}^{\infty} \left(\frac{Q_n(T)}{Q_n^2(T)+\mu} \right)^2 \left(g_{1n}^\delta - g_{1n} \right)^2 \leqslant \delta^2 \left(\sup_n A(n) \right)^2 , \tag{4.8}$$

where $g_{1n}^\delta = (g_1^\delta, \varphi_n)$ and

$$A(n) = \frac{|Q_n(T)|}{Q_n^2(T)+\mu} \leqslant \frac{1}{2\sqrt{\mu}}.$$

Thus, we get

$$\| f_\mu^\delta - f_\mu \| \leqslant \frac{\delta}{2\sqrt{\mu}}. \tag{4.9}$$

Now we estimate the second term on the right side of (4.7)

$$\begin{aligned} f_\mu(x) - f(x) &= \sum_{n=1}^{\infty} \left(\frac{Q_n(T)}{Q_n^2(T)+\mu} - \frac{1}{Q_n(T)} \right) g_{1n} \varphi_n(x) \\ &= \sum_{n=1}^{\infty} \frac{g_{1n}}{Q_n(T)} \frac{-\mu}{Q_n^2(T)+\mu} \varphi_n(x). \end{aligned} \tag{4.10}$$

Applying the a priori bound condition (3.6), we obtain

$$\begin{aligned} \| f_\mu(x) - f(x) \|^2 &= \sum_{n=1}^{\infty} \frac{g_{1n}^2}{Q_n^2(T)} \lambda_n^q \left(\frac{-\mu}{Q_n^2(T)+\mu} \right)^2 \frac{1}{\lambda_n^q} \\ &\leqslant \| f \|^2_{\mathcal{H}_0^{\frac{\beta q}{2}}(\Omega)} \left(\sup_n B(n) \right)^2 \\ &\leqslant E^2 \left(\sup_n B(n) \right)^2 , \end{aligned} \tag{4.11}$$

where $B(n) = \frac{\mu \lambda_n^{-\frac{q}{2}}}{Q_n^2(T)+\mu}$. From (3.4) and Lemma 2.5, we have

$$B(n) \leqslant \frac{\mu \lambda_n^{2-\frac{q}{2}}}{(C_1 p_0)^2 + \mu \lambda_n^2}. \tag{4.12}$$

With the help of Lemma 2.6, it follows that

$$B(n) \leqslant \begin{cases} C_2 \mu^{\frac{q}{4}}, & 0 < q < 4, \\ C_3 \mu, & q \geq 4, \end{cases} \tag{4.13}$$

where $C_2(\alpha, T, \lambda_1, p_0, q) > 0$ and $C_3(\alpha, T, \lambda_1, p_0, q) > 0$. Based on the above inequalities, we obtain

$$\| f_\mu^\delta - f \| \leqslant \frac{1}{2} \frac{\delta}{\sqrt{\mu}} + \begin{cases} C_2 E \mu^{\frac{q}{4}}, & 0 < q < 4, \\ C_3 E \mu, & q \geq 4. \end{cases} \tag{4.14}$$

Choosing the regularization parameter μ by

$$\mu = \begin{cases} \left(\frac{\delta}{E}\right)^{\frac{4}{q+2}}, & 0 < q < 4, \\ \left(\frac{\delta}{E}\right)^{\frac{2}{3}}, & q \geq 4. \end{cases} \tag{4.15}$$

Then, we conclude

$$\| f_\mu^\delta - f \| \leqslant \begin{cases} C_7 E^{\frac{2}{q+2}} \delta^{\frac{q}{q+2}}, & 0 < q < 4, \\ C_8 E^{\frac{1}{3}} \delta^{\frac{2}{3}}, & q \geq 4, \end{cases} \tag{4.16}$$

where $C_7 = C_7(\alpha, T, \lambda_1, p_0, q) > 0$ and $C_8 = C_8(\alpha, T, \lambda_1, p_0, q) > 0$, which completes the proof. □

4.2. Convergence rate estimate under an a posteriori regularization parameter choice rule

In this subsection, we use a posterior regularization parameter principle (namely the Morozov's discrepancy principle) to select the regularization parameter μ in the equation (4.4). By using the conditional stability estimate in Theorem 3.3, we can derive a convergence rate for the regularized solution (4.4).

Define the orthogonal project operator $F : L^2(\Omega) \to \overline{R(K)}$. Then according to the equation (1.3), we have

$$\| Fg_1^\delta - Fg_1 \| \leqslant \| g_1^\delta - g_1 \| = \| g^\delta - g \| \leqslant \delta. \tag{4.17}$$

The Morozov discrepancy principle here is to find μ such that

$$\| Kf_\mu^\delta - Fg_1^\delta \| = \tau\delta, \tag{4.18}$$

where $\tau > 1$ is a constant. According to the following lemma, we know there exists a unique solution for (4.18) if $\| Fg_1^\delta \| > \tau\delta$.

Lemma 4.1. *Setting $\rho(\mu) = \| Kf_\mu^\delta - Fg_1^\delta \|$, then the following results hold*

(i) $\rho(\mu)$ is a continuous function;

(ii) $\lim_{\mu \to 0} \rho(\mu) = 0$;

(iii) $\lim_{\mu \to +\infty} \rho(\mu) = \| Fg_1^\delta \|$;

(iv) $\rho(\mu)$ is a strictly increasing function over $\mu \in (0, \infty)$.

Proof. The proof can be straightforward given by the expression of

$$\rho(\mu) = \left(\sum_{n=1}^{\infty} \left(\frac{\mu}{Q_n^2(T) + \mu} \right)^2 \left(g_{1n}^\delta\right)^2 \right)^{\frac{1}{2}}.$$

□

Theorem 4.2. *Let $p(t) \in C[0, T]$, $p(t) \geq p_0 > 0$, $\forall t \in [0, T]$. Suppose that the priori condition* (3.6) *and the noise assumption* (1.3) *hold, and there exists $\tau > 1$ such that $\| Fg_1^\delta \| > \tau\delta > 0$. Additionally, the regularization parameter $\mu > 0$ is chosen by the Morozov's discrepancy principle* (4.18). *Then* *(i) if* $0 < q < 2$*, we get*

$$\| f_\mu^\delta - f \| \leqslant C_9 E^{\frac{2}{q+2}} \delta^{\frac{q}{q+2}}; \tag{4.19}$$

(ii) if $q \geqslant 2$*, we obtain*

$$\| f_\mu^\delta - f \| \leqslant C_{10} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, \tag{4.20}$$

where $C_9 = C_9(\alpha, T, \lambda_1, p_0, q, \tau, p_1)$ *and* $C_{10} = C_{10}(\alpha, T, \lambda_1, p_0, q, \tau, p_1)$ *are positive constants.*

Proof. From the triangle inequality, we have

$$\| f_\mu^\delta - f \| \leqslant \| f_\mu^\delta - f_\mu \| + \| f_\mu - f \| . \tag{4.21}$$

Firstly, we give an estimate for the second term on the right side of (4.21). Following proof process of Theorem 4.3 in [30], we get

$$\begin{aligned} K\big(f_\mu(x) - f(x)\big) &= \sum_{n=1}^{\infty} g_{1n} \frac{-\mu}{Q_n^2(T) + \mu} \varphi_n(x) \\ &= \sum_{n=1}^{\infty} \left(g_{1n} - g_{1n}^\delta\right) \frac{-\mu}{Q_n^2(T) + \mu} \varphi_n(x) + \sum_{n=1}^{\infty} g_{1n}^\delta \frac{-\mu}{Q_n^2(T) + \mu} \varphi_n(x). \end{aligned} \tag{4.22}$$

Combining (4.17) with (4.18), we have

$$\| K\left(f_\mu(x) - f(x)\right) \| \leqslant \delta + \tau\delta = (\tau + 1)\delta. \tag{4.23}$$

In addition, by applying the prior condition (3.6), it yields that

$$\begin{aligned} \| f_\mu(x) - f(x) \|_{\mathcal{H}_0^{\frac{\beta q}{2}}(\Omega)} &= \left(\sum_{n=1}^{\infty} \left(\frac{g_{1n}}{Q_n(T)} \frac{-\mu}{Q_n^2(T) + \mu} \lambda_n^{\frac{q}{2}} \right)^2 \right)^{\frac{1}{2}} \\ &\leqslant \left(\sum_{n=1}^{\infty} \left(\frac{g_{1n}}{Q_n(T)} \right)^2 \lambda_n^q \right)^{\frac{1}{2}} \leqslant E. \end{aligned} \tag{4.24}$$

By Theorem 3.3, we deduce that

$$\| f_\mu(x) - f(x) \| \leqslant C_6 (\tau + 1)^{\frac{q}{q+2}} E^{\frac{2}{q+2}} \delta^{\frac{q}{q+2}}, \quad \forall q > 0, \tag{4.25}$$

where $C_6 = (p_0 C_1)^{-\frac{q}{q+2}}$. Now we give an estimate for the first term on the right side of (4.21). Similar to (4.9), we have

$$\| f_\mu^\delta - f_\mu \| \leqslant \frac{\delta}{2\sqrt{\mu}}. \tag{4.26}$$

From (4.18), we have

$$\begin{aligned} \tau\delta &= \| \sum_{n=1}^{\infty} \frac{\mu}{Q_n^2(T) + \mu} g_{1n}^\delta \varphi_n(x) \| \\ &\leqslant \| \sum_{n=1}^{\infty} \frac{\mu}{Q_n^2(T) + \mu} \left(g_{1n}^\delta - g_{1n}\right) \varphi_n(x) \| + \| \sum_{n=1}^{\infty} \frac{\mu}{Q_n^2(T) + \mu} g_{1n} \varphi_n(x) \| \\ &\leqslant \delta + J. \end{aligned} \tag{4.27}$$

Using the priori bound condition (3.6), we obtain

$$\begin{aligned} J =& \| \sum_{n=1}^{\infty} \frac{\mu Q_n(T)}{Q_n^2(T)+\mu} \frac{1}{\lambda_n^{\frac{q}{2}}} \frac{g_{1n}}{Q_n(T)} \lambda_n^{\frac{q}{2}} \varphi_n(x) \| \\ \leqslant& \| f \|_{\mathcal{H}_0^{\frac{\beta q}{2}}(\Omega)} \left(\sup_n C(n) \right) \\ \leqslant& E \sup_n C(n), \end{aligned} \tag{4.28}$$

where $C(n) = \frac{\mu Q_n(T)}{Q_n^2(T)+\mu} \frac{1}{\lambda_n^{\frac{q}{2}}}$. Due to Lemma 2.5 and (3.5), we obtain

$$C(n) \leqslant \frac{\mu \frac{p_1}{\lambda_n}}{\frac{(C_1 p_0)^2}{\lambda_n^2} + \mu} \frac{1}{\lambda_n^{\frac{q}{2}}} \leqslant \frac{p_1 \mu \lambda_n^{1-\frac{q}{2}}}{(C_1 p_0)^2 + \mu \lambda_n^2}, \tag{4.29}$$

where $p_1 = \| p \|_{C[0,T]}$. By Lemma 2.6, we can conclude that

$$C(n) \leqslant \begin{cases} p_1 C_4 \mu^{\frac{2+q}{4}}, & 0 < q < 2, \\ p_1 C_5 \mu, & q \geqslant 2, \end{cases} \tag{4.30}$$

where $C_4(\alpha, T, \lambda_1, p_0, q) > 0$ and $C_5(\alpha, T, \lambda_1, p_0, q) > 0$. Substitute (4.28) and (4.30) into (4.27), we obtain

$$\frac{1}{\mu} \leqslant \begin{cases} (\frac{p_1 C_4}{\tau - 1})^{\frac{4}{q+2}} (\frac{E}{\delta})^{\frac{4}{q+2}}, & 0 < q < 2, \\ \frac{p_1 C_5}{\tau - 1} \frac{E}{\delta}, & q \geqslant 2. \end{cases} \tag{4.31}$$

Substitute (4.31) into (4.26), we conclude

$$\| f_\mu^\delta - f \| \leq \begin{cases} C_9 E^{\frac{2}{q+2}} \delta^{\frac{q}{q+2}}, & 0 < q < 2, \\ C_{10} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & q \geq 2, \end{cases} \tag{4.32}$$

where $C_9 = C_9(\alpha, T, \lambda_1, p_0, q, \tau, p_1)$ and $C_{10} = C_{10}(\alpha, T, \lambda_1, p_0, q, \tau, p_1)$ are positive constants, which completes the proof. □

5. Numerical example

In this section, we present an example to verify the effectiveness of the previous regularization method.

We assume that the source function $f(x) = cos(\pi x)$, $p(t) = 1$ in (1.1). Let $\phi(x) = x^2(1-x)^2$, $T = 1$, $\Omega = (0,1)$. Then one obtains the kernel $k(x_i, \xi_j) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} E_{\alpha,\alpha+1}(-\lambda_n T^\alpha) \varphi_n(x_i) \varphi_n(\xi_j)$ in (3.3). During the computation, the Mittag-Leffler function is calculated with an accuracy of 10^{-6}.

To estimate the error of the numerical solution, we calculate the relative root-mean-square error as follows:

$$\varepsilon(f) = \left(\frac{\sum_{i=1}^n \left((f_\mu^\delta)_i - f(x_i) \right)^2}{\sum_{i=1}^n f(x_i)^2} \right)^{\frac{1}{2}}, \tag{5.1}$$

where i is uniformly distributed over the interval [0,1] with $n = 100$. The noisy data is generated by adding a random perturbation, i.e.

$$g_1^\delta = g_1 + \delta g_1 \cdot (2 \cdot rand(size(g_1)) - 1),$$

where δ is the relative noise level.

Figures 1 shows the numerical results of the inverse source term $f(x)$ for different values of α and β with relative noise levels $\delta = 0, 1\%, 5\%, 10\%$ respectively. It can be seen that the proposed method can provide a good approximation of $f(x)$ though the precision is relatively poor at the endpoints due to the existence of singularities. Table 1 presents the relative error of the inverse source term $f(x)$ for different values of α and β. From this table, we see that a higher noise in the data leads to a smaller error in the results. Our results show that the Tikhonov regularization method based on the compound trapezoidal formula method is effective for identifying the source term in the time-space fractional diffusion equation.

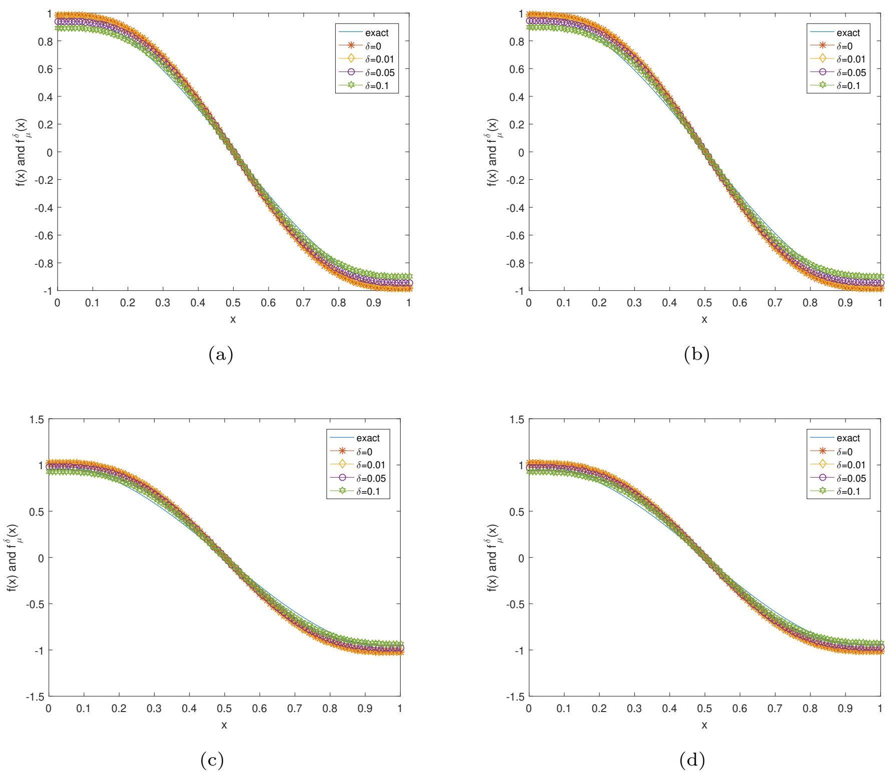

Figure 1. The numerical results of the inverse source term $f(x)$ for (a) $\alpha = 0.3, \beta = 1.2$; (b) $\alpha = 0.3, \beta = 1.8$; (c) $\alpha = 0.7, \beta = 1.2$; and (d) $\alpha = 0.7, \beta = 1.8$ with different noise levels $\delta = 0, 0.01, 0.05$ *and* 0.1

6. Concluding remarks

In this paper, we investigate an inverse problem of determining the space-dependent source term for the time-space diffusion fractional equation with initial-boundary

Table 1. The relative errors of the inverse source term $\epsilon(f)$ for different values of α and β

	$\alpha,\ \beta$ \ δ	0	0.0100	0.0500	0.1000
$\varepsilon(f)$	0.3, 1.2	0.0911	0.0842	0.0647	0.0695
	0.3, 1.8	0.0969	0.0900	0.0695	0.0709
	0.7, 1.2	0.1259	0.1172	0.0857	0.0630
	0.7, 1.8	0.1228	0.1144	0.0848	0.0660

data and additional measurement data at the final time point. We adopt the Tikhonov regularization method to obtain the regularized solution and prove the convergence rates under the priori regularization parameter selection rule and Moriozov's discrepancy principle, respectively. The numerical example shows that our proposed method is effective. For the future work, we will consider the non-uniform grid method to improve the precision at the endpoint.

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